

A note on the prime radical

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§ 1. Introduction.

In 1943 R. Baer introduced the lower nil radical, which is commonly called the prime radical, as a radical built from nilpotent rings [1]. N. McCoy first considered the intersection of prime ideals of a ring [7] and then J. Levitzki showed that the prime radical was the intersection of the prime ideals of a ring [6]. Elementwise characterizations of the prime radical were given first by N. Jacobson in terms of m -sequences [4, p. 195] and then recently by J. Lambek in terms of strongly nilpotent elements [5, p. 55]. This latter characterization enables us to describe it in terms of annihilators. The primary purpose of this paper is to give necessary and sufficient conditions in terms of annihilators for the prime radical to be nilpotent (see Corollary 3). These conditions follow from Theorem 2. This also proves that the prime radical of a ring with the minimum condition on (two-sided) ideals is nilpotent (see Corollary 5).

§ 2. The results.

R will always denote a ring and $r(S)$ ($l(S)$) the right (left) annihilator of a subset S of R . For $b \in R$ we write $r(b)$ instead of $r(\{b\})$. Also, bR means the right ideal generated by b .

A decreasing sequence of sets $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$ of R is said to have a *right (left) constant annihilator* provided that the corresponding increasing sequence of right (left) annihilators becomes constant, that is, $r(S_n) = r(S_{n+j})$ for some fixed n and all $j \geq 1$. ($l(S_n) = l(S_{n+j})$). In particular we say that a sequence of elements $\{x_i\}$ has a *right constant annihilator* if $Rx_1 \supseteq Rx_2x_1 \supseteq Rx_3x_2x_1 \supseteq \dots$ has a right constant annihilator.

PROPOSITION 1. *The prime radical is the set S of elements x such that xR is nil and each sequence $\{x_i\}$ where $x_1 = x$, $x_{k+1} \in x_k \cdots x_1 R$ has a right constant annihilator.*

PROOF. All strongly nilpotent elements belong to S . Let $y \in S$ and let $\{y_i\}$ be a sequence where $y_1 = y$, $y_{k+1} = y_k \cdots y_1 p_k$ for some $p_k \in R$ and for all

$k \geq 2$. If $y_{k+1}y_k \cdots y_1 \neq 0$, then $r(y_k \cdots y_1) \subseteq r(y_{k+1}y_k \cdots y_1)$ because $p_k(y_k \cdots y_1)$ and $y_k \cdots y_1$ are nilpotent elements. Since $y \in S$, the sequence $y_1, y_2y_1, y_3y_2y_1, \dots$ is ultimately zero and y is strongly nilpotent.

We say that a subring N of R has a right constant annihilator if the sequence $N \supseteq N^2 \supseteq N^3 \supseteq \dots$ has a right constant annihilator.

THEOREM 2. *If the prime radical P of a ring R has a right constant annihilator but is not nilpotent, then there is a sequence $\{x_i\}$ of P such that for all $k \geq 1$ $x_k \cdots x_1R \neq 0$ and $x_kRx_k \cdots x_1R = 0$.*

PROOF. Let $K = r(P^t) = r(P^{t+1})$ for some $t (\geq 1)$. Since P is not nilpotent, we have $P \not\subseteq K$, so that $R \neq K$. There exists an element b_1 of P such that $b_1R \not\subseteq K$ since $PR \subseteq K$. If $b_1Rb_1R \not\subseteq K$, then we can select $b_2 \in b_1Rb_1$ such that $b_2R \not\subseteq K$. If $b_2Rb_2R \not\subseteq K$, then select $b_3 \in b_2Rb_2$ such that $b_3R \not\subseteq K$. This process can not continue since b_1 is strongly nilpotent. We conclude that there exists an element y_1 of P such that $y_1R \not\subseteq K$ and $y_1Ry_1R \subseteq K$. Since $y_1R \not\subseteq K$, we have $Py_1R \not\subseteq K$. Repeating the similar process as above, we conclude that there exists an element y_2 of P such that $y_2y_1R \not\subseteq K$ and $y_2Ry_2y_1R \subseteq K$. Continuing in this manner, we conclude that there is an infinite sequence $\{y_i\}$ of P such that $y_k \cdots y_1R \not\subseteq K$ and $y_kRy_k \cdots y_1R \subseteq K$, for all $k \geq 1$. Let $h = t + 1$ and let $x_k = y_{kh} \cdots y_{(k-1)h+1}$, $k = 1, 2, \dots$. Then we have $x_k \cdots x_1R \not\subseteq K$ and $x_kRx_k \cdots x_1R = 0$ for all $k \geq 1$.

COROLLARY 3. *The prime radical P of a ring R is nilpotent if and only if P has a right constant annihilator and for each sequence $\{x_i\}$ of P , the sequence $Rx_1R \supseteq Rx_2x_1R \supseteq Rx_3x_2x_1R \cdots$ has a left constant annihilator.*

PROOF. Assume that P has a right constant annihilator but is not nilpotent. By Theorem 2, we can conclude that there exists an infinite sequence $\{x_i\}$ of P such that $x_k \cdots x_1R \neq 0$ and $x_kRx_k \cdots x_1R = 0$ for any $k \geq 1$. Then the sequence $Rx_1R \supseteq Rx_2x_1R \supseteq Rx_3x_2x_1R \supseteq \dots$ has no left constant annihilator since $x_{k+2} \in l(Rx_{k+2} \cdots x_1R)$ but $x_{k+2} \notin l(Rx_k \cdots x_1R)$. The rest of the proof is obvious.

COROLLARY 4. *Assume that R is a nil ring. Then R is nilpotent if and only if R has a right constant annihilator and, for each sequence $\{x_i\}$ of R , the sequence $Rx_1R \supseteq Rx_2x_1R \supseteq Rx_3x_2x_1R \supseteq \dots$ has right and left constant annihilators.*

PROOF. It follows from Proposition 1 that the prime radical of R is R . Corollary 2 implies that R is nilpotent.

COROLLARY 5. *The prime radical of a ring with the minimum condition on ideals is nilpotent.*

PROOF. The proof is clear.

COROLLARY 6. *Assume that each sequence of elements of R and each nil subring of R have right constant annihilators. If R is a right finite dimensional*

ring, then each nil subring is nilpotent.

PROOF. By Proposition 1 the prime radical of a nil subring N is N . If N is not nilpotent, then there does exist a sequence $\{x_i\}$ of N such that $x_n \cdots x_1 \neq 0$ and $x_n R x_n \cdots x_1 = 0$ for all $n \geq 1$. By hypothesis $r(x_j \cdots x_1) = r(x_{j+k} \cdots x_1)$ for some fixed j and all $k \geq 1$; for notational purposes assume that $r(x_1) = r(x_k \cdots x_1)$ for all $k \geq 1$. The sum $x_1 R + x_3 x_2 x_1 R$ is direct for if $0 \neq y = x_1 r_1 = x_3 x_2 x_1 r_2$ where $r_1, r_2 \in R$ then by multiplying on the left by $x_3 x_2$ we have $0 \neq x_3 x_2 y = x_3 x_2 x_3 x_2 x_1 r_2 = 0$ since $r(x_3 x_2 x_1) = r(x_1)$ and $x_3 R x_3 x_2 x_1 = 0$. Continuing in this manner we conclude that the sum $x_1 R + x_3 x_2 x_1 R + x_5 x_4 x_3 x_2 x_1 R + \cdots$ is direct, a contradiction. Therefore N is nilpotent.

Recall that a right Goldie ring is a right finite dimensional ring with the maximum condition on right annihilators. All right Goldie rings satisfy the hypothesis of Corollary 5, but not conversely. The following ring, which is not a right Goldie ring, satisfies the hypothesis of Corollary 5. Let R be the commutative ring generated by $p, e_1, e_2, \dots, a_1, a_2, \dots$ with the relation that all products are zero except those of the form: $a_i^2 \neq 0, a_i e_i \neq 0, a_i^2 e_i = p$. R has dimension one since pR is essential, but $r(S_1) \subseteq r(S_2) \subseteq r(S_3) \subseteq \cdots$ where $S_i = \{a_k : k \geq i\}$.

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