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Normal parts of certain operators

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1. Only bounded operators T on a Hilbert space $\mathfrak{H}$ will be considered. A compact set X of complex numbers containing $sp(T)$ is said to be a spectral set of T (von Neumann [8]) if $\|f(T)\| \leqq \sup_{z \in X} |f(z)|$, where $f(z)$ is a rational function having no poles on X; cf. Riesz and Sz.-Nagy [**12**], p. 435. For any compact set X let $C(X)$ denote the space of continuous functions on X and $R(X)$ the uniform closure of the set of rational functions with poles off X. It was shown by von Neumann that if X is a spectral set of T and if $C(X) = R(X)$ then T must be normal; see also Lebow [**6**], p. 73. It may be noted that $C(X) = R(X)$ holds when X has Lebesgue plane measure 0; this result is due to Hartogs and Rosenthal (cf. Gamelin [**4**], p. 47).

An operator T is said to be hyponormal if

$$T^*T - TT^* \geqq 0. \tag{1.1}$$

It is well-known that a subnormal operator, that is, an operator having a normal extension on a larger Hilbert space, is hyponormal, but that the converse need not hold. Further, if T is subnormal then $sp(T)$ is a spectral set of T. On the other hand, if T is only hyponormal, this need not be the case; see Clancey [**1**].

Let T be hyponormal and let D denote an open disk satisfying

$$sp(T) \cap D \neq \emptyset. \tag{1.2}$$

In case the set $sp(T) \cap D$ has planar measure zero then T has a normal part, that is,

$$T = T_1 \oplus N, \qquad N = \text{normal}; \tag{1.3}$$

see Putnam [**9**]. Whether every compact set X with the property that

$$X \cap D \neq \emptyset \Rightarrow \operatorname{meas}_2(X \cap D) > 0 \qquad (D = \text{open disk}) \tag{1.4}$$

is the spectrum of a completely hyponormal operator (hyponormal and having no non-trivial reducing space on which it is normal) is not known. In this connection, see [**3**], [**11**]. As to subnormal operators, however, the authors

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have shown in [2] that a compact set X is the spectrum of a completely subnormal operator (subnormal and completely hyponormal) if and only if

$$X \cap D \neq \emptyset \ \Rightarrow\ R(X \cap \bar{D}) \neq C(X \cap \bar{D}), \tag{1.5}$$

where D denotes an open disk. (The closure of a set A is denoted by $\bar{A}$.)

In case T is subnormal, then polynomials in T and, in fact, rational functions of T are also subnormal. On the other hand, if T is assumed only to be polynomially hyponormal, so that all polynomials in T are hyponormal, it seems to be unknown whether all rational functions of T must also be hyponormal. Further, it is also apparently not known whether T must be subnormal if all rational functions of T are hyponormal.

It may be noted that if T is hyponormal (and invertible) then so also is its inverse; Stampfli [13]. Also, there exist hyponormal operators T which are not subnormal but are such that all powers $T^2, T^3, \cdots$ are subnormal; Stampfli [14]. In addition, for every positive integer n there exists a hyponormal operator T which is not subnormal and such that all polynomials in T of degree not exceeding n are hyponormal; Joshi [5].

If T is hyponormal then $\|T\| = \sup\{|z| : z \in sp(T)\}$. It follows that if all rational functions of T are hyponormal then $sp(T)$ is a spectral set of T. Further, if T is hyponormal and if all polynomials in T are hyponormal and if, in addition, $sp(T)$ does not separate the plane, then all rational functions of T are also hyponormal. This is easily deduced from Mergelyan's theorem. (See Lebow [6], p. 66, where it is shown that if X is a compact set which does not separate the plane and if for an operator T, $\|p(T)\| \leqq \sup_{z \in X} |p(z)|$ holds for any polynomial $p(z)$, then X is a spectral set of T.)

It will be shown in the present paper that certain results on subnormal operators obtained in [2] and [10] can be extended to operators T for which $sp(T)$ is a spectral set or to operators T which are polynomially hyponormal.

THEOREM 1. *Let $sp(T)$ be a spectral set of T. Suppose that D is an open disk satisfying* (1.2) *and for which*

$$R(sp(T) \cap \bar{D}) = C(sp(T) \cap \bar{D}). \tag{1.6}$$

Then T has a normal part, so that (1.3) *holds.*

In the special case in which T is subnormal, the above result was proved in [2].

For any simple closed curve C, not necessarily having zero Lebesgue plane measure, denote its open interior by $\operatorname{int}(C)$ and its open exterior by $\operatorname{ext}(C)$. The following generalizes a result of [10].

THEOREM 2. *Let T be polynomially hyponormal. Let C be a simple closed curve such that*

(1.7) $$sp(T) \subset \{C \cup \operatorname{int}(C)\}$$

and suppose that

(1.8) $$\{sp(T) \cap C\} - \{sp(T) \cap \operatorname{int}(C)\}^{-} \neq \emptyset .$$

Then T has a normal part, so that (1.3) *holds.*

It may be noted that if T is supposed only to be hyponormal, rather than polynomially hyponormal, then T may be completely hyponormal even though its spectrum is a subset of a simple closed curve; see [**10**]. In fact, T can be chosen so that $T^*T - TT^*$ has rank one and hence is even irreducible; cf. [**10**], [**11**].

A dual of Theorem 2 is the following.

THEOREM 2′. *Let T be hyponormal and invertible and suppose that T^{-1} is polynomially hyponormal. Let C be a simple closed curve for which*

(1.7)′ $$sp(T) \subset \{C \cup \operatorname{ext}(C)\}$$

and

(1.8)′ $$\{sp(T) \cap C\} - \{sp(T) \cap \operatorname{ext}(C)\}^{-} \neq \emptyset .$$

Then T has a normal part.

The above is of course a corollary of Theorem 2 by virtue of the mapping $w = 1/z$.

2. PROOF OF THEOREM 1. In view of (1.2) it is clear that one can choose concentric open disks $D_1 \subset D_2 \subset D$ centered at z_0 with corresponding radii $r_1 < r_2 < r$ and such that $sp(T) \cap D_1 \neq \emptyset$. Let A denote the closed annulus with hole D and outer radius so large that A contains that part of $sp(T)$ lying outside D. Then put $Y = A \cup \{sp(T) \cap \bar{D}\}$. Let $f(z)$ be defined by; $f(z) = 1$ on $\bar{D}_1$, $f(z) = (R - r_2)/(r_1 - r_2)$ if $|z - z_0| = R$ and $r_1 < R < r_2$, and $f(z) = 0$ outside D_2. Thus f is continuous in the plane and, in particular, $f|_Y \in C(Y)$. Further, in view of (1.6), it is clear that $f|_Y$ is locally in $R(Y)$ so that, by Bishop's theorem (see Gamelin [**4**], p. 51 or Zalcman [**15**], p. 124), $f|_Y \in R(Y)$. (Cf. the similar argument in [**2**].)

Hence there exists a sequence $\{r_n(z)\}$, $n = 1, 2, \cdots$, of rational functions in $R(Y)$ converging uniformly on Y to $f(z)$. Since $sp(T)$, hence also Y, is a spectral set of T, it follows that $\{r_n(T)\}$ converges in the uniform topology to an operator $f(T)$. If $\mathfrak{H}_0$ is defined by

(2.1) $$\mathfrak{H}_0 = (f(T)\mathfrak{H})^{-} ,$$

then clearly $\mathfrak{H}_0$ is invariant under T. Let $T_0 = T|\mathfrak{H}_0$.

Next, we show that $\mathfrak{H}_0$ reduces T. By von Neumann [**8**], p. 266, the image of $sp(T)$ under f is a spectral set of $f(T)$. But this set is real, so that by von Neumann's theorem $f(T)$ is self-adjoint. Since T commutes with $f(T)$, so also does T^*, and hence $\mathfrak{H}_0$ reduces T. Since $\|r_n(T)-f(T)\|\to 0$ and, by the spectral mapping theorem, $sp(r_n(T))=r_n(sp(T))$, it follows that $sp(f(T)) \supset f(sp(T)) \neq \{0\}$, so that, in particular, $\mathfrak{H}_0 \neq 0$-space. (Since $f(sp(T))$ is a spectral set of $f(T)$ then, in fact, $sp(f(T))=f(sp(T))$.) Thus,

$$T=T_1\oplus T_0\,, \qquad T_0=T|\mathfrak{H}_0\,. \tag{2.2}$$

It will next be shown that T_0 is normal.

Since the spectrum of T is a spectral set it follows that for every $x\neq 0$ in $\mathfrak{H}$ there is a positive measure $\mu[x, x]$ supported on $sp(T)$ such that

$$(g(T)x, x)=\int_{sp(T)} g(t)d\mu[x, x] \tag{2.3}$$

for every g in $R(sp(T))$; see Lebow [**6**], pp. 70-71. Since $\bar{z}f(z)$ is in $R(sp(T))$, just as $f(z)$, there exists a sequence $\{s_n(z)\}$ of functions in $R(sp(T))$ converging uniformly to $\bar{z}f(z)$ and hence $\{s_n(T)\}$ converges uniformly to an operator S. By (2.3),

$$\begin{aligned}(Sx, x)&=\int \bar{t}f(t)\,d\mu[x, x]=\Big(\int tf(t)\,d\mu[x, x]\Big)^*=(f(T)Tx, x)^*\\&=(x, f(T)Tx)=(T^*f(T)x, x)\end{aligned}$$

(cf. Lebow [**6**], p. 73). Hence $S=T^*f(T)$ and so $T^*f(T)$ commutes with T. Since $f(T)$ also commutes with T, then $T^*Tf(T)=T^*f(T)T=TT^*f(T)$, so that T_0 is normal, and the proof of Theorem 1 is complete.

3. LEMMA. *Let $\{T_n\}$ be a sequence of hyponormal operators converging uniformly to the (hyponormal) operator T, so that*

$$\|T_n-T\|\to 0 \quad \text{as} \quad n\to\infty\,. \tag{3.1}$$

Then $z_0\in sp(T)$ if and only if there exists a sequence $\{z_n\}$, $z_n\in sp(T_n)$, such that $z_n\to z_0$.

PROOF. The "if" part clearly holds for any bounded operators T_n, T satisfying (3.1). In order to prove the "only if," let $z_0\in sp(T)$. If the assertion is false, then there exists a constant $\delta>0$ and a sequence $\{n_k\}$ of positive integers satisfying $n_1<n_2<\cdots$ for which $sp(T_{n_k})\cap\{z: |z-z_0|<\delta\}=\emptyset$. Since T_{n_k} is hyponormal, then $\|(T_{n_k}-z_0I)x\|\geqq\|(T_{n_k}-z_0I)^*x\|\geqq\delta\|x\|$ for all x in $\mathfrak{H}$. On letting $n_k\to\infty$, one obtains similar inequalities with T_{n_k} replaced by T, so that $z_0\notin sp(T)$, a contradiction.

4. PROOF OF THEOREM 2. By the Riemann mapping theorem, the set $C\cup\operatorname{int}(C)$ can be mapped homeomorphically onto $|w|\leqq 1$ by $w=f(z)$, where $f(z)$ is analytic in $\operatorname{int}(C)$. By Mergelyan's theorem ([**7**]) there exist polynomials $\{p_n(z)\}$, $n=1, 2, \cdots$, such that $p_n(z) \to f(z)$ uniformly on $C \cup \operatorname{int}(C)$. Since the operators $p_n(T)$ are hyponormal, then $p_n(T)$ converges in the uniform topology to a hyponormal operator $f(T)$. According to the spectral mapping theorem, $sp(p_n(T)) = p_n(sp(T))$ and it now follows from the Lemma that $sp(f(T)) = f(sp(T))$. Further, if z_1 is in the set of (1.8), then $f(z_1)$ is in $sp(f(T)) \cap C'$, where $C' = \{w: |w|=1\}$, and $f(z_1)$ is not in the closure of $sp(f(T)) \cap \operatorname{int}(C')$. It follows from [**9**] that $f(T)$ has a normal part $M = f(T) | \mathfrak{H}_0$, $\mathfrak{H}_0 \neq 0$, so that $f(T) = S \oplus M$, where M is normal on $\mathfrak{H}_0 \neq 0$. Since Mergelyan's theorem can be used again (cf. [**10**]) to recover T as $T = g(f(T)) = g(S) \oplus g(M)$, where g is the inverse of f, it follows that $g(M)$ is also normal (on $\mathfrak{H}_0$) and the proof is complete.

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