J. Math. Soc. Japan
Vol. 22, No. 3, 1970

# Cut elimination theorem for second order arithmetic with the $\Pi_1^1$-comprehension axiom and the $\omega$-rule[1)]

By Mariko YASUGI

(Received May 19, 1969)
(Revised Dec. 8, 1969)

## Introduction.

In [1] Schütte introduced the constructive $\omega$-rule to first order arithmetic and proved the (complete) cut elimination theorem for the first order arithmetic, by translating it into a cut-free subsystem of the system with the constructive $\omega$-rule. Takeuti extended this idea in [6] and showed that second order arithmetic with the $\Pi_1^1$-comprehension axiom can be translated into a cut free subsystem of second order arithmetic with the $\Pi_1^1$-comprehension axiom and the constructive $\omega$-rule. This was done by modifying his consistency proof of the system **SINN** (cf. [5]), using the same system of ordinal diagrams.

In this article we shall prove the (complete) cut elimination theorem for second order arithmetic with the $\Pi_1^1$-comprehension axiom and the (general) $\omega$-rule, using all countable ordinals. The proof of the theorem indicates that the reduction method which is used for the consistency proof of **SINN** works for the system with an infinite rule as well, although the system of ordinal diagrams which corresponds to the latter is no longer constructive.

At the end, we remark that if we restrict the $\omega$-rule to the constructive one, then the cut elimination theorem holds within the system with the constructive $\omega$-rule.[2)]

## § 1. The formulation of the system.

In this section the system of second order arithmetic with the $\Pi_1^1$-comprehension axiom and the $\omega$-rule is formulated. It is a modification of the system **SINN** in [5] and shall be called the system $\mathfrak{W}$.

1) Part of this work was done while the author was at the University of Bristol. The work was partially supported by NSF GU-2056. The author thanks the referee for his valuable comment.
2) The author thanks Dr. J. Cleave and Professor G. Takeuti for their valuable discussions.

**1.1.** The language and the rules of inference. (cf. Chapters 1 and 2 of [**5**]). The language and the formulas of $\mathfrak{W}$ are those of **SINN**. The sequences are defined as those of **SINN** except that we admit only those sequences which do not have any occurrence of a free $t$-variable (a first order variable). If a formula or a sequence has no free $t$-variable, then it may be called '$t$-closed'.

The beginning sequences of $\mathfrak{W}$ are the $t$-closed beginning sequences of **SINN** and the rules of inference of **SINN** *except* the induction rule and the $\forall$ right rule on a $t$-variable are adopted in $\mathfrak{W}$. $\mathfrak{W}$ has also the following rule of inference, called the '$\omega$-rule':

$\omega$-rule

$$\frac{\Gamma \rightarrow \Delta,\ F(i) \qquad i<\omega}{\Gamma \rightarrow \Delta,\ \forall x F(x)}$$

where '$\Gamma \rightarrow \Delta,\ F(i)\ i<\omega$' expresses the fact that $\Gamma \rightarrow \Delta,\ F(i)$ is given for every natural number $i$. Each $\Gamma \rightarrow \Delta,\ F(i)$ is called the $i$-th upper sequence and $\Gamma \rightarrow \Delta,\ \forall x F(x)$ is called the lower sequence of an $\omega$-rule. $F(i)$ is called a subformula and $\forall x F(x)$ is called the principal formula of the rule.

Following Schütte's terminology [**1**], we shall call the inferences weakening, exchange and contraction 'weak inferences' and all others 'strong inferences'.

**1.2.** Proof-figures. The tree form proof-figure of $\mathfrak{W}$ is defined like the proof-figure of **SINN** (cf. 13.3 of Chapter 1, [**5**]), changing the concept of inferences to the one in 1.1. The concepts concerning the proof-figures of **SINN** may be translated into the concepts concerning the proof-figure of $\mathfrak{W}$ in an obvious manner. For example, an $\omega$-rule is implicit if a descendant of its principal formula is a cut formula. Also, an $\omega$-rule can be a boundary inference. A sequence is said to be $\mathfrak{W}$-provable if it is the end sequence of a proof-figure of $\mathfrak{W}$.

In the following a 'proof' or a '$\mathfrak{W}$-proof' means a proof-figure of $\mathfrak{W}$.

## § 2. Cut elimination theorem.

Our main purpose is to prove the following

THEOREM. *If a sequence $S$ is $\mathfrak{W}$-provable, then $S$ is $\mathfrak{W}$-provable without cut.*

We prove the theorem in a more generalized form.

**2.1.** The system $\mathfrak{W}'$. First we introduce a rule of inference, called 'substitution' (cf. 3.1 of Chapter 2 in [**5**]), into $\mathfrak{W}$. Substitution is a rule of inference of the form

$$\frac{A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n}{A_1\binom{V}{\alpha}, \cdots, A_m\binom{V}{\alpha} \rightarrow B_1\binom{V}{\alpha}, \cdots, B_n\binom{V}{\alpha}},$$

where $V$ is an arbitrary ($t$-closed) semi-isolated variety and is substituted for all occurrences of $\alpha$ in the sequence concerned. The definition of proof in 1.1 is changed so that substitution is allowed as a rule of inference.

Let $P$ be a proof in the present extended sense. $P$ is called a $\mathfrak{W}'$-proof if there is no logical inference (including the $\omega$-rule) in $P$ under a substitution. This implies that every substitution is in the end piece of $P$ and hence the number of substitutions in a proof is finite.

The system $\mathfrak{W}'$ is the collection of $\mathfrak{W}'$-proofs and the end sequence of a $\mathfrak{W}'$-proof is said to be $\mathfrak{W}'$-provable. It is easily seen that a $\mathfrak{W}'$-proof is a $\mathfrak{W}$-proof if and only if it has no substitution.

Substitution is redundant in $\mathfrak{W}$.

**2.2.** The $\omega$-complexity of a $\mathfrak{W}'$-proof $P$, which is given as a countable ordinal and is denoted by $\omega(P)$, is defined as follows.

1) If $P$ consists of a beginning sequence only, then $\omega(P)=0$.

2) Let $P$ be of the form $\frac{P_1}{S}$ or $\frac{P_1 P_2}{S}$. Then $\omega(P)=\omega(P_1)$ or $\omega(P)=\max(\omega(P_1), \omega(P_2))$ accordingly, where $\max(\delta_1, \delta_2)$ is the maximum of $\delta_1$ and $\delta_2$ in the sense of ordinal arithmetic.

3) Let $P$ be of the form $\frac{P_i \ i<\omega}{S}$, where '$P_i \ i<\omega$' expresses that a proof $P_i$ is given for every natural number $i$. Then $\omega(P)=\sup_{i<\omega}\omega(P_i)$, where $\sup_{i<\omega}\delta_i$ is the strict supremum of $\delta_i$ for all $i<\omega$ in the sense of ordinal arithmetic.

It is obvious that $\omega(P)=0$ if and only if $P$ contains no application of the $\omega$-rule and, if $Q$ is a subproof of $P$, then $\omega(Q)\leqq\omega(P)$.

If $Q$ is a subproof of $P$ and $S$ is the end sequence of $Q$, then $\omega(Q)$ is sometimes denoted by $\omega(S:P)$.

**2.3.** $\mathfrak{W}'_\Omega$-proofs. Let $\Omega$ be a countable (non-zero) ordinal. Let $P$ be a proof of $\mathfrak{W}'$ which satisfies $\omega(P)<\Omega$. Then $P$ is called a $\mathfrak{W}'_\Omega$-proof and the end sequence of $P$ is said to be $\mathfrak{W}'_\Omega$-provable. It is obvious that every $\mathfrak{W}'$-proof (and hence every $\mathfrak{W}$-proof) is a $\mathfrak{W}'_\Omega$-proof for some $\Omega$.

**2.4.** In order to prove our theorem (stated at the beginning of § 2), we shall first define the concept of $\mathfrak{W}'_\Omega$-proof with degree in such a manner that every $\mathfrak{W}$-proof is a $\mathfrak{W}'_\Omega$-proof with degree for some $\Omega$, and prove the following

PROPOSITION. *Let $P$ be a $\mathfrak{W}'_\Omega$-proof with degree. Then there is a cut-free $\mathfrak{W}$-proof of the end sequence of $P$.*

The theorem then follows immediately: Let $S$ be provable with a $\mathfrak{W}$-proof $P$. Then, as a special case of the above proposition, there is a cut free $\mathfrak{W}$-proof of $S$.

**2.5.** The definition of $\mathfrak{W}'_\Omega$-proofs with degree and the system of ordinal diagrams $\boldsymbol{O}(\omega+1, \Omega\times\omega^3)$, where $\Omega$ is an arbitrary, countable ordinal. The notions of $\gamma$-degree, grade, and degree are defined as in Chapter 2 of [5]. Notice that the number of free $f$-variables which are used as eigenvariables of the inferences $\forall$ right on an $f$-variable under a sequence is finite. Also, the degree of a proof is well defined since the number of substitutions in a $\mathfrak{W}'_\Omega$-proof is finite. A $\mathfrak{W}'_\Omega$-proof $P$ is called a $\mathfrak{W}'_\Omega$-proof with degree if there is a degree for $P$ which satisfies the conditions in 4 of Chapter 2 in [5].

Let $\Omega \times \omega^3$ be the cartesian product of $\Omega$ and $\omega^3$ which is ordered lexicographically. Then the system of ordinal diagrams (abbreviated by o.d.'s) $\boldsymbol{O}(\omega+1, \Omega \times \omega^3)$ is defined as in [4]. We sometimes denote it by $\boldsymbol{O}(\Omega)$. For the sake of simplicity, we call the o.d.'s of $\boldsymbol{O}(\Omega)$ simply the o.d.'s. The o.d.'s are mainly denoted by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \cdots$. The elements of $\Omega \times \omega^3$ are denoted by $[u, a]$ etc., where $u < \Omega$ and $a < \omega^3$.

An o.d. of $\boldsymbol{O}(\omega+1, \Omega \times \omega^3)$ is assigned to every $\mathfrak{W}'_\Omega$-proof with degree, as in Chapter 2 of [5]. Preceding the assignment of o.d.'s, we define $s(\boldsymbol{a})$ for every o.d. $\boldsymbol{a}$ as follows. If $\boldsymbol{a}$ is $[u, a]$, then $s(\boldsymbol{a})$ is $u$. If $\boldsymbol{a}$ is $(j\,; [u, a], \boldsymbol{b})$, then $s(\boldsymbol{a})$ is $\max(u, s(\boldsymbol{b}))$. If $\boldsymbol{a}$ is $\boldsymbol{a}_1 \# \cdots \# \boldsymbol{a}_i$, then $s(\boldsymbol{a})$ is

$$\max(s(\boldsymbol{a}_1), \cdots, s(\boldsymbol{a}_i)) .$$

Let $P$ be an arbitrary $\mathfrak{W}'_\Omega$-proof with degree. The grade of an occurrence of a formula $D$ in $P$, defined as in 2.3 of Chapter I in [5], is denoted by $g(D:P)$ (or $g(D)$ when $P$ is fixed). We first assign o.d.'s of $\boldsymbol{O}(\Omega)$ to the sequences in $P$.

1) The o.d. of a beginning sequence (in $P$) is $[0, 0]$.

2) If $S_1$ and $S_2$ are the upper sequence and the lower sequence of a weak inference, then the o.d. of $S_2$ is identical with that of $S_1$.

3) If $S$ is the lower sequence of one of the inferences $\neg$, $\wedge$ left, $\forall$ left on a $t$-variable, $\forall$ right on an $f$-variable and explicit $\forall$ left on an $f$-variable, then the o.d. of $S$ is $(\omega\,; [0, 0], \boldsymbol{a})$, where $\boldsymbol{a}$ is the o.d. of the upper sequence.

4) If $S$ is the lower sequence of an inference $\wedge$ right, then the o.d. of $S$ is $(\omega\,; [0, 0], \boldsymbol{a} \# \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the o.d.'s of upper sequences.

5) If $S$ is the lower sequence of an implicit $\forall$ left on an $f$-variable of the form

$$\frac{F(V),\ \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi),\ \Gamma \rightarrow \Delta},$$

then the o.d. of $S$ is $(\omega\,; [0, g(F(V))+2], \boldsymbol{a})$, where $\boldsymbol{a}$ is the o.d. of the upper sequence.

6) If $S$ is the lower sequence of a cut, then the o.d. of $S$ is $(\omega\,; [0, m+1], \boldsymbol{a} \# \boldsymbol{b})$, where $m$ is the grade of the cut formula, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are the o.d.'s of the upper sequences.

7) Let $S$ be the lower sequence of an $\omega$-rule, and let $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_i, \cdots,$

$i<\omega$ be the o.d.'s assigned to its upper sequences. Then the o.d. of $S$ is $(\omega; [\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0])$.

8) If $S$ is the lower sequence of a substitution with degree $i$, then the o.d. of $S$ is $(i; [0, 0], \boldsymbol{a})$, where $\boldsymbol{a}$ is the o.d. of the upper sequence.

The o.d. of a sequence $S$ in a $\mathfrak{W}'_\Omega$-proof with degree, say $P$, is denoted by $w(S:P)$ or, sometimes abbreviated to $w(S)$. The o.d. of $P$ is defined as the o.d. of the end sequence of $P$, which is sometimes denoted by $w(P)$.

**2.6.** Some consequences of the definition in 2.5. The following are obvious from the definition.

COROLLARY. 1) *Let $S$ be in a $\mathfrak{W}'_\Omega$-proof $P$. Then $\omega(S; P)=s(w(S; P))$.*

2) *Define the index elements of an o.d. as follows. $[u, a]$ has no index element; the index elements of $(j; [u, a], \boldsymbol{b})$ are $j$ and those of $\boldsymbol{b}$; the index elements of $\boldsymbol{a}_1 \# \cdots \# \boldsymbol{a}_i$ are those of $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_i$. If there is no substitution above a sequence $S$ in $P$, then all index elements of $w(S:P)$ are $\omega$.*

3) *If $S_1$ is under $S_2$ in a proof $P$, then*

$$w(S_2:P) \leqq_0 w(S_1:P)\,.$$

*$<_0$ holds if and only if there is a strong inference between $S_1$ and $S_2$.*

NOTE. Due to 1) above, we could have defined $w(S:P)$ using $\omega(Q)$ for subproofs $Q$ of $P$ instead of using $s(\boldsymbol{a})$. It is, however, more convenient to use $s(\boldsymbol{a})$ in stating and proving certain lemmas for the o.d.'s. (See below.)

$\omega(P)$ or, equivalently, $s(w(P))$ is sometimes denoted by $s(P)$.

Clause 7) of the definition in 2.5 makes sense since, by 1) of the corollary,

$$\sup_{i<\omega} s(\boldsymbol{a}_i)=\sup_{i<\omega} \omega(S_i:P)=\omega(S:P)<\Omega\,,$$

where $S_0, S_1, \cdots, S_i, \cdots$ are the upper sequences of $S$.

The following lemmas are useful for the proof of the Proposition in 2.4.

LEMMA. 1) *If there is a component of an o.d. $\boldsymbol{b}$ of the form $[u, b]$ or $(i; [u, b], \boldsymbol{d})$, then $u$ is called an outermost second element of $\boldsymbol{b}$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be o.d.'s whose index elements (if there are any) are all $\omega$. If $v$ is the maximum of the outermost second elements of $\boldsymbol{b}$ and $s(\boldsymbol{a})<v$, then $\boldsymbol{a}<_j \boldsymbol{b}$ for all $j$ ($j \leqq \omega$ or $j$ is $\infty$).*

2) *Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be o.d.'s such that there exist three finite lists of o.d.'s,*

$$\{\boldsymbol{a}_0(=\boldsymbol{a}), \boldsymbol{a}_1, \cdots, \boldsymbol{a}_m\}\,,$$

$$\{\boldsymbol{b}_0(=\boldsymbol{b}), \boldsymbol{b}_1, \cdots, \boldsymbol{b}_m\}\,,$$

$$\{\boldsymbol{c}_0(=\boldsymbol{c}), \boldsymbol{c}_1, \cdots, \boldsymbol{c}_m\}\,,$$

*satisfying the following conditions:*

(1) *$\boldsymbol{a}_i$ ($i<m$) is of one of the forms $(k; [0, a], \boldsymbol{a}_{i+1})$, $(k; [0, a], \boldsymbol{a}_{i+1} \# \boldsymbol{d})$ or $(k; [0, a], \boldsymbol{d} \# \boldsymbol{a}_{i+1})$ and $\boldsymbol{b}_i$ and $\boldsymbol{c}_i$ are of the corresponding forms, i.e. $\boldsymbol{b}_i$ is*

$(k\,;[0,a],\boldsymbol{b}_{i+1})$, $(k\,;[0,a],\boldsymbol{b}_{i+1}\#\boldsymbol{d})$ *or* $(k\,;[0,a],\boldsymbol{d}\#\boldsymbol{b}_{i+1})$, *and similarly for* $\boldsymbol{c}_i$.

(2) $\boldsymbol{c}_m$ *is of the form* $(l\,;[0,a],\boldsymbol{a}_m\#\boldsymbol{b}_m)$.

*Then* $s(\boldsymbol{c})=\max(s(\boldsymbol{a}),s(\boldsymbol{b}))$.

NOTE. We may omit $\boldsymbol{b}$ in the above definition. In that case the conclusion is $s(\boldsymbol{c})=s(\boldsymbol{a})$.

3) *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be o.d.'s such that there exist two finite lists of o.d.'s*

$$\{\boldsymbol{a}_0(=\boldsymbol{a}),\boldsymbol{a}_1,\cdots,\boldsymbol{a}_m(=\boldsymbol{c})\}$$

*and*

$$\{\boldsymbol{b}_0(=\boldsymbol{b}),\boldsymbol{b}_1,\cdots,\boldsymbol{b}_m(=\boldsymbol{c})\}$$

*satisfying the following conditions:*

$\boldsymbol{a}_i$ $(i<m)$ *is of one of the forms* $(k\,;[0,a],\boldsymbol{a}_{i+1})$, $(k\,;[0,a],\boldsymbol{a}_{i+1}\#\boldsymbol{d})$, $(k\,;[0,a],\boldsymbol{d}\#\boldsymbol{a}_{i+1})$ *or* $(k\,;[u,0],[0,0])$, *where* $u>s(\boldsymbol{a}_{i+1})$ *and* $\boldsymbol{b}_i$ *has a corresponding form, namely one of the forms* $(k\,;[0,b],\boldsymbol{b}_{i+1})$, $(k\,;[0,b],\boldsymbol{b}_{i+1}\#\boldsymbol{d})$, $(k\,;[0,b],\boldsymbol{d}\#\boldsymbol{b}_{i+1})$ *or* $(k\,;[u,0],[0,0])$, *where* $u>s(\boldsymbol{b}_{i+1})$, *and* $b\leqq a$. *Then* $\boldsymbol{b}\leqq_j\boldsymbol{a}$ *for all* $j$ *and, for every* $j<\omega$, *for every* $j$*-section of* $\boldsymbol{b}$, *say* $\boldsymbol{e}$, *there exists a* $j$*-section of* $\boldsymbol{a}$, *say* $\boldsymbol{e}'$, *such that* $\boldsymbol{e}\leqq_j\boldsymbol{e}'$.

4) (*cf. Lemma* 1 *of Appendix to* 10.1.1.2 *of* § 4 *in* [**5**].) *Let* $p$ *be any natural number, and let* $\boldsymbol{c}$ *and* $\boldsymbol{d}$ *be o.d.'s such that there exist two finite lists*

$$\{\boldsymbol{c}_0(=\boldsymbol{c}),\boldsymbol{c}_1,\cdots,\boldsymbol{c}_m\}$$

*and*

$$\{\boldsymbol{d}_0(=\boldsymbol{d}),\boldsymbol{d}_1,\cdots,\boldsymbol{d}_m\}$$

*of o.d.'s satisfying the following conditions* (1)—(4):

(1) *Every* $\boldsymbol{c}_l$ $(l<m)$ *is of one of the forms* $(k\,;[0,0],\boldsymbol{c}_{l+1})$, *where* $k\geqq p$, $(\omega\,;[0,a+1],\boldsymbol{c}_{l+1}\#\boldsymbol{e})$, *or* $(\omega\,;[0,a+1],\boldsymbol{e}\#\boldsymbol{c}_{l+1})$.

(2) *Every* $\boldsymbol{d}_l$ $(l<m)$ *is* $(k\,;[0,0],\boldsymbol{d}_{l+1})$ *or* $(\omega\,;[0,a+1],\boldsymbol{d}_{l+1}\#\boldsymbol{e})$ *or* $(\omega\,;[0,a+1],\boldsymbol{e}\#\boldsymbol{d}_{l+1})$ *according as* $\boldsymbol{c}_l$ *is* $(k\,;[0,0],\boldsymbol{c}_{l+1})$, *or* $(\omega\,;[0,a+1],\boldsymbol{c}_{l+1}\#\boldsymbol{e})$ *or* $(\omega\,;[0,a+1],\boldsymbol{e}\#\boldsymbol{c}_{l+1})$.

(3) $\boldsymbol{d}_m<_j\boldsymbol{c}_m$ *for any* $j$ *such that* $p\leqq j\leqq\omega$.

(4) *For any* $j$ *such that* $p\leqq j<\omega$, *and for any* $j$*-section* $\boldsymbol{a}$ *of* $\boldsymbol{d}_m$, *there exists a* $j$*-section* $\boldsymbol{b}$ *of* $\boldsymbol{c}_m$ *such that* $\boldsymbol{a}\leqq_j\boldsymbol{b}$.

*Then,* $\boldsymbol{d}<_j\boldsymbol{c}$ *for any* $j$ *such that* $p\leqq j\leqq\omega$: *and for any* $j$ *such that* $p\leqq j<\omega$, *and for any* $j$*-section* $\boldsymbol{a}$ *of* $\boldsymbol{d}$, *there exists a* $j$*-section* $\boldsymbol{b}$ *of* $\boldsymbol{c}$ *such that* $\boldsymbol{a}\leqq_j\boldsymbol{b}$.

PROOF. 1) The proof is by induction on $m(\boldsymbol{a},\boldsymbol{b})$, where $m(\boldsymbol{a},\boldsymbol{b})$ is the sum of the numbers of ( )'s and #'s in $\boldsymbol{a}$ and $\boldsymbol{b}$.

1°. $m(\boldsymbol{a},\boldsymbol{b})=0$. Let $\boldsymbol{a}$ be $[u,a]$ and $\boldsymbol{b}$ be $[v,b]$. Then $u<v$ by hypothesis. Therefore $\boldsymbol{a}<_j\boldsymbol{b}$ for all $j$ (by definition). Suppose $m(\boldsymbol{a},\boldsymbol{b})>0$.

2°. $\boldsymbol{a}$ is $(\omega\,;[u,a],\boldsymbol{c})$ and $\boldsymbol{b}$ is $[v,b]$.

2.1°. $\boldsymbol{a}<_\infty\boldsymbol{b}$ if and only if $[u,a]<[v,b]$. But from the hypothesis $u<v$.

2.2°. $\boldsymbol{a}<_\omega\boldsymbol{b}$ if $\boldsymbol{c}<_\omega[v,b]$ and $\boldsymbol{a}<_\infty\boldsymbol{b}$. The latter is true from 2.1° and

$\boldsymbol{c} <_\omega [v, b]$ since $s(\boldsymbol{a}) < v$ implies $s(\boldsymbol{c}) < v$ and, as $m(\boldsymbol{c}, \boldsymbol{b}) < m(\boldsymbol{a}, \boldsymbol{b})$, the inductive hypothesis holds.

2.3°. $j < \omega$. Since all index elements of $\boldsymbol{a}$ and $\boldsymbol{b}$ are $\omega$, there is no $j$-section in either $\boldsymbol{a}$ or $\boldsymbol{b}$ if $j < \omega$. Therefore, $\boldsymbol{a} <_j \boldsymbol{b}$ if $\boldsymbol{a} <_\omega \boldsymbol{b}$, which is 2.2°.

3°. $\boldsymbol{a}$ is $[u, a]$ and $\boldsymbol{b}$ is $(\omega; [v, b], \boldsymbol{d})$. Similarly.

4°. $\boldsymbol{a}$ is $(\omega; [u, a], \boldsymbol{c})$ and $\boldsymbol{b}$ is $(\omega; [v, b], \boldsymbol{d})$.

4.1°. $\boldsymbol{a} <_\infty \boldsymbol{b}$ since $[u, a] < [v, b]$.

4.2°. $\boldsymbol{a} <_\omega \boldsymbol{b}$ if $\boldsymbol{c} <_\omega \boldsymbol{b}$ and $\boldsymbol{a} <_\infty \boldsymbol{b}$. The latter is 4.1° and $\boldsymbol{c} <_\omega \boldsymbol{b}$ holds since $s(\boldsymbol{a}) < v$ implies $s(\boldsymbol{c}) < v$ and, as $m(\boldsymbol{c}, \boldsymbol{b}) < m(\boldsymbol{a}, \boldsymbol{b})$, the inductive hypothesis holds.

4.3°. $\boldsymbol{a} <_j \boldsymbol{b}$ for $j < \omega$ from 4.2°.

5°. $\boldsymbol{a}$ or $\boldsymbol{b}$ is of form $\boldsymbol{a}_1 \# \cdots \# \boldsymbol{a}_k$, $k > 1$. Obvious from the inductive hypothesis.

The proof of 2) is omitted.

3) We prove the following for every $i \leqq m$ by induction on $m-i$:

(*) $\boldsymbol{b}_i \leqq_j \boldsymbol{a}_i$ for all $j$ and, for every $j < \omega$, for every $j$-section of $\boldsymbol{b}_i$, say $\boldsymbol{d}$, there exists a $j$-section of $\boldsymbol{a}_i$, say $\boldsymbol{d}'$, such that $\boldsymbol{d} \leqq_j \boldsymbol{d}'$.

1°. $i = m$. Both $\boldsymbol{a}_m$ and $\boldsymbol{b}_m$ are $\boldsymbol{c}$. So (*) trivially holds.

2°. Assume (*) for $i+1$. As an example, take the case where $\boldsymbol{a}_i$ is $(k; [0, a], \boldsymbol{a}_{i+1} \# \boldsymbol{d})$ and $\boldsymbol{b}_i$ is $(k; [0, b], \boldsymbol{b}_{i+1} \# \boldsymbol{d})$.

2.1°. $a = b$ and $\boldsymbol{a}_{i+1} = \boldsymbol{b}_{i+1}$. Then $\boldsymbol{a}_i = \boldsymbol{b}_i$ and the second part of (*) follows from a property of the general theory of o.d.'s.

2.2°. $b < a$. $\boldsymbol{b}_i <_\infty \boldsymbol{a}_i$ since $b < a$.

(1) $k = \omega$. $\boldsymbol{b}_i <_\omega \boldsymbol{a}_i$ since $\boldsymbol{b}_{i+1} \# \boldsymbol{d} \leqq_\omega \boldsymbol{a}_{i+1} \# \boldsymbol{d}$ (by the inductive hypothesis) $<_\omega \boldsymbol{a}_i$ (an $\omega$-section), and $\boldsymbol{b}_i <_\infty \boldsymbol{a}_i$.

Suppose $j < \omega$. If $\boldsymbol{e}$ is a $j$-section of $\boldsymbol{b}_i$, then $\boldsymbol{e}$ is either a $j$-section of $\boldsymbol{b}_{i+1}$ or $\boldsymbol{d}$. If $\boldsymbol{e}$ is a $j$-section of $\boldsymbol{d}$, then $\boldsymbol{e}$ is a $j$-section of $\boldsymbol{a}_i$. If $\boldsymbol{e}$ is a $j$-section of $\boldsymbol{b}_{i+1}$, then by the inductive hypothesis there is a $j$-section of $\boldsymbol{a}_{i+1}$, say $\boldsymbol{e}'$, such that $\boldsymbol{e} \leqq_j \boldsymbol{e}'$, $\boldsymbol{e}' <_j \boldsymbol{a}_i$ and so, $\boldsymbol{e} <_j \boldsymbol{a}_i$. Let $j_0$ be the least $l$ such that $l > j$ and $l$ is an index of $\boldsymbol{b}_i$ and/or $\boldsymbol{a}_i$. Then $\boldsymbol{b}_i <_{j_0} \boldsymbol{a}_i$ by the inductive hypothesis. Therefore $\boldsymbol{b}_i <_j \boldsymbol{a}_i$.

(2) $k < \omega$. For $j > k$, $\boldsymbol{b}_i <_j \boldsymbol{a}_i$ since $\boldsymbol{b}_i <_\infty \boldsymbol{a}_i$. $\boldsymbol{b}_i <_k \boldsymbol{a}_i$ since $\boldsymbol{b}_{i+1} \# \boldsymbol{d} \leqq_k \boldsymbol{a}_{i+1} \# \boldsymbol{d}$ (by the inductive hypothesis) $<_k \boldsymbol{a}_i$ ($k$-section), and $\boldsymbol{b}_i <_\infty \boldsymbol{a}_i$. $\boldsymbol{b}_{i+1} \# \boldsymbol{d}$ is the only $k$-section of $\boldsymbol{b}_i$ and $\boldsymbol{a}_{i+1} \# \boldsymbol{d}$ is a $k$-section of $\boldsymbol{a}_i$. For $j < k$, the argument in 1) for $j < \omega$ goes through.

4) See the proof of Lemma 1 of Appendix to 10.1.1.2 of § 4 in [**5**].

**2.7.** Proof of proposition in 2.4. The proposition is proved by transfinite induction on the o.d.'s of $\mathfrak{W}'_{\Omega}$-proofs along the ordering $<_0$ of o.d.'s (cf. 3) of Corollary in 2.6.). We more or less follow the consistency proof of Chapter 2 of [**5**]. Hence, we shall demonstrate the detailed proofs only for a few cases.

First we introduce another rule of inference, 'term replacement', in $\mathfrak{W}'_{\Omega}$-proofs. (cf. 8.1 of Chapter 2 in [5].)

The o.d.'s of the upper sequence and the lower sequence of a term replacement are identical. A term replacement is redundant in $\mathfrak{W}'_{\Omega}$.

In the following, an o.d. which is placed above a sequence denotes the o.d. of that sequence in the proof under consideration.

1°. There is an explicit logical inference in the end piece of $P$. Let $I$ be a last such inference.

1.1°. $I$ is an $\omega$-rule. Let $P$ be of the form

$$\begin{array}{l} \qquad\qquad \vdots \\ \qquad \Gamma \xrightarrow{\boldsymbol{a}_i} \theta,\ F(i) \quad i<\omega \\ I\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\ \qquad (\omega\,;[\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0]) \\ \qquad \Gamma \longrightarrow \theta,\ \forall x F(x) \\ \qquad\qquad \vdots \\ \qquad \Pi \xrightarrow{\boldsymbol{b}} \Lambda, \end{array}$$

where $\Lambda$ contains $\forall x\tilde{F}(x)$. ($\tilde{A}$ is either $A$ itself or is obtained from $A$ by one or more substitutions.) Define $P_i$ for each $i<\omega$, copying $P$, as follows.

$$\begin{array}{ll} \qquad \vdots & \\ \Gamma \xrightarrow{\boldsymbol{a}_i} \theta,\ F(i) & \\ \hline\hline \Gamma \longrightarrow F(i),\ \theta,\ \forall x F(x) & \text{weakening, exchange} \\ \qquad \vdots & \\ \Pi \xrightarrow{\boldsymbol{c}_i} \tilde{F}(i),\ \Lambda. & \end{array}$$

To each substitution in $P_i$ the same degree as to the corresponding substitution in $P$ is assigned.

First, $\boldsymbol{a}_i <_j (\omega\,;[\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0])$ holds for all $j$ by 1) of Lemma in 2.6. (Recall that all index elements of $\boldsymbol{a}_i$, if there are any, are $\omega$: cf. Corollary 2) in 2.6.) Therefore, by letting $\boldsymbol{a}_i$ and $(\omega\,;[\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0])$ be $\boldsymbol{d}_m$ and $\boldsymbol{c}_m$ respectively, and $\boldsymbol{c}_i$ and $\boldsymbol{b}$ be $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively, (1)-(4) in 4) of Lemma in

2.6 hold. (There is no $j$-section of $\boldsymbol{a}_i$ if $j<\omega$.) Thus $\boldsymbol{c}_i <_0 \boldsymbol{b}$ from 4) of the Lemma in 2.6, and hence, by the induction hypothesis, there is a cut-free $\mathfrak{W}$-proof $P_i'$ of $\Pi \to \Lambda, \tilde{F}(i)$. Define $P'$ as

$$I' \quad \frac{\dfrac{P_i' \qquad i<\omega}{\Pi \longrightarrow \Lambda, \forall x \tilde{F}(x)}}{\Pi \longrightarrow \Lambda .} \text{ exchange, contraction}$$

Since no substitution and no cut are introduced, $P'$ is a cut free $\mathfrak{W}$-proof.

1.2°. $I$ is $\forall$ left on an $f$-variable. Let $P$ be of the form

$$\begin{array}{c} \vdots \\ I \quad \dfrac{F(V),\ \Gamma \xrightarrow{\boldsymbol{a}} \theta}{\forall \varphi F(\varphi),\ \Gamma \xrightarrow{(\omega\,;\,[0,\,0],\,\boldsymbol{a})} \theta} \\ \vdots \\ \Pi \xrightarrow{\boldsymbol{b}} \Lambda . \end{array}$$

Define $Q$ from $P$:

$$\begin{array}{c} \vdots \\ \dfrac{F(V),\ \Gamma \xrightarrow{\boldsymbol{a}} \theta}{\forall \varphi F(\varphi),\ \Gamma,\ F(V) \longrightarrow \theta} \\ \vdots \\ \Pi,\ F(V) \xrightarrow{\boldsymbol{c}} \Lambda . \end{array}$$

Since $\boldsymbol{a} <_j (\omega\,;\,[0,0],\boldsymbol{a})$ for any $j \leqq \omega$ and there is no $j$-section of $\boldsymbol{a}$ if $j<\omega$, the conditions in 4) of the Lemma in 2.6 hold for $\boldsymbol{a}$, $\boldsymbol{c}$, $(\omega\,;\,[0,0],\boldsymbol{a})$ and $\boldsymbol{b}$. Therefore $\boldsymbol{c} <_0 \boldsymbol{b}$, and hence, by the induction hypothesis, there is a cut-free $\mathfrak{W}$-proof of $\Pi, F(V) \to \Lambda$, say $Q'$. Define $P'$ as

$$\frac{\dfrac{Q'}{\forall \varphi F(\varphi),\ \Pi \longrightarrow \Lambda}}{\Pi \longrightarrow \Lambda}$$

which is a cut-free $\mathfrak{W}$-proof.

1.3°. $I$ is $\forall$ right on an $f$-variable. Similarly to 1.2°. Use 3) of Lemma in 2.6.

2°. The case where there is no explicit logical inference in the end piece of $P$ but there is an equality axiom as a beginning sequence in the end piece of $P$. The reduction for this case is carried out like that in 8.4 of Chapter II in [**5**].

3°. The case where there is no explicit logical inference and no equality axiom in the end piece of $P$, but there is a logical beginning sequence in the end piece of $P$. The reduction is carried out as in 8.5 in [**5**].

4°. Elimination of weakenings in the end piece of $P$. We may assume besides the conditions in 3° that the end piece of $P$ does not contain any logical beginning sequences. We can define another $\mathfrak{W}'_{\Omega}$-proof with degree, say $P^*$, eliminating weakenings in the end piece of $P$ by mathematical induction on the number of inferences in the end piece of $P$. (Note that, although $P$ may be an infinite proof, the end piece of $P$ is now finite under the above conditions.) The elimination of weakenings is carried out exactly like 8.6 in Chapter 2 of [**5**]. As a consequence, we can show that for every $j$-section $\boldsymbol{a}$ of $w(P^*)$ there is a $j$-section $\boldsymbol{b}$ of $w(P)$ such that $\boldsymbol{a} \leqq_j \boldsymbol{b}$ for $0 \leqq j < \omega$, and $w(P^*) \leqq_j w(P)$ for $0 \leqq j \leqq \omega$. In particular, $w(P^*) \leqq_0 w(P)$.

If $w(P^*) <_0 w(P)$, then apply the inductive hypothesis to $P^*$ and obtain a cut-free $\mathfrak{W}$-proof $P^{*\prime}$ of the same end sequence. P′ is defined by

$$\frac{P^{*\prime}}{\text{the end sequence of } P} \text{ weakening, exchange}$$

If $w(P^*) = w(P)$, then proceed to the next step.

5°. Essential Reduction. In the following we shall assume that the end piece of a $\mathfrak{W}'_{\Omega}$-proof with degree contains no explicit logical inferences, no logical beginning sequences and no weakenings. We may also assume that $P$ is distinct from its end piece.

The existence of a suitable cut is proved as in 9 of Chapter 2 of [**5**], since the end piece of $P$ is finite under the assumption of 5°.

Now we shall define the essential reduction according to the outermost logical symbol of the cut formula of a suitable cut. We shall find a $\mathfrak{W}'_{\Omega}$-proof with degree (say $Q$) of the end sequence of $P$ such that $w(Q) <_0 w(P)$. Then, by induction hypothesis, there is a cut-free $\mathfrak{W}$-proof $Q'$ of the end sequence of $Q$. Thus, taking $Q'$ as $P'$, we complete the proof. The reduction of $P$ to $Q$ is carried out exactly as in 10 of Chapter 2 in [**5**] except the case where the outermost logical symbol of the cut formula is $\forall$ on a $t$-variable, which shall be treated separately. The required properties on the o.d.'s are easily proved. (In applying the Lemmas in the Appendix to 10.1.1.2 in [**5**], read $[0, a]$ instead of $a$.)

The case where the outermost logical symbol is $\forall$ on a $t$-variable is treated as follows. $P$ is of the form

$$
\begin{array}{cc}
\Downarrow & \Downarrow \\
\dfrac{\Gamma_1 \xrightarrow{\boldsymbol{a}_i} \theta_1,\ F_1(i) \quad i<\omega}{\begin{array}{c}(\omega\,;\,[\sup\limits_{i<\omega} s(\boldsymbol{a}_i),\ 0],\ [0,\ 0]) \\ \Gamma_1 \longrightarrow \theta_1,\ \forall x F_1(x)\end{array}} & \dfrac{F_2(s),\ \Gamma_2 \xrightarrow{\boldsymbol{b}} \theta_2}{\begin{array}{c}(\omega\,;\,[0,\ 0],\ \boldsymbol{b}) \\ \forall x F_2(x),\ \Gamma_2 \longrightarrow \theta_2\end{array}} \\
\Downarrow & \Downarrow \\
\Pi_1 \xrightarrow{\boldsymbol{c}} \Lambda_1,\ \forall x F(x) & \forall x F(x),\ \Pi_2 \xrightarrow{\boldsymbol{d}} \Lambda_2 \\
\hline
\multicolumn{2}{c}{(\omega\,;\,[0,\ g(\forall x F(x))+1],\ \boldsymbol{c}\sharp\boldsymbol{d})} \\
\multicolumn{2}{c}{\Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2} \\
\multicolumn{2}{c}{\Downarrow} \\
\multicolumn{2}{c}{\Xi \xrightarrow{\boldsymbol{e}} \Delta\,.}
\end{array}
$$

There is an $i$ such that $s=i$ is true. Define $P_1$ and $P_2$ as follows, and then $Q$ is defined in terms of $P_1$ and $P_2$. In the following two figures $P_1$ and $P_2$, the o.d.'s above the sequences are relative to $Q$.

$P_1$:

$$
\begin{array}{cc}
\Downarrow & \\
\Gamma_1 \xrightarrow{\boldsymbol{a}_i} \theta_1,\ F_1(i) & \\
\hline\hline
\Gamma_1 \xrightarrow{\boldsymbol{a}_i} F_1(i),\ \theta_1,\ \forall x F_1(x) & \\
\Downarrow & \Downarrow \\
\Pi_1 \xrightarrow{\boldsymbol{c}'} F(i),\ \Lambda_1,\ \forall x F(x) & \forall x F(x),\ \Pi_2 \xrightarrow{\boldsymbol{d}} \Lambda_2 \\
\hline
\multicolumn{2}{c}{(\omega\,;\,[0,\ g(\forall x F(x))+1],\ \boldsymbol{c}'\sharp\boldsymbol{d})} \\
\multicolumn{2}{c}{\Pi_1,\ \Pi_2 \longrightarrow F(i),\ \Lambda_1,\ \Lambda_2} \\
\hline\hline
\multicolumn{2}{c}{\Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2,\ F(i)} \\
\hline
\multicolumn{2}{c}{\Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2,\ F(s)}
\end{array}
\quad \text{term replacement}
$$

$P_2$ :

$$
\cfrac{
\begin{array}{c}
\vdots \\ \Pi_1 \xrightarrow{\ \boldsymbol{c}\ } \Lambda_1,\ \forall x F(x)
\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\dfrac{F_2(s),\ \Gamma_2 \xrightarrow{\ \boldsymbol{b}\ } \theta_2}{\dfrac{}{\forall x F_2(x),\ \Gamma_2,\ F_2(s) \xrightarrow{\ \boldsymbol{b}\ } \theta_2}} \\
\vdots \\
\forall x F(x),\ \Pi_2,\ F(s) \xrightarrow{\ \boldsymbol{d}'\ } \Lambda_2
\end{array}
}{
\begin{array}{c}
(\omega\,;[0,\ g(\forall x F(x))+1],\ \boldsymbol{c} \,\#\, \boldsymbol{d}') \\
\dfrac{\Pi_1,\ \Pi_2,\ F(s) \longrightarrow \Lambda_1,\ \Lambda_2}{\dfrac{}{F(s),\ \Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2}}
\end{array}
}
$$

$Q$ :

$$
\begin{array}{c}
\dfrac{P_1 P_2}{\Pi_1,\ \Pi_2,\ \Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2,\ \Lambda_1,\ \Lambda_2}\ \text{cut} \\
\dfrac{}{\Pi_1,\ \Pi_2 \longrightarrow \Lambda_1,\ \Lambda_2} \\
\vdots \\
\Xi \xrightarrow{\ \boldsymbol{e}'\ } \Delta\,.
\end{array}
$$

Every substitution in $Q$ is given the same degree as the degree of the corresponding substitution in $P$.

The proof of $\boldsymbol{e}' <_0 \boldsymbol{e}$ goes as follows. Let us call the sequence $\Pi_1, \Pi_2 \to \Lambda_1, \Lambda_2$ in $P$ $S_1$, and the sequence $\Pi_1, \Pi_2, \Pi_1, \Pi_2 \to \Lambda_1, \Lambda_2, \Lambda_1, \Lambda_2$ in $Q$ $S_2$. $\boldsymbol{a}_i <_j (\omega : [\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0])$ for all $0 \leqq j \leqq \omega$ by 1) of the Lemma in 2.6. (Recall that all index elements of $\boldsymbol{a}_i$ are $\omega$, as there is no substitution above $\Gamma_1 \to \theta_1, F_1(i)$ in $P$. cf. Corollary 2) in 2.6.) Therefore $\boldsymbol{a}_i$ and $(\omega : [\sup_{i<\omega} s(\boldsymbol{a}_i), 0], [0, 0])$ satisfy the condition for $\boldsymbol{d}_m$ and $\boldsymbol{c}_m$ in 4) of the Lemma in 2.6. ((4) holds trivially, since $\boldsymbol{a}_i$ has no $j$-section if $j < \omega$.) Hence $\boldsymbol{c}' <_j \boldsymbol{c}$ for $0 \leqq j \leqq \omega$ and, for every $j$-section of $\boldsymbol{c}'$, say $\boldsymbol{f}$, where $0 \leqq j < \omega$, there is a $j$-section of $\boldsymbol{c}$, say $\boldsymbol{g}$, such that $\boldsymbol{f} \leqq_j \boldsymbol{g}$. Thus

$$(\omega\,;[0,\ g(\forall x F(x))+1],\ \boldsymbol{c}' \,\#\, \boldsymbol{d}) <_j (\omega\,;[0,\ g(\forall x F(x))+1],\ \boldsymbol{c} \,\#\, \boldsymbol{d})$$

for all $0 \leqq j \leqq \omega$ and, for $0 \leqq j < \omega$, for every $j$-section of $(\omega\,;[0,\ g(\forall x F(x))+1],$

$\boldsymbol{c}'\sharp\boldsymbol{d}$), say $\boldsymbol{f}$, there is a $j$-section of $(\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}\sharp\boldsymbol{d})$, say $\boldsymbol{g}$, such that $\boldsymbol{f}\leqq_j\boldsymbol{g}$.

By the definition of $Q$,

$$w(S_2:Q)=$$
$$(\omega; [0, g(F(s))+1], (\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}'\sharp\boldsymbol{d})\sharp(\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}\sharp\boldsymbol{d}')),$$

while $w(S_1:P)=(\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}\sharp\boldsymbol{d})$. $w(S_2:Q)<_\infty w(S_1:P)$ is obvious, since $g(F(s))<g(\forall xF(x))$. (There is no $\forall$ right on an $f$-variable under those sequences in either $P$ or $Q$.) $w(S_2:Q)<_\omega w(S_1:P)$, since each component of the $\omega$-section of $w(S_2:Q)$, say $\boldsymbol{f}$, satisfies $\boldsymbol{f}<_\omega w(S_7:P)$ from above and $w(S_2:Q)<_\infty w(S_1:P)$. Suppose $0\leqq j<\omega$. If $\boldsymbol{f}$ is a $j$-section of $w(S_2:Q)$, then it is a $j$-section of $(\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}'\sharp\boldsymbol{d})$ or of $(\omega; [0, g(\forall xF(x))+1], \boldsymbol{c}\sharp\boldsymbol{d}')$. In any case, there is a $j$-section of $w(S_1:P)$, say $\boldsymbol{g}$, such that $\boldsymbol{f}\leqq_j\boldsymbol{g}$. Therefore, $\boldsymbol{f}<_j w(S_1:P)$. So by the induction hypothesis, $w(S_2:Q)<_j w(S_1:P)$. Therefore, by 4) of the Lemma in 2.6, $\boldsymbol{e}'<_0\boldsymbol{e}$.

## § 3. Remark on the system with the constructive $\omega$-rule.[3)]

**3.1.** The definitions of the system and $\omega$-complexity. The system of second order arithmetic with the $\Pi_1^1$-comprehension axiom and the constructive $\omega$-rule is defined by an inductive definition in terms of Gödel numbering (see [2] and [6]). We shall call this system $\mathfrak{Z}$ (which is actually a set of numbers). In particular, the constructive $\omega$-rule is described as follows.

Let $e$ be (Gödel number of) a recursive function such that $\{e\}(i)$ gives a proof of a sequence of the form $\Gamma\to\theta$, $F(i)$ for every $i$. Then we may conclude $\Gamma\to\theta$, $\forall xF(x)$.

We shall use the notations $\ulcorner A\urcorner$, $\ulcorner P\urcorner$ etc. in order to denote the concepts of a formula $A$, a proof $P$, etc., though actually we have only the numbers.

The $\omega$-complexity of a proof of $\mathfrak{Z}$, say $\ulcorner P\urcorner$, is defined as in 2.2, and it is easily shown that $\omega(\ulcorner P\urcorner)<\omega_1$ for every proof $\ulcorner P\urcorner$ of $\mathfrak{Z}$, where $\omega_1$ is the first non-constructive ordinal. Thus, for the $\Omega$ in 2.3, we only have to consider $\Omega<\omega_1$. In fact we can give the $\omega$-complexities in the set $\boldsymbol{O}_1$ (a linearly ordered subset of the set $\boldsymbol{O}$ of constructive ordinals which has the order type $\omega_1$).[4)]

The subsystem of $\mathfrak{Z}$ which consists of all the proofs $\ulcorner P\urcorner$ such that $\omega(\ulcorner P\urcorner)<_{\boldsymbol{O}}\Omega$ for an $\Omega$ in $\boldsymbol{O}_1$ is denoted by $\mathfrak{Z}_\Omega$, where $<_{\boldsymbol{O}}$ is the ordering of $\boldsymbol{O}$.

3) It should be noted that, like the case of first order arithmetic (cf. [2]), the constructive $\omega$-rule is adequate for any second order arithmetic. This has been proved by Takahashi in [3]. Hence, mathematically, it suffices to deal with the system with the constructive $\omega$-rule.

4) In fact, the length of any proof in $\mathfrak{Z}$ is less than $\omega_1$; more precisely, it can be defined in $\boldsymbol{O}_1$.

**3.2.** We may extend $\mathfrak{Z}$ so that 'substitution' is allowed as a rule of inference. The condition on the degree is recursive since the number of substitutions in a proof is finite (cf. 2.5.). Thus we can define the set of proofs with degree, say $\mathfrak{Z}'$, as in 2.5. The grade of a formula $\ulcorner A \urcorner$ in a $\mathfrak{Z}'$-proof $\ulcorner P \urcorner$ is defined as a recursive function of $\ulcorner A \urcorner$ and $\ulcorner P \urcorner$. It is easily shown as before that $\mathfrak{Z}$ is a subset of $\mathfrak{Z}'$.

**3.3.** The concept of 'a cut-free proof of $\mathfrak{Z}$' is defined in an obvious manner.

LEMMA. *There exists a partial recursive function $f$ such that $f$ is defined for all proofs with degree (of $\mathfrak{Z}'$) and, if $\ulcorner P \urcorner$ is a member of $\mathfrak{Z}'$, then $f(\ulcorner P \urcorner)$ is a cut-free $\mathfrak{Z}$-proof of the end sequence of $\ulcorner P \urcorner$.*

From the lemma follows the

THEOREM. (*Cut Elimination Theorem*). *If a sequence is $\mathfrak{Z}$-provable, then it is $\mathfrak{Z}$-provable without cut.*

We only outline the proof of the lemma.

**3.4.** The function $f$ is defined by examining the reductions which are carried out in 2.7. Let $q(e, p)$ be a partial recursive function of $e$ and $p$ such that if $e$ actually gives the function $f$ and $p$ denotes a proof of $\mathfrak{Z}'$, then $q(e, p)$ gives the result of the reduction.

The crucial cases are 1.1° and 5° (of 2.7; the cases where the outermost logical symbols are $\forall$ on a $t$-variable). For 1.1° $q(e, p)$ is expressed as $\xi(r(e, p), p)$, where $r(e, p)$ corresponds to a recursive function which produces the cut-free proof of $\Pi \to F(i), \Lambda$ for every $i$ and $\xi$ is a recursive function (cf. 1.1° of 2.7). For 5°, $q(e, p)$ is expressed as $\{e\}(\tau_i(p))$, where $\tau_i(p)$ corresponds to the $Q$ in 5° of 2.7 and $i$ can be found recursively from $p$.

Thus, by recursion theorem, there is a number $e_0$ such that

$$\{e_0\}(p) \simeq q(e_0, p) .$$

The partial recursive function which is represented by $e_0$ shall be called $f$.

**3.5.** We may define the system of o.d.'s $\boldsymbol{O}(\omega+1, \boldsymbol{O}_1 \times \omega^3)$ and the well orderings $<_j$ for $j \leqq \omega$ and $<_\infty$, where $\boldsymbol{O}_1 \times \omega^3$ is ordered lexicographically. If $p$ is in $\mathfrak{Z}'$, then we can assign an o.d. of the above system to $p$, say $w(p)$, as in 2.5 in terms of the degree and the grade (cf. 3.2). We can then prove the lemma in 3.3 for the function $f$, which has been defined in 3.4, by transfinite induction on $w(p)$ along $<_0$ of the above system. The computation on the o.d.'s is carried out as in 2.7, using the lemmas in 2.6. We shall only remark that $r(e_0, p)$ indeed represents the required recursive function, for: let $\eta$ be a recursive function such that $\eta(i, \ulcorner P \urcorner) = \ulcorner P_i \urcorner$ in 1.1°. Then $r(e_0, \ulcorner P \urcorner)$ is defined as $\Lambda i(\{e_0\}(\eta(i, \ulcorner P \urcorner)))$ where $\Lambda i(\{e_0\}(\eta(i, \ulcorner P \urcorner)))$ represents the Gödel number of a function of $i$ whose value is $\{e_0\}(\eta(i, \ulcorner P \urcorner))$ for each $i$. On the other hand,

$w(\eta(i, \ulcorner P \urcorner)) <_0 w(\ulcorner P \urcorner)$ holds, and hence $\{e_0\}(\eta(i, \ulcorner P \urcorner))$ is defined for every $i$ by the induction hypothesis.

**3.6.** A translation of the system **SINN**. The system **SINN** is translated into $\mathfrak{Z}_{\tilde{\omega}}$, where $\tilde{\omega}$ is the notation for $\omega$ in $\boldsymbol{O}_1$.

PROPOSITION. *Let $S$ be a t-closed sequence (of* **SINN**). *If $S$ is* **SINN**-*provable, then $S$ is $\mathfrak{Z}_{\tilde{\omega}}$-provable.*

PROOF. A proof-figure of **SINN** is called regular if it satisfies the following conditions: all eigen variables are distinct from one another and, if a variable $a$ ($\alpha$) is the eigen variable of a $\forall$ right on a $t$-variable ($f$-variable), say $I$, then $a$ ($\alpha$) does not occur under $I$ or in any string which does not contain the upper sequence of $I$. It suffices to prove the proposition for regular proofs (of **SINN**).

Let $P$ be a proof-figure of **SINN**. Let $\pi(S; P)$ and $\pi(P)$ be defined as follows. If $S$ is a beginning sequence in $P$, then $\pi(S; P) = 1$. If $S$ is the lower sequence of a $\forall$ right on a $t$-variable, and $S_1$ is its upper sequence, then

$$\pi(S; P) = \pi(S_1; P) + 1 .$$

If $S$ is the lower sequence of other inferences, then $\pi(S; P) = \pi(S_1; P)$ or $= \max(\pi(S_1; P), \pi(S_2; P))$ respectively, where $S_1$ and $S_2$ are upper sequences. $\pi(P)$ is defined as $\pi$(the end sequence of $P; P$). ($\pi(P) < \omega$ is obvious.)

Now we shall prove the proposition in a stricter form:

(*) Let $P(b_1, \cdots, b_k)$ be an arbitrary regular proof-figure of **SINN**, where $b_1, \cdots, b_k$ in $P$ indicate all occurrences of free $t$-variables in $P$ which are not used as eigenvariables. Then there is a recursive function $\phi$ of $k$ arguments such that for an arbitrary $k$-tuple of natural numbers $i_1, \cdots, i_k$, $\phi(i_1, \cdots, i_k)$ is a $\mathfrak{Z}_{\pi(P(b_1,\cdots,b_k))}$-proof whose end sequence is (the Gödel number of) that of $P(i_1, \cdots, i_k)$, where $P(i_1, \cdots, i_k)$ is obtained from $P(b_1, \cdots, b_k)$ by replacing $b_1, \cdots, b_k$ by $i_1, \cdots, i_k$ respectively.

First we introduce the rule 'term replacement' to the system and prove (*) by mathematical induction on the number, say $l$, of two rules of inference, $\forall$ right on a $t$-variable and induction in $P$.

0) $l = 0$, i.e. $P$ has neither an induction nor a $\forall$ right on a $t$-variable. Define $\phi$ as $\phi(i_1, \cdots, i_k) = \ulcorner P(i_1, \cdots, i_k) \urcorner$ for all $(i_1, \cdots, i_k)$. It is easily seen that, for an arbitrary $(i_1, \cdots, i_k)$, $\phi(i_1, \cdots, i_k)$ is a $\mathfrak{Z}_1$-proof.

In the following $l > 0$ is assumed and, in order to simplify the notation, we shall assume $k = 1$ and denote $b_1$ and $i_1$ simply by $b$ and $i$ respectively. There are three cases.

1) There is an inference $I$ in $P$ which has two upper sequences and satisfies the following.

(a) There is neither an induction nor a $\forall$ right on a $t$-variable under $I$.

(b) Let $P$ be of the form

$$I\,\frac{P_1(b) \quad P_2(b)}{R(b)}$$

where $P_1$ and $P_2$ are subproofs of $P$ and $R$ is the part of $P$ under $I$. Then both $P_1$ and $P_2$ have either an induction or a $\forall$ right on a $t$-variable.

From (b) the number of inductions and $\forall$ rights on a $t$-variable in each of $P_1$ and $P_2$ is less than $l$, so that, by the inductive hypothesis, there are recursive functions $\phi_1(i)$ and $\phi_2(i)$ corresponding to $P_1$ and $P_2$ respectively. Let $\phi_j(i)=\ulcorner P_j'(i)\urcorner$ for $j=1, 2$. Then define $\phi(i)$ as the Gödel number of

$$\frac{P_1'(i) \quad P_2'(i)}{R(i)} \;.$$

Evidently $\phi$ is recursive. That $\phi(i)$ is a $\mathfrak{Z}_{\pi(P(b))}$-proof follows from the induction hypothesis.

2) 1) is not the case and the lowermost inference in $P$, say $I$, which is either induction or $\forall$ right on a $t$-variable, is induction. Let $P$ be of the form

$$\begin{array}{ll} Q(a, b)\left\{\begin{array}{c} \vdots \\ F(a),\, \Gamma \longrightarrow \theta,\, F(a') \end{array}\right. \\ I \quad \overline{\hphantom{F(a),\, \Gamma \longrightarrow \theta,\, F(a')}} \\ R(b)\left\{\begin{array}{c} F(0),\, \Gamma \longrightarrow \theta,\, F(s) \\ \vdots \\ S \end{array}\right. \end{array}$$

We may assume that $s$ does not contain $a$. The number of inductions and $\forall$ rights on a $t$-variable in $Q(a, b)$ is less than $l$, and hence the inductive hypothesis applies. Namely, there is a recursive function $\psi$ corresponding to $Q(a, b)$ and, for each $(n, i)$, $\psi(n, i)$ is a $\mathfrak{Z}_{\pi(Q(a,b))}$-proof whose end sequence is $\ulcorner F(n, i),\, \Gamma(i)\to\theta(i),\, F(n', i)\urcorner$, where $\Gamma(i)$ etc. is obtained from $\Gamma$ etc. by replacing $b$ by $i$, and $F(n, i)$ is an abbreviation of $F(n)(i)$. In particular, for all $i$, for a fixed $n$, this is so. Let $s^*$ be obtained from the term $s$ by replacing $b$ by $i$. As $s^*$ is closed, there is a numeral $m$ such that $s^*=m$ is true. Using the above facts and the inductive hypothesis, $\phi(i)$ is defined as the Gödel number of the reduction of a proof-figure with respect to an induction for the consistency proof (cf. 8.3 in Chapter 2 of [**5**]).

3) 1) is not the case and the lowermost such inference is a $\forall$ right on a $t$-variable. Let $P$ be of the form

$$\dfrac{Q(a,b)\left\{\begin{array}{c}\vdots\\ \Gamma \longrightarrow \theta,\ F(a)\end{array}\right.}{R(b)\left\{\begin{array}{c}\Gamma \longrightarrow \theta,\ \forall x F(x)\\ \vdots \\ S\end{array}\right.}\ .$$

The number of such inferences in $Q(a, b)$ is less than $l$, and hence the inductive hypothesis applies. Namely, there is a recursive function $\psi$ corresponding to $Q(a, b)$ and, for every $(n, i)$, $\psi(n, i)$ is a $\mathfrak{Z}_{\pi(Q(a,b))}$-proof. Let $\psi(n, i)=\ulcorner Q'(n, i)\urcorner$. The Gödel number of

$$I\ \dfrac{Q'(n,i)\left\{\begin{array}{c}\vdots\\ \Gamma(i) \longrightarrow \theta(i),\ F'(n,i)\end{array}\right. \quad n<\omega}{\Gamma(i) \longrightarrow \theta(i),\ \forall x F(x,i)}$$

is given as $3.5^{\Lambda n\psi(n,i)}.7^{\ulcorner\Gamma(i)\to\theta(i),\ \forall xF(x)\urcorner}$, where $\Gamma(i)$ etc. indicates the substitution of $i$ for $b$ in $\Gamma$, etc. This is a recursive function of $i$, which we call $\chi(i)$. $\phi(i)$ is defined in terms of $\chi$, by adding the part $R(i)$. $\phi$ is recursive and $\phi(i)$ is a $\mathfrak{Z}_{\pi(P(b))}$-proof of $\ulcorner S(i)\urcorner$.

Carnegie-Mellon University

## References

[1] K. Schütte, Beweistheorie, Springer, (1960), pp. x+355.
[2] J. Shoenfield, On a restricted $\omega$-rule, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys., **7** (1959), 405–407.
[3] M. Takahashi, A theorem on second order arithmetic with the $\omega$-rule, J. Math. Soc. Japan, **22** (1970), 15–24.
[4] G. Takeuti, Ordinal diagrams II, J. Math. Soc. Japan, **12** (1960), 385–391.
[5] G. Takeuti, Consistency proofs of subsystems of classical analysis, Ann. of Math., **86** (1967), 299–348.
[6] G. Takeuti, The $\Pi_1^1$-comprehension schema and $\omega$-rules, Lecture Notes in Mathematics, **70** (1968), Proceedings of the Summer School in Logic, pp. 303–330.