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Multiplications in cohomology theories with coefficient maps

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Introduction

S. Araki and H. Toda [3] discussed the multiplicative structures in mod q generalized cohomology theories. The mod q (reduced) cohomology of a cohomology h was defined by $\tilde{h}^i(\ ;Z_q)=\tilde{h}^{i+2}(\ \wedge M_q)$ where M_q is a co-Moore space of type $(Z_q, 2)$. In this paper we consider the cohomology theories with stable maps of spheres as coefficients, i. e., let C_α be a mapping cone of a stable map $\alpha\in\{S^{r+k-1}, S^r\}$ then α-coefficient (reduced) cohomology of a cohomology h is defined by $\tilde{h}^i(\ ;\alpha)=\tilde{h}^{i+r+k}(\ \wedge C_\alpha)$. And we discuss the multiplicative structures in α-coefficient cohomology theories by postulating three axioms (A_1), (A_2) and (A_3) as in [3]. (A_1) and (A_3) are quite similar with these of [3], but (A_2) is not a routine generalization of that of [3] but contains an extra element. A multiplication in an α-coefficient cohomology theory satisfying (A_1), (A_2) and (A_3) is called "an admissible multiplication".

One of the important examples of α-coefficient cohomologies other than mod q theories is the case $\alpha=\eta$ (a stable class of the Hopf map $S^3\to S^2$) and $h=KO$. In the case $\alpha=\eta$ for any h a sufficient condition for existence of admissible multiplications in $\tilde{h}(\ ;\eta)$ is obtained (Theorem 4.11). Obviously $\widetilde{KO}$-theory satisfies this condition. In this case $\widetilde{KO}(\ ;\eta)$ can be identified with the cohomology $\widetilde{KU}$ by Wood isomorphism [2], where an admissible multiplication corresponds to the multiplication in $\widetilde{KU}$ defined by tensor products (Theorem 5.3).

Some uniqueness type theorems of admissible multiplications are discussed in general, which states that admissible multiplications are in a one-to-one correspondence with elements of a group which is specific to the considered cohomology theory under the assumption that the original multiplication is commutative and associative (Corollary 2.8). In the case of $h=KO$ and $\alpha=\eta$ this group is isomorphic to Z, hence there are countably many different admissible multiplications in $\widetilde{KO}(\ ;\eta)$.

In § 1 we exhibit some elements of α-coefficient cohomology theory: reduction mod α, Bockstein homomorphism, etc., and define the notion of admissible multiplications in α-coefficient cohomology theories. Uniqueness-type theorems of admissible multiplication in α-coefficient cohomologies are discussed in § 2. In § 3 we compute some stable homotopy groups and make preparations to the existence theorem of admissible multiplications for case $\alpha=\eta$ from homotopy theoretical point of view. § 4 is devoted to the proof of the existence theorem of admissible multiplications in the case $\alpha=\eta$ by constructing a multiplication. In § 5 we discuss the admissible multiplications in $\widetilde{KO}(\ ;\eta)$ and their relations with Wood isomorphism [2].

Finally the author wishes to express his hearty thanks to Professor S. Araki for his many valuable suggestions and discussions.

§ 1. α-coefficient cohomology theories

In this section we define the α-coefficient cohomology theories and state the axioms for admissible multiplications.

1.1. We use the same notations as [3], p. 73; for examples,

$X\wedge Y$ the reduced join of two spaces X and Y with base points,

$SX=X\wedge S^1$ the reduced suspension of X,

$T=T(A, B): A\wedge B\rightarrow B\wedge A$ a map switching factors and

$\{X, Y\}$ the stable homotopy groups of CW-complexes X and Y with base points.

For a map $f: S^nX\rightarrow S^nY$, $n\geqq 0$, we denote by the same letter f the stable homotopy class represented by f, when there arises no confusion.

1.2. By a cohomology theory we understand, throughout the present work, a reduced general cohomology theory $\{\tilde{h}, \sigma\}$ defined on the category of finite CW-complexes with base points (or of the same homotopy type).

For the sake of simplicity $\tilde{h}^*(f)$ is denoted by

$$f^*: \tilde{h}^*(Y)\rightarrow\tilde{h}^*(X)$$

for any map $f: X\rightarrow Y$ preserving base points. By the naturality of suspension σ, f^* depends only on the stable homotopy class of f. Let A and B be finite CW-complexes with base points, for any element α of $\{A, B\}$, we define a homomorphism

$$\alpha^{**}: \tilde{h}^*(X\wedge B)\rightarrow\tilde{h}^*(X\wedge A)$$

by the formula

$$\alpha^{**}=(\sigma^n)^{-1}(1_X\wedge f)^*\sigma^n$$

where $f: S^nA\rightarrow S^nB$ in a map representing α. The definition does not depend on the choice of f.

1.3. Let α be a stable homotopy class of $g: S^{r+k-1}\rightarrow S^r$, $k>0$. For the stable homotopy type of the mapping cone of g depends only on α we denote as

$$C_\alpha = S^r \bigcup_g C(S^{r+k-1}).$$

Let $\{\tilde{h}, \sigma\}$ be a cohomology theory. The *α-coefficient cohomology theory* $\{\tilde{h}(\ ;\alpha), \sigma_\alpha\}$ is defined by

$$\tilde{h}^i(X;\alpha) = \tilde{h}^{i+r+k}(X \wedge C_\alpha) \qquad \text{for all } i$$

and the *suspension* isomorphism

$$\sigma_\alpha : \tilde{h}^i(X;\alpha) \to \tilde{h}^{i+1}(SX;\alpha) \qquad \text{for all } i$$

is defined as the composition

$$\sigma_\alpha = (1_X \wedge T)^* \sigma : \tilde{h}^{i+r+k}(X \wedge C_\alpha) \to \tilde{h}^{i+r+k+1}(X \wedge C_\alpha \wedge S^1) \to \tilde{h}^{i+r+k+1}(X \wedge S^1 \wedge C_\alpha),$$

where $T = T(S^1, C_\alpha)$. Since σ is natural, σ_α is also natural.

For a map $f: X \to Y$, $\tilde{h}^*(f;\alpha)$ is defined by

$$\tilde{h}^*(f;\alpha) = (f \wedge 1_{C_\alpha})^*$$

and, for the sake of simplicity, sometimes denoted by

$$f^* : \tilde{h}^*(Y;\alpha) \to \tilde{h}^*(X;\alpha).$$

Obviously $\{\tilde{h}(\ ;\alpha), \sigma_\alpha\}$ becomes a reduced cohomology theory and depends only on the stable homotopy class α of g.

Denote by

$$i_\alpha(=i): S^r \to C_\alpha \quad \text{and} \quad \pi_\alpha(=\pi): C_\alpha \to S^{r+k},$$

the canonical inclusion and the map collapsing S^r to a point. We put

$$\rho_\alpha = (-1)^{i(r+k)}(1 \wedge \pi)^* \sigma^{r+k} : \tilde{h}^i(X) \to \tilde{h}^i(X;\alpha),$$

$$\delta_{\alpha,0} = (-1)^{i(r+k)} \sigma^{-r}(1 \wedge i)^* : \tilde{h}^i(X;\alpha) \to \tilde{h}^{i+k}(X)$$

and

$$\delta_\alpha = \rho_\alpha \delta_{\alpha,0} = (-1)^{k(r+k)}(1 \wedge \pi)^* \sigma^k (1 \wedge i)^* : \tilde{h}^i(X;\alpha) \to \tilde{h}^{i+k}(X;\alpha)$$

which are natural and called the *reduction* "mod α", the *Bockstein homomorphism* and the "mod α" *Bockstein homomorphism* respectively. The following relations are easily seen.

$$\sigma_\alpha \rho_\alpha = \rho_\alpha \sigma, \quad \delta_{\alpha,0}\sigma_\alpha = (-1)^k \sigma \delta_{\alpha,0}, \quad \delta_{\alpha,0}\rho_\alpha = 0,$$

$$\sigma_\alpha \delta_\alpha = (-1)^k \delta_\alpha \sigma_\alpha, \quad \delta_\alpha \rho_\alpha = 0 \quad \textit{and} \quad \delta_\alpha \delta_\alpha = 0.$$

From the exact sequence of $\tilde{h}$ associated with the cofibration

$$X \wedge S^r \xrightarrow{1\wedge i} X \wedge C_\alpha \xrightarrow{1\wedge\pi} X \wedge S^{r+k} \tag{1.1}$$

and the definition of ρ_α and $\delta_{\alpha,0}$, we obtain the following exact sequence

$$\cdots \xrightarrow{\alpha^{**}} \tilde{h}^i(X) \xrightarrow{\rho_\alpha} \tilde{h}^i(X;\alpha) \xrightarrow{\delta_{\alpha,0}} \tilde{h}^{i+k}(X) \xrightarrow{\alpha^{**}} \tilde{h}^{i+1}(X) \longrightarrow \cdots$$

for all i.

1.4. A cohomology theory $\tilde{h}$ is said to be *multiplicative*, if it is equipped with a map

$$\mu: \tilde{h}^i(X)\otimes\tilde{h}^j(Y)\to\tilde{h}^{i+j}(X\wedge Y)$$

for all i and j, which is

(M_1) *linear,*

(M_2) *natural with respect to both variables,*

(M_3) *has a bilateral unit* $1\in\tilde{h}^0(S^0)$, *i.e.,* $\mu(1\otimes x)=\mu(x\otimes 1)=x$ *for any* $x\in\tilde{h}^i(X)$ *and*

(M_4) *compatible with the suspension isomorphism* σ, *i.e.,*

$$\sigma\mu(x\otimes y)=(1\wedge T)^*\mu(\sigma x\otimes y)=(-1)^i\mu(x\otimes\sigma y)$$

for any $x\in\tilde{h}^i(X)$ *and* $y\in\tilde{h}^j(Y)$ *where* $T=T(Y, S^1)$.

If μ is associative, i.e., satisfies

(M_5) $\mu(\mu\otimes 1)=\mu(1\otimes\mu)$,

or if commutative, i.e., satisfies

(M_6) $T^*\mu(x\otimes y)=(-1)^{ij}\mu(y\otimes x)$

for $x\in\tilde{h}^i(X)$ and $y\in\tilde{h}^j(Y)$, where $T=T(Y, X)$ is a map switching factors, then we say that μ is an *associative*, or *commutative multiplication.*

1.5. Let $\tilde{h}$ be a multiplicative cohomology theory with multiplication μ. Similarly to [3], the multiplications

$$\mu_L: \tilde{h}^i(X)\otimes\tilde{h}^j(Y;\alpha)\to\tilde{h}^{i+j}(X\wedge Y;\alpha),$$
$$\mu_R: \tilde{h}^i(X;\alpha)\otimes\tilde{h}^j(Y)\to\tilde{h}^{i+j}(X\wedge Y;\alpha)$$

for all i and j, are canonically associated with μ by setting

$$\begin{aligned}\mu_L=\mu: \tilde{h}^i(X)\otimes\tilde{h}^j(Y;\alpha)&=\tilde{h}^i(X)\otimes\tilde{h}^{j+r+k}(Y\wedge C_\alpha)\\ &\to\tilde{h}^{i+j+r+k}(X\wedge Y\wedge C_\alpha)=\tilde{h}^{i+j}(X\wedge Y;\alpha),\\ \mu_R=(-1)^{j(r+k)}(1\wedge T)^*\mu&: \tilde{h}^i(X,\alpha)\otimes\tilde{h}^j(Y)\\ &=\tilde{h}^{i+r+k}(X\wedge C_\alpha)\otimes\tilde{h}^j(Y)\to\tilde{h}^{i+j+r+k}(X\wedge C_\alpha\wedge Y)\\ &\to\tilde{h}^{i+j+r+k}(X\wedge Y\wedge C_\alpha)=\tilde{h}^{i+j}(X\wedge Y;\alpha)\end{aligned}$$

and they satisfy the following properties:

(H_1) *linear;*

(H_2) *natural*;

(H_3) 1 *is a left unit for* μ_L *and a right unit for* μ_R;

(H_4) *compatible with the suspension isomorphisms in the sense that*

$$\sigma_\alpha\mu_L(x\otimes y)=(1\wedge T)^*\mu_L(\sigma x\otimes y)=(-1)^i\mu_L(x\otimes\sigma_\alpha y)\,,$$

$$\sigma_\alpha\mu_R(x\otimes y)=(1\wedge T)^*\mu_R(\sigma_\alpha x\otimes y)=(-1)^i\mu_R(x\otimes\sigma y)$$

for $\deg x=i$;

(H_5) *compatible with the reduction* $\bmod\,\alpha$ *in the sense that*

$$\mu_R(\rho_\alpha\otimes 1)=\rho_\alpha\mu=\mu_L(1\otimes\rho_\alpha)\,;$$

(H_6) *compatible with the Bockstein homomorphisms in the sense that*

$$\delta_{\alpha,0}\mu_L(x\otimes y)=(-1)^{ik}\mu(x\otimes\delta_{\alpha,0}y)\,,$$

$$\delta_{\alpha,0}\mu_R(x\otimes y)=\mu(\delta_{\alpha,0}x\otimes y)\,,$$

$$\delta_\alpha\mu_L(x\otimes y)=(-1)^{ik}\mu_L(x\otimes\delta_\alpha y)\,,$$

$$\delta_\alpha\mu_R(x\otimes y)=\mu_R(\delta_\alpha x\otimes y)$$

for $\deg x=i$.

If μ is associative, then the following *associativity*

$$(H_7)\qquad \begin{aligned}\mu_R(\mu_R\otimes 1)&=\mu_R(1\otimes\mu)\,,\\ \mu_R(\mu_L\otimes 1)&=\mu_L(1\otimes\mu_R)\,,\\ \mu_L(\mu\otimes 1)&=\mu_L(1\otimes\mu_L)\end{aligned}$$

holds; and if μ is commutative, then the *commutativity*

$$(H_8)\qquad T^*\mu_L(x\otimes y)=(-1)^{ij}\mu_R(y\otimes x)$$

holds for $x\in\tilde h^i(X)$ and $y\in\tilde h^j(Y;\alpha)$, where $T=T(Y,X)$.

1.6. Let $\tilde h$ be a multiplicative cohomology theory with an associative and commutative multiplication μ. We shall discuss multiplications

$$\mu_\alpha\colon\ \tilde h^i(X;\alpha)\otimes\tilde h^j(Y;\alpha)\to\tilde h^{i+j}(X\wedge Y;\alpha)$$

in $\tilde h(\ ;\alpha)$ by postulating the following properties:

(Λ_0) μ_α *is a multiplication, i. e., satisfies* $(M_1)\sim(M_4)$ *for the cohomology theory* $\tilde h(\ ;\alpha)$ *and*

(Λ_1) *compatible with* μ_L *and* μ_R *through the reduction* $\bmod\,\alpha$ *i. e.,*

$$\mu_L=\mu_\alpha(\rho_\alpha\otimes 1)\quad\textit{and}\quad \mu_R=\mu_\alpha(1\otimes\rho_\alpha)\,;$$

(Λ_2) *there exists a cohomology operation* $\chi_\alpha\colon\tilde h^*(\ )\to\tilde h^*(\ ;\alpha)$ *of degree* $-k$ *satisfying the relation*

$$(1.2)\qquad \chi_\alpha\mu(x\otimes y)=(-1)^{ki}\mu_L(x\otimes\chi_\alpha(y))=\mu_R(\chi_\alpha(x)\otimes y)$$

for $\deg x=i$ *and it is related to* μ_α *by the following relation*

$$\delta_\alpha\mu_\alpha(x\otimes y)=\mu_L(\delta_{\alpha,0}x\otimes y)+(-1)^{ki}\mu_R(x\otimes\delta_{\alpha,0}y)-\chi_\alpha\mu(\delta_{\alpha,0}x\otimes\delta_{\alpha,0}y)$$

for $x\in\tilde{h}^i(X;\alpha)$ *and* $y\in\tilde{h}^j(Y;\alpha)$;

(A_3) *it is quasi-associative in the sense that*

(i) $$\mu_\alpha(\mu_L\otimes 1)=\mu_L(1\otimes\mu_\alpha),$$

(ii) $$\mu_\alpha(\mu_R\otimes 1)=\mu_\alpha(1\otimes\mu_L),$$

(iii) $$\mu_R(\mu_\alpha\otimes 1)=\mu_\alpha(1\otimes\mu_R).$$

Except (A_2), all axioms are routine generalizations of the corresponding ones of [3]. As to (A_2), putting $\chi_\alpha=0$ and $\alpha=$ " the stable class of maps of degree q", our (A_2) reduces to the corresponding one of [3].

A multiplication μ_α satisfying (A_0), (A_1), (A_2) and (A_3) is called an *admissible multiplication* and the cohomology operation of (A_2) is called the *associated cohomology operation* of μ_α.

For any $x\in\tilde{h}^i(X)$

$$\chi_\alpha(x)=\chi_\alpha\mu(1\otimes x)=\mu_R(\chi_\alpha(1)\otimes x)$$
$$=\chi_\alpha\mu(x\otimes 1)=(-1)^{ki}\mu_L(x\otimes\chi_\alpha(1)). \tag{1.3}$$

This means that the associated cohomology operation χ_α is determined uniquely by $\chi_\alpha(1)\in\tilde{h}^{-k}(S^0;\alpha)$.

$\delta_{\alpha,0}\delta_\alpha=0$ imply that

$$\delta_{\alpha,0}\chi_\alpha\mu(\delta_{\alpha,0}\otimes\delta_{\alpha,0})=2\varepsilon\mu(\delta_{\alpha,0}\otimes\delta_{\alpha,0}) \tag{1.4}$$

where $\varepsilon=0$ *if* k *is odd, or* $=1$ *if* k *is even.*

(A_1) and (H_5) imply that

(A_1') μ_α *is compatible with* μ *through the reduction* mod α *in the sense that*

$$\mu_\alpha(\rho_\alpha\otimes\rho_\alpha)=\rho_\alpha\mu.$$

If a multiplication μ_α satisfies (A_1), then (A_2), or (A_3), are equivalent to the following relation (A_2'), or (A_3'):

(A_2') δ_α *is a modified derivation in the sense that*

$$\delta_\alpha\mu_\alpha(x\otimes y)=\mu_\alpha(\delta_\alpha x\otimes y)+(-1)^{ki}\mu_\alpha(x\otimes\delta_\alpha y)-\mu_\alpha(\chi_\alpha\delta_{\alpha,0}x\otimes\delta_\alpha y)$$
$$=\mu_\alpha(\delta_\alpha x\otimes y)+(-1)^{ki}\mu_\alpha(x\otimes\delta_\alpha y)-(-1)^{k(i+1)}\mu_\alpha(\delta_\alpha x\otimes\chi_\alpha\delta_{\alpha,0}y)$$

for $\deg x=i$;

(A_3') *if at least one element of* $\{x, y, z\}$ *is in* ρ_α*-images, then the associativity*

$$\mu_\alpha(\mu_\alpha(x\otimes y)\otimes z)=\mu_\alpha(x\otimes\mu_\alpha(y\otimes z))$$

holds.

Denote by 1_α the bilateral unit of μ_α. From (A_1) and (H_3) we see easily that

$\rho_\alpha(1)$ is bilateral unit of μ_α. Then from the uniqueness of bilateral units we have

PROPOSITION 1.1. *If a multiplication μ_α satisfies (Λ_1), then*

$$(\Lambda_1'') \qquad 1_\alpha = \rho_\alpha(1).$$

PROPOSITION 1.2. *If a multiplication μ_α satisfies (Λ_1''), then the exact sequence of $\tilde{h}(\ ;\alpha)$ associated with the cofibration (1.1) breaks into short exact sequences*

$$0 \longrightarrow \tilde{h}^j(X\wedge S^{r+k};\alpha) \xrightarrow{(1\wedge\pi)^*} \tilde{h}^j(X\wedge C_\alpha;\alpha) \xrightarrow{(1\wedge i)^*} \tilde{h}^j(X\wedge S^r;\alpha) \longrightarrow 0$$

for any X and j.

For any $x\in\tilde{h}^*(X\wedge S^t;\alpha)$,

$$\begin{aligned}(1\wedge\alpha)^*x &= (1\wedge\alpha)^*\mu_\alpha(x\otimes 1_\alpha) = (-1)^{t(k-1)}\mu_\alpha(x\otimes\alpha^* 1_\alpha)\\ &= (-1)^{t(k-1)}\mu_\alpha(x\otimes\rho_\alpha\alpha^{**}(1)) = 0\end{aligned}$$

because $\rho_\alpha\alpha^{**}=0$. Thus Proposition 1.2 was obtained.

§ 2. Uniqueness theorems of admissible multiplications μ_α

Let μ be an associative and commutative multiplication in $\tilde{h}$. We fix μ once for all throughout this section and shall discuss relations between different admissible multiplications in $\tilde{h}(\ ;\alpha)$.

2.1. Assuming the existence of μ_α satisfying (Λ_1''), put

$$(2.1) \qquad \kappa_2 = (S^{k-r}\pi)^*\sigma_\alpha^{2k}1_\alpha \in \tilde{h}^{2k}(S^{k-r}C_\alpha;\alpha).$$

There exists an element $\kappa_1\in\tilde{h}^k(S^{k-r}C_\alpha;\alpha)$ such that

$$(2.2) \qquad \delta_\alpha\kappa_1=\kappa_2 \quad \textit{and} \quad (S^{k-r}i)^*\kappa_1 = -\sigma_\alpha^k 1_\alpha.$$

This fact can be proved by an entirely parallel argument to [**3**], p. 86, and we omit the details. The choice of κ_1 is not unique. Nevertheless we fix κ_1 once for all.

PROPOSITION 2.1. *If a multiplication μ_α satisfies (Λ_1''), then for any X and $x\in\tilde{h}^i(X\wedge S^{k-r}C_\alpha;\alpha)$ it can be expressed uniquely as a sum*

$$x=\mu_\alpha(x_1\otimes\kappa_1)+\mu_\alpha(x_2\otimes\kappa_2)$$

with $x_1\in\tilde{h}^{i-k}(X;\alpha)$ and $x_2\in\tilde{h}^{i-2k}(X;\alpha)$.

PROOF. Define a homomorphism

$$\lambda:\ \tilde{h}^i(X\wedge S^k;\alpha)\to\tilde{h}^i(X\wedge S^{k-r}C_\alpha;\alpha)$$

by putting

$$\lambda(y) = (-1)^{k(i+1)+1}\mu_\alpha(\sigma_\alpha^{-k}y\otimes\kappa_1)$$

for $y\in\tilde{h}^i(X\wedge S^k;\alpha)$. Then λ gives a splitting of the exact sequence of Proposition 1.2 since $(1\wedge S^{k-r}i)^*\lambda$ is an identity map. Thus, for any $x\in\tilde{h}^i(X\wedge S^{k-r}C_\alpha;\alpha)$ two elements $y\in\tilde{h}^i(X\wedge S^k;\alpha)$ and $y'\in\tilde{h}^i(X\wedge S^{2k};\alpha)$ are determined uniquely so as to satisfy

$$x=(1\wedge S^{k-r}\pi)^*y'+\lambda(y).$$

Put

$$x_1=(-1)^{k(i+1)+1}\sigma_\alpha^{-k}y \quad\text{and}\quad x_2=\sigma_\alpha^{-2k}y'.$$

Then

$$\begin{aligned}\mu_\alpha(x_1\otimes\kappa_1)&+\mu_\alpha(x_2\otimes\kappa_2)\\&=\mu_\alpha(\sigma_\alpha^{-2k}y'\otimes\kappa_2)+(-1)^{k(i+1)+1}\mu_\alpha(\sigma_\alpha^{-k}y\otimes\kappa_1)\\&=(1\wedge S^{k-r}\pi)^*\sigma_\alpha^{2k}\mu_\alpha(\sigma_\alpha^{-2k}y'\otimes 1_\alpha)+(-1)^{k(i+1)+1}\mu_\alpha(\sigma_\alpha^{-k}y\otimes\kappa_1)\\&=(1\wedge S^{k-r}\pi)^*y'+\lambda(y)=x.\end{aligned}$$

The uniqueness of x_1 and x_2 follows also from the exact sequence of Proposition 1.2 and definitions of κ_1 and κ_2. q. e. d.

2.2. Let μ_α and μ'_α be the admissible multiplications with associated cohomology operations χ_α and χ'_α respectively.

To simplify notations we put

$$\mu_\alpha(x\otimes y)=x\wedge y \quad\text{and}\quad \mu'_\alpha(x\otimes y)=x\wedge' y.$$

We see by (A_1) that

(2.3) $\quad x\wedge y=x\wedge' y$ *if either* x *or* y *is in* ρ_α*-images.*

In particular,

(2.4) $\quad \delta_\alpha x\wedge y=\delta_\alpha x\wedge' y$ *and* $x\wedge\delta_\alpha y=x\wedge'\delta_\alpha y$

for any x *and* y.

Also, by (H_8) and (A_1) we obtain

(2.5) $\quad T^*(x\wedge y)=(-1)^{ij}y\wedge x$ *if either* x *or* y *is in* ρ_α*-images*

where $\deg x=i$ *and* $\deg y=j$.

In particular, since *via* the identification $X\wedge S^0=S^0\wedge X=X$ we get $T(S^0,X)=1_X$, we see that

(2.6) $\quad x\wedge a=T^*(x\wedge a)=(-1)^{ij}a\wedge x$ *if either* x *or* a *is in* ρ_α*-images*

where $a\in\tilde{h}^j(S^0;\alpha)$ *and* $x\in\tilde{h}^i(X;\alpha)$.

By (2.1)~(2.5) we obtain

(2.7) $\quad \kappa_i\wedge\kappa_j=\kappa_i\wedge'\kappa_j$ *if* $i=2$ *or* $j=2$,

$\qquad T^*(\kappa_i\wedge\kappa_j)=\kappa_j\wedge\kappa_i$ *if* $i=2$ *or* $j=2$.

2.3. By (1.3) the associated cohomology operation χ_α is determined uniquely by $\bar{\chi}_\alpha = \chi_\alpha(1)$.

Conversely if given any element $\bar{\chi}_\alpha$ of $\tilde{h}^{-k}(S^0; \alpha)$, assuming the multiplication μ is associative and commutative, we put

$$\chi_\alpha(x) = \mu_R(\bar{\chi}_\alpha \otimes x) = (-1)^{ki}\mu_L(x \otimes \bar{\chi}_\alpha)$$

for $x \in \tilde{h}^i(X)$. Then, for any $x \in \tilde{h}^i(X)$ and $y \in \tilde{h}^j(Y)$, we obtain

$$\begin{aligned}\chi_\alpha \mu(x \otimes y) &= \mu_R(\bar{\chi}_\alpha \otimes \mu(x \otimes y)) = \mu_R(\mu_R(\bar{\chi}_\alpha \otimes x) \otimes y) \\ &= \mu_R(\chi_\alpha(x) \otimes y)\end{aligned}$$

and

$$\begin{aligned}\chi_\alpha \mu(x \otimes y) &= (-1)^{k(i+j)}\mu_L(\mu(x \otimes y) \otimes \bar{\chi}_\alpha) \\ &= (-1)^{ki}\mu_L(x \otimes (-1)^{kj}\mu_L(y \otimes \bar{\chi}_\alpha)) = (-1)^{kj}\mu_L(x \otimes \chi_\alpha(y)),\end{aligned}$$

i.e., χ_α is a cohomology operation: $\tilde{h}^*(\ ) \to \tilde{h}^*(\ ; \alpha)$ of degree $-k$ satisfying (1.2) with $\chi_\alpha(1) = \bar{\chi}_\alpha$.

Thus we see that *the cohomology operations satisfying the relation* (1.2) *are in a one-to-one correspondence with the elements of* $\tilde{h}^{-k}(S^0; \alpha)$.

2.4. By the relation (1.3) and (Λ_1) we obtain

$$\chi_\alpha \delta_{\alpha,0} x = \mu_R(\bar{\chi}_\alpha \otimes \delta_{\alpha,0} x) = \mu_\alpha(\bar{\chi}_\alpha \otimes \delta_\alpha x) = \bar{\chi}_\alpha \wedge \delta_\alpha x \tag{2.8}$$

for any $x \in \tilde{h}^*(X; \alpha)$. Thus, by (Λ_2) and (2.4) we obtain

$$\delta_\alpha(x \wedge y) - \delta_\alpha(x \wedge' y) = (\bar{\chi}'_\alpha - \bar{\chi}_\alpha) \wedge \delta_\alpha x \wedge \delta_\alpha y \tag{2.9}$$

for any x and y.

LEMMA 2.2. *Let μ_α and μ'_α be admissible multiplications with associated cohomology operations χ_α and χ'_α respectively. There exists the elements $a(\mu_\alpha)$ and $b(\mu_\alpha, \mu'_\alpha)$ in $\tilde{h}^{-2k}(S^0; \alpha)$ depending only on μ_α and the pair $(\mu_\alpha, \mu'_\alpha)$ respectively determined by relations*

$$T^*(\kappa_1 \wedge \kappa_1) = (-1)^k \kappa_1 \wedge \kappa_1 + a(\mu_\alpha) \wedge \kappa_2 \wedge \kappa_2 \tag{a}$$

and

$$\kappa_1 \wedge \kappa_1 - \kappa_1 \wedge' \kappa_1 = b(\mu_\alpha, \mu'_\alpha) \wedge \kappa_2 \wedge \kappa_2. \tag{b}$$

Further we have relations

$$\delta_\alpha a(\mu_\alpha) \wedge \kappa_2 \wedge \kappa_2 = -2\varepsilon' \chi_\alpha \mu(\delta_{\alpha,0}\kappa_1 \otimes \delta_{\alpha,0}\kappa_1), \tag{i}$$

$$\delta_\alpha b(\mu_\alpha, \mu'_\alpha) = \bar{\chi}'_\alpha - \bar{\chi}_\alpha, \tag{ii}$$

$$b(\mu_\alpha, \mu''_\alpha) = b(\mu_\alpha, \mu'_\alpha) + b(\mu'_\alpha, \mu''_\alpha), \tag{iii}$$

$$a(\mu_\alpha) - a(\mu'_\alpha) = 2\varepsilon' b(\mu_\alpha, \mu'_\alpha) \tag{iv}$$

and

(v) $$\delta_\alpha(x \wedge y) - \delta_\alpha(x \wedge' y) = \delta_\alpha b(\mu_\alpha, \mu'_\alpha) \wedge \delta_\alpha x \wedge \delta_\alpha y\,,$$

where $\varepsilon' = 0$ *if* k *is even, or* $=1$ *if* k *is odd.*

PROOF. By Proposition 2.1 every element $x \in \tilde{h}^i(X \wedge S^{k-r}C_\alpha \wedge S^{k-r}C_\alpha\,;\alpha)$ can be expressed uniquely as

$$x = (x_1 \wedge \kappa_1) \wedge \kappa_1 + (x_2 \wedge \kappa_2) \wedge \kappa_1 + (x_3 \wedge \kappa_1) \wedge \kappa_2 + (x_4 \wedge \kappa_2) \wedge \kappa_2$$

with $x_1 \in \tilde{h}^{i-2k}(X\,;\alpha)$, x_2, $x_3 \in \tilde{h}^{i-3k}(X\,;\alpha)$ and $x_4 \in \tilde{h}^{i-4k}(X\,;\alpha)$.

(a) Put

(*1) $$T^*(\kappa_1 \wedge \kappa_1) = (a_1 \wedge \kappa_1) \wedge \kappa_1 + (a_2 \wedge \kappa_2) \wedge \kappa_1 + (a_3 \wedge \kappa_1) \wedge \kappa_2 + (a_4 \wedge \kappa_2) \wedge \kappa_2$$

with $a_1 \in \tilde{h}^0(S^0\,;\alpha)$, a_2, $a_3 \in \tilde{h}^{-k}(S^0\,;\alpha)$ and $a_4 \in \tilde{h}^{-2k}(S^0\,;\alpha)$. Apply $(S^{k-r}i \wedge 1)^*$ on both sides of (*1), then, by (2.1), (2.2) and (2.5), we obtain

$$(-1)^{k+1}(\sigma_\alpha^k 1_\alpha \wedge \kappa_1) = -\sigma_\alpha^k a_1 \wedge \kappa_1 - \sigma_\alpha^k a_3 \wedge \kappa_2\,.$$

Thus, by Proposition 2.1, we see that

$$a_1 = (-1)^k 1_\alpha \quad \text{and} \quad a_3 = 0\,.$$

Similarly, applying $(1 \wedge S^{k-r}i)^*$ on both sides of (*1), we see that

$$a_2 = 0\,.$$

Finally, making use of (Λ_3'), we get

$$T^*(\kappa_1 \wedge \kappa_1) = (-1)^k \kappa_1 \wedge \kappa_1 + a(\mu_\alpha) \wedge \kappa_2 \wedge \kappa_2$$

with $a(\mu_\alpha) \in \tilde{h}^{-2k}(S^0\,;\alpha)$. From Proposition 2.1 $a(\mu_\alpha)$ is uniquely determined by μ_α.

(b) Put

(*2) $$\kappa_1 \wedge' \kappa_1 = (b_1 \wedge \kappa_1) \wedge \kappa_1 + (b_2 \wedge \kappa_2) \wedge \kappa_1 + (b_3 \wedge \kappa_1) \wedge \kappa_2 + (b_4 \wedge \kappa_2) \wedge \kappa_2$$

with $b_1 \in \tilde{h}^0(S^0\,;\alpha)$, b_2, $b_3 \in \tilde{h}^{-k}(S^0\,;\alpha)$ and $b_4 \in \tilde{h}^{-2k}(S^0\,;\alpha)$. Apply $(S^{k-r}i \wedge 1)^*$ on both sides of (*2), then, by (2.1), (2.2), (2.3) and Proposition 2.1, we see that

$$b_1 = 1_\alpha \quad \text{and} \quad b_3 = 0\,.$$

Similarly, applying $(1 \wedge S^{k-r}i)^*$ on both sides of (*2), we see that

$$b_2 = 0\,.$$

Thus, making use of (Λ_3'), we get

$$\kappa_1 \wedge \kappa_1 - \kappa_1 \wedge' \kappa_1 = b(\mu_\alpha, \mu'_\alpha) \wedge (\kappa_2 \wedge \kappa_2)$$

with $b(\mu_\alpha, \mu'_\alpha) \in \tilde{h}^{-2k}(S^0\,;\alpha)$ determined uniquely by $(\mu_\alpha, \mu'_\alpha)$.

(i) Apply δ_α on both sides of (a), then, by (Λ_2), (2.1), (2.2) and (2.7), we see that

$$\delta_\alpha a(\mu_\alpha)\wedge\kappa_2\wedge\kappa_2=\{(-1)^k-1\}\chi_\alpha\delta_{\alpha,0}\kappa_1\wedge\kappa_2\,.$$

(ii) Apply δ_α on both sides of (b), then, by (Λ_2), (2.1), (2.2) and (2.9), we see that

$$\delta_\alpha b(\mu_\alpha,\mu'_\alpha)\wedge\kappa_2\wedge\kappa_2=(\bar{\chi}'_\alpha-\bar{\chi}_\alpha)\wedge\kappa_2\wedge\kappa_2\,.$$

Thus, by Proposition 2.1, (ii) follows.

(iv) Apply T^* on both sides of (b) and make use of (a);

$$\{(-1)^{k+1}+1\}b(\mu_\alpha,\mu'_\alpha)\wedge\kappa_2\wedge\kappa_2=(a(\mu_\alpha)-a(\mu'_\alpha))\wedge\kappa_2\wedge\kappa_2\,.$$

Then, by Proposition 2.1, we get the relation (iv).

(iii) Obvious.

(v) Obviously seen by (2.9) and (ii). q. e. d.

2.5. The following theorem shows that $b(\mu_\alpha,\mu'_\alpha)$ measures the difference of μ_α from μ'_α.

THEOREM 2.3. *Let μ_α and μ'_α be admissible multiplications in $\tilde{h}(\ ;\alpha)$. Then*

$$x\wedge y-x\wedge' y=(-1)^{k(i+1)}b(\mu_\alpha,\mu'_\alpha)\wedge(\delta_\alpha x\wedge\delta_\alpha y)$$

for any $x\in\tilde{h}^i(X;\alpha)$ and $y\in\tilde{h}^j(Y;\alpha)$.

PROOF. To simplify notations we put

$$\tilde{z}=(-1)^{ks}\chi_\alpha\delta_{\alpha,0}z,\quad \tilde{z}'=(-1)^{ks}\chi'_\alpha\delta_{\alpha,0}z\quad\text{and}\quad b=b(\mu_\alpha,\mu'_\alpha)$$

for any $z\in\tilde{h}^s(Z;\alpha)$.

By (Λ_2), (2.4), (2.8), (2.9), Lemma 2.2 (ii) and (v), we obtain

$$\tilde{z}'=(-1)^{ks}(\bar{\chi}'_\alpha\wedge\delta_\alpha z)=\tilde{z}+(-1)^{ks}\delta_\alpha b\wedge\delta_\alpha z\,,$$

$$(-1)^{ks}\delta_\alpha(z\wedge\kappa_1)=(-1)^{ks}\delta_\alpha z\wedge\kappa_1+z\wedge\kappa_2-\tilde{z}\wedge\kappa_2$$

and

$$(-1)^{ks}\delta_\alpha(z\wedge'\kappa_1)+\tilde{z}'\wedge'\kappa_2=(-1)^{ks}\delta_\alpha(z\wedge\kappa_1)+\tilde{z}\wedge\kappa_2\,.$$

For any $x\in\tilde{h}^i(X;\alpha)$ and $y\in\tilde{h}^j(Y;\alpha)$ making use of (Λ_3), (2.3)~(2.6) and Lemma 2.2 (b) we get

$$\begin{aligned}
b\wedge\delta_\alpha x\wedge\delta_\alpha y\wedge\kappa_2\wedge\kappa_2&=\delta_\alpha x\wedge\delta_\alpha y\wedge b\wedge\kappa_2\wedge\kappa_2\\
&=\delta_\alpha x\wedge\delta_\alpha y\wedge(\kappa_1\wedge\kappa_1-\kappa_1\wedge'\kappa_1)\\
&=(-1)^{k(j+1)}(1\wedge T\wedge 1)^*(\delta_\alpha x\wedge\kappa_1\wedge\delta_\alpha y\wedge\kappa_1-\delta_\alpha x\wedge'\kappa_1\wedge'\delta_\alpha y\wedge'\kappa_1)\\
&=(-1)^{k(j+1)+k(i+j)}(1\wedge T\wedge 1)^*\{((-1)^{ki}\delta_\alpha(x\wedge\kappa_1)-x\wedge\kappa_2+\tilde{x}\wedge\kappa_2)\\
&\quad\wedge((-1)^{kj}\delta_\alpha(y\wedge\kappa_1)-y\wedge\kappa_2+\tilde{y}\wedge\kappa_2)-((-1)^{ki}\delta_\alpha(x\wedge'\kappa_1)\\
&\quad-x\wedge'\kappa_2+\tilde{x}'\wedge'\kappa_2)\wedge'((-1)^{kj}\delta_\alpha(y\wedge'\kappa_1)-y\wedge'\kappa_2+\tilde{y}'\wedge'\kappa_2)\}
\end{aligned}$$

$$=(-1)^{k(i+1)}(1\wedge T\wedge 1)^*\{(x\wedge\kappa_2-\tilde{x}\wedge\kappa_2)$$
$$\wedge(y\wedge\kappa_2-\tilde{y}\wedge\kappa_2)-(x\wedge\kappa_2-\tilde{x}\wedge\kappa_2)\wedge'(y\wedge\kappa_2-\tilde{y}\wedge\kappa_2)\}$$
$$=(-1)^{k(i+1)}\{(x-\tilde{x})\wedge(y-\tilde{y})-(x-\tilde{x})\wedge'(y-\tilde{y})\}\wedge\kappa_2\wedge\kappa_2$$
$$=(-1)^{k(i+1)}\{(x\wedge y-x\wedge' y)-(\tilde{x}\wedge y-\tilde{x}\wedge' y)$$
$$-(x\wedge\tilde{y}-x\wedge'\tilde{y})+(\tilde{x}\wedge\tilde{y}-\tilde{x}\wedge'\tilde{y})\}\wedge\kappa_2\wedge\kappa_2$$

where $T=T(S^{k-r}C_\alpha, Y)$. We put

$$D(a, b)=a\wedge b-a\wedge' b$$

then, from Proposition 2.1, we see that

$$D(x, y)=(-1)^{k(i+1)}b\wedge\delta_\alpha x\wedge\delta_\alpha y+D(\tilde{x}, y)+D(x, \tilde{y})-D(\tilde{x}, \tilde{y})\,. \tag{*3}$$

In the case $x=\tilde{x}\wedge\kappa_2$ of (*3), by (H_6), (A_1), (1.2) and (1.4) we obtain

$$\widetilde{\tilde{x}\wedge\kappa_2}=(-1)^{ki}2\varepsilon(x\wedge\kappa_2)$$

and

$$\delta_\alpha(\tilde{x}\wedge\kappa_2)=\delta_\alpha\tilde{x}\wedge\kappa_2=(-1)^{ki}2\varepsilon(\delta_\alpha x\wedge\kappa_2)$$

where $\varepsilon=0$ if k is even, or $=1$ if k is odd, from which we have

$$D(\tilde{x}\wedge\kappa_2, y)=-2\varepsilon b\wedge\delta_\alpha x\wedge\kappa_2\wedge\delta_\alpha y+D(\tilde{x}\wedge\kappa_2, \tilde{y})\,. \tag{*4}$$

Apply $(1\wedge T)^*$ on both sides of (*4), then, by (2.5) and Proposition 2.1, we obtain

$$D(\tilde{x}, y)=-2\varepsilon b\wedge\delta_\alpha x\wedge\delta_\alpha y+D(\tilde{x}, \tilde{y})\,. \tag{*5}$$

Similarly $D(x, \tilde{y})$ and $D(\tilde{x}, \tilde{y})$ can be calculated as

$$D(x, \tilde{y})=-2\varepsilon b\wedge\delta_\alpha x\wedge\delta_\alpha y+D(\tilde{x}, \tilde{y}) \tag{*6}$$

and

$$D(\tilde{x}, \tilde{y})=4\varepsilon b\wedge\delta_\alpha x\wedge\delta_\alpha y\,. \tag{*7}$$

From (*5), (*6) and (*7), we see that the sum of the last three terms of (*3) is zero. q. e. d.

2.6. The next theorem shows that $a(\mu_\alpha)$ measures the deficiency of μ_α from the commutativity.

THEOREM 2.4. *Let μ_α be an admissible multiplication in $\tilde{h}(\ ;\alpha)$. Then*

$$T^*(y\wedge x)=(-1)^{ij}\{x\wedge y+(-1)^{ki}a(\mu_\alpha)\wedge(\delta_\alpha x\wedge\delta_\alpha y)\}$$

for any $x\in\tilde{h}^i(X;\alpha)$ and $y\in\tilde{h}^j(Y;\alpha)$, where $T=T(X, Y)$.

PROOF. Put $\mu'_\alpha(x\otimes y)=(-1)^{ij}T^*(y\wedge x)$, then it is a routine matter to see that μ'_α is also an admissible multiplication with the same associated cohomology operation as μ_α. Lemma 2.2 (a) shows that

$$\kappa_1 \wedge' \kappa_1 = \kappa_1 \wedge \kappa_1 + (-1)^k a(\mu_\alpha) \wedge (\kappa_2 \wedge \kappa_2).$$

Hence the theorem follows from Theorem 2.3.

THEOREM 2.5. *Let μ_α be an admissible multiplication in $\tilde{h}(\ ;\alpha)$ with the associated cohomology operation χ_α and any $b \in \tilde{h}^{-2k}(S^0;\alpha)$ given. If we put*

$$\mu'_\alpha(x \otimes y) = x \wedge y + (-1)^{k(i+1)+1} b \wedge \delta_\alpha x \wedge \delta_\alpha y$$

for $x \in \tilde{h}^i(X;\alpha)$ and $y \in \tilde{h}^j(Y;\alpha)$, then μ'_α is also an admissible multiplication with associated cohomology operation χ'_α such that $\bar{\chi}'_\alpha = \delta_\alpha b + \bar{\chi}_\alpha$ and $b(\mu_\alpha, \mu'_\alpha) = b$.

PROOF. It is straightforward to see that μ'_α satisfies (A_0) and (A_1). Put $\chi'_\alpha(x) = \mu_R((\delta_\alpha b + \bar{\chi}_\alpha) \otimes x) = (-1)^{ki} \mu_L(x \otimes (\delta_\alpha b + \bar{\chi}_\alpha))$ for any $x \in \tilde{h}^i(X)$ then μ'_α satisfy (A_2) with associated cohomology operation χ'_α and since $\delta_\alpha b = \bar{\chi}'_\alpha - \bar{\chi}_\alpha$, we obtain $b(\mu_\alpha, \mu'_\alpha) = b$. By a simple calculation we see that

$$(\sharp) \qquad (x \wedge' y) \wedge' z - x \wedge' (y \wedge' z) = (x \wedge y) \wedge z - x \wedge (y \wedge z)$$
$$+ (-1)^{k(j+1)+1} (b \wedge x - x \wedge b) \wedge \delta_\alpha y \wedge \delta_\alpha z$$
$$+ \varepsilon'' b' \wedge \delta_\alpha x \wedge \chi_\alpha \mu(\delta_{\alpha,0} y \otimes \delta_{\alpha,0} z)$$

where $j = \deg y$ and $\varepsilon'' = 0$ if k is even, or $= (-1)^{j+1} - (-1)^{i+1}$ if k is odd. If x or b is in ρ_α-images, then

$$b \wedge x = x \wedge b \quad \text{and} \quad \delta_\alpha x = 0$$

by (2.6). If y or z is in ρ_α-images, then

$$\delta_\alpha y \wedge \delta_\alpha z = 0 \quad \text{and} \quad \delta_{\alpha,0} y \otimes \delta_{\alpha,0} z = 0.$$

Thus, if x or y, or z, is in ρ_α-images, then the second term and the third term of the right side of $(\sharp)$ vanishe, and the first term also vanishes by (A_3') for μ_α, i.e., (A_3) for μ'_α was proved. q. e. d.

2.7. From Theorems 2.3, 2.4, 2.5 and Lemma 2.2 we obtain the following corollaries.

COROLLARY 2.6. *Let μ_α, μ'_α be two admissible multiplications in $\tilde{h}(\ ;\alpha)$ and χ_α, χ'_α be the associated cohomology operations with μ_α, μ'_α respectively. The following conditions are equivalent.*

(i) $$\delta_\alpha b(\mu_\alpha, \mu'_\alpha) = 0,$$

(ii) $$\chi_\alpha = \chi'_\alpha,$$

(iii) $$\chi_\alpha(1) = \chi'_\alpha(1).$$

COROLLARY 2.7. *Let μ_α and μ'_α be two admissible multiplications. The following conditions are equivalent.*

(i) $$\mu_\alpha = \mu'_\alpha,$$

(ii) $$b(\mu_\alpha, \mu'_\alpha) = 0,$$

(iii) μ_α *coincides with* μ'_α *for the case of* $X=Y=S^{k-r}C_\alpha$.

COROLLARY 2.8. *If there exists an admissible multiplication with associated cohomology operation* χ_α *in* $\tilde{h}(\ ;\alpha)$, *then admissible multiplications in* $\tilde{h}(\ ;\alpha)$ *are in one-to-one correspondence with the elements of* $\tilde{h}^{-2k}(S^0;\alpha)$. *In particular, admissible multiplications in* $\tilde{h}(\ ;\alpha)$ *which have the same cohomology operation are in one-to-one correspondence with the elements of* $\tilde{h}^{-2k}(S^0;\alpha)\cap\delta_\alpha^{-1}(0)$.

COROLLARY 2.9. (i) *If k is even and there exists an admissible multiplication in* $\tilde{h}(\ ;\alpha)$, *then either there is no commutative one, or every one is commutative.*

(ii) *If k is odd and there exists an admissible multiplication, then either there is no commutative one, or commutative ones are in one-to-one correspondence with the elements of* $\mathrm{Tor}(\tilde{h}^{-2k}(S^0;\alpha), Z_2)$.

COROLLARY 2.10. (i) *If k is even,* $\tilde{h}^{-2k}(S^0;\alpha)=\rho_\alpha(\tilde{h}^{-2k}(S^0))$ *and there exists an associative admissible multiplication in* $\tilde{h}(\ ;\alpha)$, *then every admissible one in* $\tilde{h}(\ ;\alpha)$ *is associative.*

(ii) *If k is odd,* $\tilde{h}^{-2k}(S^0;\alpha)=\rho_\alpha(\tilde{h}^{-2k}(S^0))$ *and there exists an associative admissible multiplication in* $\tilde{h}(\ ;\alpha)$ *of which the associated cohomology operation satisfies* $\chi_\alpha\mu(\delta_{\alpha,0}\otimes\delta_{\alpha,0})=0$, *then every admissible one is associative.*

PROOF. See the formula (♯) in the proof of Theorem 2.5, Corollary 2.10 follows from (2.6).

§ 3. Stable homotopy of some elementary complexes

3.1. The results in the following table are well-known.

	$i<0$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4, 5$	$i=6$
$\{S^{n+i}, S^n\}$	0	Z	Z_2	Z_2	Z_{24}	0	Z_2
generators		1	η	$\eta^2=\eta\cdot\eta$	ν		ν^2

Let P be the complex projective plane, i.e., $P=S^2\cup_\eta e^4$. Let $i: S^2\to P$ and $\pi: P\to S^4$ be the inclusion and the map collapsing S^2. We have a cofibration

$$S^2 \xrightarrow{i} P \xrightarrow{\pi} S^4. \tag{3.1}$$

From Puppe's exact sequence and its dual associated with (3.1) we obtain

LEMMA 3.1. *The groups* $\{S^{n+k+3}, S^nP\}$ *and* $\{S^nP, S^{n-k+3}\}$ *are both isomorphic to the corresponding groups in the following table:*

generators of	$k\leqq -2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
	0	Z	0	Z	Z_{12}	0	Z_{24}	Z_2
$\{S^{n+k+3}, S^nP\}$		i		$\tilde{\zeta}$	$i\nu$		$\tilde{\nu}$	$i\nu^2$
$\{S^nP, S^{n-k+3}\}$		π		$\bar{\zeta}$	$\nu\pi$		$\bar{\nu}$	$\nu^2\pi$

where $\tilde{\zeta}$, $\bar{\zeta}$, $\tilde{\nu}$ *and* $\bar{\nu}$ *are defined by*

(3.2) $$\pi\tilde{\zeta}=2\cdot 1_{S^{n+4}},\quad \bar{\zeta}i=2\cdot 1_{S^{n+2}},\quad \pi\tilde{\nu}=\nu \quad and \quad \bar{\nu}i=\nu .$$

LEMMA 3.2. *The groups* $\{P, S^iP\}$, $0\leqq i$, *and* $\{S^jP, P\}$, $j=1, 2$, *are isomorphic to the corresponding groups in the following table:*

		generators
$\{P, S^iP\}$, $i\geqq 3$	0	
$\{P, S^2P\}$	Z	$(S^2i)\pi$
$\{P, SP\}$	0	
$\{P, P\}$	$Z+Z$	1_P, $\tilde{\zeta}\pi$(*or* $i\bar{\zeta}$)
$\{SP, P\}$	Z_6	$i\nu(S\pi)$
$\{S^2P, P\}$	Z	$\tilde{\zeta}(S^2\bar{\zeta})$

We have relations

(3.3) $$i\bar{\zeta}+\tilde{\zeta}\pi=2\cdot 1_P ,$$

(3.4) $$1_P\wedge\eta=3i\nu(S\pi) ,$$

(3.5) $$\tilde{\zeta}\eta=6i\nu \quad and \quad \eta\bar{\zeta}=6\nu\pi .$$

PROOF. From Lemma 3.1 and Puppe's exact sequence, we see easily the results for $\{P, S^iP\}$, $i\geqq 0$. Then, by (3.2), we obtain the relation (3.3).

We sketch the proof of $\{SP, P\}\cong Z_6$ and (3.4) (for details, see [3] Theorem 8.1). In the exact sequence

(*) $$0\longrightarrow\{S^2P, P\}\xrightarrow{(S^2i)^*}\{S^4, P\}\xrightarrow{\eta^*}\{S^5, P\}\xrightarrow{(S\pi)^*}\{SP, P\}\xrightarrow{(Si)^*}0$$

η^*-image is at most of order 2. Thus, making use of Lemma 3.1, we obtain $\{SP, P\}\cong Z_6$ or Z_{12}, with generator $i\nu(S\pi)$, and we can put $1_P\wedge\eta=a(Si)\nu(S^2\pi)$ $\in\{S^2P, SP\}$ for some integer a. $Sq^4\neq 0$ in $P\wedge M_2\wedge M_2$ (where $M_2=S^1\cup_2 e^2$) implies that a is odd. By (3.1), (3.3) and (*), $2a\cdot i\nu\pi=(S\pi)^*\eta^*\tilde{\zeta}=0$, which implies that $a=3$ and $\{SP, P\}\cong Z_6$.

Since η^*-image is of order 2 in the exact sequence (*), we obtain $\bar{\zeta}\eta=6i\nu$ and $\{S^2P, P\}\cong Z$ with generator $\bar{\zeta}(S^2\bar{\zeta})$. Similarly in the exact sequence

$$0 \longrightarrow \{S^2P, P\} \xrightarrow{\pi_*} \{SP, S^3\} \xrightarrow{\eta_*} \{SP, S^2\} \xrightarrow{i_*} \{SP, P\} \longrightarrow 0$$

η_*-image is of order 2, then making use of Lemma 3.1, we obtain $\eta\bar{\zeta}=6\nu\pi$.

q. e. d.

3.2. We shall see that $P\wedge P$ is homotopy equivalent in stable range to the following mapping cone

$$\bar{N}_\eta = S^2P \bigcup_{\bar{g}} C(S^3P)\,, \tag{3.6}$$

where

$$\bar{g}=3(S^2i)\nu(S^3\pi): S^3P \to S^7 \to S^4 \subset S^2P\,.$$

We denote also by N_η a subcomplex of $\bar{N}_\eta$ obtained by removing the 6-cell S^2P-S^4, i. e.,

$$N_\eta = S^4 \bigcup_{g} C(S^3P)\,, \tag{3.6$'$}$$

where

$$g=3\nu(S^3\pi): S^3P\to S^7\to S^4\,.$$

The cell structures of $\bar{N}_\eta$ and N_η can be interpreted as follows:

$$\bar{N}_\eta=(S^2P\vee S^6)\cup e^8 \quad \textit{and} \quad N_\eta=(S^4\vee S^6)\cup e^8\,, \tag{3.7}$$

where e^8 is attached to $S^4\vee S^6$ by a map representing the sum of $3\nu\in\{S^7, S^4\}$ and $S^4\eta\in\{S^7, S^6\}$.

We use the following notations:

(3.8) $j: N_\eta\subset\bar{N}_\eta$ *the inclusion;*
$p;\ N_\eta\to S^6$ *the map collapsing* N_η;
$\bar{i}_0: S^2P\subset\bar{N}_\eta$, $i_0: S^4\subset N_\eta$ *the inclusions;*
$\bar{\pi}_0: \bar{N}_\eta\to S^4P$, $\pi_0: N_\eta\to S^4P$ *the map collapsing* S^2P *or* S^4;
$\bar{i}_1: S^6\subset\bar{N}_\eta$, $i_1: S^6\subset N_\eta$ *the inclusions;*
$\bar{\pi}_1: \bar{N}_\eta\to S^2P\cup e^8$, $\pi_1: N_\eta\to S^4\cup e^8$ *the map collapsing* S^6.

Hereafter, these mappings will be fixed as to satisfy the following relations:

$$ji_0=\bar{i}_0(S^2i),\quad \bar{i}_1=ji_1\quad p\bar{i}_0=S^2\pi,\quad \pi_0=\bar{\pi}_0 j \tag{3.8$'$}$$
$$\textit{and}\quad \bar{\pi}_0\bar{i}_1=\pi_0 i_1=S^4 i\,.$$

By (3.4), $1_P\wedge\eta$ is homotopic to $\bar{g}$ in stable range. On the other hand $P\wedge P$ is mapping cone of $1_P\wedge\eta$. Therefore we obtain

LEMMA 3.3. *There exists an element $\bar{\alpha}$ of $\{\bar{N}_\eta, P\wedge P\}$ satisfying the* *following conditions:*

(3.9) (i) $\bar{\alpha}$ *is a homotopy equivalence, i.e., there is an inverse* $\bar{\beta}\in\{P\wedge P, \bar{N}_\eta\}$ *of* $\bar{\alpha}$ *such that* $\bar{\alpha}\bar{\beta}=1$ *and* $\bar{\beta}\bar{\alpha}=1$,

(ii) $\bar{\alpha}\bar{i}_0=1_P\wedge i$, *thus* $\bar{\beta}(1_P\wedge i)=\bar{i}_0$

and

(iii) $(1_P\wedge\pi)\bar{\alpha}=\bar{\pi}_0$, *thus* $\bar{\pi}_0\bar{\beta}=1_P\wedge\pi$.

Put

$$\alpha_0=\bar{\alpha}\bar{i}_1\in\{S^6, P\wedge P\} \quad and \quad \beta_0=p\bar{\beta}\in\{P\wedge P, S^6\}\,. \tag{3.10}$$

It follows from (ii), (iii) of (3.9)

$$(1_P\wedge\pi)\alpha_0=S^4 i \quad and \quad \beta_0(1_P\wedge i)=S^2\pi\,. \tag{3.10$'$}$$

LEMMA 3.4. (i) *Let* $\bar{\alpha}\in\{\bar{N}_\eta, P\wedge P\}$ *be an element satisfying* (3.9). *Any element* $\bar{\alpha}'\in\{\bar{N}_\eta, P\wedge P\}$ *satisfies* (3.9) *if and only if*

$$\bar{\alpha}'=\bar{\alpha}+k(1_P\wedge i)(S^2\zeta)(S^2\bar{\zeta})\bar{\pi}_0 \qquad \textit{for some integer } k.$$

(ii) *Put* $\alpha_0=\bar{\alpha}\bar{i}_1$ *and* $\alpha_0'=\bar{\alpha}'\bar{i}_1$. *If* $\bar{\alpha}$ *and* $\bar{\alpha}'$ *satisfy* (3.9), *then*

$$\alpha_0'=\alpha_0+2k(\zeta\wedge i) \qquad \textit{for some integer } k.$$

PROOF. (i) Assume that $\bar{\alpha}$ and $\bar{\alpha}'$ satisfy (3.9). There exists $\gamma'\in\{\bar{N}_\eta, S^2P\}$ such that $(1_P\wedge i)\gamma'=\bar{\alpha}'-\bar{\alpha}$ since $(1_P\wedge\pi)(\bar{\alpha}'-\bar{\alpha})=0$. Then $(1_P\wedge i)(\gamma'\bar{i}_0)=(\bar{\alpha}'-\bar{\alpha})\bar{i}_0=0$. The kernel of $(1_P\wedge i)_*:\{S^2P, S^2P\}\to\{S^2P, P\wedge P\}$ vanishes since $\{S^2P, S^3P\}=0$. We have $\gamma'\bar{i}_0=0$. Thus γ' is contained in the image of $\bar{\pi}_0^*:\{S^4P, S^2P\}\to\{\bar{N}_\eta, S^2P\}$. Then there exists an integer k such that $\gamma'=k(S^2\zeta)(S^4\bar{\zeta})\bar{\pi}_0$, and $\bar{\alpha}'=\bar{\alpha}+(1_P\wedge i)\gamma'=\bar{\alpha}+k(1_P\wedge i)(S^2\zeta)(S^4\bar{\zeta})\bar{\pi}_0$.

Conversely, if $\bar{\alpha}$ satisfies (ii), (iii) of (3.9), then so does $\bar{\alpha}'$. Put

$$\bar{\beta}'=\bar{\beta}-k\cdot\bar{i}_0(S^2\zeta)(S^4\bar{\zeta})(1_P\wedge\pi)\,,$$

then $\bar{\alpha}'$ is a homotopy equivalence with an inverse $\bar{\beta}'$.

(ii) Making use of (3.8) and (3.2), it follows from (i). q.e.d.

3.3. From the Puppe's exact sequence associated with a cofibration

$$P\wedge S^2\xrightarrow{1_P\wedge i}P\wedge P\xrightarrow{1_P\wedge\pi}P\wedge S^4\,, \tag{3.11}$$

making use of Lemma 3.1, (3.5) and (3.10) we obtain

LEMMA 3.5. *The groups* $\{S^{6-k}, P\wedge P\}$ *and* $\{P\wedge P, S^{6+k}\}$, $k\geqq -2$, *are both isomorphic to the corresponding groups in the following table:*

generators of	$k\geqq 3$	$k=2$	$k=1$	$k=0$	$k=-1$	$k=-2$
	0	Z	0	$Z+Z$	Z_6	Z
$\{S^{6-k}, P\wedge P\}$		$i\wedge i$		$\tilde{\zeta}\wedge i,\ \alpha_0$	$i\nu\wedge i$	$\tilde{\zeta}\wedge\tilde{\zeta}$
$\{P\wedge P, S^{6+k}\}$		$\pi\wedge\pi$		$\bar{\zeta}\wedge\pi,\ \beta_0$	$\nu\pi\wedge\pi$	$\bar{\zeta}\wedge\bar{\zeta}$

where α_0 *and* β_0 *are elements satisfying* $(1_P\wedge\pi)\alpha_0=S^4 i$ *and* $\beta_0(1_P\wedge i)=S^2\pi$.

LEMMA 3.6. *The groups* $\{S^{2-k}, P\wedge P\}$ *and* $\{P\wedge P, S^{4+k}P\}$ *are isomorphic to the corresponding groups in the following table:*

generators of	$k\geqq 3$	$k=2$	$k=1$	$k=0$	$k=-1$	$k=-2$
	0	Z	0	$Z+Z+Z$	Z_3	$Z+Z+Z$
$\{S^{2-k}P, P\wedge P\}$		$(i\wedge i)\pi$		$i_P\wedge i,\ \alpha_0(S^2\pi),$ $i\bar{\zeta}\wedge i$ *(or* $\tilde{\zeta}\pi\wedge i$*)*	$i\nu(S\pi)\wedge i$	$(S^2\bar{\zeta})\wedge i,$ $\tilde{\alpha}_1,\ \bar{\alpha}_1$
$\{P\wedge P, S^{4+k}P\}$		$(S^2 i)\pi\wedge\pi$		$1_P\wedge\pi,\ (S^4 i)\beta_0,$ $i\bar{\zeta}\wedge\pi$ *(or* $\tilde{\zeta}\pi\wedge\pi$*)*	$i\nu(S\pi)\wedge\pi$	$\tilde{\zeta}(S^2\bar{\zeta})(1_P\wedge\pi),$ $\bar{\beta}_1,\ \tilde{\beta}_1$

where $\tilde{\alpha}_1$, $\bar{\alpha}_1$, $\tilde{\beta}_1$ *and* $\bar{\beta}_1$ *are elements satisfying*

$$(1_P\wedge\pi)\tilde{\alpha}_1=\tilde{\zeta}\pi,\quad (1_P\wedge\pi)\bar{\alpha}_1=i\bar{\zeta},\quad \tilde{\beta}_1(1_P\wedge i)=\tilde{\zeta}\pi$$
$$\text{and}\quad \bar{\beta}_1(1_P\wedge i)=i\bar{\zeta}\,.$$

We have relations

$$\alpha_0\eta=3(i\nu\wedge i)\quad\text{and}\quad \eta\beta_0=3(\nu\pi\wedge\pi)\,. \tag{3.12}$$

PROOF. From the Puppe's exact sequence associated with (3.11) and Lemma 3.5, we see easily the table. In the exact sequence

$$\xrightarrow{(S^4 i)^*}\{S^6, P\wedge P\}\xrightarrow{\eta^*}\{S^7, P\wedge P\}\xrightarrow{(S^3\pi)^*}\{S^3P, P\wedge P\}\xrightarrow{(S^3 i)^*}0$$

η^*-image is of order 2 since $\{S^7, P\wedge P\}\cong Z_6$ and $\{S^3P, P\wedge P\}\cong Z_3$. Put $\alpha_0\eta=a(i\nu\wedge i)$ then $2a\equiv 0$ (mod. 6). On the other hand, η^*-image of the another generator of $\{S^6, P\wedge P\}$ vanishes since

$$\eta^*(\tilde{\zeta}\wedge i)=6(i\nu\wedge i)=0\quad \text{in}\ \{S^7, P\wedge P\}\,. \tag{3.12'}$$

Thus we obtain $a\equiv 3$ (mod. 6) and $\alpha_0\eta=3(i\nu\wedge i)$. Similarly we obtain $\eta\beta_0=3(\nu\pi\wedge\pi)$. q. e. d.

3.4. Next we put

$$Q=S^3\underset{3\nu}{\cup}e^7 \tag{3.14}$$

and denote by

(3.15) $$i': S^3 \longrightarrow Q \quad and \quad \pi': Q \longrightarrow S^7$$

the canonical inclusion and the map collapsing S^3 to a point. From (3.8) and (3.15) we have following three cofibrations:

(3.15′) $$S^3 \xrightarrow{i'} Q \xrightarrow{\pi'} S^7,$$

(3.16) $$S^4 \xrightarrow{i_0} N_\eta \xrightarrow{\pi_0} S^4P,$$

(3.17) $$S^6 \xrightarrow{i_1} N_\eta \xrightarrow{\pi_1} SQ,$$

where $C(S^3P)$ is attached to S^4 by a map representing $3\nu\pi \in \{S^3P, S^4\}$ and CQ is attached to S^6 by a map representing $\eta\pi' \in \{Q, S^6\}$.

The following tables (3.18), (3.19) and (3.20) are verified from the Puppe's exact sequence associated with the cofibrations (3.15′), (3.16), (3.17) and Lemmas 3.1, 3.2 and 3.5.

(3.18)

	$i\geqq 6$	$i=5$	$i=4$	$i=3$	$i=2$	$i=1$
groups $\{Q, S^iP\}$	0	Z	0	Z	Z_3	Z
generators		$i\pi'$		$\xi\pi'$	$i\nu\pi'$	ε'

where ε' *is defined by* $\varepsilon' i' = 4i$.

(3.19)

	$i\geqq 5$	$i=4$	$i=3$	$i=2$
groups $\{N_\eta, S^iP\}$	0	$Z+Z$	Z_3	$Z+Z$
generators		π_0, $\xi\pi\pi_0=\xi\pi'\pi_1$ (*or* $i\xi\pi_0$)	$i\nu\pi\pi_0$ $=i\nu\pi'\pi_1$	$\xi(S^2\bar{\xi})\pi_0$, ε_0

where ε_0 *is defined by* $\varepsilon_0 i_0 = 2i$.

(3.20)

	$\{Q, P\wedge P\}$	$\{SQ, P\wedge P\}$
	Z_3	$Z+Z$
generators	$(i\nu\wedge i)\pi'$	$(\xi\wedge\xi)\pi'$, ρ

where ρ *is defined by* $\rho i' = i\wedge i$.

Making use of (3.5), (3.12) and (3.12′) we get

(3.21) $$(\zeta \wedge i)\eta\pi' = (\zeta\eta \wedge i)\pi' = 6(i\nu \wedge i)\pi' = 0\,,$$
$$\alpha_0\eta\pi' = 3(i\nu \wedge i)\pi' = 0$$

in $\{Q, P\wedge P\}$. Then, from the Puppe's exact sequence associated with (3.17), Lemma 3.5 and (3.20) we have the following table

(3.22)

	$\{N_\eta, P\wedge P\}$	$\{N_\eta, S(P\wedge P)\}$
	$Z+Z+Z+Z$	Z_3
generators	$(\zeta\wedge\zeta)\pi'\pi_1 = (\zeta\wedge\zeta)\pi\pi_0$, $\rho\pi_1$, u, v,	$(i\nu\wedge i)\pi'\pi_1 = (i\nu\wedge i)\pi\pi_0$

where u and v are elements satisfying $ui_1 = \alpha_0$ and $vi_1 = \zeta\wedge i$.

LEMMA 3.7. *For any $\omega_0 = a\alpha_0 + b(\zeta\wedge i) \in \{S^6, P\wedge P\}$*

(i) *there exists $\omega \in \{N_\eta, P\wedge P\}$ satisfying*

(3.23) $$\omega_0 = \omega i_1\,,$$

(ii) *any $\omega \in \{N_\eta, P\wedge P\}$ satisfies* (3.23) *if and only if*

(3.23′) $$\omega = c_1(\zeta \wedge \zeta)\pi\pi_0 + c_2\rho\pi_1 + au + bv \qquad \textit{for some integers } c_1,\ c_2,$$

(iii) *there exists $\omega \in \{N_\eta, P\wedge P\}$ satisfying* (3.23) *and*

(3.24) $$(1_P \wedge \pi)\omega = a\pi_0\,.$$

PROOF. (i) From (3.21) we obtain

$$(\eta\pi')^*\omega_0 = 0 \in \{Q, P \wedge P\}\,.$$

Thus, there exists an element $\omega \in \{N_\eta, P\wedge P\}$ satisfying (3.23).

(ii) Making use of (3.22) any element $\omega \in \{N_\eta, P\wedge P\}$ can be expressed as

$$\omega = c_1(\zeta \wedge \zeta)\pi'\pi_1 + c_2\rho\pi_1 + c_3 u + c_4 v$$

for some integers c_i. If ω satisfies (3.23) we obtain

$$\omega_0 = \omega i_1 = c_3 u i_1 + c_4 v i_1 = c_3\alpha_0 + c_4(\zeta \wedge i)\,.$$

Then from Lemma 3.5 we have

$$c_3 = a \quad \text{and} \quad c_4 = b\,.$$

Conversely, if ω satisfies (3.23′) for any integer c_1 and c_2 then

$$\omega i_1 = \omega_0\,.$$

(iii) Making use of (3.18), (3.19), (3.20) and (3.22) we have the following commutative diagram associated with (3.11) and (3.17), in which all rows and columns are exact, and the middle column, the right column and the third row splits.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{SQ, S^2P\} \arrow[r] \arrow[d] & \{SQ, P\wedge P\} \arrow[r] \arrow[d, "\pi_1^*"] & \{SQ, S^4P\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{N_7, S^2P\} \arrow[r, "(1_p\wedge i)_*"] \arrow[d] & \{N_7, P\wedge P\} \arrow[r, "(1_p\wedge \pi)_*"] \arrow[d, "i_1^*"] & \{N_7, S^4P\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{S^6, S^2P\} \arrow[r] \arrow[d] & \{S^6, P\wedge P\} \arrow[r] \arrow[d] & \{S^6, S^4P\} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From (3.2), we obtain

$$(1_P\wedge\pi)(\xi\wedge\zeta)\pi'=2\xi\pi'$$

then, for the other generator ρ of $\{SQ, P\wedge P\}$, we have

$$(1_P\wedge\pi)\rho=l\xi\pi' \qquad \text{for some odd integer } l$$

since the first row of the above diagram is exact. And since the diagram is commutative, we obtain

$$(1_P\wedge\pi)_*(\xi\wedge\zeta)\pi'\pi_1=2\xi\pi'\pi_1=2\xi\pi\pi_0\,,$$

$$(1_P\wedge\pi)_*\rho\pi_1=l\xi\pi'\pi_1=l\xi\pi\pi_0\,,$$

$$(1_P\wedge\pi)_*u=\pi_0\,,$$

$$(1_P\wedge\pi)_*v=m\xi\pi\pi_0 \qquad \text{for some integer } m\,.$$

We can take the integers c_1 and c_2 satisfying

$$2c_1+lc_2+mb=0$$

since l is odd, and we put

$$\omega=c_1(\xi\wedge\zeta)\pi\pi_0+c_2\rho\pi_1+au+bv\,.$$

Then, from (ii), ω satisfies (3.23) and

$$(1_P\wedge\pi)_*\omega=(2c_1+lc_2+mb)\xi\pi\pi_0+a\pi_0=a\pi_0\,. \qquad \text{q. e. d.}$$

3.5. We consider the ordinary homology maps induced by the elements of $\{S^6, P\wedge P\}$. Let s_6 be a generator of the group $\tilde{H}_6(S^6)$ and let $(e_2\wedge e_2, e_4\wedge e_2, e_2\wedge e_4, e_4\wedge e_4)$ be the generators of groups $\tilde{H}_*(P\wedge P)$, where $e_i\wedge e_j$ is a generator of $(i+j)$-dim. groups, $e_4\wedge e_2$ is the generator represented by the 6-cell of S^2P if we put $P\wedge P=S^2P\cup C(S^3P)$, and $e_2\wedge e_4$ is the other 6-dim. generator.

The element of $f\in\{S^6, P\wedge P\}$ is called to be of type (k, l) if induced homology map is

$$f_*(s_6)=k(e_4\wedge e_2)+l(e_2\wedge e_4) \qquad \text{for some integers } k,\ l.$$

Making use of the relations $(1_P\wedge\pi)\alpha_0=S^4i$, $(1_P\wedge\pi)(\zeta\wedge i)=0$ and $(\pi\wedge 1_P)(\zeta\wedge i)=2(1_{S^4}\wedge i)$, we obtain that α_0 is of type $(k, 1)$ and $\zeta\wedge i$ is of type $(2, 0)$. For any element $\omega_0\in\{S^6, P\wedge P\}$ we can put

$$\omega_0=a\cdot\alpha_0+b(\zeta\wedge i)$$

then ω_0 is of type $(ak+2b, a)$. On the other hand there is $T\alpha_0$ of type $(1, k)$ where $T=T(P, P)$, then easily seen that k is odd. We put $k=2k'+1$ and

$$\omega_0'=\alpha_0-k'(\zeta\wedge i) \tag{3.25}$$

then ω_0' is of type $(1, 1)$ and $T\omega_0'$ is so.

PROPOSITION 3.8. *There exists an element $\alpha\in\{N_\eta, P\wedge P\}$ satisfying the relations*

$$\text{(i)} \qquad (1_P\wedge\pi)T\alpha=(1_P\wedge\pi)\alpha=\pi_0$$

and

$$\text{(ii)} \qquad T(1_P\wedge i)+(1_P\wedge i)=\alpha i_1(S^2\pi)+i\zeta\wedge i.$$

PROOF. We take $\omega_0'\in\{S^6, P\wedge P\}$ of type $(1, 1)$. From Lemma 3.7 (iii) and (3.25), there exists $\alpha\in\{N_\eta, P\wedge P\}$ (which is extended from ω_0') satisfying the relation (i). Let s_i be a generator of $\tilde{H}_i(S^i)$ and (e_2, e_4) be the generators of $\tilde{H}_2(P)$ and $\tilde{H}_4(P)$. The homology maps i_*, π_* and ζ_* can put

$$i_*:\ \tilde{H}_*(S^2)\longrightarrow\tilde{H}_*(P):\ i_*(s_2)=e_2,$$

$$\pi_*:\ \tilde{H}_*(P)\longrightarrow\tilde{H}_*(S^4):\ \pi_*(e_2, e_4)=(0, s_4)$$

and

$$\zeta_*:\ \tilde{H}_*(P)\longrightarrow\tilde{H}_*(S^2):\ \zeta_*(e_2, e_4)=(2s_2, 0)$$

since $\zeta i=2\cdot 1_{S^2}$. We consider the following maps $\tilde{H}_*(S^2P)\longrightarrow\tilde{H}_*(P\wedge P)$:

$$(1_P\wedge i)_*:\ (\sigma^2e_2, \sigma^2e_4)\longrightarrow(e_2\wedge e_2, e_4\wedge e_2),$$

$$T_*(1_P\wedge i)_*:\ (\sigma^2e_2, \sigma^2e_4)\longrightarrow(e_2\wedge e_2, e_2\wedge e_4),$$

$$(\alpha i_1(S^2\pi))_*:\ (\sigma^2e_2, \sigma^2e_4)\longrightarrow(0, e_4\wedge e_2+e_2\wedge e_4)$$

and

$$(i\zeta\wedge i)_*:\ (\sigma^2e_2, \sigma^2e_4)\longrightarrow(2e_2\wedge e_2, 0).$$

Then we obtain

$$(1_P \wedge i)_* + T_*(1_P \wedge i)_* = (\alpha i_1(S^2\pi))_* + (i\bar{\zeta} \wedge i)_* .$$

From Lemma 3.6 we see easily that the stable homotopy class of any element $f \in \{S^2P, P \wedge P\}$ is determined by its homology type, and then (ii) follows.

q. e. d.

§ 4. Existence of admissible multiplication in $\tilde{h}(\ ;\eta)$

Let μ be an associative and commutative multiplication in a reduced cohomology theory $\tilde{h}$. In this section we define a multiplication μ_η in $\tilde{h}(\ ;\eta)$ and prove that μ_η is admissible. Thereby, we need some assumptions on $\tilde{h}$.

4.1. Using the notation of **3.2**, the cofibration

$$S^4 \xrightarrow{i_0} N_\eta \xrightarrow{\pi_0} S^4P$$

yields, for any finite CW-complex W with a base point, a cofibration

$$W \wedge S^4 \xrightarrow{1\wedge i_0} W \wedge N_\eta \xrightarrow{1\wedge\pi_0} W \wedge S^4P .$$

In the exact sequence of $\tilde{h}$ associated with this cofibration,

$$(1_W \wedge S^n g)^* : \tilde{h}^k(W \wedge S^{n+4}) \longrightarrow \tilde{h}^k(W \wedge P \wedge S^{n+3})$$

becomes a trivial map if $3(\nu\pi)^{**} = 0$. Thus we obtain

LEMMA 4.1. *If $3(\nu\pi)^{**}=0$ in $\tilde{h}$, then the $\tilde{h}$-cohomology sequence associated with the above cofibration breaks into the following short exact sequences*

$$(4.1) \qquad 0 \longrightarrow \tilde{h}^k(W \wedge S^4P) \xrightarrow{(1\wedge\pi_0)^*} \tilde{h}^k(W \wedge N_\eta) \xrightarrow{(1\wedge i_0)^*} \tilde{h}^k(W \wedge S^4) \longrightarrow 0 .$$

By (4.1) for $W=S^0$ and $k=4$, there exists $\gamma_0' \in \tilde{h}^4(N_\eta)$ such that $i_0^*\gamma_0' = \sigma^4 1$. If $3\nu^{**}=0$ in $\tilde{h}$, $(i_1(S^4\eta))^*\gamma_0' = 0$ since $i_1\eta$ is homotopic to $3i_0\nu$. From the exactness of the sequence

$$\tilde{h}^4(S^4P) \xrightarrow{(S^4i)^*} \tilde{h}^4(S^6) \xrightarrow{(S^4\eta)^*} \tilde{h}^4(S^7)$$

follows that $(S^2i)^*x = i_1^*\gamma_0'$ for some $x \in \tilde{h}^4(S^4P)$. Put

$$\gamma_0 = \gamma_0' - \pi_0^* x .$$

Then we have

$$i_0^*\gamma_0 = i_0^*\gamma_0' - i_0^*\pi_0^*x = i_0^*\gamma_0' = \sigma^4 1 ,$$

$$i_1^*\gamma_0 = i_1^*\gamma_0' - i_1^*\pi_0^*x = (S^4i)^*x - (\pi_0 i_1)^*x = 0$$

by (3.9). Thus

LEMMA 4.2. (i) *If $3(\nu\pi)^{**}=0$ in $\tilde{h}$, then there exists $\gamma_0 \in \tilde{h}^4(N_\eta)$ satisfying*

$$(4.2) \qquad i_0^*\gamma_0 = \sigma^4 1 .$$

(ii) *If* $3\nu^{**}=0$ *in* $\tilde{h}$, *then there exists* $\gamma_0 \in \tilde{h}^4(N_\eta)$ *satisfying*

$$i_0^* \gamma_0 = \sigma^4 1 \quad and \quad i_1^* \gamma_0 = 0\,. \tag{4.2'}$$

4.2. Making use of γ_0 of Lemma 4.2, hence at least under the assumption of $3(\nu\pi)^{**}=0$, we define a homomorphism

$$\gamma = \gamma_W : \tilde{h}^k(W \wedge N_\eta) \to \tilde{h}^k(W \wedge S^4 P)$$

by the formula

$$\gamma_W(x) = (1_W \wedge \pi_0)^{*-1}(x - \mu(\sigma^{-4}(1_W \wedge i_0)^* x \otimes \gamma_0)) \tag{4.3}$$

for $x \in \tilde{h}^k(W \wedge N_\eta)$. Since

$$\begin{aligned}(1_W \wedge i_0)^* \mu(\sigma^{-4}(1_W \wedge i_0)^* x \otimes \gamma_0) &= \mu(\sigma^{-4}(1_W \wedge i_0)^* x \otimes \sigma^4 1) \\ &= (1_W \wedge i_0)^* x\,,\end{aligned}$$

$x - \mu(\sigma^{-4}(1 \wedge i_0)^* x \otimes \gamma_0)$ is in the kernel of $(1_W \wedge i_0)^*$. By (4.1), $(1_W \wedge \pi_0)^*$ is monomorphic. Thus the map γ is a well-defined homomorphism.

LEMMA 4.3. (i) γ_W *is a left inverse of* $(1_W \wedge \pi_0)^*$, *i. e.*, $\gamma_W(1_W \wedge \pi_0)^* =$ *an identity map; hence the sequence of* (4.1) *splits:*

$$\tilde{h}^k(W \wedge N_\eta) = \tilde{h}^k(W \wedge S^4 P) + \tilde{h}^k(W \wedge S^4)\,.$$

(ii) γ *is natural in the sense that*

$$(f \wedge S^n 1_P)^* \gamma_W = \gamma_{W'}(f \wedge 1_{N_\eta})^*,$$

where $f: W' \to W$.

(iii) γ *is compatible with the suspension in the sense that*

$$(1_W \wedge T'')^* \sigma \gamma_W = \gamma_{SW}(1_W \wedge T')^* \sigma\,,$$

where $T' = T(S^1, N_\eta)$ *and* $T'' = T(S^1, S^4 P)$.

(iv) *The relation*

$$\mu(y \otimes \gamma_W(x)) = \gamma_{Y \wedge W}\, \mu(y \otimes x)$$

holds, where $x \in \tilde{h}^k(W \wedge N_\eta)$ *and* $y \in \tilde{h}^j(Y)$.

PROOF. (i) If $x = (1 \wedge \pi_0)^* y$, then $(1 \wedge i_0)^* x = 0$ and (i) follows from (4.3).

(ii) Since $(1_{W'} \wedge \pi_0)^*$ is monomorphic, it is sufficient to prove the equality

$$(1_{W'} \wedge \pi_0)^*(f \wedge S^4 1_P)^* \gamma_W(x) = (1_{W'} \wedge \pi_0)^* \gamma_{W'}(f \wedge 1_{N_\eta})^*(x)\,.$$

Now, the left side $= (f \wedge \pi_0)^* \gamma_W(x) = (f \wedge 1_{N_\eta})^*(1_W \wedge \pi_0)^* \gamma_W(x)$

$$\begin{aligned}&= (f \wedge 1_{N_\eta})^* x - \mu(\sigma^{-4}(S^4 f)^*(1_W \wedge i_0)^* x \otimes \gamma_0) \\ &= (f \wedge 1_{N_\eta})^* x - \mu(\sigma^{-4}(1_{W'} \wedge i_0)^*(f \wedge 1_{N_\eta})^* x \otimes \gamma_0) \\ &= \text{the right side}.\end{aligned}$$

(iii) Since $T_0 = T(S^1, S^4)$ is a map of degree 1, we have

$$(1_{SW} \wedge i_0)^*(1_W \wedge T')^*\sigma = (1_W \wedge T_0)^*\sigma(1_W \wedge i_0)^*$$
$$= \sigma(1_W \wedge i_0)^* .$$

Making use of this identity

$$\begin{aligned}
&(1_{SW} \wedge \pi_0)^*\gamma_{SW}(1_W \wedge T')^*\sigma(x) \\
&\quad = (1_W \wedge T')^*\sigma x - \mu(\sigma^{-4}(1_{SW} \wedge i_0)^*(1_W \wedge T')^*\sigma x \otimes \gamma_0) \\
&\quad = (1_W \wedge T')^*\sigma x - (1_W \wedge T')^*\sigma\mu(\sigma^{-4}(1_W \wedge i_0)^* x \otimes \gamma_0) \\
&\quad = (1_W \wedge T')^*\sigma(1_W \wedge \pi_0)^*\gamma_W(x) \\
&\quad = (1_{SW} \wedge \pi_0)^*(1_W \wedge T'')^*\sigma\gamma_W(x) ,
\end{aligned}$$

from which (iii) follows since $(1_{SW} \wedge \pi_0)^*$ is monomorphic.

(iv)
$$\begin{aligned}
&(1_{Y\wedge W} \wedge \pi_0)^*\gamma_{Y\wedge W}\mu(y \otimes x) \\
&\quad = \mu(y \otimes x) - \mu(\sigma^{-4}(1_{Y\wedge W} \wedge i_0)^*\mu(y \otimes x) \otimes \gamma_0) \\
&\quad = \mu(y \otimes x) - \mu(\mu(y \otimes \sigma^{-4}(1_W \wedge i_0)^* x) \otimes \gamma_0) \\
&\quad = \mu(y \otimes (1_W \wedge \pi_0)^*\gamma_W(x)) \\
&\quad = (1_{Y\wedge W} \wedge \pi_0)^*\mu(y \otimes \gamma_W(x)) ,
\end{aligned}$$

from which (iv) follows. q. e. d.

LEMMA 4.4. *If γ_0 satisfies (4.2′), then the relation*

$$(1_W \wedge S^4 i)^*\gamma_W = (1_W \wedge i_1)^*$$

holds for the inclusion $i: S^2 \subset P$ and $i_1: S^6 \subset N_\eta$.

PROOF. Since $S^4 i = \pi_0 i_1$ by (3.8′) and $i_1^*\gamma_0 = 0$ by (4.2′), we see that

$$\begin{aligned}
(1_W \wedge S^4 i)^*\gamma_W(x) &= (1_W \wedge i_1)^*(1_W \wedge \pi_0)^*\gamma_W(x) \\
&= (1_W \wedge i_1)^* x - \mu(\sigma^{-4}(1_W \wedge i_0)^* x \otimes i_1^*\gamma_0) \\
&= (1_W \wedge i)^* x .
\end{aligned}$$

q. e. d.

If μ is commutative, we can easily see that

LEMMA 4.5. *If μ is commutative, then there holds*

$$\mu(z \otimes \gamma_0) = T'^*\mu(\gamma_0 \otimes z) \tag{4.4}$$

for any $z \in \tilde{h}^i(Z)$, where $T' = T(Z, N_\eta)$.

LEMMA 4.6. *If γ_0 satisfies (4.4), then there holds a relation*

$$(1_W \wedge T'')^*\mu(\gamma_W(x) \otimes z) = \gamma_{W\wedge Z}(1_W \wedge T')^*\mu(x \otimes z)$$

for any $x \in \tilde{h}^k(W \wedge N_\eta)$ and $z \in \tilde{h}^i(Z)$, where $T' = T(Z, N_\eta)$ and $T'' = T(Z, S^4 P)$.

PROOF.

$$(1_{W\wedge Z}\wedge\pi_0)^*(1_W\wedge T'')^*\mu(\gamma_W(x)\otimes z)$$
$$=(1_W\wedge T')^*(1_W\wedge\pi_0\wedge 1_Z)^*\mu(\gamma_W(x)\otimes z)$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-(1_W\wedge T')^*\mu(\mu(\sigma^{-4}(1_W\wedge i_0)^*x\otimes\gamma_0)\otimes z)$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\sigma^{-4}(1_W\wedge i_0)^*x\otimes T'^*\mu(\gamma_0\otimes z))$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\sigma^{-4}(1_W\wedge i_0)^*x\otimes\mu(z\otimes\gamma_0))\quad\text{by (4.4),}$$
$$=(1_W\wedge T')^*\mu(x\otimes z)-\mu(\mu(\sigma^{-4}(1_W\wedge i_0)^*x\otimes z)\otimes\gamma_0)$$
$$=(1_{W\wedge Z}\wedge\pi_0)^*\gamma_{W\wedge Z}(1_W\wedge T')^*\mu(x\otimes z)\,.\qquad\text{q.e.d.}$$

4.3. Making use of the homomorphism γ_W defined by (4.3) and the element $\alpha\in\{N,P\wedge P\}$ of Proposition 3.8, we define a map

$$\mu_\eta:\ \tilde{h}^i(X;\eta)\otimes\tilde{h}^j(Y;\eta)\to\tilde{h}^{i+j}(X\wedge Y;\eta)\tag{4.5}$$

as the composition

$$\begin{aligned}\mu_\eta=\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu:&\\ &\tilde{h}^i(X;\eta)\otimes\tilde{h}^j(Y;\eta)=\tilde{h}^{i+4}(X\wedge P)\otimes\tilde{h}^{j+4}(Y\wedge P)\\ &\to\tilde{h}^{i+j+8}(X\wedge P\wedge Y\wedge P)\to\tilde{h}^{i+j+8}(X\wedge Y\wedge P\wedge P)\\ &\to\tilde{h}^{i+j+8}(X\wedge Y\wedge N_\eta)\to\tilde{h}^{i+j+8}(X\wedge Y\wedge P\wedge S^4)\\ &\to\tilde{h}^{i+j+4}(X\wedge Y\wedge P)=\tilde{h}^{i+j}(X\wedge Y;\eta)\,,\end{aligned}$$

where $T=T(Y,P)$.

μ_η is defined only if $3(\nu\pi)^{**}=0$.

The definition of μ_η depends on the choices of γ_0 and α which are but fixed during the subsequent proofs of properties of an admissible multiplication.

4.4. THEOREM 4.7. *The map μ_η of* (4.5) *is a multiplication satisfying* (Λ_1).

PROOF. The linearity and naturality of μ_η are obvious.

To prove (Λ_1): putting $T'=T(Y,P)$, $T_1=T(S^4,Y\wedge P)$, $T_2=T(S^4,P)$ and $T=T(P,P)$, by definitions of ρ_η and μ_η we have

$$\begin{aligned}\mu_\eta(\rho_\eta\otimes 1)&=\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_P)^*\mu((1_X\wedge\pi)^*\sigma^4\otimes 1_{Y\wedge P})\\ &=\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_P)^*(1_X\wedge\pi\wedge 1_{Y\wedge P})^*(1_X\wedge T_1)^*\sigma^4\mu\\ &=\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_{X\wedge Y}\wedge\pi\wedge 1_P)^*(1_{X\wedge Y}\wedge T_2)^*\sigma^4\mu\\ &=\sigma^{-4}\gamma_{X\wedge Y}((1_P\wedge\pi)T\alpha)^{**}\sigma^4\mu\\ &=\sigma^{-4}\gamma_{X\wedge Y}\pi_0^{**}\sigma^4\mu &&\text{by Proposition 3.8, (i),}\\ &=\sigma^{-4}1_{X\wedge Y\wedge S^4P}\sigma^4\mu=\mu=\mu_L &&\text{by Lemma 4.3, (i).}\end{aligned}$$

Similarly, making use of Proposition 3.8, (i), and Lemma 4.3, (i), we obtain

$$\mu_\eta(1\otimes\rho_\eta)=(1_X\wedge T')^*\mu=\mu_R$$

i. e., (A_1) was proved.

From (A_1) and (H_3) it follows that $\rho_\eta(1)$ is the bilateral unit of μ_η, i. e., the existence of 1_η is obtained.

To prove the compatibility of μ_η with σ_η, putting $T=T(Y,P)$, $T_1=T(Y,S^1)$, $T_2=T(S^1,P)$, $T_3=T(S^1,Y\wedge N_\eta)$, $T'=T(S^1,N_\eta)$ and $T''=T(S^1,S^4P)$, by definitions of μ_η and σ_η we have

$$\begin{aligned}
&(1_X\wedge T_1)^*\mu_\eta(\sigma_\eta\otimes 1)\\
&\quad=(1_X\wedge T_1\wedge 1_P)\sigma^{-4}\gamma_{SX\wedge Y}\alpha^{**}(1_{SX}\wedge T\wedge 1_P)^*\mu((1_X\wedge T_2)^*\sigma\otimes 1)\\
&\quad=\sigma^{-4}(1_X\wedge T_1\wedge S^4 1_P)^*\gamma_{SX\wedge Y}(1_X\wedge T_3)^*\sigma\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu\\
&\quad=\sigma^{-4}\gamma_{X\wedge SY}(1_X\wedge T_1\wedge 1_{N_\eta})^*(1_X\wedge T_3)^*\sigma\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu\\
&\qquad\qquad\qquad\qquad\text{by Lemma 4.3, (ii),}\\
&\quad=\sigma^{-4}\gamma_{S(X\wedge Y)}(1_{X\wedge Y}\wedge T')^*\sigma\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu\\
&\quad=\sigma^{-4}(1_{X\wedge Y}\wedge T'')^*\sigma\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu \quad\text{by Lemma 4.3, (iii),}\\
&\quad=(1_{X\wedge Y}\wedge T_2)^*\sigma\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu=\sigma_\eta\mu_\eta\,.
\end{aligned}$$

Similarly we see that

$$\begin{aligned}
\mu_\eta(1\otimes\sigma_\eta)&=\sigma^{-4}\gamma_{X\wedge SY}\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu(1_X\otimes(1_Y\wedge T_2)^*\sigma)\\
&=\sigma^{-4}\gamma_{X\wedge SY}(1_{X\wedge Y}\wedge T')^*\sigma\alpha^{**}(1_X\wedge T\wedge 1_P)^*\mu\\
&=\sigma_\eta\mu_\eta\,. \qquad\qquad \text{q. e. d.}
\end{aligned}$$

COROLLARY 4.8. *The condition that* $3(\nu\pi)^{**}=0$ *in* $\tilde{h}$ *is necessary and sufficient for the existence of a multiplication* μ_η *satisfying* (A_1).

PROOF. We need only to prove the necessity. Making use of Proposition 1.2 for $\alpha=\eta$, $X=P$ and $j=4$, (2.2) and (3.4), we see that

$$\begin{aligned}
0=(1_P\wedge\eta)^*\sigma_\eta{}^2\kappa_1&=3((S^2i)\nu(S^3\pi))^*\sigma_\eta{}^2\kappa_1\\
&=3\sigma_\eta{}^2(S\pi)^*\nu^*i^*\kappa_1=-3\sigma_\eta{}^4(S\pi)^*\nu^*1_\eta\,.
\end{aligned}$$

Now σ_η and $(S\pi)^*$ are monomorphic (by Proposition 1.2). Hence

$$3\nu^*1_\eta=3\rho_\eta\nu^*(1)=0\,.$$

For any $x\in\tilde{h}^j(X\wedge S^4)$,

$$\begin{aligned}
3(\nu\pi)^{**}x&=3(\nu\pi)^{**}\mu(\sigma^{-4}x\otimes\sigma^4 1)=\mu(\sigma^{-4}x\otimes 3(S^3\pi)^*\nu^*\sigma^4 1)\\
&=\mu(\sigma^{-4}x\otimes 3T^*(1\wedge\pi)^*\sigma^4\nu^*1)=\mu(\sigma^{-4}x\otimes T^*(3\rho_\eta)\nu^*1)\\
&=0\,,
\end{aligned}$$

where $T=T(P, S^3)$. q. e. d.

THEOREM 4.9. *If γ_0 satisfies (4.2′), then the multiplication μ_η of (4.5) satisfies (A_2) with associated cohomology operation $\bar{\zeta}^{**}\sigma^2$: $\tilde{h}^i(\ ) \to \tilde{h}^{i-2}(\ ;\eta)$.*

PROOF. By Theorem 4.6 we can use (A_1) for μ_η. Putting $T=T(P, P)$, $T'=T(Y, P)$ and $T''=T(Y\wedge P, S^2)$, we have on $\tilde{h}^i(X;\eta)\otimes\tilde{h}^j(Y;\eta)$

$$\begin{aligned}
&\mu_L(\delta_{\eta,0}\otimes 1)+\mu_R(1\otimes\delta_{\eta,0})\\
&\quad=\mu(\sigma^{-2}(1_X\wedge i)^*\otimes 1)+(1_X\wedge T')^*\mu(1\otimes\sigma^{-2}(1_Y\wedge i)^*)\\
&\quad=\sigma^{-2}\{(1_X\wedge T'')^*(1_X\wedge i\wedge 1_{Y\wedge P})^*+(1_X\wedge T'\wedge 1_{S^2})^*(1_{X\wedge P\wedge Y}\wedge i)^*\}\mu\\
&\quad=\sigma^{-2}\{(1_{X\wedge Y\wedge P}\wedge i)^*(1_{X\wedge Y}\wedge T)^*+(1_{X\wedge Y\wedge P}\wedge i)^*\}(1_X\wedge T'\wedge 1_P)^*\mu\\
&\quad=\sigma^{-2}(T(1_P\wedge i)+1_P\wedge i)^{**}(1_X\wedge T'\wedge 1_P)^*\mu\\
&\quad=\{\sigma^{-2}(\alpha i_1\pi)^{**}+\sigma^{-2}(i\bar{\zeta}\wedge i)^{**}\}(1_X\wedge T'\wedge 1_P)^*\mu
\end{aligned}$$

by Proposition 3.8, (ii),

$$\begin{aligned}
&\quad=\pi^{**}\sigma^2 i^{**}\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T'\wedge 1_P)^*\mu\\
&\qquad\qquad+\sigma^{-2}\bar{\zeta}^{**}\sigma^4\mu(\sigma^{-2}i^{**}\otimes\sigma^{-2}i^{**}) \qquad \text{by Lemma 4.4,}\\
&\quad=\rho_\eta\delta_{\eta,0}\mu_\eta+\bar{\zeta}^{**}\sigma^2\mu(\delta_{\eta,0}\otimes\delta_{\eta,0})\,.
\end{aligned}$$

Here we put $\chi_\eta=\bar{\zeta}^*\sigma^2$, then

$$\delta_\eta\mu_\eta=\mu_L(\delta_{\eta,0}\otimes 1)+\mu_R(1\otimes\delta_{\eta,0})-\chi_\eta\mu(\delta_{\eta,0}\otimes\delta_{\eta,0})\,.$$

Clearly χ_η is a cohomology operation and the relation

$$\chi_\eta\mu=\mu_R(\chi_\eta\otimes 1)=\mu_L(1\otimes\chi_\eta)$$

holds. Moreover we obtain

$$\delta_{\eta,0}\chi_\eta=\sigma^{-2}i^{**}\bar{\zeta}^{**}\sigma^2=2$$

since $\bar{\zeta}i=2\cdot 1_{S^2}$ (by (3.2)). q. e. d.

THEOREM 4.10. *If γ_0 satisfies (4.4), then the multiplication μ_η of (4.5) satisfies (A_3).*

PROOF. To prove (i) of (A_3), discussing on $\tilde{h}^*(X)\otimes\tilde{h}^*(Y;\eta)\otimes\tilde{h}^*(Z;\eta)$ and putting $T_1=T(Z, P)$, we have

$$\begin{aligned}
\mu_\eta(\mu_L\otimes 1)&=\sigma^{-4}\gamma_{X\wedge Y\wedge Z}\alpha^{**}(1_{X\wedge Y}\wedge T_1\wedge 1_P)^*\mu(\mu\otimes 1)\\
&=\sigma^{-4}\gamma_{X\wedge Y\wedge Z}\mu(1\otimes\alpha^{**}(1_Y\wedge T_1\wedge 1_P)^*\mu)\\
&=\sigma^{-4}\mu(1\otimes\gamma_{Y\wedge Z}\alpha^{**}(1_Y\wedge T_1\wedge 1_P)^*\mu) \quad \text{by Lemma 4.3, (iv),}\\
&=\mu(1\otimes\sigma^{-4}\gamma_{X\wedge Z}\alpha^{**}(1_Y\wedge T_1\wedge 1_P)^*\mu)\\
&=\mu_L(1\otimes\mu_\eta)\,.
\end{aligned}$$

In a similar way we can easily see (ii) of (Λ_3) on $\tilde{h}^*(X;\eta)\otimes\tilde{h}^*(Y)\otimes\tilde{h}^*(Z;\eta)$. To prove (iii) of (Λ_3), discussing on $\tilde{h}^*(X;\eta)\otimes\tilde{h}^*(Y;\eta)\otimes\tilde{h}^*(Z)$ and putting $T_1=T(Z,P)$, $T_2=T(P\wedge Z,P)$, $T_3=T(Y,P)$, $T'=T(Z,N_\eta)$ and $T''=T(Z,P\wedge S^4)$, we have

$$\begin{aligned}\mu_\eta(1\otimes\mu_R) &= \sigma^{-4}\gamma_{X\wedge Y\wedge Z}\alpha^{**}(1_X\wedge T_2\wedge 1_P)^*\mu(1\otimes(1_Y\wedge T_1)^*\mu)\\ &= \sigma^{-4}\gamma_{X\wedge Y\wedge Z}\alpha^{**}(1_X\wedge T_2\wedge 1_P)^*(1_Y\wedge T_1)^*\mu(\mu\otimes 1)\\ &= \sigma^{-4}\gamma_{X\wedge Y\wedge Z}(1_{X\wedge Y}\wedge T')^*\mu(\alpha^{**}(1_X\wedge T_3\wedge 1_P)^*\mu\otimes 1)\\ &= \sigma^{-4}(1_{X\wedge Y}\wedge T'')^*\mu(\gamma_{X\wedge Y}\otimes 1)(\alpha^{**}(1_X\wedge T_3\wedge 1_P)^*\mu\otimes 1)\\ &\qquad\qquad\text{by Lemma 4.6,}\\ &= (1_{X\wedge Y}\wedge T_1)^*\mu(\sigma^{-4}\gamma_{X\wedge Y}\alpha^{**}(1_X\wedge T_3\wedge 1_P)^*\mu\otimes 1)\\ &= \mu_R(\mu_\eta\otimes 1)\,. \qquad\qquad \text{q. e. d.}\end{aligned}$$

4.5. As a corollary of Theorems 4.6, 4.9, 4.10, Lemma 4.2 and Lemma 4.5 we obtain

THEOREM 4.11. *If we assume that* $3\nu^{**}=0$ *in* $\tilde{h}$, *then admissible multiplications* μ_η *exist in* $\tilde{h}^*(\ ;\eta)$.

§ 5. Admissible multiplications in $\widetilde{KO}(\ ;\eta)$-theory

5.1. The $\widetilde{KO}$ (or $\widetilde{KU}$)-cohomology theory of real (or complex) vector bundles has the commutative and associative multiplication μ_0 (or μ_U) defined by tensor product.

First we recall [1] that $\widetilde{KO}^0(S^q)$ and $\widetilde{KU}^0(S^q)$ are both isomorphic to corresponding groups in the following table:

(5.1)

q (mod 8)	0	1	2	3	4	5	6	7
$\widetilde{KO}^{-q}(S^0)=\widetilde{KO}^0(S^q)$	Z	Z_2	Z_2	0	Z	0	0	0
$\widetilde{KU}^{-q}(S^0)=\widetilde{KU}^0(S^q)$	Z	0	Z	0	Z	0	Z	0

And the operations—the complexification $\varepsilon:\widetilde{KO}^i(X)\to\widetilde{KU}^i(X)$, the real restriction $\gamma:\widetilde{KU}^i(X)\to\widetilde{KO}^i(X)$ and the conjugation $*:\widetilde{KU}^i(X)\to\widetilde{KU}^i(X)$—have the following relations (c. f. [5]):

$$\text{(5.2)}\qquad \gamma\varepsilon=2:\widetilde{KO}^i(X)\to\widetilde{KO}^i(X)\,,$$

$$\varepsilon\gamma=1+*:\widetilde{KU}^i(X)\to\widetilde{KU}^i(X)\,.$$

The operations are natural with respect to maps and ring homomorphism with respect to tensor product multiplication, excepting γ which is a homomorphism of groups.

Let g be a generator of $\widetilde{KU}^0(S^2)$, given by the reduced Hopf bundle. Bott's isomorphism

$$\beta_U : \widetilde{KU}^i(X) \xrightarrow{\cong} \widetilde{KU}^{i-2}(X)$$

is given by the formula $\beta_U = \sigma^{-2}\mu_U(\ \otimes g)$.

We can easily obtain the following relations

$$\beta_U * = -*\beta_U\,, \tag{5.3}$$

$$\beta_U\mu_U(x\otimes y) = \mu_U(\beta_U x\otimes y) = \mu_U(x\otimes\beta_U y) \quad \textit{for } x\in\widetilde{KU}^*(X),\ y\in\widetilde{KU}^*(Y) \tag{5.4}$$

and

$$\gamma\mu_U(x\otimes\varepsilon y) = \mu_0(\gamma x\otimes y) \qquad \textit{for } x\in\widetilde{KU}^*(X) \textit{ and } y\in\widetilde{KO}^*(Y)\,. \tag{5.5}$$

5.2. Let H be the complex Hopf bundle over P (complex projective plane) and let $h=[H]-1$.

Define $W: \widetilde{KU}^n(X)\to\widetilde{KO}^n(X;\eta)$ by

$$W(x) = \gamma\mu_U(\beta_U^{-2}x\otimes h)\,. \tag{5.6}$$

Making use of exact sequence associated with the cofibration

$$X\wedge S^2 \xrightarrow{1\wedge i} X\wedge P \xrightarrow{1\wedge\pi} X\wedge S^4\,,$$

we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{KO}^i(X) \arrow[r, "\rho_\eta"] \arrow[dr, "\varepsilon"'] & \widetilde{KO}^i(X;\eta) \arrow[r, "\delta_{\eta,0}"] & \widetilde{KO}^{i+2}(X) \arrow[r, "\eta^{**}"] & \widetilde{KO}^{i+1}(X) \arrow[r] & \cdots \\
& & \widetilde{KU}^i(X) \arrow[u, "W"'] \arrow[ur, "\gamma\beta_U^{-1}"'] & & &
\end{tikzcd}$$

D.W. Anderson [**2**] proved that W is an isomorphism and the above diagram is commutative. Thus

THEOREM 5.1 (Anderson). *We have an exact sequence*

$$\cdots \longrightarrow \widetilde{KO}^i(X) \xrightarrow{\varepsilon} \widetilde{KU}^i(X) \xrightarrow{\gamma\beta_U^{-1}} \widetilde{KO}^{i+2}(X) \xrightarrow{\eta^{**}} \widetilde{KO}^{i+1}(X) \xrightarrow{\varepsilon} \cdots$$

and the relations

$$W\varepsilon = \rho_\eta \quad \textit{and} \quad \gamma\beta_U^{-1} = \delta_{\eta,0}W \tag{5.7}$$

where W is an isomorphism.

W is called the *Wood isomorphism*.

5.3. THEOREM 5.2 *There exists an admissible multiplication μ_η in $\widetilde{KO}(\ ;\eta)$. The admissible multiplications in $\widetilde{KO}(\ ;\eta)$ are in one-to-one correspondence with the elements of the group $Z \cong \widetilde{KU}^{-4}(S^0)$.*

PROOF. From the Bott's isomorphism in $\widetilde{KO}$-theory we obtain

$$\widetilde{KO}^i(X) \cong \widetilde{KO}^{i+8}(X) \otimes \widetilde{KO}^0(S^8)$$

and since ν^*-image in $\widetilde{KO}^{-8}(S^0)$ vanishes, then ν^*-image in $\widetilde{KO}^i(X)$ also vanishes. Thus by Theorem 4.10, the $\widetilde{KO}(\ ;\eta)$ has an admissible multiplication μ_η constructed by tensor product multiplication. From (5.1) and Wood isomorphism,

$$\delta_\eta = \rho_\eta \delta_{\eta,0} : \widetilde{KO}^{-4}(S^0\,;\eta) \to \widetilde{KO}^{-2}(S^0) \to \widetilde{KO}^{-2}(S^0\,;\eta)$$

is a zero map. Thus δ_η-image of $\widetilde{KO}^{-4}(S^0\,;\eta)$ vanishes and $\delta_\eta^{-1}(0) = \widetilde{KO}^{-4}(S^0\,;\eta) \cong \widetilde{KU}^{-4}(S^0) \cong Z$. Then any admissible multiplication in $\widetilde{KO}(\ ;\eta)$ have the same cohomology operation and the theorem follows from Corollary 2.8.

5.4. To simplify notation we put

$$\mu_0(x \otimes y) = x \wedge_0 y, \quad \mu_U(x \otimes y) = x \wedge_U y, \quad \mu_\eta(x \otimes y) = x \wedge y, \quad \text{etc.}$$

We define the maps μ'_R and μ'_L

$$\mu'_R : \widetilde{KU}^i(X) \otimes \widetilde{KO}^j(Y) \to \widetilde{KU}^{i+j}(X \wedge Y)$$

$$\mu'_L : \widetilde{KO}^i(X) \otimes \widetilde{KU}^j(Y) \to \widetilde{KU}^{i+j}(X \wedge Y)$$

by

(5.8) $$\mu'_R(x \otimes y) = x \wedge_U \varepsilon y \quad \textit{and} \quad \mu'_L(x \otimes y) = \varepsilon x \wedge_U y\,.$$

From (5.4) the following diagrams are commutative since μ_0 and μ_U are commutative associative multiplications:

(5.8′)
$$\begin{tikzcd}
\widetilde{KU}^i(X) \otimes \widetilde{KO}^j(Y) \arrow[r, "\mu'_R"] \arrow[d, "W \otimes 1"] & \widetilde{KU}^{i+j}(X \wedge Y) \arrow[d, "W"'] \\
\widetilde{KO}^i(X\,;\eta) \otimes \widetilde{KO}^j(Y) \arrow[r, "\mu_R"] & \widetilde{KO}^{i+j}(X \wedge Y\,;\eta),
\end{tikzcd}$$

$$\begin{tikzcd} \widetilde{KO}^i(X)\otimes\widetilde{KU}^j(Y) \arrow[r, "\mu'_L"] \arrow[d, "1\otimes W"] & \widetilde{KU}^{i+j}(X\wedge Y) \arrow[d, "W"'] \\ \widetilde{KO}^i(X)\otimes\widetilde{KO}^j(X;\eta) \arrow[r, "\mu_L"] & \widetilde{KO}^{i+j}(X\wedge Y;\eta)\,. \end{tikzcd}$$

We put

(5.9) $$\delta_{U,0}=\gamma\beta_U^{-1},\quad \rho_U=\varepsilon \quad \textit{and} \quad \delta_U=\rho_U\delta_{U,0}=(1+*)\beta_U^{-1}\,.$$

From (5.7) we obtain

(5.10) $$\delta_{U,0}=\delta_{\eta,0}W,\quad \rho_U=W^{-1}\rho_\eta \quad \textit{and} \quad \delta_U=W^{-1}\delta_\eta W\,.$$

Making use of the relations (5.5), (5.9) and (5.10), we obtain that the multiplications μ'_R and μ'_L satisfy $(H_1)\sim(H_8)$ with respect to $\delta_{U,0}$ and ρ_U. Moreover we can easily obtain that the tensor product multiplication μ_U satisfies (Λ_0), (Λ_1) and (Λ_3) with respect to ρ_U, μ'_R and μ'_L. And we obtain

$$x\wedge_U y-\bar{x}\wedge_U\bar{y}=(x-\bar{x})\wedge_U y+x\wedge_U(y-\bar{y})-(x-\bar{x})\wedge_U(y-\bar{y})$$

for any $x\in\widetilde{KU}^*(X)$ and $y\in\widetilde{KU}^*(Y)$ where $\bar{z}$ is denoted by $\bar{z}=*z$, then, from (5.3), (5.8) and (5.9)

$$\begin{aligned}\beta_U^{-1}(1-*)(x\wedge_U y)&=\beta_U^{-1}(1-*)x\wedge_U y+x\wedge_U\beta_U^{-1}(1-*)y\\&\quad-\beta_U(\beta_U^{-1}(1-*)x\wedge_U\beta_U^{-1}(1-*)y)\\&=\mu'_L(\gamma\beta_U^{-1}x\otimes y)+\mu'_R(x\otimes\gamma\beta_U^{-1}y)-\beta_U\varepsilon(\gamma\beta_U^{-1}x\wedge_0\gamma\beta_U^{-1}y)\,.\end{aligned}$$

Here we put $\chi=\beta_U\varepsilon$, then

$$\begin{aligned}\beta_U\varepsilon(x\wedge_0 y)&=\beta_U\varepsilon x\wedge_U\varepsilon y=\varepsilon x\wedge_U\beta_U\varepsilon y\\&=\mu'_R(\beta_U\varepsilon x\otimes y)=\mu'_L(x\otimes\beta_U\varepsilon y)\end{aligned}$$

and

$$\delta_{U,0}\chi=\gamma\beta_U^{-1}\beta_U\varepsilon=\gamma\varepsilon=2$$

by (5.2), (5.4) and (5.8).

Thus μ_U is an admissible multiplication in $\widetilde{KU}$ with associated cohomology operation $\chi=\beta_U\varepsilon$.

Next we put

$$\tilde{\mu}_\eta=W\mu_U(W^{-1}\otimes W^{-1}):\ \widetilde{KO}^i(X;\eta)\otimes\widetilde{KO}^j(Y;\eta)\to\widetilde{KO}^{i+j}(X\wedge Y;\eta)$$

then, making use of (5.8′), (5.9) and (5.10) we obtain that $\tilde{\mu}_\eta$ is an admissible multiplication in $\widetilde{KO}(\ ;\eta)$ with associated cohomology operation $W\beta_U\varepsilon$. Thus

THEOREM 5.3. *$\widetilde{KO}(\ ;\eta)$ has an admissible multiplication (associated with* *the tensor product multiplication in* $\widetilde{KO}$) *which corresponds to the tensor product multiplication* μ_U *in* $\widetilde{KU}$ *by Wood isomorphism.*

COROLLARY 5.4. *The associated cohomology operation* $\zeta^*\sigma^{-2}$ *with admissible multiplication in* $\widetilde{KO}(\ ;\eta)$ *is regarded as*

$$\zeta^*\sigma^{-2}(x) = x \wedge_0 \gamma\beta_U^{-1}h \qquad \text{for any } x \in \widetilde{KO}^i(X),$$

and a bilateral unit $1_\eta \in \widetilde{KO}^0(S^0;\eta)$ *is*

$$1_\eta = \gamma\beta_U^{-2}h.$$

COROLLARY 5.5. *Any admissible multiplication* μ_η *in* $\widetilde{KO}(\ ;\eta)$ *can be expressed as*

$$\mu_\eta(x\otimes y) = \tilde{\mu}_\eta(x\otimes y) + k\cdot\tilde{\mu}_\eta(\gamma h\otimes\tilde{\mu}_\eta(\delta_\eta x\otimes\delta_\eta y))$$

for some integer k, *where* $\tilde{\mu}_\eta$ *is the multiplication corresponding to* μ_U.

PROOF. Since $\widetilde{KO}^{-4}(S^0;\eta)\cong\widetilde{KU}^{-4}(S^0)\cong Z$ and a generator of $\widetilde{KU}^{-4}(S^0)$ is $\beta_U^2(1_U)$, a generator of $\widetilde{KO}^{-4}(S^0;\eta)$ is $W\beta_U^2(1_U)=\gamma h$. Then the corollary follows from Theorem 2.3.

COROLLARY 5.6. *The admissible multiplications in* $\widetilde{KU}$ *(identified with* $\widetilde{KO}(\ ;\eta)$*) have the same associated cohomology operation* $\beta_U\varepsilon$ *and are in one-to-one correspondence with the elements of the group* $Z\cong\widetilde{KU}^{-4}(S^0)$.

COROLLARY 5.7. *The multiplication* μ'_U *in* $\widetilde{KU}$ *is admissible if and only if*

$$\mu'_U(x\otimes y) = x\wedge_U y - k(x-\bar{x})\wedge_U(y-\bar{y})$$

for some integer k.

PROOF. From Theorem 2.3, through the Wood isomorphism we obtain that any admissible multiplication can be expressed as

$$\begin{aligned} x\wedge'_U y &= x\wedge_U y - k\cdot\beta_U^2(1_U)\wedge_U(\beta_U^{-1}(1-*)x\wedge_U\beta_U^{-1}(1-*)y) \\ &= x\wedge_U y - k(x-\bar{x})\wedge_U(y-\bar{y}) \end{aligned}$$

for some integer k. Conversely it is clear that μ'_U satisfies $(\Lambda_0)\sim(\Lambda_8)$ with respect to ρ_U, $\delta_{U,0}$, μ'_R and μ'_L. q.e.d.

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