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On complex manifolds with positive tangent bundles

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§ 1. Introduction

Let E be a holomorphic vector bundle over a complex manifold M. To each point x of M we assign the complex projective space of complex 1-dimensional subspaces in the fibre E_x. Let $P(E)$ be the resulting fibre bundle over M with fibre $P_{r-1}(\boldsymbol{C})$, where r is the fibre dimension of E. To each point of $P(E)$ which is a complex line in a fibre of E, we assign that complex line. The resulting complex line bundle over $P(E)$ will be denoted by $L(E)$. In order to prevent any misunderstanding, we emphasize that E minus its zero section is the principal bundle associated to $L(E)$ and that, when M reduces to a point, $L(E)$ is a line bundle over $P_{r-1}(\boldsymbol{C})$ without any non-trivial holomorphic section.

For a complex line bundle we have a universally accepted notion of positivity or negativity. We say that a line bundle L over M is *semi-negative* and write $L \leqq 0$ if, for every proper holomorphic map π of a complex manifold Y into M and for every negative line bundle F over Y, the line bundle $\pi^*L^k \cdot F$ over Y is negative for every positive integer k.[1] In this paper, we say that a complex vector bundle E is *negative* and write $E<0$ if the line bundle $L(E)$ over $P(E)$ is negative. We say that E is *semi-negative* and write $E \leqq 0$ if $L(E)$ is *semi-negative* and if $L(E)^k \pi^* F$ is negative for every positive integer k and every negative line bundle F over M, where π denotes the projection $P(E) \to M$. We say that E is *positive* (resp. *semi-positive*) and write $E>0$ (resp. $E \geqq 0$) if its dual bundle E^* is negative (resp. semi-negative). It has been pointed out to us by P. Kiernan that E is negative in our sense if and only if the zero section of E has a strongly pseudo-convex neighborhood in E, i.e., weakly negative in the sense of Grauert [5]. But we omit in this paper the cumbersome adverb "weakly".

Combining results of Leray and Bott with Kodaira's vanishing theorems, we obtain vanishing theorems for positive or negative vector bundles. From these vanishing theorems and Riemann-Roch-Hirzebruch theorem we prove

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1) We may adopt a weaker "semi-negativity" by considering only holomorphic bundles $\pi: Y \to M$ whose fibres are complex projective spaces.

that every algebraic surface with positive tangent bundle (resp. semi-positive tangent bundle) and positive first Chern class admits at least 3 (resp. at least 2) linearly independent holomorphic vector fields and that every algebraic threefold with positive tangent bundle (resp. semi-positive tangent bundle) and positive first Chern class admits at least 6 (resp. at least 5) linearly independent holomorphic vector fields. Since a Kaehler manifold with positive holomorphic bisectional curvature has positive tangent bundle, our result shows in particular that a compact 3-dimensional Kaehler manifold with positive holomorphic bisectional curvature admits at least 6 linearly independent holomorphic vector fields, thus giving another supporting evidence to the conjecture that such a manifold must be biholomorphic to a projective space. As a matter of fact, we started this work in an attempt to prove this conjecture. As another application of our vanishing theorems and Riemann-Roch-Hirzebruch theorem, we determine with a few exceptions those complete intersection submanifolds of projective spaces which have positive or semi-positive tangent bundles. We shall prove also that no complete intersection submanifolds can admit Kaehler metrics of positive holomorphic bisectional curvature except the projective spaces themselves and possibly the quadrics of odd dimension. We have been unable to decide if a quadric of odd dimension (even in 3-dimension) admits such a metric.

Finally, we like to remark that our definition of positivity or negativity of a vector bundle is algebraic rather than differential geometric since it depends only on the notion of negativity for a line bundle and this latter can be characterized by ampleness of a certain negative power.

§ 2. Vanishing theorems

Throughout this section, let E be a holomorphic complex vector bundle over a compact complex manifold M with fibre dimension r. Let $E-0$ denote the bundle space E minus its zero section. We denote by $P(E)$ the quotient $(E-0)/\boldsymbol{C}^*$ by the multiplicative group $\boldsymbol{C}^*$ of nonzero complex numbers acting on $E-0$. Clearly, $E-0$ is a principal bundle over $P(E)$ with group $\boldsymbol{C}^*$. Let $L(E)$ be the associated complex line bundle over $P(E)$.

The following theorem relates cohomology of M with cohomology of $P(E)$.

THEOREM 2.1. *Let E and W be holomorphic complex vector bundles over M. Then*

$$H^*(M; W\otimes S^kE^*) = H^*(P(E); \pi^*W\otimes L(E)^{-k}) \qquad \textit{for } k=0, 1, 2, \cdots,$$

where S^kE^* denotes the k-th symmetric tensor power of the dual E^* of E and π^*W is the pull-back bundle of W by the projection $\pi \cdot P(E)\rightarrow M$.

We note that the fibre of S^kE^* over $x\in M$ is the space of homogeneous polynomials of degree k on E_x.

PROOF. To avoid an inessential and only technical complication, we prove this theorem for $k=1$ and for the trivial line bundle W. It is easy to see the proof for the general case from the reasoning below.

We recall the following theorem of Leray. Let $\pi: Y \to X$ be a continuous, proper map of paracompact, locally compact spaces. Let $\mathcal{F}$ be a sheaf over Y. For each open set U of X, let

$$\mathcal{H}_U^q = H^q(\pi^{-1}(U) : \mathcal{F} \mid \pi^{-1}(U)) .$$

We denote by $\mathcal{H}^q$ the sheaf over X defined by the presheaf $\mathcal{H}_U^q$.

THEOREM OF LERAY. *If $\mathcal{H}^q = 0$ except for $q = p$, then*

$$H^i(Y ; \mathcal{F}) = H^{i-p}(X ; \mathcal{H}^p) \qquad \textit{for all } i .$$

For a proof, see [**13**]. We use this theorem as follows.

$$Y = P(E), \quad X = M, \quad \pi : P(E) \to M, \quad \mathcal{F} = \Omega(L(E)^{-1}) ,$$

where Ω denotes the functor "the sheaf of germs of holomorphic sections in $\cdots$". Choose a small open polydisk U in M and let

$$\pi^{-1}(U) \approx U \times P(V) ,$$

where V is the standard fibre of E and $P(V)$ is the projective space of 1-dimensional subspaces in V, i.e., $P(V) = (V-0)/\boldsymbol{C}^*$. We denote by $L(V)$ the line bundle over $P(V)$ associated with the principal bundle $V-0$ over $P(V)$ with group $\boldsymbol{C}^*$. In other words, $L(V)$ is obtained by specializing the construction of $L(E)$ to the case where the base space M is a point. Then

$$\Omega(L(E)^{-1}) \mid \pi^{-1}U \approx \mathcal{O}_U \widehat{\otimes} \Omega(L(V)^{-1}) ,$$

where $\mathcal{O}_U$ denotes the sheaf of germs of holomorphic functions over U. By Künneth formula (see, for instance, [**7**]),

$$H^q(\pi^{-1}(U) ; L(E)^{-1} \mid \pi^{-1}(U)) \approx H^0(U ; \mathcal{O}_U) \widehat{\otimes} H^q(P(V) ; L(V)^{-1}) ,$$

since U is a domain of holomorphy and hence

$$H^i(U ; \mathcal{O}_U) = 0 \qquad \text{for } i > 0 .$$

On the other hand, $L(V)^{-1}$ is a positive line bundle over $P(V)$ and the canonical line bundle K of $P(V)$ is given by $L(V)^r$, $r = \dim V$, and is negative. Hence, $L(V)^{-1}K^{-1} = L(V)^{-r-1}$ is positive. We invoke now the following vanishing theorem of Kodaira, [**10**].

THEOREM OF KODAIRA. *If X is a compact complex manifold with canonical line bundle K and if F is a line bundle over X such that FK^{-1} is positive, then*

$$H^i(X ; F) = 0 \qquad \textit{for } i \geqq 1 .$$

If we apply this theorem to

$$X = P(V) \quad \text{and} \quad F = L(V)^{-1},$$

then we obtain

$$H^i(P(V); L(V)^{-1}) = 0 \quad \text{for } i \geqq 1.$$

On the other hand,

$$H^0(P(V); L(V)^{-1}) = V^*,$$

where V^* is the dual space of V. (To prove the theorem in its full generality, we have only to observe the fact $H^0(P(V); L(V)^{-k}) = S^k V^*$). Hence,

$$H^q(\pi^{-1}(U); L(E)^{-1}|\pi^{-1}(U)) = 0 \quad \text{for } q \geqq 1,$$

$$H^0(\pi^{-1}(U); L(E)^{-1}|\pi^{-1}(U)) = H^0(U; \mathcal{O}_U) \otimes V^*.$$

In order to conclude

$$H^*(P(E); L(E)^{-1}) = H^*(M; E^*)$$

from the theorem of Leray, we have to establish a natural isomorphism

$$f: H^0(\pi^{-1}(U); L(E)^{-1}|\pi^{-1}(U)) \longrightarrow H^0(U; E^*).$$

Let $\eta \in H^0(\pi^{-1}(U); L(E)^{-1}|\pi^{-1}(U))$. We want to define $f(\eta) \in H^0(U; E^*)$. An element of $H^0(U; E^*)$ should define, for each point x of U, a linear functional on the fibre E_x. We want to define $f(\eta)$ as a linear functional on E_x. Let $e \in E_x$. If $e = 0$, then $f(\eta) \cdot e = 0$. If $e \neq 0$, then consider the point y of $\pi^{-1}(U) \subset P(E)$ represented by e. Since e is in the fibre of $L(E)$ over y and since $\eta(y) \in L(E)^{-1}$ is a linear functional on the fibre of $L(E)$ over y, we can set

$$f(\eta) \cdot e = \langle \eta(y), e \rangle.$$

It is easy to verify that f is an isomorphism. QED.

REMARK. This is a special case of Bott's theorem VI [**2**; p. 238]. It seems that Bott's proposition 13.1 [**2**; p. 242] does not agree with the theorem just proved. He seems to assert $H^*(M, E) = H^*(P(E), L(E))$ with our notation.

In order to be able to apply Kodaira's vanishing theorem to a line bundle over $P(E)$, we have to know the canonical line bundle $K_{P(E)}$ of $P(E)$. We denote by $\det E$ the line bundle $\wedge^r E$ over M, where r is the fibre dimension of E. Then

PROPOSITION 2.2. *Let E be a holomorphic complex vector bundle over M with fibre dimension r. Let $K_{P(E)}$ and K_M be the canonical line bundles of $P(E)$ and M, respectively. Then*

$$K_{P(E)} = L(E)^r \cdot \pi^*(K_M \cdot \det E^*),$$

where $\pi: P(E) \to M$ is the projection.

PROOF. Consider the following three vector bundles over $P(E)$:

$T = T(P(E))$ = the tangent bundle of $P(E)$;

T' = the subbundle of T consisting of vectors tangent to fibres of the fibration $P(E) \to M$;

$T'' = \pi^* T(M)$ = the pull-back of $T(M)$ by π.

Their duals are denoted by T^*, T'^*, T''^*. From the exact sequence

$$0 \longrightarrow T''^* \longrightarrow T^* \longrightarrow T'^* \longrightarrow 0$$

we obtain

$$\det T^* = (\det T'^*) \cdot (\det T''^*) ,$$

which may be rewritten in the form

$$K_{P(E)} = (\det T'^*)(\pi^* K_M) .$$

To complete the proof, we have only to show that

$$\det T'^* = L(E)^r \cdot \pi^*(\det E^*) ,$$

i. e.,

$$\det T' = L(E)^{-r} \cdot \pi^*(\det E) .$$

In other words, it suffices to construct a non-degenerate dual pairing

$$\mu : \det T' \times L(E)^r \longrightarrow \pi^*(\det E) .$$

Let $u \in P(E)$ and let

$$\zeta = Z_2 \wedge \cdots \wedge Z_r \in \wedge^{r-1} T' = \det T'$$

be an element over u, where $Z_2, \cdots, Z_r \in T'$. Represent u by a nonzero element $e_1 \in E_x$, where $x = \pi(u)$. Since E_x is a vector space, we identify the tangent space $T_{e_1}(E_x)$ at e_1 with E_x itself in a natural manner. Let $e_2, \cdots, e_r$ be elements of E_x which, considered as elements in $T_{e_1}(E_x)$, are mapped onto $Z_2, \cdots, Z_r$ by the differential of the projection $E_x - 0 \to P(E)_x$. Let $\varphi \in L(E)^r$ be an element over $u \in P(E)$. Then φ is of the form

$$\varphi = a \cdot e_1 \otimes \cdots \otimes e_1 \qquad (e_1 \colon r \text{ times}), \quad a \in \boldsymbol{C} .$$

We define

$$\mu(\zeta, \varphi) = a \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_r .$$

It is straightforward to verify that $\mu(\zeta, \varphi)$ is well-defined, independent of all the choices made above. QED.

We are now in a position to prove the following vanishing theorem.

THEOREM 2.3. *Let E be a holomorphic complex vector bundle over a compact complex manifold M with fibre dimension r. Let F be a line bundle over M. Let k be a non-negative integer. If the line bundle*

$$L(E^*)^{-(r+k)} \cdot \pi^*(K_M^{-1} \cdot \det E^* \cdot F)$$

over $P(E^*)$ *is positive, then*

$$H^i(M; S^kE\otimes F)=0 \qquad \textit{for } i\geqq 1,$$

where S^kE *is the k-th symmetric tensor power of* E.

PROOF. By Theorem 2.1, we have an isomorphism

$$H^i(M; S^kE\otimes F)=H^i(P(E^*);\ L(E^*)^{-k}\cdot\pi^*F) \qquad \text{for all } i.$$

The right hand side vanishes for $i\geqq 1$ by Kodaira's vanishing theorem since

$$L(E^*)^{-k}\cdot\pi^*F\cdot K_{P(E)}^{-1}=L(E^*)^{-(r+k)}\cdot\pi^*(K_M^{-1}\cdot\det E^*\cdot F)$$

by Proposition 2.2. QED.

We shall list a few immediate consequences.

COROLLARY 2.4. *If either*

(i) $$E>0 \qquad \textit{and} \qquad K_M\cdot\det E\cdot F^{-1}\leqq 0$$

or

(ii) $$E\geqq 0 \qquad \textit{and} \qquad K_M\cdot\det E\cdot F^{-1}<0,$$

then

$$H^i(M; S^kE\otimes F)=0 \qquad \textit{for } i\geqq 1 \textit{ and } k=0, 1, 2, \cdots.$$

COROLLARY 2.5. *Let* M *be a compact complex manifold such that either*

(i) $$T(M)>0 \qquad \textit{and} \qquad F\geqq 0$$

or

(ii) $$T(M)\geqq 0 \qquad \textit{and} \qquad F>0.$$

Then

$$H^i(M; S^kT\otimes F)=0 \qquad \textit{for } i\geqq 1 \textit{ and } k=0, 1, 2, \cdots,$$

where $T=T(M)$.

We remark here that if $T(M)>0$, then $H^1(M, T(M))=0$ so that M has no infinitesimal deformations of the complex structure.

COROLLARY 2.6. *If* $L(E)^{-(r+k)}\cdot\pi^*(\det E\cdot F^{-1})>0$, *then*

$$H^j(M; S^kE\otimes F)=0 \qquad \textit{for } j\leqq n-1\ ,$$

where $n=\dim M$.

PROOF. If we replace F by $F\cdot K_M$ in Theorem 2.3, then

$$L(E^*)^{-(r+k)}\cdot\pi^*(\det E^*\cdot F)>0\Rightarrow H^i(M; S^kE\otimes F\cdot K_M)=0 \quad \text{for } i\geqq 1.$$

On the other hand, by Serre's duality theorem, we have

$$H^i(M; S^kE\otimes F\cdot K_M)\underset{\text{dual}}{\sim}H^{n-i}(M; S^kE^*\otimes F^{-1}).$$

Replacing E^* by E and F^{-1} by F, we obtain the corollary. QED.

COROLLARY 2.7. *If either*

(i) $$E<0 \quad \text{and} \quad \det E\cdot F^{-1}\geqq 0$$

or

(ii) $$E\leqq 0 \quad \text{and} \quad \det E\cdot F^{-1}>0\,,$$

then

$$H^j(M; S^kE\otimes F)=0 \qquad \text{for } j\leqq n-1 \text{ and } k=0, 1, 2, \cdots.$$

REMARK. In relation to Corollary 2.5, we mention the following

PROPOSITION 2.8. *Let M be a compact complex manifold with canonical line bundle K_M. If F is a line bundle such that $K_M^{-1}\cdot F>0$, then*

$$H^i(M; T(M)\otimes F)=0 \qquad \text{for } i\geqq 2\,.$$

PROOF. By Serre's duality theorem, we obtain (setting $T=T(M)$)

$$H^i(M; T\otimes F)\underset{\text{dual}}{\sim} H^{n-i}(M; T^*\otimes K_M\cdot F^{-1})=H^{n-i,1}(M; K_M\cdot F^{-1})\,.$$

Now our proposition follows from the following vanishing theorem of Nakano [**12**]:

If F is a negative line bundle over a compact complex manifold M, then

$$H^{p,q}(M; F)=0 \qquad p+q\leqq n-1, \ (\text{where } n=\dim M)\,.$$

QED.

§ 3. Positive vector bundles over algebraic surfaces

Let E be a holomorphic complex vector bundle over a compact complex manifold M of dimension n. Let g denote the first Chern class of the line bundle $L(E^*)^{-1}$ over $P(E^*)$. Denote by d_i the i-th Chern class of the vector bundle E. Then

$$g^r-d_1g^{r-1}+d_2g^{r-2}-\cdots+(-1)^rd_r=0\,,$$

where r is the fibre dimension of E. Using this identity, we can reduce any polynomial in g to a polynomial of degree less than r (whose coefficients are polynomials in the Chern classes d_i). In particular, we can reduce g^{n+r-1} to such a polynomial. Using the fact that the integral of g^{r-1} along a fibre of $P(E^*)$ is 1 and the integral of a term whose degree in g is less than $r-1$ along a fibre is zero, we can express $g^{n+r-1}[P(E^*)]$ in terms of Chern numbers of E.

For instance, if $\dim M=2$, then we obtain by a simple calculation

$$g^{2+r-1}[P(E^*)]=(d_1^2-d_2)[M]\,.$$

As a consequence, we obtain

THEOREM 3.1. *Let E be a holomorphic complex vector bundle over an alge-*

braic surface M. Then

(i) *If* $E>0$, *then* $(d_1^2-d_2)[M]>0$;

(ii) *If* $E\geqq 0$, *then* $(d_1^2-d_2)[M]\geqq 0$.

PROOF. (i). If $E>0$, then $L(E^*)^{-1}$ is positive so that g can be represented by a positive definite closed (1, 1)-form φ. The integral of φ^{r+1} on $P(E^*)$ is therefore positive, i.e., $g^{r+1}[P(E^*)]>0$.

(ii). Let F be a positive line bundle over M; such a bundle exists since M is algebraic. Then $L(E^*)^{-k}\cdot\pi^*F$ is a positive line bundle over $P(E^*)$; this is immediate from the definition of $E\geqq 0$. Let f denote the first Chern class of π^*F. Represent f and g by closed (1, 1)-forms ψ and φ, respectively. Since $kg+f$ is positive for every positive integer k, it follows that the integral of $(k\varphi+\psi)^{r+1}$ over $P(E^*)$ is positive. Hence, the integral of $\left(\varphi+\frac{\psi}{k}\right)^{r+1}$ over $P(E^*)$ is also positive. Letting k go to infinity, we see that the integral of φ^{r+1} is positive or zero. QED.

REMARK. Although we did not make use of the assumption that M be algebraic in (i), the assumption that E be positive implies M is algebraic.

THEOREM 3.2. *Let E be a holomorphic complex vector bundle over an algebraic surface M with fibre dimension r. Then*

(i) *If* $E>0$, $\det E>0$ *and* $K_M\cdot\det E\leqq 0$, *then* $\dim H^0(M; E)>r$;

(ii) *If* $E\geqq 0$, $\det E\geqq 0$ *and* $K_M\cdot\det E<0$, *then* $\dim H^0(M; E)\geqq r$.

PROOF. (i). From Corollary 2.4, we obtain

$$H^i(M; E)=0 \qquad \text{for } i\geqq 1$$

so that

$$H^0(M; E)=\chi(M; E).$$

For an algebraic surface M and a holomorphic vector bundle E over M, $\chi(M; E)$ is given by (see [6])

$$\chi(M; E)=\left\{\frac{1}{2}(d_1^2-2d_2)+\frac{1}{2}d_1c_1+\frac{r}{12}(c_1^2+c_2)\right\}[M],$$

which may be rewritten as follows:

$$\chi(M; E)=(d_1^2-d_2)[M]+\left\{\frac{1}{2}d_1(c_1-d_1)+\frac{r}{12}(c_1^2+c_2)\right\}[M].$$

Since $\det E>0$, its characteristic class d_1 can be represented by a positive definite closed (1, 1)-form. Since $(K_M\cdot\det E)^{-1}\geqq 0$, $(K_M\cdot\det E)^{-k}\cdot F$ is positive for every positive integer k and every positive line bundle F. Let f denote the Chern class of F. Then $k(c_1-d_1)+f$ can be represented by a positive definite closed (1, 1)-form. Hence, $d_1(k(c_1-d_1)+f)[M]>0$. Dividing the both sides by k and letting k go to infinity, we obtain

$$d_1(c_1-d_1)[M]\geqq 0\,.$$

Since $\det E>0$ and $K_M\cdot\det E\leqq 0$, it follows that $K_M<0$ so that

$$H^{i,0}(M\,;\boldsymbol{C})=0 \qquad \text{for } i>1$$

by Kodaira's vanishing theorem. This implies that arithmetic genus $\chi(M)=\frac{1}{12}(c_1^2+c_2)[M]$ is equal to 1. Hence, by Theorem 3.1, we have

$$\chi(M\,;E)\geqq(d_1^2-d_2)[M]+r>r\,.$$

(iii). The proof for (ii) is similar and hence omitted. QED.

REMARK. It is very likely that $E>0$ implies $\det E>0$ and that $E\geqq 0$ implies $\det E\geqq 0$.

THEOREM 3.3. *Let M be an algebraic surface with $K_M<0$.*

(i) *If $T(M)>0$, then $\dim H^0(M\,;T(M))>2$ and $\dim H^{1,1}(M\,;C)<4$,*

(ii) *If $T(M)\geqq 0$, then $\dim H^0(M\,;T(M))\geqq 2$ and $\dim H^{1,1}(M\,;C)\leqq 4$.*

PROOF. Since $K_M<0$, Kodaira's vanishing theorem implies $H^{i,0}(M\,;\boldsymbol{C})=0$ for $i\geqq 1$ so that

$$c_2[M]=2+\dim H^{1,1}(M\,;\boldsymbol{C})\,,$$

$$(c_1^2+c_2)[M]=12\cdot\chi(M)=12\,.$$

Hence,

$$8-2\dim H^{1,1}(M\,;\boldsymbol{C})=(c_1^2-c_2)[M]\,,$$

which is positive or non-negative according as $T(M)>0$ or $T(M)\geqq 0$ by Theorem 3.1. Since

$$\chi(M\,;T(M))=\left\{c_1^2-c_2+\frac{1}{6}(c_1^2+c_2)\right\}[M]=(c_1^2-c_2)[M]+2\,,$$

$\chi(M\,;T(M))>2$ or $\geqq 2$ according as $T(M)>0$ or $T(M)\geqq 0$. Since $H^i(M\,;T(M))=0$ for $i\geqq 2$ by Proposition 2.8, we have

$$\dim H^0(M\,;T(M))=\chi(M\,;T(M))+\dim H^1(M\,;T(M))\geqq\chi(M\,;T(M))\,.$$

Hence, $\dim H^0(M\,;T(M))>2$ or $\geqq 2$ according as $T(M)>0$ or $T(M)\geqq 0$. QED.

In Theorem 3.2, let $E=T(M)$. Then $\det E=K_M^{-1}$. So part of Theorem 3.3 may be derived from Theorem 3.2. We do not know if $T(M)>0$ implies $K_M<0$. The assumption K_M was needed only to prove $\chi(M)=1$. Note that we proved $\dim H^0(M\,;T(M))\geqq 10-2\dim H^{1,1}(M\,;\boldsymbol{C})$.

§ 4. Positive vector bundles over algebraic threefolds

Some of the results in the preceding section can be extended to algebraic threefolds. Let E be a holomorphic complex vector bundle over a compact complex manifold M of dimension 3. Let g be the first Chern class of the line bundle $L(E^*)^{-1}$ over $P(E^*)$. Let d_i be the i-th Chern class of E. Then the method described in §3 gives

$$g^{3+r-1}[P(E^*)]=(d_1^3-2d_1d_2+d_3)[M],$$

where r is the fibre dimension of E. As a consequence, we have

THEOREM 4.1. *Let E be a holomorphic complex vector bundle over a 3-dimensional compact complex manifold M. Then*

(i) *If $E>0$, then* $(d_1^3-2d_1d_2+d_3)[M]>0$;

(ii) *If $E\geqq 0$, then* $(d_1^3-2d_1d_2+d_3)[M]\geqq 0$.

The proof is similar to that of Theorem 3.1 and hence is omitted.

If M is an algebraic threefold, then Riemann-Roch-Hirzebruch theorem states

$$\chi(M;E)=\left\{\frac{1}{6}(d_1^3-3d_1d_2+3d_3)+\frac{1}{4}(d_1^2-2d_2)c_1+\frac{1}{12}(c_1^2+c_2)d_1+\frac{r}{24}c_1c_2\right\}[M].$$

Although we have been unable to extend Theorem 3.2 to threefolds, we have the following partial generalization of Theorem 3.3.

THEOREM 4.2. *Let M be an algebraic threefold with $K_M<0$. Then*

(i) *If $T(M)>0$, then* $\dim H^0(M;T(M))>5$;

(ii) *If $T(M)\geqq 0$, then* $\dim H^0(M;T(M))\geqq 5$.

PROOF. We have

$$\chi(M;T(M))=\left(\frac{1}{2}c_1^3-\frac{19}{24}c_1c_2+\frac{1}{2}c_3\right)[M]$$

$$=\left\{\frac{1}{2}(c_1^3-2c_1c_2+c_3)+\frac{5}{24}c_1c_2\right\}[M].$$

By Theorem 4.1, $\chi(M;T(M))>\frac{5}{24}c_1c_2[M]$ or $\geqq\frac{5}{24}c_1c_2[M]$ according as $T(M)>0$ or $T(M)\geqq 0$. On the other hand, $K_M<0$ implies that the arithmetic genus $\chi(M)=\frac{1}{24}c_1c_2[M]$ is equal to 1. Hence, $\chi(M;T(M))>5$ or $\geqq 5$ according as $T(M)>0$ or $T(M)\geqq 0$. By Proposition 2.8, $\dim H^0(M;T(M))=\chi(M;T(M))+\dim H^1(M;T(M))$. Hence $\dim H^0(M;T(M))>5$ or $\geqq 5$ according as $T(M)>0$ or $T(M)\geqq 0$. QED.

Again, the assumption $K_M<0$ was used only to prove that the arithmetic genus of M is 1.

We conclude this section by remarking that by computation we obtain the following formula for an algebraic threefold M:

$$\chi(M;S^kT(M))=\left(\frac{3k}{40}+\frac{3k^2}{16}+\frac{k^3}{6}+\frac{k^4}{16}+\frac{k^5}{120}\right)(c_1^3+c_3)[M]$$

$$-\left(-1+\frac{43k}{30}+\frac{15k^2}{2}+\frac{23k^3}{3}+3k^4+\frac{2k^5}{5}\right)\frac{c_1c_2}{24}[M]$$

$$=\left(\frac{3k}{40}+\frac{3k^2}{16}+\frac{k^3}{6}+\frac{k^4}{16}+\frac{k^5}{120}\right)(c_1^3-2c_1c_2+c_3)[M]$$

$$+\left(1+\frac{13k}{6}+\frac{3k^2}{2}+\frac{k^3}{3}\right)\frac{c_1c_2}{24}[M].$$

This implies:

$$\dim H^0(M; S^kT(M)) > 1+\frac{13}{6}k+\frac{3}{2}k^2+\frac{1}{3}k^3 \quad \text{if } T(M)>0 \text{ and } K_M<0.$$

§ 5. Complete intersections with (semi-) positive tangent bundle[2)]

Let M be a complete intersection of r hypersurfaces of degree $a_1, \cdots, a_r$ in $P_{n+r}(\boldsymbol{C})$ with $a_i \geqq 2$. We want to determine those M with $T(M)>0$ or $T(M) \geqq 0$. Let H denote the (positive) line bundle over $P_{n+r}(\boldsymbol{C})$ defined by a hyperplane. Since $T(M)>0$ (resp. $T(M) \geqq 0$) implies $H^1(M; T(M))=0$ (resp. $H^1(M; T(M)\otimes H)=0$), we want to first compute the cohomology groups $H^1(M; T(M))$ and $H^1(M; T(M)\otimes H)$. We denote by M^k the intersection of the first k hypersurfaces (of degree $a_1, \cdots, a_k$) used in the definition of M. In particular, $M^0=P_{n+r}(\boldsymbol{C})$ and $M^r=M$. Let N^k denote the normal bundle of M^k in $P_{n+r}(C)$. We write also N for N^r. For M^k, the superscript k coincides with its codimension in $P_{n+r}(\boldsymbol{C})$. For N^k, k coincides with its fibre dimension.

We need the following theorem of Bott, [**2**]:

$$H^q(P_n; \Omega^p(H^k))=0$$

except for the following three cases: (1) $p=q$ and $k=0$, (2) $q=0$ and $k>p$, (3) $q=n$ and $k<p-n$. ($\Omega^p(H^k)$ denotes the sheaf of germs of holomorphic p-forms with coefficients in H^k). Setting $p=1$ and using the Serre duality theorem and also setting $p=0$, we obtain

THEOREM 5.1. *For $P_n=P_n(\boldsymbol{C})$, we have*

(i) $H^i(P_n; TP_n\otimes H^m)=0$ *for* $1\leqq i\leqq n-1$, *all* m *except for the case* $i=n-1$ *and* $m=-(n+1)$;

(ii) $H^0(P_n; TP_n\otimes H^m)=0$ *for* $m\leqq -2$ *except for* $n=1$, $m=-2$;

(iii) $H^i(P_n; H^m)=0$ *for* $1\leqq i\leqq n-1$ *and all* m;

(iv) $H^0(P_n; H^m)=0$ *for* $m\leqq -1$.

REMARK. (ii) and (iv) will also follow from (ii) of Corollary 7.2. (iii) may be easily obtained from the vanishing theorem of Kodaira. Part of (i),

2) Throughout this paper, we consider only non-singular intersections of non-singular hypersurfaces.

i.e., for $n+m\geqq 0$, may be also obtained from (i) of Corollary 7.2.

From Theorem 5.1, we prove

PROPOSITION 5.2. *Let* $n\geqq 2$. *Then*

(i) $H^i(M^k; TP_{n+r}\otimes H^m)=0$ *for* $1\leqq i\leqq r-k$ *and all* m;

(ii) $H^0(M^k; TP_{n+r}\otimes H^m)=0$ *for* $m\leqq -2$;

(iii) $H^i(M^k; H^m)=0$ *for* $1\leqq i\leqq r-k+1$ *and all* m;

(iv) $H^0(M^k; H^m)=0$ *for* $m\leqq -1$.

PROOF. The proof is by induction on k. The case $k=0$ is Theorem 5.1. Assume Proposition 5.2 for $k-1$. We write P for P_{n+r}.

(i). Since H^{a_k} is the normal bundle of M^k in M^{k-1}, we have the following exact sequence

$$(1)\qquad \longrightarrow H^i(M^{k-1}; TP\otimes H^m)\longrightarrow H^i(M^k; TP\otimes H^m)\longrightarrow H^{i+1}(M^{k-1}; TP\otimes H^{m-a_k})\longrightarrow .$$

From this, we obtain (i).

(ii). In the sequence (1), let $i=0$ and use (i).

(iii). In the same way as (1), we obtain the following exact sequence:

$$(2)\qquad \longrightarrow H^i(M^{k-1}; H^m)\longrightarrow H^i(M^k; H^m)\longrightarrow H^{i+1}(M^{k-1}; H^{m-a_k})\longrightarrow .$$

From this, we obtain (iii).

(iv). In the sequence (2), let $i=0$ and use (iii). QED.

From the sequence (1) and (i) of Proposition 5.2, we obtain the following exact sequence:

$$(3)\qquad 0\longrightarrow H^0(M^{k-1}; TP\otimes H^{m-a_k})\longrightarrow H^0(M^{k-1}; TP\otimes H^m)\longrightarrow H^0(M^k; TP\otimes H^m)\longrightarrow 0.$$

Hence,

PROPOSITION 5.3.

$$\dim H^0(M; TP_{n+r}\otimes H^m)\leqq \dim H^0(P_{n+r}; TP_{n+r}\otimes H^m)\qquad \textit{for all } m.$$

From the sequence (2) and (iii) of Proposition 5.2, we obtain the following exact sequence:

$$(4)\qquad 0\longrightarrow H^0(M^{k-1}; H^{m-a_k})\longrightarrow H^0(M^{k-1}; H^m)\longrightarrow H^0(M^k; H^m)\longrightarrow 0\qquad \text{for all } m.$$

Hence,

PROPOSITION 5.4.

$$\dim H^0(M^k; H^m)=\dim H^0(M^{k-1}; H^m)-\dim H^0(M^{k-1}; H^{m-a_k}).$$

From the exact sequence

$$0 \longrightarrow H^{a_k} \longrightarrow N^k \longrightarrow N^{k-1} \longrightarrow 0$$

and from (iii) of Proposition 5.2, we obtain the following exact sequence

$$0 \longrightarrow H^0(M; H^{a_k+m}) \longrightarrow H^0(M; N^k \otimes H^m) \longrightarrow H^0(M; N^{k-1} \otimes H^m) \longrightarrow 0.$$

Hence,

$$\dim H^0(M; N^k \otimes H^m) = \dim H^0(M; N^{k-1} \otimes H^m) + \dim H^0(M; H^{a_k+m}),$$

which implies

PROPOSITION 5.5.

$$\dim H^0(M; N \otimes H^m) = \sum_{i=1}^{r} H^0(M; H^{a_i+m}) \qquad \textit{for all } m.$$

If we denote by $M \cap H$ a hyperplane section of M and if W is a holomorphic vector bundle over M, then we have the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(M; W \otimes H^{m-1}) &\longrightarrow H^0(M; W \otimes H^m) \\ &\longrightarrow H^0(M \cap H; W \otimes H^m) \longrightarrow H^1(M; W \otimes H^{m-1}) \end{aligned}$$

which is valid for any closed complex submanifold M in $P_{n+r}(\boldsymbol{C})$. Hence,

PROPOSITION 5.6. *If M is a closed complex submanifold of $P_{n+r}(\boldsymbol{C})$, and if W is a holomorphic vector bundle over M, then*

$$\dim H^0(M; W \otimes H^{m-1}) \leqq \dim H^0(M; W \otimes H^m) \qquad \textit{for all } m.$$

The following proposition is useful in computing $\dim H^0(P_n; TP_n \otimes H^m)$.

PROPOSITION 5.7. *Let $P_n = P_n(\boldsymbol{C})$. For $n+m \geqq 1$, we have*

$$\begin{aligned} \dim H^0(P_n; TP_n \otimes H^m) &= \dim H^0(P_{n-1}; TP_{n-1} \otimes H^m) \\ &\quad + \dim H^0(P_n; TP_n \otimes H^{m-1}) + \dim H^0(P_{n-1}; H^{m+1}). \end{aligned}$$

PROOF. Since P_{n-1} is a hyperplane in P_n with normal bundle H and since $H^1(P_n; TP_n \otimes H^{m-1}) = 0$ for $n+m \geqq 1$ by (i) of Theorem 5.1, we have the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(P_n; TP_n \otimes H^{m-1}) &\longrightarrow H^0(P_n; TP_n \otimes H^m) \\ &\longrightarrow H^0(P_{n-1}; TP_n \otimes H^m) \longrightarrow 0. \end{aligned}$$

On the other hand, from the exact sequence

$$0 \longrightarrow TP_{n-1} \longrightarrow (TP_n)|_{P_{n-1}} \longrightarrow H|_{P_{n-1}} \longrightarrow 0$$

and from $H^1(P_{n-1}; TP_{n-1} \otimes H^m) = 0$ (see (i) of Theorem 5.1), we obtain the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(P_{n-1}; TP_{n-1} \otimes H^m) &\longrightarrow H^0(P_{n-1}; TP_n \otimes H^m) \\ &\longrightarrow H^0(P_{n-1}; H^{m+1}) \longrightarrow 0. \end{aligned}$$

From these two cohomology exact sequences, we obtain the desired formula. QED.

From Proposition 5.7, we can compute $\dim H^0(P_n; TP_n \otimes H^m)$ inductively. In particular, we have

PROPOSITION 5.8. *Let* $P_n = P_n(\boldsymbol{C})$. *Then*

(i) $\dim H^0(P_n; TP_n) = n(n+2)$;

(ii) $\dim H^0(P_n; TP_n \otimes H) = \frac{1}{2} n(n+1)(n+3)$;

(iii) $\dim H^0(P_n; TP_n \otimes H^{-1}) = n+1$;

(iv) $\dim H^0(P_n; TP_n \otimes H^m) = 0$ *for* $m \leqq -2$ *and* $n \geqq 2$.

PROOF. (i). $H^0(P_n; TP_n)$ is the space of holomorphic vector fields on P_n and it is a well known fact that it is of dimension $n(n+2)$.

(ii). In Proposition 5.7, let $m=1$. Then use (i) and the formula $\dim H^0(P_{n-1}; H^2) = \frac{1}{2} n(n+1)$, which can be found on p. 165 in [**6**]. Then

$$\dim H^0(P_n; TP_n \otimes H) = \dim H^0(P_{n-1}; TP_{n-1} \otimes H) + \frac{1}{2} n(3n+5).$$

Hence,

$$\dim H^0(P_n; TP_n \otimes H) = \sum_{k=1}^{n} \frac{1}{2} k(3k+5) = \frac{1}{2} n(n+1)(n+3).$$

(iii). In Proposition 5.7, let $m=0$. Then use (i) and the formula $\dim H^0(P_{n-1}; H) = n$.

(iv). Although this is stated in (ii) of Theorem 5.1, we give here a direct proof. In Proposition 5.7, let $m=-1$ and use (iii). Then $H^0(P_n, TP_n \otimes H^{-2}) = 0$. The rest follows from Proposition 5.6. QED.

From the exact sequence

$$0 \longrightarrow TM \longrightarrow (TP_{n+r})|_M \longrightarrow N \longrightarrow 0,$$

we obtain the following exact sequence:

$$\begin{aligned} (4) \qquad & 0 \longrightarrow H^0(M; TM \otimes H^m) \longrightarrow H^0(M; TP_{n+r} \otimes H^m) \\ & \longrightarrow H^0(M; N \otimes H^m) \longrightarrow H^1(M; TM \otimes H^m) \longrightarrow . \end{aligned}$$

We are now in a position to establish the following fact which has been communicated to us by S. Iitaka.[3)]

THEOREM 5.9. *Let M be a complete intersection of r hypersurfaces of degree $a_1, \cdots, a_r$ in $P_{n+r}(\boldsymbol{C})$ with $a_i \geqq 2$ and $n \geqq 2$. Then $H^1(M; TM) \neq 0$ except for the case $r=1$ and $a_1 = 2$ (i.e., the case where M is a quadric in $P_{n+1}(\boldsymbol{C})$).*

3) For a hypersurface, this has been proved in [**11**].

PROOF. First, we estimate $\dim H^0(M; H^2)$. In Proposition 5.4, let $m=2$. Then $\dim H^0(M^{k-1}; H^{2-a_k})=0$ or 1 according as $a_k\geqq 2$ or $a_k=2$ (see (iv) of Proposition 5.2). Proposition 5.4 implies therefore the following inequalities:

$$\dim H^0(M^k; H^2)\geqq \dim H^0(M^{k-1}; H^2)-1 \qquad \text{for } k=1, \cdots, r.$$

By telescoping these inequalities, we obtain

$$\dim H^0(M; H^2)\geqq \dim H^0(P_{n+r}; H^2)-r=\frac{1}{2}(n+r+2)(n+r+1)-r.$$

We used here the formula $\dim H^0(P_s; H^m)=\binom{s+m}{m}$, (see p. 165 [6]). This inequality together with Propositions 5.5 and 5.6 implies

$$\dim H^0(M; N)\geqq \frac{1}{2}r(n+r+2)(n+r+1)-r^2.$$

On the other hand, from (i) of Proposition 5.8, we have

$$\dim H^0(P_{n+r}; TP_{n+r})=(n+r)(n+r+2).$$

It is now easy to verify $\dim H^0(M; N)>\dim H^0(P_{n+r}; TP_{n+r})$ for $r\geqq 2$. From the exact sequence (4) for $m=0$, we may conclude $H^1(M; TM)\neq 0$ for $r\geqq 2$.

Assume now $r=1$ and $a_1\geqq 3$. In this case, $M=M^1$ and $N=H^{a_1}$. From Proposition 5.4, we obtain

$$\dim H^0(M; N)=\dim H^0(M^1; H^{a_1})=\dim H^0(P_{n+1}; H^{a_1})-1$$
$$=\binom{n+1+a_1}{a_1}-1.$$

It is now easy to verify that $\dim H^0(M; N)>\dim H^0(P_{n+1}; TP_{n+1})$. From the exact sequence (4) for $m=0$, we can again conclude $H^1(M; TM)\neq 0$ for $a_1\geqq 3$.

From the results so far obtained, it is not difficult to show

$$\dim H^1(M; TM)=\binom{n+1+a_1}{a_1}-1-(n+1)(n+3)$$

for a hypersurface M of degree a_1. But this is already known, [11]. QED.

We want to find now M with $H^1(M; TM\otimes H)=0$. We begin with an estimate of $\dim H^0(M; H^3)$. From Proposition 5.4, we obtain

$$\dim H^0(M^k; H^3)=\dim H^0(M^{k-1}; H^3)-\dim H^0(M^{k-1}; H^{3-a_k}) \quad \text{for } k=1, \cdots, r.$$

Since $3-a_k\leqq 1$, Proposition 5.6 implies

$$\dim H^0(M^k; H^3)\geqq \dim H^0(M^{k-1}; H^3)-\dim H^0(M^{k-1}; H) \qquad \text{for } k=1, \cdots, r.$$

By telescoping these inequalities, we obtain

$$\dim H^0(M; H^3)\geqq \dim H^0(P_{n+r}; H^3)-\sum_{k=1}^{r}\dim H^0(M^{k-1}; H).$$

If we set $m=1$ in Proposition 5.4 and use (iv) of Proposition 5.2, then we obtain

$$\dim H^0(M^k;H)=\dim H^0(M^{k-1};H) \qquad \text{for } k=1,\cdots,r,$$

and hence

$$\dim H^0(M^k;H)=\dim H^0(P_{n+r};H)=n+r+1.$$

Hence,

$$\begin{aligned}\dim H^0(M;H^3) &\geqq \dim H^0(P_{n+r};H^3)-r(n+r+1)\\ &=\frac{1}{6}(n+r+3)(n+r+2)(n+r+1)-r(n+r+1).\end{aligned}$$

From Propositions 5.5 and 5.6, we obtain

$$\begin{aligned}\dim H^0(M;N\otimes H) &\geqq r\cdot\dim H^0(M;H^3)\\ &\geqq \frac{r}{6}(n+r+3)(n+r+2)(n+r+1)-r^2(n+r+1).\end{aligned}$$

On the other hand, from (ii) of Proposition 5.8, we have

$$\dim H^0(P_{n+r};TP_{n+r}\otimes H)=\frac{1}{2}(n+r)(n+r+1)(n+r+3).$$

It is easy to verify that $\dim H^0(M;N\otimes H)>\dim H^0(P_{n+r};TP_{n+r}\otimes H)$ for $r\geqq 3$. From the exact sequence (4) for $m=1$, we may conclude that $H^1(M;TM\otimes H)\neq 0$ for $r\geqq 3$.

We consider the case $r=2$ and $a_1\geqq 2$, $a_2\geqq 3$. Using Proposition 5.4 twice, we obtain

$$\begin{aligned}\dim H^0(M^2;H^3) &=\dim H^0(P_{n+2};H^3)-\dim H^0(P_{n+2};H^{3-a_1})\\ &\quad -\dim H^0(P_{n+2};H^{3-a_2})+\dim H^0(P_{n+2};H^{3-a_1-a_2})\\ &\geqq \frac{1}{6}(n+5)(n+4)(n+3)-(n+3)-1.\end{aligned}$$

To estimate $\dim H^0(M;H^4)$, we compute $\dim H^0(M;H^2)$ using Proposition 5.4 twice and obtain

$$\begin{aligned}\dim H^0(M^2;H^2) &=\dim H^0(P_{n+2};H^2)-\dim H^0(P_{n+2};H^{2-a_1})\\ &\quad -\dim H^0(P_{n+2};H^{2-a_2})+\dim H^0(P_{n+2};H^{2-a_1-a_2})\\ &\leqq \frac{1}{2}(n+4)(n+3).\end{aligned}$$

Now we estimate $\dim H^0(M;H^4)$ as follows:

$$\begin{aligned}\dim H^0(M^2\,;H^4)&=\dim H^0(P_{n+2}\,;H^4)-\dim H^0(P_{n+2}\,;H^{4-a_1})\\&\quad-\dim H^0(P_{n+2}\,;H^{4-a_2})+\dim H^0(P_{n+2}\,;H^{4-a_1-a_2})\\&\geqq\frac{1}{24}(n+6)(n+5)(n+4)(n+3)-\frac{1}{2}(n+4)(n+3)-(n+3)\,.\end{aligned}$$

From Propositionswe 5.5 and 5.6, obtain

$$\begin{aligned}\dim H^0(M\,;N\otimes H)&=\dim H^0(M\,;H^{a_1+1})+\dim H^0(M\,;H^{a_2+1})\\&\geqq\dim H^0(M\,;H^3)+\dim H^0(M\,;H^4)\\&\geqq\frac{1}{24}(n+10)(n+5)(n+4)(n+3)-\frac{1}{2}(n+8)(n+3)-1\,.\end{aligned}$$

On the other hand, by (ii) of Proposition 5.8, we have

$$\dim H^0(P_{n+2}\,;TP_{n+2}\otimes H)=\frac{1}{2}(n+2)(n+3)(n+5)\,.$$

It is now easy to verify that $\dim H^0(M\,;N\otimes H)>\dim H^0(P_{n+2}\,;TP_{n+2}\otimes H)$.

We consider now the case $r=1$, i.e., the case where M is a hypersurface of degree a in $P_{n+1}(C)$. From Proposition 5.4, we obtain

$$\begin{aligned}\dim H^0(M\,;N\otimes H)&=\dim H^0(M^1\,;H^{a+1})\\&=\dim H^0(P_{n+1}\,;H^{a+1})-\dim H^0(P_{n+1}\,;H)\\&=\binom{n+a+2}{a+1}-(n+2)\,.\end{aligned}$$

On the other hand, from (ii) of Proposition 5.8, we have

$$\dim H^0(P_{n+1}\,;TP_{n+1}\otimes H)=\frac{1}{2}(n+1)(n+2)(n+4)\,.$$

It is easy to verify that $\dim H^0(M;N\otimes H)>\dim H^0(P_{n+1}\,;TP_{n+1}\otimes H)$ in the following two cases: (1) $a\geqq 4$, (2) $a=3$ and $n\geqq 5$. It follows that $H^1(M;TM\otimes H)\neq 0$ in these two cases.

We summarize what we have proved in the following

THEOREM 5.10. *Let M be a complete intersection of r hypersurfaces of degree $a_1,\cdots,a_r$ in $P_{n+r}(C)$ with $a_i\geqq 2$ and $n\geqq 2$. Then $H^1(M;TM\otimes H)\neq 0$ except for the following cases:* (1) *$r=1$ and $a_1=2$,* (*i.e., quadrics*), (2) *$r=1$, $a_1=3$ and $n\leqq 4$,* (*i.e., cubics of dimension $\leqq 4$*), (3) *$r=2$ and $a_1=a_2=2$,* (*i.e., intersection of two quadrics*).

Although we know that $H^1(M;TM\otimes H)=0$ for a quadric M, we do not know if the same is actually true for the cases (2) and (3).

THEOREM 5.11. *Let M be a complete intersection of r hypersurfaces of degree $a_1,\cdots,a_r$ in $P_{n+r}(C)$ with $a_i\geqq 2$ and $n\geqq 2$. Then*

(i) *Its tangent bundle* $T(M)$ *is not positive except possibly for the case* $r=1$, $a_1=2$;

(ii) *Its tangent bundle* $T(M)$ *is not semi-positive except for the case* $r=1$, $a_1=2$ *and possibly for the case* $r=2$, $a_1=a_2=2$.

PROOF. (i). This follows from Corollary 2.5 and Theorem 5.9. But we do not know if a quadric has positive tangent bundle.

(ii). We combine Corollary 2.5 and Theorem 5.10. The only thing we have to prove is that a cubic of dimension $\leqq 4$ does not have semi-positive tangent bundle. Let M be a cubic of dimension n in $P_{n+1}(C)$. Let h be the first Chern class of H. Let c_i denote the i-th Chern class of M. A simple calculation shows the following:

$$c_1^2-c_2=-2h^2 \qquad \text{for } n=2,$$

$$c_1^3-2c_1c_2+c_3=-10h^3 \qquad \text{for } n=3,$$

$$c_1^4-3c_1^2c_2+2c_1c_3+c_2^2-c_4=-42h^4 \qquad \text{for } n=4.$$

From Theorems 3.1 and 4.1 we may conclude that $T(M)$ is not semi-positive for $n=2$ or $n=3$. In the same way as we proved Theorems 3.1 and 4.1, we can prove that if E is a holomorphic vector bundle over an algebraic manifold M of dimension 4, then

$$(d_1^4-3d_1^2d_2+2d_1d_3+d_2^2-d_4)[M]>0 \quad \text{or} \quad \geqq 0$$

according as E is positive or semi-positive, where d_i denotes the i-th Chern class of E. QED.

A similar calculation eliminates the case $r=2$, $a_1=a_2=2$ for $n\leqq 4$ in (ii). We have been unable to perform the calculation for dimension n. We shall see in §6 that a quadric has semi-positive tangent bundle.

§6. Hermitian vector bundles

In this section we shall show that a holomorphic vector bundle is negative or semi-negative if it admits a hermitian metric with negative or semi-negative curvature (in the sense to be made precise below).

Let E be a hermitian vector bundle over a complex manifold M, i.e., a holomorphic vector bundle over M with hermitian fibre metric h. We follow notations in §10 of Chapter IX, [8]. Taking local holomorphic sections $s_1, \cdots, s_r$ which form a basis for each fibre, we define

$$h_{\alpha\bar{\beta}}=h(s_\alpha, \bar{s}_\beta) \qquad \alpha, \beta=1, \cdots, r.$$

We start with the following trivial analogue of the existence of normal coordinate systems.

PROPOSITION 6.1. *Given a point $o \in M$, there exist local holomorphic sections $s_1, \cdots, s_r$ around o such that*

$$h_{\alpha\bar{\beta}} = \delta_{\alpha\beta} \quad \text{and} \quad dh_{\alpha\bar{\beta}} = 0 \quad \text{at } o.$$

PROOF. Choose local holomorphic sections $t_1, \cdots, t_r$ around o which are orthonormal at o. We set

$$s_\alpha = \sum a_\alpha^\beta t_\beta \qquad (a_\alpha^\beta: \text{holomorphic})$$

and try to find (a_α^β) such that $a_\alpha^\beta = \delta_\alpha^\beta$ at o and $s_1, \cdots, s_r$ satisfy the required second condition. If we set

$$g_{\alpha\bar{\beta}} = h(t_\alpha, \bar{t}_\beta),$$

then

$$h_{\alpha\bar{\beta}} = h(s_\alpha, \bar{s}_\beta) = \sum a_\alpha^\gamma g_{\gamma\bar{\delta}} \bar{a}_\beta^\delta,$$

or in matrix form

$$H = {}^tA \cdot G \cdot \bar{A}.$$

We want to find A such that $A = I$ at o and $dH = 0$ at o. Since

$$\partial H = \partial {}^tA \cdot G \cdot \bar{A} + {}^tA \cdot \partial G \cdot \bar{A},$$

it suffices to set

$$a_\alpha^\beta = \delta_\alpha^\beta - \sum \left(\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^j} \right)_0 \cdot z^j,$$

where $z^1, \cdots, z^n$ is a local coordinate system with origin o. QED.

The Christoffel symbols of the hermitian connection are given by

$$\Gamma_{i\beta}^\alpha = \sum h^{\alpha\bar{\gamma}} \frac{\partial h_{\beta\bar{\gamma}}}{\partial z^i} \qquad \alpha, \beta = 1, \cdots, r; \quad i = 1, \cdots, n,$$

and the components of the curvature are given by

$$K_{\beta i\bar{j}}^\alpha = -\partial \Gamma_{i\beta}^\alpha / \partial \bar{z}^j = -\sum h^{\alpha\bar{\gamma}} \frac{\partial^2 h_{\beta\bar{\gamma}}}{\partial z^i \bar{z}^j} + \sum h^{\alpha\bar{\delta}} h^{\varepsilon\bar{\gamma}} \frac{\partial h_{\beta\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\varepsilon\bar{\delta}}}{\partial \bar{z}^j}.$$

We set

$$K_{\alpha\bar{\beta}i\bar{j}} = -K_{\bar{\beta}\alpha i\bar{j}} = -\sum h_{\gamma\bar{\beta}} K_{\alpha i\bar{j}}^\gamma.$$

Then

$$K_{\alpha\bar{\beta}i\bar{j}} = \frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} - \sum h^{\varepsilon\bar{\gamma}} \frac{\partial h_{\alpha\bar{\gamma}}}{\partial z^i} \frac{\partial h_{\varepsilon\bar{\beta}}}{\partial \bar{z}^j}.$$

The Ricci curvature is defined by

$$K_{i\bar{j}} = \sum K_{\alpha i\bar{j}}^\alpha (= -\sum h^{\alpha\bar{\beta}} K_{\alpha\bar{\beta}i\bar{j}}).$$

Then

$$K_{i\bar{j}} = -\frac{\partial^2 \log \det (h_{\alpha\bar{\beta}})}{\partial z^i \partial \bar{z}^j}.$$

The hermitian vector bundle E is said to have *positive* (resp. *semi-positive, negative* or *semi-negative*) *curvature* if

$$-\sum K_{\alpha\bar{\beta}i\bar{j}}\xi^{\alpha}\bar{\xi}^{\beta}u^{i}\bar{u}^{j}>0 \qquad (\text{resp. } \geqq 0,\ <0 \text{ or } \leqq 0)$$

for all nonzero ξ and nonzero u. The hermitian vector bundle E is said to have *positive* (resp. *semi-positive, negative,* or *semi-negative*) *Ricci curvature* if

$$\sum K_{i\bar{j}}u^{i}\bar{u}^{j}>0 \qquad (\text{resp. } \geqq 0,\ <0 \text{ or } \leqq 0)$$

for all nonzero u. If E has positive (resp. semi-positive, negative or semi-negative) curvature, it has positive (resp. semi-positive, negative or semi-negative) Ricci curvature. For a hermitian line bundle, its curvature has the same sign as its Ricci curvature.

PROPOSITION 6.2. *A hermitian vector bundle E has positive (resp. semi-positive) curvature if and only if its dual hermitian vector bundle E^* has negative (resp. semi-negative) curvature. Similarly, for the Ricci curvature.*

PROOF. Given a point o in the base manifold M, we choose holomorphic sections $s_1, \cdots, s_r$ as in Proposition 6.1. Let $s^1, \cdots, s^r$ be the dual system of cross sections of the dual hermitian vector bundle E^*. If we denote the induced hermitian metric in E^* by the same letter h and set

$$h^{\alpha\bar{\beta}}=h(s^{\alpha}, \bar{s}^{\beta}),$$

then $(h^{\alpha\bar{\beta}})$ is the inverse matrix of $(h_{\alpha\bar{\beta}})$. Differentiating the identity $\sum h^{\alpha\bar{\beta}}h_{\gamma\bar{\beta}}=\delta^{\alpha}_{\gamma}$, we obtain

$$\sum\frac{\partial^2 h^{\alpha\bar{\beta}}}{\partial z^i\partial\bar{z}^j}h_{\gamma\bar{\beta}}+\sum h^{\alpha\bar{\beta}}\frac{\partial^2 h_{\gamma\bar{\beta}}}{\partial z^i\partial\bar{z}^j}+\sum\frac{\partial h^{\alpha\bar{\beta}}}{\partial z^i}\frac{\partial h_{\gamma\bar{\beta}}}{\partial\bar{z}^j}+\sum\frac{\partial h^{\alpha\bar{\beta}}}{\partial\bar{z}^j}\frac{\partial h_{\gamma\bar{\beta}}}{\partial z^i}.$$

From the properties of $s_1, \cdots, s_r$ (see Proposition 6.1), we obtain

$$\frac{\partial^2 h^{\alpha\bar{\gamma}}}{\partial z^i\partial\bar{z}^j}+\frac{\partial^2 h_{\gamma\bar{\alpha}}}{\partial z^i\partial\bar{z}^j}=0 \qquad \text{at } o.$$

From the formula for the components of the curvature tensor, we see that $\partial^2 h_{\alpha\bar{\beta}}/\partial z^i\partial\bar{z}^j$ coincides with $K_{\alpha\bar{\beta}i\bar{j}}$ at o. Similarly, $\partial^2 h^{\alpha\bar{\beta}}/\partial z^i\partial\bar{z}^j$ coincides with the curvature tensor of E^*. The proof for the Ricci curvature is similar.

QED.

In §§1 and 2, we associated to each holomorphic vector bundle E a line bundle $L(E)$ over $P(E)$.

PROPOSITION 6.3. *If E is a hermitian vector bundle with negative curvature (resp. semi-negative curvature), then the line bundle $L(E)$ over $P(E)$ with the induced hermitian metric has negative (resp. semi-negative) curvature.*

PROOF. The naturally induced hermitian metric $\tilde{h}$ in $L(E)$ may be described as follows. Since $L(E)$ minus its zero section is naturally isomorphic to E minus its zero section, every nonzero element X of $L(E)$ may be identified with an element of E, and

$$\tilde{h}(X, X) = h(X, X).$$

Fixing a point o in the base manifold M, we choose holomorphic sections $s_1, \cdots, s_r$ in a neighborhood of o with the properties stated in Proposition 6.1. Then we may write

$$h(X, X) = \sum h_{\alpha\bar{\beta}} \xi^{\alpha} \bar{\xi}^{\beta} \qquad \text{for } X = \sum \xi^{\alpha} s_{\alpha}.$$

We shall compute the Ricci tensor of the line bundle $L(E)$ at an arbitrarily fixed point of $P(E)$ which lies over $o \in M$. This point is represented by a unit vector $X_0 \in E$. Applying a unitary transformation to $s_1, \cdots, s_r$, we may assume that $X_0 = s_r(o)$. We take $z^1, \cdots, z^n, \xi^1, \cdots, \xi^{r-1}$ as a local coordinate system around $[X_0]$ in $P(E)$, $[X_0]$ denotes the point of $P(E)$ represented by X_0. Then the components of the Ricci tensor of $L(E)$ at $[X_0]$ are given by

$$\begin{pmatrix} -\dfrac{\partial^2 \log h(X, X)}{\partial z^i \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X, X)}{\partial z^i \partial \bar{\xi}^{\beta}} \\ -\dfrac{\partial^2 \log h(X, X)}{\partial \xi^{\alpha} \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X, X)}{\partial \xi^{\alpha} \partial \bar{\xi}^{\beta}} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2 h_{\alpha\beta}}{\partial z^i \partial \bar{z}^j} & 0 \\ 0 & -\delta^{\alpha}_{\beta} \end{pmatrix}$$

where $i, j = 1, \cdots, n$ and $\alpha, \beta = 1, \cdots, r-1$. It is clear that this matrix is negative (semi-) definite if the curvature of E is (semi-) negative. QED.

REMARK. If E has (semi-) positive curvature, its dual E^* has (semi-) negative curvature by Proposition 6.2 and hence the line bundle $L(E^*)$ over $P(E^*)$ has (semi-) negative curvature by Proposition 6.3 and its dual $L(E^*)^{-1} = L(E^*)^*$ has (semi-) positive curvature. But $L(E)$ itself does not have (semi-) positive curvature.

From Proposition 6.3, we obtain immediately the following

THEOREM 6.4. *A hermitian vector bundle E with negative (resp. semi-negative, positive, or semi-positive) curvature is negative (resp. semi-negative, positive, or semi-positive).*

We do not know if the converse is true, e. g., if a negative vector bundle E admits a hermitian metric with negative curvature. For a line bundle E, by definition E is negative (resp. positive) if and only if it admits a hermitian metric with negative (resp. positive) curvature. It is, however, not clear if a semi-negative (resp. semi-positive) line bundle admits a hermitian metric with semi-negative (resp. semi-positive) curvature.

§7. The tangent bundle of $P_n(C)$

We shall apply Theorem 2.3 to the tangent bundle of $P_n(C)$. As we know the curvature of $P_n(C)$ explicitly, we can obtain sharper results than those we can from Corollaries 2.4 to 2.7. We continue to denote by H the positive line bundle over $P_n(C)$ defined by a hyperplane in $P_n(C)$.

PROPOSITION 7.1. *Let TP and T^*P denote the tangent and cotangent bundles of $P=P_n(C)$. Then the line bundle $L(T^*P)^{-p}\cdot\pi^*H^m$ over $P(T^*P)$ is positive if $p+m\geqq 1$ and $p\geqq 1$, where $\pi: P(T^*P)\to P$ is the projection.*

PROOF. For the Fubini-Study metric on $P_n(C)$ with holomorphic sectional curvature c, the curvature tensor is given by

$$K_{i\bar{j}k\bar{l}}=-\frac{c}{2}(h_{i\bar{j}}h_{k\bar{l}}+h_{i\bar{l}}h_{k\bar{j}}).$$

Given a point o in $P_n(C)$, we may always choose a local coordinate system around o so that the metric tensor $h_{i\bar{j}}$ coincides with δ_{ij} at o. Then the curvature of the cotangent bundle is given by

$$\frac{c}{2}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj}).$$

Note that the sign changes when we pass from TP to T^*P. The matrix representing the Ricci curvature of $L(T^*P)$ in the proof of Proposition 6.3 reduces in this case to the following:

$$\begin{pmatrix} -\frac{c}{2}(\delta_{ij}+\delta_{in}\delta_{jn}) & 0 \\ 0 & -\delta_{\alpha\beta} \end{pmatrix}_{\substack{i,j=1,\cdots,n \\ \alpha,\beta=1,\cdots,n-1}}.$$

The Ricci curvature of $L(T^*P)^{-1}$ is obtained from that of $L(T^*P)$ by changing its sign. On the other hand, the Ricci curvature of P at o (which is nothing but the Ricci curvature of $K_P^{-1}=H^{n+1}$) is given by $K_{i\bar{j}}=-\frac{c}{2}(n+1)\delta_{ij}$. Hence, the Ricci curvature of H is given by

$$-\frac{1}{n+1}K_{i\bar{j}}=\frac{c}{2}\delta_{ij}.$$

Consequently, the Ricci curvature of $L(T^*P)^{-p}\cdot\pi^*H^m$ can be expressed by the following matrix:

$$\begin{pmatrix} p\frac{c}{2}(\delta_{ij}+\delta_{in}\delta_{jn}) & 0 \\ 0 & p\delta_{\alpha\beta} \end{pmatrix}+\begin{pmatrix} m\frac{c}{2}\delta_{ij} & 0 \\ 0 & 0 \end{pmatrix},$$

which is clearly positive if $m+p\geqq 1$ and $p\geqq 1$. QED.

COROLLARY 7.2. *Let* $P_n = P_n(\boldsymbol{C})$. *Then*

(i) $H^i(P_n; S^kTP_n \otimes H^m) = 0$ *for* $n+k+m \geqq 1$;

(ii) $H^0(P_n; S^kTP_n \otimes H^m) = 0$ *for* $k(n+1)+mn < 0$ *and* $k = 0, 1, 2, \cdots$.

PROOF. (i). This is immediate from Theorem 2.3 and Proposition 7.1.

(ii). By Theorem 2.1, $H^0(P_n; S^kTP_n \otimes H^m) = H^0(P(T^*P_n); L(T^*P_n)^{-k} \cdot \pi^*H^m)$. We consider the matrix representing the Ricci curvature of the line bundle $L(T^*P_n)^{-k} \cdot \pi^*H^m$. From the matrix given at the end of the proof of Proposition 7.1, we see that its trace is equal to $\frac{c}{2}(k(n+1)+mn)+k(n-1)$. If we take c sufficiently large, this trace is negative when $k(n+1)+mn$ is negative. Now (ii) follows from the following fact, [**9**]: A hermitian line bundle over a compact complex manifold admits no holomorphic section other than the zero section if its Ricci curvature has negative trace. QED.

COROLLARY 7.3. *Let M be an n-dimensional closed complex submanifold of $P_{n+r}(\boldsymbol{C})$ such that its canonical line bundle K_M satisfies*

$$K_M^{-1} \cdot H^{m+k-2} \geqq 0, \qquad (k \geqq 0).$$

Then

$$H^i(M; S^kTP_{n+r} \otimes H^m) = 0 \qquad \textit{for } i \geqq 1.$$

PROOF. This follows from Theorem 2.3, Proposition 7.1 and $\det(T^*P_{n+r}) = H^{-(n+r+1)}$. QED.

COROLLARY 7.4. *Let M be a complete intersection of r hypersurfaces of degree $a_1, \cdots, a_r$ in $P_{n+r}(\boldsymbol{C})$. If $n+r+m+k-1 \geqq \sum a_i$ and $k \geqq 0$, then*

$$H^i(M; S^kTP_{n+r} \otimes H^m) = 0 \qquad \textit{for } i \geqq 1.$$

PROOF. Since $K_M^{-1} = H^{n+r+1-\Sigma a_i}$, this follows from Corollary 7.3. QED.

§ 8. Kaehler manifolds with positive holomorphic bisectional curvature

Let M be a Kaehler manifold. Its tangent bundle $T(M)$ is a hermitian vector bundle in a natural manner. This hermitian vector bundle has positive curvature in the sense defined in § 6 if and only if M has positive holomorphic bisectional curvature, (see [4] for the concept of holomorphic bisectional curvature). The following results are known, [**1**], [**3**], [**4**].

THEOREM 8.1. *Let M be a compact Kaehler manifold of dimension n with positive holomorphic bisectional curvature. Then*

(i) $\dim H^{1,1}(M; \boldsymbol{C}) = 1$;

(ii) *If M' and M'' are closed complex submanifolds of M such that* $\dim M' + \dim M'' \geqq \dim M$, *then $M' \cap M''$ is non-empty.*

If M has positive holomorphic bisectional curvature, then it has positive

Ricci tensor so that the canonical line bundle K_M is negative. In § 3, we proved that an algebraic surface M with $T(M)>0$ and $K_M<0$ satisfies

$$c_2[M]=2+\dim H^{1,1}(M;\boldsymbol{C}),$$
$$(c_1^2+c_2)[M]=12,$$
$$\dim H^0(M;T(M))=\chi(M;T(M))=(c_1^2-c_2)[M]+2.$$

If we combine this with (i) of Theorem 8.1, we obtain

PROPOSITION 8.2. *If M is a 2-dimensional compact Kaehler manifold with positive holomorphic bisectional curvature, then*

$$\dim H^{p,q}(M;\boldsymbol{C})=\delta_{pq},\quad c_1^2[M]=9,\quad c_2[M]=3,\quad \dim H^0(M;T(M))=8.$$

Using the classification of algebraic surfaces and (ii) of Theorem 8.1, Andreotti and Frankel [3] have shown that a 2-dimensional compact Kaehler manifold with positive holomorphic bisectional curvature is biholomorphic with $P_2(\boldsymbol{C})$.

In Theorem 5.11 we could not decide whether a quadric has actually positive tangent bundle or not. Making use of (ii) of Theorem 8.1, we can prove

PROPOSITION 8.3. *Let M be a quadric in $P_{n+1}(\boldsymbol{C})$. If n $(=\dim M)$ is even, then M cannot admit a Kaehler metric with positive holomorphic bisectional curvature.*

PROOF. Set $2m=n+2$ and let $z^1, z^2, \cdots, z^{2m}$ be a homogeneous coordinate system in $P_{n+1}(\boldsymbol{C})$. We may assume that a quadric M is defined by

$$M:\quad z^1z^{m+1}+z^2z^{m+2}+\cdots+z^mz^{2m}=0.$$

Let M' and M'' be the $(m-1)$-dimensional subspaces of $P_{n+1}(\boldsymbol{C})$ defined by

$$M':\quad z^1=\cdots=z^m=0;$$
$$M'':\quad z^{m+1}=\cdots=z^{2m}=0.$$

Clearly, both M' and M'' are contained in M and

$$\dim M'+\dim M''=\dim M,\qquad M'\cap M''=\emptyset.$$

Our assertion now follows from (ii) of Theorem 8.1. QED.

§ 9. Almost positive hermitian vector bundles

Let E be a hermitian vector bundle over M. As we saw in § 6, $L(E)$ is a hermitian line bundle over $P(E)$ in a natural manner. If the curvature of $L(E)$ is negative semi-definite everywhere on $P(E)$ and negative definite almost everywhere on $P(E)$ (in the measure theoretic sense), then we call E

an *almost negative* hermitian vector bundle. We say that E is *almost positive* if its dual hermitian vector bundle E^* is almost negative. We shall explain these concepts in terms of the curvature $K_{\alpha\bar{\beta}i\bar{j}}$ of E. To each element $X=\sum \xi^i s_i$ of E with unit length, we associate a quadratic form Q_X as follows:

$$Q_X = -\sum K_{\alpha\bar{\beta}i\bar{j}} \xi^\alpha \bar{\xi}^\beta dz^i d\bar{z}^j .$$

If we regard X as a point of $P(E)$, then we associate to each point X of $P(E)$ a quadratic form Q_X. According to the definitions given in § 6, E has positive (resp. semi-positive, negative, or semi-negative) curvature if and only if Q_X is positive definite (resp. positive semi-definite, negative definite, or negative semi-definite) everywhere on $P(E)$. From the proof of Proposition 6.3, we see that E is almost negative (resp. almost positive) if and only if Q_X is negative semi-definite (resp. positive semi-definite) everywhere on $P(E)$ and negative definite (resp. positive definite) almost everywhere on $P(E)$.

For example, the tangent bundle $T(M)$ of a hermitian symmetric space of compact type is almost positive. It is not difficult to see that Q_X is positive definite when (and only when) X is off the walls of the Weyl chambers.

It is also an easy matter to verify that the vanishing theorem of Kodaira can be generalized as follows:

If X is a compact complex manifold with canonical line bundle K and if F is a line bundle over X such that FK^{-1} is almost positive with respect to a suitable hermitian metric, then

$$H^i(X; F) = 0 \qquad \textit{for } i \geqq 1 .$$

Similarly, for Nakano's vanishing theorem. It is now clear that all the results in § 2 remain true when we replace "positive" or "negative" by "almost positive" or "almost negative".

In particular, from Corollary 2.5, we obtain the following result:

THEOREM 9.1. *Let M be a compact hermitian manifold with almost positive curvature, e.g., a hermitian symmetric space of compact type. Let F be a line bundle over M with $F \geqq 0$. Then*

$$H^i(M; S^k T \otimes F) = 0 \qquad \textit{for } i \geqq 1 \textit{ and } k = 0, 1, 2, \cdots .$$

This shows also a limitation of the power of Corollary 2.5 in attacking the problem of Kaehler manifolds with positive holomorphic bisectional curvature. On the other hand, Theorem 8.1 seems to rely more heavily on positivity of curvature as Proposition 8.3 testifies.

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Bibliography

[1] R. L. Bishop and S. I. Goldberg, On the second cohomology group of a Kaehler manifold of positive curvature, Proc. Amer. Math. Soc., **16** (1965), 119-122.

[2] R. Bott, Homogeneous vector bundles, Ann. of Math., **66** (1957), 203-248.

[3] T. T. Frankel, Manifolds with positive curvature, Pacific J. Math., **11** (1961), 165-174.

[4] S. I. Goldberg and S. Kobayashi, On holomorphic bisectional curvature, J. Differential Geometry, **1** (1967), 225-233.

[5] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331-368.

[6] F. Hirzebruch, Topological methods in algebraic geometry, Springer, 1966.

[7] L. Kaup, Eine Künnethformel für kohärente analytische Garben, Erlangen thesis, 1965, or Math. Z., **97** (1967), 158-168.

[8] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Interscience tract, No. 15, vol. II, 1969.

[9] S. Kobayashi and H. H. Wu, On holomorphic sections of certain hermitian vector bundles, to appear.

[10] K. Kodaira, On a differential geometric method in the theory of analytic stacks, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 1268-1273.

[11] K. Kodaira and D. C. Spencer, On deformations of complex analytic structures, II, Ann. of Math., **67** (1958), 403-466.

[12] S. Nakano, On complex analytic vector bundles, J. Math. Soc. Japan, **7** (1955), 1-12.

[13] Seminaire Henri Cartan, E. N. S. 1950-1951.

Remarks (added on May 22, 1970)

In a recent paper, Griffiths (Hermitian differential geometry, Chern classes, and positive vector bundles, Volume in honor of K. Kodaira) studies several notions of positivity of vector bundles. Whereas we take $L(E)<0$ as our definition of $E<0$, he takes the curvature-negativity as his definition of $E<0$. Theorem 6.4 in the present paper is therefore equivalent to the implication (1.8) in his paper. (His L corresponds to our $L(E)^{-1}$). Our definition seems to be more consistent with the definition of ampleness for vector bundles which is being used by algebraic geometers (see, for instance, R. Hartshorne, Ample vector bundles, Publ. I.H.E.S. **29** (1966), 319-350) and, as pointed out in the introduction, it fits well with the function theoretic definition by Grauert. Contrary to Griffiths' statement on p. 188, the algebraic (and also function theoretic) notion of positivity yields precise vanishing theorems. Actually, Corollary 2.4, (ii) in the present paper implies Griffiths' Theorem G; he assumes that E is generated by its sections while we assume $E\geqq 0$. In the case when E is the tangent bundle of a compact complex manifold M, his assumption that E is generated by its sections amounts to saying that M is a *homo-*

geneous compact complex manifold. But we are mainly interested in the non-homogeneous compact Kähler manifolds with positive curvature.

In a recent note, S. Kleiman (Ample vector bundles on algebraic surfaces, Proc. Amer. Math. Soc., **21** (1969), 673-676) proved Theorem 3.1, (i) in a more general situation.