J. Math. Soc. Japan
Vol. 24, No. 2, 1972

A topological invariant of substitution minimal sets

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(Received June 17, 1971)
(Revised Jan. 21, 1972)

§ 0. Introduction.

In this paper, we examine the ergodic properties of a bisequence over some finite set of symbols which is generated by a *substitution*. By a substitution, we mean a mapping which maps each symbol to a sequence of some common length ($\geqq 2$) of the symbols. For example, consider a substitution $\theta: 0\rightarrow 0\ 1,\ 1\rightarrow 1\ 0$ which is defined on $\{0, 1\}$. Define substitutions $\theta^2, \theta^3, \cdots$, as follows:

$$\begin{aligned}\theta^2(0)&=\theta(0\ 1)=\theta(0)\theta(1)=0\ 1\ 1\ 0\,,\\ \theta^2(1)&=\theta(1\ 0)=\theta(1)\theta(0)=1\ 0\ 0\ 1\,,\\ \theta^3(0)&=\theta(0\ 1\ 1\ 0)=\theta(0)\theta(1)\theta(1)\theta(0)\\ &=0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\,,\\ &\ \ \vdots\end{aligned}$$

A bisequence $\alpha=\cdots 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1*0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\cdots$ which is known as the Morse sequence is defined as the limit of extensions:

$$\begin{aligned}1*0&\prec\theta^2(1*0)\equiv\theta^2(1)*\theta^2(0)=1\ 0\ 0\ 1*0\ 1\ 1\ 0\\ &\prec\theta^4(1*0)\prec\theta^6(1*0)\prec\cdots,\end{aligned}$$

where "$*$" denotes the "center" of bisequences. In this case, it holds that $\theta^2(\alpha)=\alpha$. Generally speaking, given any substitution θ over some finite set D, a bisequence α over D is said to be *generated* by θ if $\theta^k(\alpha)=\alpha$ for some integer $k\geqq 1$. It is known ([3]) that for any substitution θ, there exists at least one almost periodic sequence generated by θ. Moreover, it can be proved (from Lemma 1～3) that if θ satisfies Condition $\sharp$ defined in Section 1, all almost periodic sequences generated by θ belong to a common minimal set of the shift dynamical system over D. Such a minimal set S as above is unique and characterized as a minimal set S for which $\theta(S)\subset S$ holds. In this case, denote $S=W(\theta)$. Let Θ be a set of all substitutions defined on $\{0, 1, \cdots, r-1\}$ for some integer $r\geqq 1$ which satisfy Condition $\sharp$. We introduce a computable (in the sense of the recursive function theory) function B called the branching number which is defined on Θ and takes positive integral values. We prove the followings:

I. For any positive integers b and r, there exists a substitution θ on r symbols such that $B(\theta)=b$, if and only if $b \leqq r$ (Theorem 2 and Theorem 6).

II. $B(\theta)=\min_{\alpha \in W(\theta)} \operatorname{Card}(\alpha \Lambda)$, where Λ is the trace relation of $(W(\theta), T)$ (T is the shift). Therefore, the branching number is a topological invariant (Theorem 5).

III. If there exists a homomorphism (in the sense of topological dynamics) from $(W(\theta), T)$ onto $(W(\theta'), T)$, then $B(\theta) \geqq B(\theta')$ (Corollary 2).

IV (from I and III). For any integer $r \geqq 2$, there exists a substitution minimal set on r symbols which is not a homomorphic image of any substitution minimal set on r' symbols, where $r' < r$ (Corollary 3).

V. If $B(\theta)=1$, then $(W(\theta), T)$ is measure isomorphic to the divided system $(W(\theta)/\Lambda, T/\Lambda)$, where Λ is the trace relation of $(W(\theta), T)$. Therefore, in this case T has a rational pure point spectrum (Theorem 7).

§ 1. Formulation of the problem.

The set of integers is denoted by I. N is the set of non-negative integers. For $p \in N$, let $N_p=\{0, 1, \cdots, p-1\}$. For any pair of sets E and F, E^F denotes the set of all functions from F into E. For any non-empty finite set D, the set D^I is considered as a topological space with the following metric d:

$$d(\alpha, \beta)=\frac{1}{\min\{|i| ;\ \alpha(i) \neq \beta(i)\}+1}.$$

T denotes the shift transformation on D^I, that is, T is defined by $(T\alpha)(i)=\alpha(i+1)$. By a *compact dynamical system*, we mean a pair of a compact metric space and a homeomorphism from it onto itself. Thus (D^I, T) is a compact dynamical system. Denote by D^* the disjoint sum $\bigcup_{p \in N} D^{N_p}$. An element of D^* is called a block, which may be also represented by a finite sequence. D^{N_0} consists only of the empty block. D^{N_1} is sometimes identified with D. For example, $\xi=100$ is an element of $(N_2)^{N_3}$ such that $\xi(0)=1$, $\xi(1)=0$ and $\xi(2)=0$. For $\xi \in D^*$, the length of ξ is denoted by $l(\xi)$, that is, $l(\xi)=p$ if and only if $\xi \in D^{N_p}$. For $\xi, \eta \in D^*$, $\xi\eta$ denotes the concatenation, that is,

$$(\xi\eta)(i)=\begin{cases} \xi(i) & \cdots \text{ if } 0 \leqq i < l(\xi) \\ \eta(i-l(\xi)) & \cdots \text{ if } l(\xi) \leqq i < l(\xi)+l(\eta). \end{cases}$$

Similarly an element of D^I can be regarded as a bisequence. That is, $\cdots \alpha_{-2}\alpha_{-1}{*}\alpha_0\alpha_1 \cdots$, where $\alpha_i \in D$ $(i \in I)$, is an element α of D^I for which $\alpha(i)=\alpha_i$ $(i \in I)$. By a *substitution*, we mean a mapping from D into D^{N_p}, where

$p \geqq 2$ is any integer. D is called the domain of the substitution. Card(D) (= number of elements of D) and p are called the *size* and the *length* of the substitution, respectively. The size and the length of a substitution θ are denoted by $s(\theta)$ and $L(\theta)$, respectively. Let θ be a substitution $D \to D^{N_p}$. We define a mapping $\bar{\theta}$ from D^* into D^* as follows:

$$\bar{\theta}(\xi_0\xi_1 \cdots \xi_{k-1}) = \theta(\xi_0)\theta(\xi_1) \cdots \theta(\xi_{k-1}),$$

where $\xi_i \in D$ $(i=0, 1, \cdots, k-1)$. That is to say, for $\xi \in D^*$ and a non-negative integer $i < pl(\xi)$, we set $\bar{\theta}(\xi)(i) = \theta(\xi(j))(i-pj)$, where $j = \left[\frac{i}{p}\right]$. Also, we define a mapping $\hat{\theta}$ from D^I into D^I, as follows:

$$\hat{\theta}(\cdots \alpha_{-2}\alpha_{-1}{*}\alpha_0\alpha_1 \cdots) = \cdots \theta(\alpha_{-2})\theta(\alpha_{-1}){*}\theta(\alpha_0)\theta(\alpha_1) \cdots,$$

where $\alpha_i \in D$ $(i \in I)$, that is, for $\alpha \in D^I$ and $i \in I$, we set $\hat{\theta}(\alpha)(i) = \theta(\alpha(j))(i-pj)$, where $j = \left[\frac{i}{p}\right]$. For a substitution θ and a positive integer k, $\bar{\theta}^k$ or $\hat{\theta}^k$ denotes the k-ple composition of the mapping $\bar{\theta}$ or $\hat{\theta}$, respectively. The restriction of $\bar{\theta}^k$ to the domain of θ is denoted by θ^k.

DEFINITION 1. A subset S of D^I is called a *minimal set* of the compact dynamical system (D^I, T) if $S = \overline{\mathrm{Orb}}(\alpha)$ (= the closure of $\{T^i\alpha\,;\, i \in I\}$) for any $\alpha \in S$. An element α of D^I is called an *almost periodic sequence* if $\overline{\mathrm{Orb}}(\alpha)$ is a minimal set of (D^I, T). A minimal set S of (D^I, T) is said to be *associated with* a substitution θ whose domain is D, if $\hat{\theta}(S) \subset S$. A *substitution minimal set* is a minimal set of (D^I, T) which is associated with some substitution.

Our definition of a minimal set associated with a substitution θ is slightly different from that of Gottschalk ([3]). In fact, in 3.39 of [3], a minimal set S is said to be generated by a substitution θ if $\hat{\theta}^k(S) \subset S$ for some positive integer k. Nevertheless, these two definitions coincide if θ satisfies the following Condition $\sharp$ (see Lemma 3), concerning a substitution θ such that the domain of θ is D and $L(\theta) = p$.

CONDITION $\sharp$. There exists a positive integer k, such that for any $n, m \in D$, there exists $j \in N_{p^k}$ satisfying $\theta^k(n)(j) = m$.

LEMMA 1. (1) *Let a substitution θ satisfy Condition $\sharp$. Then there exists uniquely a minimal set associated with θ, which will be denoted by $W(\theta)$.*

(2) *For any substitution minimal set S, there exists a subset A of D and a substitution θ on A satisfying Condition $\sharp$ such that $S = W(\theta)$.*

PROOF. (1) Let a substitution $\theta: D \to D^{N_p}$ satisfy Condition $\sharp$. Let S be a minimal set of (D^I, T) such that $\hat{\theta}(S) \subset S$. Let $\alpha \in S$. Let k be a positive integer as in Condition $\sharp$ for this θ. Since $\hat{\theta}^k(\alpha)(i) = \bar{\theta}^k(\alpha(0))(i)$ for $i = 0, 1, \cdots, p^k-1$, it holds that $\{\hat{\theta}^k(\alpha)(i)\,;\, i \in I\} = D$. Since α and $\hat{\theta}^k(\alpha)$ belong to the common minimal set S, this implies that $\{\alpha(i)\,;\ i\in I\}=D$. Suppose that there exists another minimal set S' of (D^I, T) such that $\hat{\theta}(S')\subset S'$. Let $\beta\in S'$. From the above discussion, there exist $i, j\in I$ such that $\alpha(i)=\beta(j)$. For any positive integer h, let

$$\alpha' = T^{ip^h+h}\circ\hat{\theta}^h(\alpha)\,,\qquad \text{and}$$
$$\beta' = T^{jp^h+h}\circ\hat{\theta}^h(\beta)\,.$$

Then it is easy to verify that $d(\alpha'\cdot\beta')\leqq 1/h$. Since S and S' are closed sets and h is arbitrary, this implies $S\cap S'\neq\emptyset$, from which $S=S'$ follows since S and S' are minimal sets. Thus the uniqueness is proved.

Next, we prove the existence of a minimal set associated with θ. Let θ satisfy Condition $\sharp$. It was proved in 3.38 of [3] that there exist an almost periodic sequence α and a positive integer h such that $\hat{\theta}^h(\alpha)=\alpha$. Denote by S the orbit closure of α under the shift T. Denote by θ^h the restriction of $\bar{\theta}^h$ to D. Then θ^h is a substitution defined on D which also satisfies Condition $\sharp$. It is clear that $\hat{\theta}^h(S)\subset S$. It was proved in 3.41 of [3] that $\hat{\theta}(\alpha)$ is also an almost periodic sequence. Let S' be the orbit closure of $\hat{\theta}(\alpha)$ under the shift T. Then it holds that S', as well as S, is a minimal set associated with θ^h. Therefore $S=S'$. That is, $\hat{\theta}(\alpha)\in S$, which implies $\hat{\theta}(S)\subset S$ since $\hat{\theta}$ is a continuous mapping satisfying $\hat{\theta}\circ T=T^p\circ\hat{\theta}$. Thus (1) is proved.

(2) Let S be a minimal set of (D^I, T) associated with a substitution $\theta': D\rightarrow D^{N_p}$. Let $A=\{\alpha(i)\,;\ i\in I,\ \alpha\in S\}$. Denote by θ the restriction of θ' to A. Then $\hat{\theta}(S)=\hat{\theta}'(S)\subset S$. Since S is a minimal set, there exists a positive integer n such that $a\in\{\alpha(i)\,;\ j\leqq i<j+n\}$ for any $a\in A$, $\alpha\in S$ and $j\in I$ ([7]). Let k be a positive integer satisfying $n\leqq p^k$. Let $a\in A$. Let $\alpha\in S$ and $\alpha(m)=a$. Since $\bar{\theta}^k(a)(i)=\hat{\theta}^k(\alpha)(mp^k+i)$ for any $i=0, 1, \cdots, p^k-1$ and $\hat{\theta}^k(\alpha)$ belongs to S, it holds that $\{\bar{\theta}^k(a)(i)\,;\ i\in N_{pk}\}=A$. Since $a\in A$ is arbitrary, this implies that θ satisfies Condition $\sharp$. Thus $S=W(\theta)$.

LEMMA 2. *For a substitution θ satisfying Condition $\sharp$, the mapping $\hat{\theta}: W(\theta)\rightarrow W(\theta)$ is open and continuous and satisfies $\hat{\theta}\circ T=T^p\circ\hat{\theta}$.*

PROOF. It is sufficient to prove that the mapping $\hat{\theta}: W(\theta)\rightarrow W(\theta)$ is open since the other statements of the lemma are clear. Since the mapping $\hat{\theta}: D^I\rightarrow D^I$ is open, it is sufficient to prove that $\hat{\theta}(W(\theta))$ is an open set in $W(\theta)$. Since $\hat{\theta}(W(\theta))$ is a minimal set of (D^I, T^p), this follows from Lemma 15.

The proof of Lemma 1 implies the following two lemmas.

LEMMA 3. *For any substitution θ which satisfies Condition $\sharp$ and for any positive integer n, θ^n satisfies also Condition $\sharp$ and we have $W(\theta)=W(\theta^n)$.*

LEMMA 4. *If θ satisfies Condition $\sharp$ and $\alpha\in W(\theta)$, then $\alpha: I\rightarrow D$ is an onto mapping, where D is the domain of θ.*

Since any substitution minimal set S is strictly ergodic (see [5] or [6]), the pair (S, T) can be also regarded as the pair of the probability measure space S with the unique T-invariant probability measure and the measure preserving transformation T on S. The statement that such pairs are measure isomorphic (measure homomorphic) should be understood in this sense. On the contrary, topological isomorphisms (topological homomorphisms) are called simply isomorphisms (homomorphisms). That is, the statement that ψ is a *measure homomorphism* (*homomorphism*) from (S, T) to (S', T'), where S and S' are substitution minimal sets and T, T' are shifts on S, S', respectively, implies that ψ is a measure preserving mapping (continuous and onto mapping) from S to S' satisfying $\psi\circ T(\alpha)=T\psi(\alpha)$ for almost all $\alpha\in S$ (for any $\alpha\in S$). In the above, if ψ is invertible, then ψ is called a *measure isomorphism* (*isomorphism*).

Denote by Θ the set of all substitutions satisfying Condition $\sharp$ and the domains of which are one of N_r's ($r\geqq 1$). We use the common notation T for shifts on distinct D^I's so far as ambiguities can be avoided.

DEFINITION 2. By a *topological invariant* (*measure invariant*), we mean a function ψ from Θ into I, satisfying the condition that if $(W(\theta), T)$ and $(W(\theta'), T)$ are isomorphic (measure isomorphic) to each other, then $\psi(\theta)=\psi(\theta')$.

Since any substitution minimal set is strictly ergodic, it is clear that a measure invariant is a topological invariant. In the sequel, θ denotes a substitution belonging to Θ such that $s(\theta)=r$ and $L(\theta)=p$, unless stated otherwise. Therefore, θ is a mapping $N_r\rightarrow(N_r)^{N_p}$. Also, we use the common notation θ for θ, $\bar{\theta}$ and $\hat{\theta}$.

§ 2. Cyclic substitutions.

DEFINITION 3. Let θ be any substitution satisfying Condition $\sharp$. It is said to be *cyclic* if $(W(\theta), T)$ is cyclic (i. e. $W(\theta)$ is a finite set). In this case, Card$(W(\theta))$ is called the *cycle* of θ.

LEMMA 5. *Assume that $\theta\in\Theta$ is cyclic and one-to-one* (*i. e. the mapping $\theta: N_r\rightarrow(N_r)^{N_p}$ is one-to-one*). *Then it holds that*

(i) *the cycle of θ is r,*

(ii) *r and p are relatively prime to each other, and*

(iii) *for any $\alpha\in W(\theta)$, $\alpha(i)=\alpha(j)$ if and only if $i\equiv j$* (mod r).

PROOF. Let c be the cycle of θ. Let $\alpha\in W(\theta)$. Then α is a cyclic function with the least cycle c from I *onto* N_r (by Lemma 4). Since $\theta(\alpha)\in W(\theta)$, there exists an integer k such that $\theta(\alpha)=T^k\alpha$, that is,

$$\theta(\alpha(i))(j)=\alpha(k+pi+j) \tag{1}$$

for any $i\in I$ and $j\in N_p$. We may and do assume $p\geqq c$, since otherwise, we consider θ^n for sufficiently large n instead of θ, and have the conclusions (i) and (iii) (by Lemma 3) and that r and p^n are relatively prime to each other, from which (ii) follows. Let e be the greatest common factor of c and p. Since $p \geqq c$, the statement that $\alpha(k+pi+j)=\alpha(k+pi'+j)$ for any $j \in N_p$ is equivalent to $k+pi \equiv k+pi'$ (mod c), or $i \equiv i'$ (mod c/e). On the other hand, since θ is one-to-one, the statement that $\theta(\alpha(i))(j)=\theta(\alpha(i'))(j)$ for any $j \in N_p$ is equivalent to $\alpha(i)=\alpha(i')$. Using (1), we arrive at the conclusion that $\alpha(i)=\alpha(i')$ if and only if $i \equiv i'$ (mod c/e). This implies that c/e is a cycle of α, from which it follows that $e=1$. Also, the above implies that $c/e=r$, since $\{\alpha(i)\,;\, i \in I\}=N_r$. Thus we complete the proof.

LEMMA 6. *Let $\theta \in \Theta$ be cyclic and one-to-one. Then for any integers $k \geqq 1$ and $j \in N_{p^k}$, the mapping $n \to \theta^k(n)(j)$ is a bijection from N_r to N_r.*

PROOF. Let $\theta \in \Theta$ be cyclic and one-to-one. Let $k \geqq 1$ and $0 \leqq j < p^k$ be any integers. Let $n, m \in N_r$ and $n \neq m$. Let $\alpha \in W(\theta)$, $\alpha(h)=n$ and $\alpha(i)=m$. By (iii) of Lemma 5, $h \not\equiv i$ (mod r). It holds that $\theta^k(n)(j)=\theta^k(\alpha)(hp^k+j)$ and $\theta^k(m)(j)=\theta^k(\alpha)(ip^k+j)$. Since $hp^k+j \not\equiv ip^k+j$ (mod r) by (ii) of Lemma 5 and $\theta^k(\alpha)$ belongs to $W(\theta)$, we have $\theta^k(n)(j) \neq \theta^k(m)(j)$ by (iii) of Lemma 5. Thus the mapping $n \to \theta^k(n)(j)$ is an injection, and therefore, a bijection from N_r to N_r.

LEMMA 7. *Let $\theta \in \Theta$ be one-to-one and $s(\theta)=r \geqq 2$. Assume that there exist integers $n \in N_r$, $k \geqq 1$ and $0 \leqq j < p^k-1$ such that $\theta^k(n)(j)=\theta^k(n)(j+1)$. Then θ is not cyclic.*

PROOF. Clear from (iii) of Lemma 5.

LEMMA 8. *There exists an algorithm on θ to decide whether or not $\theta \in \Theta$ is cyclic and one-to-one.*

PROOF. That θ is one-to-one is clearly decidable (in the sense of the recursive function theory). Let $\theta \in \Theta$ be one-to-one. If $\{\theta(n)(0)\,;\, n \in N_r\} \neq N_r$ or $\{\theta(n)(1)\,;\, n \in N_r\} \neq N_r$, then θ is not cyclic by Lemma 6. Assume that

$$\{\theta(n)(0)\,;\, n \in N_r\} = \{\theta(n)(1)\,;\, n \in N_r\} = N_r\,.$$

Define a bijective mapping $\psi : N_r \to N_r$ by $\psi(\theta(n)(0))=\theta(n)(1)$ ($n \in N_r$). We prove that θ is cyclic if and only if

(1) $\psi^j(\theta(n)(0))=\theta(n)(j)$, and

(2) $\psi^p(\theta(n)(0))=\theta(\psi(n))(0)$

for any $n \in N_r$ and $j \in N_p$. Let θ be cyclic and $\alpha \in W(\theta)$. Then exists an integer k, such that

$$\theta(\alpha(i))(j)=\alpha(k+pi+j)$$

for any $i \in I$ and $j \in N_p$. Since r and p are relatively prime to each other (by Lemma 5), for any integer h, there exists an integer i such that $h \equiv k+pi$ (mod r). Since r is the least cycle of α (by Lemma 5),

$$\begin{aligned}\psi(\alpha(h)) &= \psi(\alpha(k+pi)) \\ &= \psi(\theta(\alpha(i))(0)) \\ &= \theta(\alpha(i))(1) \\ &= \alpha(k+pi+1) \\ &= \alpha(h+1)\,.\end{aligned}$$

Thus, $\psi(\alpha(h))=\alpha(h+1)$ for any integer h. Since $\alpha: I \to N_r$ is an onto mapping, for any $n \in N_r$, we can find h such that $n=\alpha(h)$. Then,

$$\begin{aligned}\psi^j(\theta(n)(0)) &= \psi^j(\theta(\alpha(h))(0)) \\ &= \psi^j(\alpha(k+ph)) \\ &= \alpha(k+ph+j) \\ &= \theta(\alpha(h))(j) \\ &= \theta(n)(j)\end{aligned}$$

for any $j \in N_p$. Also,

$$\begin{aligned}\psi^p(\theta(n)(0)) &= \psi^p(\theta(\alpha(h))(0)) \\ &= \psi^p(\alpha(k+ph)) \\ &= \alpha(k+ph+p) \\ &= \theta(\alpha(h+1))(0) \\ &= \theta(\psi(\alpha(h)))(0) \\ &= \theta(\psi(n))(0)\,.\end{aligned}$$

Conversely, suppose (1) and (2) hold. We prove that $\psi^j(\theta^k(n)(0))=\theta^k(n)(j)$ for any integer $k \geqq 1$ and $j=0, 1, \cdots, p^k-1$ by induction about j. If $0 \leqq j < p$, then using (1) we have

$$\begin{aligned}\psi^j(\theta^k(n)(0)) &= \psi^j(\theta(\theta^{k-1}(n)(0))(0)) \\ &= \theta(\theta^{k-1}(n)(0))(j) \\ &= \theta^k(n)(j)\,.\end{aligned}$$

Assume that the above equality holds for any j' less than j and for any k such that $p \leqq j < p^k$. Let $j=ap^c+b$, where a, b and c are integers such that $0<a<p$, $0<c<k$ and $0 \leqq b < p^c$.

Using (1) (2) and the assumption of induction, we have

$$\begin{aligned}\psi^j(\theta^k(n)(0)) &= \psi^b \circ \psi^{ap^c}(\theta^c(\theta^{k-c}(n)(0))(0)) \\ &= \psi^b(\theta^c(\psi^a(\theta^{k-c}(n)(0)))(0))\end{aligned}$$

$$= \psi^b(\theta^c(\theta^{k-c}(n)(a))(0))$$
$$= \theta^c(\theta^{k-c}(n)(a))(b)$$
$$= \theta^k(n)(ap^c+b)$$
$$= \theta^k(n)(j)\,.$$

Let k be a positive integer as in Condition $\sharp$ for this θ. Then, $\{\psi^j(\theta^k(n)(0))\,;\ j\in I\}=N_r$ from the above equality. Therefore, $\{\psi^j(0)\,;\ j\in I\}=N_r$. Define $\alpha\in(N_r)^I$ by $\alpha(i)=\psi^i(0)$ $(i\in I)$. Then α is cyclic and has the least cycle r. From the above equality, for any integers $n\in N_r$ and $h\geqq 1$ it holds that $\theta^h(n)$ is a section of length p^h of α. This implies that $\alpha\in W(\theta)$ and θ is cyclic. Thus that θ is cyclic is decidable.

REMARK 1. If θ satisfies Condition $\sharp$, then we can select k as in Condition $\sharp$ to be $r!+r^2$, where r is the size of θ. This fact proves the decidability of $\theta\in\Theta$.

§ 3. The branching number.

DEFINITION 4. Let $\theta\in\Theta$. A partition π of N_r (i.e. π is a family $\{S_0, S_1, \cdots, S_{q-1}\}$ of non-empty subsets of N_r, such that $\bigcup_{i=0}^{q-1} S_i=N_r$ and $S_i\cap S_j=\emptyset$ if $i\neq j$) is said to be *consistent* (w.r.t. θ) if $n\sim m$ (π) (i.e. there exists an element of π which contains both n and m) implies $\theta(n)(j)\sim\theta(m)(j)$ (π) for any $n, m\in N_r$ and $j\in N_p$. Let π be a consistent partition. Then, a substitution θ^π from π into π^{N_p} can be well defined as follows:

$\theta^\pi(S)(j)=S'$ if $\theta(n)(j)\in S'$ for some $n\in S$, where $S, S'\in\pi$ and $j\in N_p$.

LEMMA 9. *Let $\theta\in\Theta$. Let π be a consistent partition and $\tilde{\pi}$ be the projection $N_r\to\pi$. For $\alpha\in(N_r)^I$, define $\hat{\pi}(\alpha)\in\pi^I$ by $\hat{\pi}(\alpha)(i)=\tilde{\pi}(\alpha(i))$ $(i\in I)$. Then it holds that*

(1) *θ^π satisfies Condition $\sharp$,*

(2) *$W(\theta^\pi)=\hat{\pi}(W(\theta))$, and*

(3) *if θ^π is cyclic and the cycle is relatively prime with p, then θ^π is one-to-one (as the mapping $\pi\to\pi^p$).*

PROOF. (1) is clear. Since $\hat{\pi}\circ T=T\circ\hat{\pi}$, $\hat{\pi}(W(\theta))$ is a minimal set of (π^I, T). Therefore, (2) follows from the fact that $\theta^\pi\circ\hat{\pi}(W(\theta))=\hat{\pi}\circ\theta(W(\theta))\subset\hat{\pi}(W(\theta))$. Let $\mathrm{Card}(W(\theta^\pi))=k$. Let T^π be the shift on $W(\theta^\pi)$. Since $\theta^\pi\circ T^\pi=(T^\pi)^p\circ\theta^\pi$ and p is relatively prime with k, $\theta^\pi: W(\theta^\pi)\to W(\theta^\pi)$ must be a bijection. Suppose that $\theta^\pi(S)=\theta^\pi(S')$ for some $S\neq S'$ $(\in\pi)$. There exist E and F belonging to $W(\theta^\pi)$ such that $E(0)=S$ and $F(0)=S'$. Then for any large integer n, $d((\theta^\pi)^n\circ T^\pi\circ\theta^\pi(E),\ (\theta^\pi)^n\circ T^\pi\circ\theta^\pi(F))\leqq p^{-n}$, where d is the metric on π^I, which is a contradiction since both θ^π and T^π are bijections on $W(\theta^\pi)$ and $W(\theta^\pi)$ is a finite set.

LEMMA 10. *Let $\theta \in \Theta$. Let π and ψ be consistent partitions such that both θ^{π} and θ^{ψ} are cyclic and one-to-one. Let τ be the least common refinement of π and ψ. Then* Card(τ) *is the least common multiple of* Card(π) *and* Card(ψ). *Therefore, if* Card$(\pi)=$Card(ψ), *then $\pi=\psi$.*

PROOF. Let $\alpha \in W(\theta)$. Let $\tilde{\pi}$ and $\tilde{\psi}$ be the projections from N_r onto π and ψ, respectively. Then $\tilde{\pi}\circ\alpha$ and $\tilde{\psi}\circ\alpha$ are cyclic functions with the least cycles Card(π) and Card(ψ), respectively (by Lemma 5 and Lemma 9). For any $i \in I$, consider the pair $(\tilde{\pi}(\alpha(i)), \tilde{\psi}(\alpha(i)))$. The number of all distinct pairs is the least common multiple of Card(π) and Card(ψ) (by (iii) of Lemma 5), which completes the proof since the image of α is N_r.

LEMMA 11. *Let $\theta \in \Theta$. There exists a unique consistent partition π, such that*

(1) *θ^{π} is cyclic and one-to-one (as the mapping $\pi \to \pi^p$), and*

(2) *π has the greatest* Card(π) *among consistent partitions satisfying* (1) *above.*

PROOF. The existence is clear since the trivial partition $\{N_r\}$ satisfies (1) above. The uniqueness follows from Lemma 10.

DEFINITION 5. Let $\theta \in \Theta$. The consistent partition π satisfying the conditions (1) and (2) of Lemma 11 is called the *partial cycle partition* of θ. Card(π) is called the *partial cycle* of θ and denoted by $P(\theta)$.

DEFINITION 6. Let $\theta \in \Theta$. Let π be the partial cycle partition of θ. Define $B(\theta)$ by

$$B(\theta)=\min_{0\leqq j<p^{2^r}}\min_{S\in\pi}\operatorname{Card}(\{\theta^{2^r}(n)(j)\,;\ n\in S\})\,,$$

which will be called the *branching number* of θ.

LEMMA 12. *In the same situation as in Definition* 6, *we have*

$$B(\theta)=\min_{k\in N}\min_{0\leqq j<p^k}\min_{S\in\pi}\operatorname{Card}(\{\theta^{k}(n)(j)\,;\ n\in S\}) \tag{1}$$

$$=\min_{0\leqq j<p^{2^r}}\min_{S\in\pi}\operatorname{Card}(\{\theta^{2^r}(n)(j)\,;\ n\in N_r\}\cap S) \tag{2}$$

$$=\min_{k\in N}\min_{0\leqq j<p^k}\min_{S\in\pi}\operatorname{Card}(\{\theta^{k}(n)(j)\,;\ n\in N_r\}\cap S)\,. \tag{3}$$

PROOF. For $k=1, 2, \cdots$, let

$$B_k=\min_{0\leqq j<p^k}\min_{S\in\pi}\operatorname{Card}(\{\theta^{k}(n)(j)\,;\ n\in S\})\,.$$

It is clear that $B_1 \geqq B_2 \geqq \cdots$. Let $\mathcal{S}$ be the class of all subsets of N_r. For $j \in N_p$, define a mapping θ_j from $\mathcal{S}$ into $\mathcal{S}$ by $\theta_j(U)=\{\theta(n)(j)\,;\ n\in U\}$, where $U\in\mathcal{S}$. Since Card$(\mathcal{S})=2^r$, for any sequence $j_1, j_2, \cdots, j_k \in N_p$ and $U \in \mathcal{S}$, we can select a subsequence $j'_1, j'_2, \cdots, j'_{k'}$, where $k' \leqq 2^r$, such that

$$\theta_{j_k}\circ\cdots\circ\theta_{j_2}\circ\theta_{j_1}(U)=\theta_{j'_{k'}}\circ\cdots\circ\theta_{j'_2}\circ\theta_{j'_1}(U)\,.$$

This implies that $B_k = B_{2r}$ for any $k \geqq 2^r$ and proves (1). To prove (2) and (3), it is sufficient to prove that for any integers $k \geqq 1$ and $j \in N_{p^k}$, it holds that $n \sim m$ (π) if and only if $\theta^k(n)(j) \sim \theta^k(m)(j)$ (π). Since this follows from Lemma 6, we complete the proof.

DEFINITION 7. Let $\theta \in \Theta$. Define $C(\theta)$ by

$$C(\theta) = \min_{0 \leqq j < p^{2^r}} \mathrm{Card}\,(\{\theta^{2^r}(n)(j)\,;\ n \in N_r\}),$$

which will be called the *column number* of θ.

Similarly as (1) of Lemma 12, we can prove the following lemma.

LEMMA 13. *Let $\theta \in \Theta$. It holds that*

$$C(\theta) = \min_{k \in N} \min_{0 \leqq j < p^k} \mathrm{Card}\,(\{\theta^k(n)(j)\,;\ n \in N_r\}).$$

THEOREM 1. *There exist algorithms (in the sense of the recursive function theory) to compute $B(\theta)$, $C(\theta)$ and $P(\theta)$ for $\theta \in \Theta$.*

PROOF. Clear from Lemma 8 and the definitions.

THEOREM 2. *For any $\theta \in \Theta$, it holds that*

(1) $1 \leqq B(\theta) \leqq C(\theta) \leqq s(\theta)$,

(2) *$B(\theta) = C(\theta)$ if and only if $P(\theta) = 1$,*

(3) *$C(\theta) = 1$ if and only if $B(\theta) = P(\theta) = 1$,*

(4) *$P(\theta) = s(\theta)$ if and only if θ is cyclic and one-to-one,*

(5) *if θ is cyclic, then $B(\theta) = 1$, and*

(6) *$P(\theta)$ is relatively prime with $L(\theta)$.*

PROOF. (1) and (4) are clear from the definitions. (3) follows from (1) and (2). Since the trace relation (see Definition 8, Section 4) of any cyclic system coincides with the diagonal, (5) follows from Theorem 5 which shall be proved in Section 5. (6) is clear from Lemma 5. The "if" part of (2) is clear from the definitions. The "only if" part of (2) is clear from the fact that for the partial cycle partition π of θ and any integers $k \geqq 1$ and $j \in N_{p^k}$, $n \sim m$ (π) if and only if $\theta^k(n)(j) \sim \theta^k(m)(j)$ (π), which follows from Lemma 6.

We use the following lemma later.

LEMMA 14. *Let $\theta \in \Theta$. Assume that there exist integers $n \in N_r$, $k \geqq 1$ and $0 \leqq j < p^k - 1$, such that $\theta^k(n)(j) = \theta^k(n)(j+1)$. Then $P(\theta) = 1$.*

PROOF. Clear from Definition 5 and Lemma 7.

§ 4. The trace relation.

Throughout this section, we fix a substitution $\theta \in \Theta$, where $s(\theta) = r$ and $L(\theta) = p$. Let $W = W(\theta)$. For this θ, we define a function λ as in the statement of the following lemma.

LEMMA 15 (Gottschalk and Hedlund [2]). *For any positive integer k, there* *exists a factor $\lambda(k)$ of k, such that W can be decomposed into $\lambda(k)$ minimal sets of (W, T^k). Therefore, each minimal set of (W, T^k) is open and closed in W.*

PROOF. Let S be a minimal set of (W, T^k). Since $T^k S=S$, $\bigcup_{i=0}^{k-1} T^i S$ is closed and T-invariant. Therefore $W=\bigcup_{i=0}^{k-1} T^i S$. Let $\lambda(k)$ be the least positive integer such that $T^{\lambda(k)}S=S$. Then it is clear that $\lambda(k)$ is a factor of k and $W=\bigcup_{i=0}^{\lambda(k)-1} T^i S$. Since $T^i S$ is a minimal set of (W, T^k) for any $i \in I$, either $T^i S=T^j S$ or $T^i S \cap T^j S=\emptyset$ holds for any $i, j \in I$. The minimality of $\lambda(k)$ means that $W=\bigcup_{i=0}^{\lambda(k)-1} T^i S$ is a disjoint sum.

DEFINITION 8 (Gottschalk and Hedlund [2]). For $k=1, 2, \cdots$, define a T-invariant equivalence relation Λ_k on W, as follows:

$$(\alpha, \beta) \in \Lambda_k \text{ if } \alpha \text{ and } \beta \text{ belong to a common minimal set of } (W, T^k).$$

The closed T-invariant equivalence relation $\Lambda=\bigcap_{k=1}^{\infty} \Lambda_k$ is called the *trace relation* (of (W, T)).

DEFINITION 9. Let R and R' be any equivalence relations on W which are closed sets of $W \times W$. They are said to be *independent* of each other if $\alpha R=\{\gamma\,;\,(\alpha, \gamma) \in R\}$ intersects with $\beta R'$ for any α and β belonging to W.

Note that since W is compact, a closed relation R on W satisfies the condition that $FR=\bigcup_{\alpha \in F} \alpha R$ is a closed set if F is a closed set of W.

LEMMA 16. *Let $R_1, R_2, \cdots$ and R be closed equivalence relations on W, such that*

(1) *$R_1 \supset R_2 \supset \cdots$, and*

(2) *R is independent of any of R_i $(i=1, 2, \cdots)$.*

Then R is independent of $\bigcap_{i=1}^{\infty} R_i$.

PROOF. Clear since W is compact.

LEMMA 17. *Let R and R' be any closed equivalence relation on W which are independent of each other. Then $W/(R \cap R')$ is homeomorphic to $(W/R) \times (W/R')$.*

PROOF. Let $f(\alpha(R \cap R'))=(\alpha R, \alpha R')$. From the definition of quotient topologies, f is clearly a continuous mapping from $W/(R \cap R')$ into $(W/R) \times (W/R')$. Also it is clear that f is an injection. From Definition 9, for any $\beta, \gamma \in W$, there exists $\alpha \in W$ such that $\alpha \in \beta R \cap \gamma R'$. Hence $f(\alpha(R \cap R'))=(\beta R, \gamma R')$ and f is an onto mapping. Since $W/(R \cap R')$ is compact and metrizable, this implies that f is a homeomorphism from $W/(R \cap R')$ onto $(W/R) \times (W/R')$.

LEMMA 18. (1) *$\Lambda_1=W \times W$.*

(2) *If* h *is a factor of* k, *then* $\Lambda_h \supset \Lambda_k$.

(3) $\Lambda_k = \Lambda_{\lambda(k)}$ *for* $k = 1, 2, \cdots$.

(4) $\lambda(\lambda(k)) = \lambda(k)$ *for* $k = 1, 2, \cdots$.

PROOF. Clear.

LEMMA 19. *If* h *and* k *are relatively prime with each other, then* Λ_h *and* Λ_k *are independent of each other. Hence,* Λ_h *is independent of* $\bigcap_{i=1}^{\infty} \Lambda_{k^i}$ *if* h *is relatively prime with* k.

PROOF. Let h and k be relatively prime with each other. Then by Lemma 15, $\lambda(h)$ and $\lambda(k)$ are relatively prime. Therefore, for any $\alpha \in W$, the elements $\alpha, T^{\lambda(k)}\alpha, T^{2\lambda(k)}\alpha, \cdots, T^{(\lambda(h)-1)\lambda(k)}\alpha$ of W, $\lambda(h)$ in number, belong to different classes of Λ_h, while they belong to a common class of Λ_k. This completes the proof.

LEMMA 20. (1) *If* $\lambda(k) = k$ *and* h *is a factor of* k, *then* $\lambda(h) = h$.

(2) *If* $\lambda(h) = h$, $\lambda(k) = k$ *and* j *is the least common multiple of* h *and* k, *then* $\lambda(j) = j$ *and* $\Lambda_j = \Lambda_h \cap \Lambda_k$.

(3) *Assume that* $\lambda(h) = h$. *Then* $\Lambda_h \supset \Lambda_k$ *if and only if* h *is a factor of* k.

PROOF. (1) Let $k = ha$. Let S be a minimal set of (W, T^k). Then $U = \bigcup_{i=0}^{a-1} T^{hi}S$ is a minimal set of (W, T^h). Since $\lambda(k) = k$, $T^iS \subset U$ implies $i \equiv 0 \pmod{h}$. Thus, $T^iU = U$ only if $i \equiv 0 \pmod{h}$, which proves $\lambda(h) = h$.

(2) Let U and V be minimal sets of (W, T^h) and (W, T^k), respectively, such that U and V intersect. Let $S = U \cap V$. Then it holds that $S, TS, \cdots, T^{j-1}S$ are closed T^j-invariant sets which are disjoint with each other. This implies $\lambda(j) \geqq j$. From $\lambda(j) \leqq j$, it follows that $\lambda(j) = j$ and that S is a minimal set of (W, T^j).

(3) It is sufficient to prove the "only if" part. Assume that $\Lambda_h \supset \Lambda_k$. Let j be the least common multiple of h and $\lambda(k)$. From (2), $\Lambda_j = \Lambda_h \cap \Lambda_{\lambda(k)} = \Lambda_k$, and hence $\lambda(j) = \lambda(k)$. This implies, however, $j = k$, since k is a factor of j and $\lambda(j) = j$.

LEMMA 21. *For* $k = 1, 2, \cdots$ *and* $0 \leqq j < p^k$, $T^j \circ \theta^k(W)$ *is a complete class of the equivalence relation* Λ_{p^k}.

PROOF. Clear since $\theta^k(W)$ is a minimal set of (W, T^{p^k}) (by (1) of Lemma 2).

LEMMA 22 (Gottschalk [3]). $\bigcap_{i=1}^{\infty} \theta^i(W)$ *is a finite set.*

PROOF. Assume that we can select r^2+1 distinct elements $\alpha_0, \alpha_1, \cdots, \alpha_{r^2}$, belonging to $\bigcap_{i=1}^{\infty} \theta^i(W)$. There exists k such that if $i \neq j$, then $\alpha_i(h) \neq \alpha_j(h)$ for some $-p^k \leqq h < p^k$. For $i = 0, 1, \cdots, r^2$, there exists $\beta_i \in W$ such that $\alpha_i = \theta^k(\beta_i)$. Then there exist i and j such that $\beta_i(-1) = \beta_j(-1)$ and $\beta_i(0) = \beta_j(0)$. This implies that $\alpha_i(h) = \alpha_j(h)$ for any h such that $-p^k \leqq h < p^k$, which is a contradiction. Thus $\mathrm{Card}\left(\bigcap_{i=1}^{\infty} \theta^i(W)\right) \leqq r^2$.

THEOREM 3. *There exists an integer $q \geqq 1$ satisfying the following three conditions:*

(1) *q is relatively prime with p,*

(2) *$\lambda(q) = q$, and*

(3) *$\Lambda = \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \Lambda_q$.*

Moreover, such q is unique and equal to $P(\theta)$.

PROOF. Let $K = \{k\,;\ k = \lambda(k)$ and k is relatively prime with $p\}$. Let $h \geqq 1$ be any integer. Let $\lambda(h) = jk$, where j is the greatest common factor of $\lambda(h)$ and p^i. Choose i so large that $\lambda(h)$ is the least common multiple of j and k. Then k is relatively prime with p and belongs to K (by (4) of Lemma 18 and (1) of Lemma 20). From Lemma 18 and Lemma 20,

$$\begin{aligned}\Lambda_h &= \Lambda_{\lambda(h)} \\ &= \Lambda_j \cap \Lambda_k \supset \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \left(\bigcap_{k \in K} \Lambda_k\right).\end{aligned}$$

Therefore

$$\Lambda = \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \left(\bigcap_{k \in K} \Lambda_k\right).$$

Let $k \in K$. From Lemma 19, Λ_k is independent of $\bigcap_{i=1}^{\infty} \Lambda_{p^i}$. Since $\bigcap_{i=1}^{\infty} \theta^i(W)$ is a complete class of $\bigcap_{i=1}^{\infty} \Lambda_{p^i}$ (by Lemma 21), any equivalence class of Λ_k intersects with the finite set $\bigcap_{i=1}^{\infty} \theta^i(W)$. Therefore $k \leqq \mathrm{Card}\left(\bigcap_{i=1}^{\infty} \theta^i(W)\right)$. That is, K is a finite set. Let q be the least common multiple of all elements of K. Then q also belongs to K and $\bigcap_{k \in K} \Lambda_k = \Lambda_q$ (by (2) of Lemma 20). Thus $\Lambda = \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \Lambda_q$, where $q = \max_{k \in K} k$. For any other $q' \in K$, suppose $\Lambda = \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \Lambda_{q'}$. Then $\Lambda_q \supset \left(\bigcap_{i=1}^{\infty} \Lambda_{p^i}\right) \cap \Lambda_{q'}$. Since Λ_i is an open and closed set of $W \times W$ for any i and $W \times W$ is compact, there exists j such that $\Lambda_q \supset \Lambda_{p^j} \cap \Lambda_{q'}$. Therefore, from Lemma 18 and Lemma 20,

$$\Lambda_q \supset \Lambda_{p^j} \cap \Lambda_{q'} = \Lambda_{\lambda(p^j)q'}\,.$$

This implies that q is a factor of $\lambda(p^j)q'$ by Lemma 20. Since q is relatively prime with $\lambda(p^j)$, q is a factor of q'. Thus $q = q'$. To complete the proof, it is sufficient to prove the following lemma.

LEMMA 23. *A necessary and sufficient condition that $k \in K$ is that there exists a consistent partition π of N_r, such that* Card$(\pi) = k$ *and θ^π is cyclic and one-to-one. Therefore, $P(\theta) = \max_{k \in K} k$.*

PROOF. To prove the sufficiency, let π be a consistent partition of N_r such that θ^π is cyclic and one-to-one. Let $k = \mathrm{Card}\,(\pi)$. From Lemma 5, k is relatively prime with p. Let λ' be the function defined in Lemma 15 for this θ^{π}. From Lemma 5, $\lambda'(k)=k$. Since $\hat{\pi}$ (defined in Lemma 9) is a continuous mapping from $W(\theta)$ onto $W(\theta^{\pi})$ which commutes with the shifts, it is easy to see that $\lambda(k)\geqq\lambda'(k)$. Since $\lambda(k)\leqq k$, this implies that $\lambda(k)=k$. When $k=1$, the necessity is clearly true. Suppose $k\in K$ and $k\geqq 2$. Let S_0 be a minimal set of (W, T^k). Let $S_i=T^iS_0$ for $i=1, 2, \cdots, k-1$. Then $S_0, S_1, \cdots, S_{k-1}$ are open and closed sets which are disjoint with each other. Let d be the metric on W. There exists a real number $\varepsilon>0$, such that for any $\alpha\in S_i$ and $\beta\in S_j$, where $i\neq j$, $d(\alpha, \beta)>\varepsilon$. For $i=0, 1, \cdots, k-1$, let $P_i=\{n\in N_r$; there exists $\alpha\in S_i$ such that $\alpha(0)=n\}$. We prove that $\pi=\{P_i; i=0, 1, \cdots, k-1\}$ is a partition of N_r. First, suppose that $n\in P_i\cap P_j$ for $i\neq j$. Let $\alpha\in S_0$. Then, there exist $i', j'\in I$ such that $i'\equiv i$, $j'\equiv j \pmod{k}$ and $\alpha(i')=\alpha(j')=n$. It is easy to verify that

$$d(\theta^h\circ T\circ\theta\circ T^{i'}(\alpha), \theta^h\circ T\circ\theta\circ T^{j'}(\alpha))<(1/p)^h$$

for any $h=1, 2, \cdots$. Let h be sufficiently large and satisfy $(1/p)^h<\varepsilon$. Then $(\theta^h\circ T\circ\theta\circ T^{i'}(\alpha), \theta^h\circ T\circ\theta\circ T^{j'}(\alpha))\in\Lambda_k$. On the other hand, since

$$\theta^h\circ T\circ\theta\circ T^{i'}(\alpha)=T^{p^h+i'p^{h+1}}\circ\theta^{h+1}(\alpha)$$

$$\theta^h\circ T\circ\theta\circ T^{j'}(\alpha)=T^{p^h+j'p^{h+1}}\circ\theta^{h+1}(\alpha)$$

and

$$p^h+i'p^{h+1}\not\equiv p^h+j'p^{h+1} \pmod{k},$$

we have

$$(\theta^h\circ T\circ\theta\circ T^{i'}(\alpha), \theta^h\circ T\circ\theta\circ T^{j'}(\alpha))\notin\Lambda_k,$$

which is a contradiction. Therefore π is a partition of N_r. Next, suppose that $n\sim m$ (π). Let $j\in N_p$. Let $\alpha\in W(\theta)$. There exist $i, i'\in I$ such that $\alpha(i)=n$ and $\alpha(i')=m$ (by Lemma 4). Then $i\equiv i' \pmod{k}$ from the definition of the partition π. Therefore, $\theta(\alpha)(ip+j)\sim\theta(\alpha)(i'p+j)$ (π), which implies $\theta(n)(j)\sim\theta(m)(j)$ (π). Thus π is consistent. It is clear from Lemma 9 that θ^{π} is cyclic and one-to-one.

LEMMA 24. *Let $H=\{k$; $\lambda(k)=k$ and k is a factor of p^i for some positive integer $i\}$. Then H is a finite set if and only if θ is cyclic.*

PROOF. The "if" part is clear. Assume that H is a finite set. Then there exists an integer $j\geqq 1$ such that $\Lambda_{p^j}=\Lambda_{p^{j+1}}=\cdots$. This implies that

$$W=\bigcup_{h=0}^{p^j-1}T^h\Big(\bigcap_{i=0}^{\infty}\theta^i(W)\Big).$$

Thus from Lemma 22, W is a finite set.

Lemma 23 and Lemma 24 lead us to the following theorem.

THEOREM 4. *Let $\theta, \theta'\in\Theta$. Assume that $(W(\theta), T)$ and $(W(\theta'), T)$ are isomorphic to each other. Then at least one of the following two cases occurs.*

(1) *Both θ and θ' are cyclic.*

(2) *$L(\theta)$ and $L(\theta')$ have a common factor* ($\geqq 2$).

PROOF. Assume that $(W(\theta), T)$ and $(W(\theta'), T)$ are isomorphic to each other and are not cyclic. Then the common function λ which is defined in Lemma 15 corresponds to both $(W(\theta), T)$ and $(W(\theta'), T)$. Let $H=\{k\,;\ \lambda(k)=k$ and k is a factor of $L(\theta)^i$ for some positive integer $i\}$ and $K'=\{k\,;\ \lambda(k)=k$ and k is relatively prime with $L(\theta')\}$. Suppose that $L(\theta)$ and $L(\theta')$ are relatively prime each other. Then $H\subset K'$. Since H is an infinite set by Lemma 24, K' is also an infinite set, which contradicts with the fact that $P(\theta')=\max_{k\in K'} k\leqq s(\theta')<\infty$.

The direct product $N_p\times N_p\times\cdots$ is considered as a compact metrizable space in the usual sense. Let $(a_0, a_1, a_2, \cdots)\in N_p\times N_p\times\cdots$. Define $(b_0, b_1, b_2, \cdots)\in N_p\times N_p\times\cdots$ inductively as follows:

$$c_0=1$$

$$c_n=\begin{cases} 1 & \text{if} \quad c_{n-1}=1 \quad \text{and} \quad a_{n-1}=p-1 \\ 0 & \text{else} \end{cases}$$

$$(n=1, 2, \cdots)$$

$$b_n=\begin{cases} a_n & \text{if} \quad c_n=0 \\ a_n+1 & \text{if} \quad c_n=1 \quad \text{and} \quad a_n\neq p-1 \\ 0 & \text{if} \quad c_n=1 \quad \text{and} \quad a_n=p-1 \end{cases}$$

$$(n=0, 1, 2, \cdots)\,.$$

Denote $(b_0, b_1, b_2, \cdots)=\phi((a_0, a_1, a_2, \cdots))$. Then ϕ is a homeomorphism from $N_p\times N_p\times\cdots$ onto itself. The compact dynamical system $(N_p\times N_p\times\cdots, \phi)$ is called the *p-adic system*. The p-adic system is the projective limit of the cyclic system with cycle p^k as $k\uparrow\infty$.

For $\theta\in\Theta$, assume that $\lambda(p^k)=p^k$ for $k=1, 2, \cdots$. Then it is clear that $(W/\Lambda', T/\Lambda')$ is isomorphic to the p-adic system, where $\Lambda'=\bigcap_{i=1}^{\infty}\Lambda_{p^i}$, $W/\Lambda'=\{\alpha\Lambda'\,;\ \alpha\in W\}$ and T/Λ' is a homeomorphism from W/Λ' onto itself defined by $(T/\Lambda')(\alpha\Lambda')=(T\alpha)\Lambda'$ for any $\alpha\in W$. Let (W, T) and (W', T') be any dynamical systems. By the direct product of (W, T) and (W', T'), we mean the dynamical system $(W\times W', T\times T')$, where $W\times W'$ is the product space and $T\times T'$ is a homeomorphism from $W\times W'$ onto itself defined by $(T\times T')((w, w'))=(Tw, T'w')$ $((w, w')\in W\times W')$. Since Λ' and $\Lambda_{P(\theta)}$ are independent of each other (by Lemma 19), it follows from Theorem 3 and Lemma 17 that $(W/\Lambda, T/\Lambda)$ is isomorphic to the direct product of the p-adic system by the cyclic system with cycle $P(\theta)$. Let $\Psi=\{\theta\in\Theta\,;\ \lambda(p^k)=p^k$ for $k=1, 2, \cdots$, where $p=L(\theta)$ and λ is the function defined in Lemma 15 for this $\theta\}$. For a positive integer

i, let $\tau(i)$ be the product of all distinct prime factors of i.

COROLLARY 1. *If we restrict our consideration to $\theta \in \Psi$, then both $\tau(L(\theta))$ and $P(\theta)$ are topological invariants.*

PROOF. Assume that p, q, p' and q' are positive integers such that p and q are relatively prime with each other and p' and q' are also relatively prime with each other. Then it holds that the direct product of the p-adic system by the cyclic system with cycle q is isomorphic to the direct product of the p'-adic system by the cyclic system with cycle q' only if $\tau(p)=\tau(p')$ and $q=q'$. To prove this, let the above two systems be isomorphic to each other. Note that they are minimal systems. Let λ be the function defined in Lemma 15 for them. Then it is easy to see that

$$\{k\,;\ \lambda(k)=k\}=\{k\,;\ k \text{ is a factor of } qp^i \text{ for some } i\in N\}$$
$$=\{k\,;\ k \text{ is a factor of } q'p'^i \text{ for some } i\in N\}\,,$$

which implies $\tau(p)=\tau(p')$ and $q=q'$. This fact proves Corollary 1.

From Lemma 24 and (1) of Lemma 20, it can be easily verified that $\theta \in \Psi$ if θ is non-cyclic and $L(\theta)$ is a power of a prime number. Also, it was proved in [3] that $\theta \in \Psi$ if θ is non-cyclic and $s(\theta)=2$.

§ 5. Main results.

Throughout this section, we fix a substitution $\theta \in \Theta$, where $s(\theta)=r$ and $L(\theta)=p$. Let $W=W(\theta)$. Let Λ be the trace relation of (W, T) and λ be the function defined in Lemma 15 for this (W, T).

LEMMA 25. *Let $q=P(\theta)$. Then $(\alpha, \beta)\in \Lambda_q$ if and only if $(\theta(\alpha), \theta(\beta))\in \Lambda_q$.*

PROOF. The "only if" part is clear since $\theta\circ T=T^p\circ\theta$ and q is relatively prime with p. The "if" part is also clear since the θ-induced mapping on W/Λ_q is one-to-one (see Lemma 23 and its proof).

LEMMA 26. *Let $q=P(\theta)$ and W_0 $(\subset W)$ be any equivalence class of Λ_q. Then for any integers $k\geqq 0$ and j (we may assume $0\leqq j<p^k$), $T^j\circ\theta^k(W_0)$ is a complete class of $\Lambda_{p^k}\cap\Lambda_q$. Conversely, any equivalence class of $\Lambda_{p^k}\cap\Lambda_q$ is expressed like this. Therefore, if S is a complete class of $\Lambda_{p^{k'}}\cap\Lambda_q$, then $T^j\circ\theta^k(S)$ is a complete class of $\Lambda_{p^{k+k'}}\cap\Lambda_q$, where k and k' are any non-negative integers and j is any integer.*

PROOF. Since the partial cycle partition is consistent, it is clear that $T^j\circ\theta^k(W_0)$ is contained in some equivalence class of $\Lambda_{p^k}\cap\Lambda_q$. Let $(\alpha, \beta)\in\Lambda_{p^k}\cap\Lambda_q$ and $\alpha\in T^j\circ\theta^k(W_0)$. Since $T^j\circ\theta^k(W)$ is a complete class of Λ_{p^k}, there exists $\beta'\in W$, such that $\beta=T^j\circ\theta^k(\beta')$. Let $\alpha=T^j\circ\theta^k(\alpha')$ for $\alpha'\in W_0$. Suppose that $(\alpha', \beta')\notin\Lambda_q$. Then from Lemma 25, we have a contradiction that $(\alpha, \beta)\notin\Lambda_q$. Therefore, we have that $\beta'\in W_0$ and $\beta\in T^j\circ\theta^k(W_0)$. This proves one half of Lemma 26. Let W_1 and W_2 be any equivalence classes of Λ_q. Assume that $T^{j_1}\circ\theta^k(W_1)=T^{j_2}\circ\theta^k(W_2)$. Then from Lemma 25 and Lemma 21, it is easy to see that $j_1\equiv j_2 \pmod{\lambda(p^k)}$ and $W_1=w_2$. Since the number of equivalence classes of $\Lambda_{p^k}\cap\Lambda_q$ is $\lambda(p^k)q$, this completes the proof.

THEOREM 5 (Main Theorem). *We have*

$$B(\theta)=\min_{\alpha\in W}\mathrm{Card}\,(\alpha\Lambda)\,.$$

Therefore, $B(\theta)$ is a topological invariant of $\theta\in\Theta$.

PROOF. Let $b=B(\theta)$ and $q=p(\theta)$. Let π be the partial cycle partition of θ. First we prove $b\leqq\mathrm{Card}\,(\alpha\Lambda)$ for any $\alpha\in W$. Let $\alpha\in W$ and k be any integer $\geqq 1$. There exists an integer $0\leqq h<p^k$ such that $\alpha\in T^h\circ\theta^k(W)$. By Definition 6 and Lemma 12, for any $S\in\pi$, it holds that

$$\mathrm{Card}\,(\{\theta^k(n)(h)\,;\ n\in S\})\geqq b\,.$$

Select $\beta_0, \beta_1, \cdots, \beta_{b-1}\in W$ which satisfy

(1) $\beta_i(0)\in S$ for $i=0, 1, \cdots, b-1$, and

(2) $\theta^k(\beta_i(0))(h)\neq\theta^k(\beta_j(0))(h)$ if $i\neq j$.

Let $\alpha_i=T^h\circ\theta^k(\beta_i)$ for $i=0, 1, \cdots, b-1$. Since we can select S and β_0 so that $\alpha_0=\alpha$, we may and do assume $\alpha_0=\alpha$. It is easy to see (by Lemma 26) that

(1) $(\alpha_i, \alpha_j)\in\Lambda_{p^k}\cap\Lambda_q$ for any $i, j=0, 1, \cdots, b-1$, and

(2) $d(\alpha_i, \alpha_j)=1$ if $i\neq j$, where d is the metric on W.

Hereafter, α_i thus obtained for k is denoted by $\alpha_{k,i}$. Let $\gamma_0=\alpha$. From the sequence $\{\alpha_{k,1}\,;\ k=1, 2, \cdots\}$, select a convergent subsequence $\{\alpha_{k_m,1}\,;\ m=1, 2, \cdots\}$, which converges to, say, γ_1. Next, from the sequence $\{\alpha_{k_m,2}\,;\ m=1, 2, \cdots\}$, select a convergent subsequence $\cdots$. Continuing this procedure, we obtain γ_0 $(=\alpha)$, $\gamma_1, \cdots, \gamma_{b-1}$. By Theorem 3 it holds that

(1) $\gamma_i\in\alpha\Lambda$ for $i=0, 1, \cdots, b-1$, and

(2) $\gamma_i\neq\gamma_j$ if $i\neq j$,

from which it follows that $b\leqq\mathrm{Card}\,(\alpha\Lambda)$ for any $\alpha\in W$.

Conversely, let

$$b=\mathrm{Card}\,(\{\theta^k(n)(j)\,;\ n\in S\})$$

for some $k\geqq 1$, $0\leqq j<p^k$ and $S\in\pi$. Since in this case,

$$b=\mathrm{Card}\,(\{\theta^{k+1}(n)(jp+i)\,;\ n\in S\})$$

for any $i=0, 1, \cdots, p-1$, we may and do assume that $k\geqq 2$ and $1\leqq j<p^k-1$. Let $W_0=\{\alpha\in W;\ \alpha(0)\in S\}$ and $W_i=T^iW_0$ for $i=1, 2, \cdots, q-1$. Then $\{W_0, W_1, \cdots, W_{q-1}\}$ is the family of all equivalence classes of Λ_q. Let $V_0=T^j\circ\theta^k(W_0)$. Then, V_0 is a complete class of $\Lambda_{p^k}\cap\Lambda_q$ (by Lemma 26) and there exist $n_1, n_2, \cdots, n_b\in N_r$, such that for any $\alpha\in V_0$, $\alpha(0)$ is equal to one of $n_1, n_2, \cdots, n_b$. For $i=1, 2, \cdots$, let $V_i=(T^j\circ\theta^k)^i(V_0)$. Since $1\leqq j<p^k-1$,

the above statement implies that V_i is covered by a family of b balls with diameter less than p^{-i+1}. Also, by Lemma 26, V_i is a complete class of $\Lambda_{p^{k(i+1)}}\cap\Lambda_q$. Therefore, there exists an integer $0\leqq h_i<q$ such that $V_i=(T^j\circ\theta^k)^{i+1}(W)\cap W_{h_i}$. We can select an infinite sequence $i_1, i_2, \cdots$, such that $h_{i_1}=h_{i_2}=\cdots$. Then $V_{i_1}\supset V_{i_2}\supset\cdots$. By Theorem 3, $\bigcap_{m=1}^{\infty} V_{i_m}$ is a complete class of Λ. Since $\bigcap_{m=1}^{\infty} V_{i_m}$ is covered by ε-balls, b in number, for any real number $\varepsilon>0$, we have $\mathrm{Card}\left(\bigcap_{m=1}^{\infty} V_{i_m}\right)\leqq b$. Q. E. D.

COROLLARY 2. *Let $\theta, \theta'\in\Theta$. If there exists a homomorphism from $(W(\theta), T)$ onto $(W(\theta'), T)$, then $B(\theta)\geqq B(\theta')$.*

PROOF. Let Λ and Λ' be the trace relations of $(W(\theta), T)$ and $(W(\theta'), T)$, respectively. Let ψ be a homomorphism from $(W(\theta), T)$ onto $(W(\theta'), T)$. Then it is well known and proved without difficulty that $\psi(\alpha\Lambda)=\psi(\alpha)\Lambda$ for any $\alpha\in W(\theta)$. This implies that

$$\min_{\alpha\in W(\theta)} \mathrm{Card}\,(\alpha\Lambda)\geqq \min_{\alpha\in W(\theta')} \mathrm{Card}\,(\alpha\Lambda'),$$

which completes the proof in virtue of Theorem 5.

EXAMPLE 1. Let r, p and b be any positive integers such that $r\geqq 2$, $p\geqq 2$ and $b\leqq r$. Define $\theta\in\Theta$ as follows:

(i) for $n\in N_r$,

$$\theta(n)(0)=\begin{cases} n & \text{if } n\leqq b-1 \\ b-1 & \text{otherwise} \end{cases}$$

(ii) for $n\in N_r$ and $j=1, 2, \cdots, p-1$,

$$\theta(n)(j)=\begin{cases} n+1 & \text{if } n\leqq r-2 \\ 0 & \text{if } n=r-1. \end{cases}$$

From Lemma 14, we have $P(\theta)=1$. Since clearly $C(\theta)=b$, we have $B(\theta)=b$. Let λ be the function defined in Lemma 15 for this θ. We prove that $\lambda(p^k)=p^k$ for $k=1, 2, \cdots$. It is sufficient to prove that $\lambda(p)=p$ since θ is one-to-one ([2]). Since $\lambda(p^2)=p^2$ implies $\lambda(p)=p$, it is sufficient to prove $\lambda(p^2)=p^2$. Let $\alpha\in\theta^2(W(\theta))$. Then it is easy to see that

$$\emptyset\neq\{i\in I;\ \alpha(i)=0,\ \alpha(i+1)=\alpha(i+2)=1 \text{ and } \alpha(i+3)=2\}\subset\{ip^2;\ i=0, \pm 1, \cdots\}.$$

This implies that $T^i\circ\theta^2(W(\theta))\cap T^j\circ\theta^2(W(\theta))=\emptyset$ if $i\not\equiv j \pmod{p^2}$. Thus $\lambda(p^2)=p^2$.

Therefore, $(W(\theta)/\Lambda, T/\Lambda)$ is isomorphic to the p-adic system, where Λ is the trace relation of $(W(\theta), T)$.

THEOREM 6. *Let r, p and b be any positive integers such that $r\geqq 2$, $p\geqq 2$ and $b\leqq r$. Then there exists $\theta\in\Theta$ such that*

(1) $s(\theta)=r$ *and* $L(\theta)=p$,

(2) $B(\theta)=b$, *and*

(3) $(W(\theta)/\Lambda, T/\Lambda)$ *is isomorphic to the p-adic system, where* Λ *is the trace relation of* $(W(\theta), T)$.

The following corollary follows from (1) of Theorem 2, Corollary 2 and Theorem 6.

COROLLARY 3. *For any integer* $r\geqq 2$, *there exists a substitution minimal set on r symbols which is not a homomorphic image of any substitution minimal set on* r' *symbols, where* $r'<r$.

§ 6. Discrete or continuous substitutions.

DEFINITION 10. A substitution $\theta\in\Theta$ is said to be *discrete* if $B(\theta)=1$.

If θ is cyclic or if $C(\theta)=1$, then θ is discrete. The following example shows a discrete substitution which is not of these types.

EXAMPLE 2.

$$\theta(0)=010$$

$$\theta(1)=102$$

$$\theta(2)=201\,.$$

Let $\theta\in\Theta$. Let $W=W(\theta)$. Let μ be the unique T-invariant probability measure on W. Let $L_2(W, \mu)$ be the L_2-space over the complex numbers. Let U be the unitary operator on $L_2(W, \mu)$ defined by $(Uf)(\alpha)=f(T\alpha)$, where $\alpha\in W$ and $f\in L_2(W, \mu)$. By the spectrum of (W, T) or W, we mean the spectrum of the unitary operator U thus defined.

THEOREM 7. *If* $\theta\in\Theta$ *is discrete, then* (W, T) *is measure isomorphic to* $(W/\Lambda, T/\Lambda)$, *where* $W=W(\theta)$ *and* Λ *is the trace relation of* (W, T). *Therefore,* $W(\theta)$ *has rational pure point spectrum.*

PROOF. Let $\theta\in\Theta$ be discrete. Let μ be the unique T-invariant probability measure on W. Let $q=P(\theta)$. From Theorem 5, there exists $\alpha_0\in W(\theta)$ such that $\alpha_0\Lambda=\{\alpha_0\}$. Let $\alpha_i=T^i\alpha_0$ for any positive integer $i<q$. Then it is clear that $\alpha_i\Lambda=\{\alpha_i\}$ for any $i<q$. Since W is compact, for any real number $\varepsilon>0$, there exists a positive integer k such that the diameter of $\alpha_i(\Lambda_{p^k}\cap\Lambda_q)$ is less than ε for any $i<q$. For $i<q$, let $0\leqq j_i<p^k$ satisfy $\alpha_i\in T^{j_i}\circ\theta^k(W_i)$, where $W_0, W_1, \cdots, W_{q-1}$ are distinct equivalence classes of Λ_q (see Lemma 26). Let $T^j\circ\theta^h(W_i)$ be any equivalence class of $\Lambda_{p^h}\cap\Lambda_q$, where $0\leqq j<p^h$ and $0\leqq i<q$. Since there exists $0\leqq i'<q$ such that $T^j\circ\theta^h(W_i)\subset W_{i'}$, there exists $0\leqq j'<p^k$ such that the diameter of $T^{j'}\circ\theta^k\circ T^j\circ\theta^h(W_i)$ is less than ε, where for example, $j'=j_{i'}$. Since

$$d(T^j\circ\theta^h(\alpha), T^j\circ\theta^h(\beta))\leqq d(\alpha, \beta)$$

for any $\alpha, \beta\in W$ if $0\leqq j<p^h$, where d is the metric on W, the above statement

implies that the ratio of the number of the equivalence classes of $\Lambda_{pmk} \cap \Lambda_q$ whose diameters are less than ε among all equivalence classes of $\Lambda_{pmk} \cap \Lambda_q$ is at least $1-(1-p^{-k})^m$. Since each equivalence class of $\Lambda_{pmk} \cap \Lambda_q$ has a common measure, this implies that

$$\begin{aligned} &\mu(\{\alpha \in W;\ \text{the diameter of } \alpha\Lambda \text{ is less than } \varepsilon\}) \\ &\geqq \mu(\{\alpha \in W;\ \text{the diameter of } \alpha(\Lambda_{pmk} \cap \Lambda_q) \text{ is less than } \varepsilon\}) \\ &\geqq 1-(1-p^{-k})^m . \end{aligned}$$

Since m and $\varepsilon > 0$ are arbitrary, we have

$$\mu(\{\alpha \in W;\ \alpha\Lambda = \{\alpha\}\}) = 1 .$$

Thus the projection from W onto W/Λ gives a measure isomorphism from (W, T) to $(W/\Lambda, T/\Lambda)$. It is easy to verify that $(W/\Lambda, T/\Lambda)$ has a pure point spectrum $\{k\text{-th roots of } 1;\ \lambda(k) = k\}$. This completes the proof.

COROLLARY 4. *If θ belongs to Ψ and is discrete, then $(W(\theta), T)$ is measure isomorphic to the direct product of the $L(\theta)$-adic system by the cyclic system with cycle $P(\theta)$.*

DEFINITION 11. A substitution $\theta \in \Theta$ is said to be *continuous* if the following two conditions are satisfied.

(1) $B(\theta) \geqq 2$.

(2) There exists a cyclic permutation σ on N_r such that $\theta(\sigma(n))(j) = \sigma(\theta(n)(j))$ for any $n \in N_r$ and $j \in N_p$.

LEMMA 27. *If $\theta \in \Theta$ is continuous, then there exists a homeomorphism $\tilde{\sigma}$ from $W(\theta)$ onto $W(\theta)$ satisfying that*

(1) *$\tilde{\sigma}^r = $ identity,*

(2) *$\tilde{\sigma}$ is measure preserving, and*

(3) *$\tilde{\sigma}$ commutes with the shift.*

PROOF. Let σ be as in Definition 11. Define a mapping $\tilde{\sigma}: W(\theta) \to W(\theta)$ by $\tilde{\sigma}(\alpha)(i) = \sigma(\alpha(i))$, where $\alpha \in W(\theta)$ and $i \in I$. Then (1) and (3) are clear. Let μ be the unique T-invariant probability measure on $W(\theta)$. For $\xi \in (N_r)^*$ and $\alpha \in (N_r)^* \cup (N_r)^I$ such that $l(\xi) \leqq l(\alpha)$, let

$$R(\xi : \alpha) = \begin{cases} \dfrac{1}{l(\alpha)-l(\xi)+1} \operatorname{Card}(\{k;\ 0 \leqq k \leqq l(\alpha)-l(\xi) \text{ and } \xi(i) = \alpha(k+i) \\ \qquad\qquad \text{for } i = 0, 1, \cdots, l(\xi)-1\}) \qquad \cdots \text{ if } \alpha \in (N_r)^* \\ \lim\limits_{n\to\infty} \dfrac{1}{2n+1} \operatorname{Card}(\{k;\ |k| \leqq n \text{ and } \xi(i) = \alpha(k+i) \\ \qquad\qquad \text{for } i = 0, 1, \cdots, l(\xi)-1\}) \qquad \cdots \text{ if } \alpha \in (N_r)^I. \end{cases}$$

Note that $R(\xi : \alpha)$ exists and is equal to $\mu(\Gamma_\xi)$ for any $\xi \in (N_r)^*$ and $\alpha \in (N_r)^I$ since $(W(\theta), T)$ is a strictly ergodic system ([7]), where

$$\Gamma_\xi=\{\beta\in W(\theta)\,;\ \beta(i)=\xi(i)\ \text{for}\ i=0,1,\cdots,l(\xi)-1\}\,.$$

For any $n\in N_r$, it holds that

$$\sum_{m\in N_r}R(m:\theta(n))=1\,,$$

and

$$\begin{aligned}\sum_{m\in N_r}R(n:\theta(m))&=\sum_{i=0}^{r-1}R(n:\theta(\sigma^i(0)))\\&=\sum_{i=0}^{r-1}R(\sigma^{-i}(n):\theta(0))\\&=\sum_{m\in N_r}R(m:\theta(0))\\&=1\,.\end{aligned}$$

Therefore, the matrix A defined by

$$A=\begin{bmatrix}R(0:\theta(0)) & R(1:\theta(0)) & \cdots & R(r-1:\theta(0))\\ R(0:\theta(1)) & R(1:\theta(1)) & \cdots & R(r-1:\theta(1))\\ \vdots & \vdots & & \vdots\\ R(0:\theta(r-1)) & R(1:\theta(r-1)) & \cdots & R(r-1:\theta(r-1))\end{bmatrix}$$

is a doubly stochastic matrix. Let $v=(\mu(\Gamma_0),\mu(\Gamma_1),\cdots,\mu(\Gamma_{r-1}))$ be a row vector. Since for any $\alpha\in W(\theta)$ and $n\in N_r$, it holds that $\mu(\Gamma_n)=R(n:\alpha)=R(n:\theta(\alpha))$, $vA=v$. On the other hand, $\left(\frac{1}{r},\frac{1}{r},\cdots,\frac{1}{r}\right)A=\left(\frac{1}{r},\frac{1}{r},\cdots,\frac{1}{r}\right)$. Since θ satisfies Condition $\sharp$, there exists k such that all components of A^k are positive. This implies that 1 is a simple proper value of A. Therefore, $v=\left(\frac{1}{r},\frac{1}{r},\cdots,\frac{1}{r}\right)$. That is, $\mu(\Gamma_n)=\frac{1}{r}$ for any $n\in N_r$. Let $\xi\in(N_r)^*$. Define $\bar{\sigma}(\xi)\in(N_r)^*$ by $\bar{\sigma}(\xi)(i)=\sigma(\xi(i))$ for $i=0,1,\cdots,l(\xi)-1$. We prove that $\mu(\Gamma_\xi)=\mu(\Gamma_{\bar{\sigma}(\xi)})$. For any real number $\varepsilon>0$, let k be an integer satisfying $\frac{l(\xi)}{p^k}<\varepsilon$. Let $\alpha\in W(\theta)$. Since $R(n:\alpha)=\frac{1}{r}$ for any $n\in N_r$, it holds that

$$\begin{aligned}\mu(\Gamma_\xi)&=R(\xi:\theta^k(\alpha))\\&\geqq\frac{1}{r}\sum_{n\in N_r}R(\xi:\theta^k(n))-\varepsilon\\&=\frac{1}{r}\sum_{n\in N_r}R(\xi:\theta^k(\sigma^{-1}(n)))-\varepsilon\\&=\frac{1}{r}\sum_{n\in N_r}R(\bar{\sigma}(\xi):\theta^k(n))-\varepsilon\\&\geqq R(\bar{\sigma}(\xi):\theta^k(\alpha))-2\varepsilon\\&=\mu(\Gamma_{\bar{\sigma}(\xi)})-2\varepsilon\,.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, $\mu(\Gamma_{\xi})\geqq\mu(\Gamma_{\tilde{\sigma}(\xi)})$. Similarly we can prove $\mu(\Gamma_{\xi})\leqq\mu(\Gamma_{\tilde{\sigma}(\xi)})$. Thus $\mu(\Gamma_{\xi})=\mu(\Gamma_{\tilde{\sigma}(\xi)})$ and $\tilde{\sigma}$ is measure preserving.

THEOREM 8. *Let $\theta\in\Theta$ be continuous. Then $W(\theta)$ has a partially continuous spectrum.*

PROOF. Let $W=W(\theta)$. Let $\tilde{\sigma}$ be as in Lemma 27. Let $L_2=L_2(W,\mu)$, where μ is the unique T-invariant probability measure on W. Let U and V be the unitary operators on L_2 defined by $(Uf)(\alpha)=f(T\alpha)$ and $(Vf)(\alpha)=f(\tilde{\sigma}(\alpha))$, where $f\in L_2$ and $\alpha\in W$. Then it is clear that V commutes with U. Since any proper value of U is simple, it is clear that any proper function of U is a proper function of V. Assume that U has a pure point spectrum. Then V also has a pure point spectrum and any proper function of V is a proper function of U. For $n\in N_r$, let $\phi(n)$ $(\in N_r)$ be the smallest non-negative integer i such that $\sigma^i(0)=n$. Let ω be any primitive r-th root of 1. Define $f\in L_2$ by $f(\alpha)=\omega^{\phi(\alpha(0))}$. Then it is clear that f is a proper function of V. But f is not a proper function of U, since (W,T) is not cyclic $(\because B(\theta)\geqq 2)$. This contradiction completes the proof.

REMARK 2. It is clear that if $s(\theta)=2$, then θ is either discrete or continuous. In this special case, Theorem 7 and Theorem 8 are obtained in [1].

ACKNOWLEDGEMENT. The author should like to express his cordial thanks to Professor Tadasi Okamoto who, reading through the manuscript carefully, gave him useful advices. The author also should like to express his cordial thanks to Professor Takesi Watanabe, Professor Nobuyuki Ikeda and Professor Haruo Totoki for their useful advices and discussions.

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