Existence of digital extensions of semi-modular state charts

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Let J and W be a finite set of indices and the set of non-negative integers, respectively. By W^J we mean the cartesian product of W, with itself $|J|$ times where $|J|$ is the number of elements of J. A semi-modular state chart (V, h) is said to be finite whenever the number of similarity classes of V is finite [7]. Moreover, we say (V, h) is digital if, for arbitrary M and N of V, $h(M) = h(N)$ implies $M \sim N$. A state chart (V^e, h^e) is called a digital extension of (V, h) when (V^e, h^e) is digital and restrictions $V^e|J$ and $h^e|J$ are equal to V and h, respectively. Then there exists a binary, distributive and digital extension of (V, h), if (V, h) is binary, finite and distributive [8]. The principal aim of the present paper is to generalize the above result under the condition of semi-modularity of (V, h), which is the affirmative solution of one of the fundamental problems proposed by D. E. Muller and W. S. Bartky [1, 2, 3] as a model of asynchronous circuits, and later mathematically reorganized by H. Noguchi [5, 6, 7] as a mathematical system constructed over relations. Terminology of the paper relies on [5, 6, 7, 8].

We prove the following theorem.

MAIN THEOREM. *A finite, binary and semi-modular state chart* (V, h) *is finitely realizable. In fact, there exists a distributive state chart* (D^e, h^e) *which induces a binary, semi-modular and digital extension* (V^e, h^e) *of* (V, h).

In § 1 a special type of extension called a separation is defined and a necessary and sufficient condition for existence of digital extensions is given (Theorem 1.2). Thus in order to prove Main Theorem we have only to show that there exists a separation of (V, h). However, in this paper we look for a wide class of digital extensions rather than giving a direct proof of Main Theorem. For this purpose, relations between semi-modular subsets and distributive subsets are investigated as follows; if V is semi-modular then $\sigma(V)$ becomes a change diagram, and $\mu(\sigma(V))$ is well defined and $(\mu(\sigma(V)), h)$ is finite if (V, h) is finite. It is to be noted that Corollary 1.13 suggests the idea of the proof of Main Theorem.

In § 2 some properties of the induced synthetic relation $\smile$ and v-similarity

classes are prepared to be used in § 3. Although Theorem 2.11 is not directly used for the proof of the existence of separations, it ensures that the method given in § 3 is reasonable.

The proof of Main Theorem will be carried out in § 3, the last section of the paper, where it will be shown that there exist separations of (V, h) using previous results.

Roughly speaking, the proof of Main Theorem is as follows: If (V, h) is finite and binary then we have a binary and distributive state chart (D, h) such that $(V, h) \subset (D, h)$. Then we consider a separation (D^k, h^k) of (V, h) for (D, h). Then, by [8] there exists a binary, distributive and digital extension (D^e, h^e) of (D^k, h^k). The induced semi-modular extension $\lambda_V(D^e, h^e)$ of (V, h) (for λ_V, see 1.3) is the desired digital extension of (V, h).

For the further development of the theory, it seems important to synthesize digital extensions of semi-modular state chart directly and to investigate minimal extensions.

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§ 1. Separations of state charts.

In this section we will define separations of state charts and show that existence of separations is necessary and sufficient for existence of digital extensions of a given state chart.

Let (V^i, h^i), $i = 1, 2$, be semi-modular state charts such that $V^1 \subset V^2$. For the sake of simplicity we write (V^1, h^2) to stand for the state chart $(V^1, h^2 | V^1)$ where $h^2 | V^1$ is the restriction of h^2 to V^1. When there is no possibility of confusion, (V^2, h^1) is used for implying the state chart (V^2, h^1) where h^1 is the canonical extension of h^1 to V^2.

DEFINITION 1.1. Let (V, h) and (D, h) be semi-modular state charts such that $(V, h) \subset (D, h)$. Then an extension (D^k, h^k) is called a *separation* of (V, h) for (D, h) if there is an extension $(V^k, h^k) \subset (D^k, h^k)$ of (V, h) such that (V^k, h^k) is *simple* in (D^k, h^k), that is, if $A^k \sim B^k$ in (D^k, h^k) for A^k and B^k of V^k then $A^k \sim B^k$ in (V^k, h^k).

THEOREM 1.2. *A binary and semi-modular state chart (V, h) has its digital extension if and only if it has a finite, binary and distributive separation (D^k, h^k) of (V, h) for some $(D, h) \supset (V, h)$.*

In order to prove the theorem we will introduce a special type of extensions called the induced extensions and supply some preliminary lemmas, although some of which are not used directly for the proof. Hence we will

provide the proof of Theorem 1.2 at the end of this section, where the more detailed form Theorem 1.2′ will be given.

DEFINITION 1.3. Let (V, h) and (D, h) be semi-modular state charts such that $(V, h) \subset (D, h)$ and let (D^k, h^k) be an extension of (D, h). We define a subset V^k of D^k by $V^k = \{A^k \mid A^k \in D^k, A^k \mid J \in V\}$. V^k will be written as $\lambda_V(D^k)$ or simply $\lambda(D^k)$ when there is no confusion. We say V^k the induced extension of V by D^k, or V^k is induced by D^k. Furthermore, let $\lambda_V(h^k) = h^k \mid \lambda_V(D^k)$. By the lemma below, if D^k is distributive then $\lambda_V(D^k)$ is semi-modular and $(\lambda_V(D^k), \lambda_V(h^k))$, written as $\lambda_V(D^k, h^k)$, is a semi-modular state chart. $\lambda_V(D^k, h^k)$ is called the *induced extension* of (V, h) by (D^k, h^k). As we forementioned at the beginning of this section, we often write (V^k, h^k) in place of $(V^k, \lambda_V(h^k))$.

LEMMA 1.4. *If* (D^k, h^k) *is a distributive extension of* (D, h), *then the induced extension* (V^k, h^k) *of* (V, h) *is a semi-modular state chart. In particular,* (V^k, h^k) *is distributive if* V *and* D^k *are distributive.*

PROOF. It is clear that $0^{J^k} \in V^k$. Moreover, if M^k and N^k are states of V^k, then $M^k \vee N^k \in D^k$ and $(M^k \vee N^k) \mid J = (M^k \mid J) \vee (N^k \mid J) \in V$ and thus $M^k \vee N^k \in V^k$. Therefore, to prove that V^k is semi-modular, it remains to show that $N^k = M^k + \delta^p$ for some $p \in J^k$ if N^k covers M^k in V^k.

Let us consider in D^k a covering sequence $M^k = C(0)^k, C(1)^k, \cdots, C(r)^k = N^k$. If $N^k \mid J = M^k \mid J$ then $C(1)^k \mid J = M^k \mid J \in V$ and hence $C(1)^k \in V^k$. This implies, however, $N^k = M^k + \delta^p$ for some $p \in J^k$. Suppose that $M^k \mid J < N^k \mid J$. Then there exists at least a P^k of D^k such that $C = P^k \mid J$ covers $M^k \mid J$ and $C \leq N^k \mid J$ in V. Now let $C^k = (M^k \vee P^k) \wedge N^k$. Then $C^k \in D^k$ and $C^k \mid J = ((M^k \mid J) \vee (P^k \mid J)) \wedge (N^k \mid J) = C \in V$, and we have $C^k \in V^k$ and $M^k < C^k \leq N^k$. Therefore, considering a covering sequence again in D^k from M^k to C^k, we have $M(1)^k$ of D^k such that $M^k \ll M(1)^k \leq C^k \leq N^k$ in D^k. From the definition of C^k, $M(1)^k \mid J$ must be equal either to $M^k \mid J$ or $C^k \mid J$ and hence $M(1)^k \in V^k$, that is, V^k is semi-modular. Therefore (V^k, h^k) is a semi-modular state chart because (D^k, h^k) is a state chart. Moreover if V and D^k are distributive and $A^k, B^k \in V^k$, then $A^k \wedge B^k \in D^k$ and $(A^k \wedge B^k) \mid J = (A^k \mid J) \wedge (B^k \mid J) \in V$. Hence $A^k \wedge B^k \in V^k$, completing the proof of the lemma.

LEMMA 1.5. *Let* $\{D^k, k \in K\}$ *be a co-intersectional family of distributive subsets of* W^{J^k} *with the intersection* $D \subset W^J$, *and let* $D \supset V$. *Then*

$$\lambda_V(\bigotimes_k D^k) = \bigotimes_k \lambda_V(D^k).$$

(For the definition of amalgamation $\otimes$, see § 6 of [7].)

PROOF. Let $V^k = \lambda_V(D^k)$, $D^e = \bigotimes_k D^k$ and $V^e = \lambda_V(D^e)$. If $A^e \in V^e$, then $A^e \mid J \in V$ and $A^e \mid J^k \in D^k$ for each $k \in K$. Hence $A^e \mid J^k \in V^k$ and so $A^e \in \bigotimes_k V^k$. Conversely, if $A^e \in \bigotimes_k V^k$, then $A^e \mid J^k \in V^k \subset D^k$, so $A^e \in D^e$ and $A^e \mid J \in V$.

Therefore $A^e \in \lambda_V(D^e)$ and we have the desired result.

LEMMA 1.6. *Let* (D^k, h^k) *be a distributive separation of* (V, h), *that is, there exists an extension* (V^k, h^k) *such that* (V^k, h^k) *is simple in* (D^k, h^k). *Then for any distributive extension* (D^e, h^e) *of* (D^k, h^k), $\lambda_{Vk}(D^e, h^e)$, *say* (V^e, h^e), *is simple in* (D^e, h^e), *that is,* (D^e, h^e) *is a separation of* (V, h). *Furthermore, if* D_0 *is a distributive subset such that* $V \subset D_0 \subset D = D^k | J$, *then* $\lambda_{Vk}(D^e, h^e) = \lambda_{Vk}(D_0^e, h^e)$, *where* $D_0^e = \lambda_{D_0}(D^e)$. *In particular,* $\lambda_{Vk}(D_0^e, h^e)$ *is simple in* (D_0^e, h^e) *if* $\lambda_{Vk}(D^e, h^e)$ *is digital.*

PROOF. By 1.4, $\lambda_{Vk}(D^e, h^e)$ is a semi-modular extension of (V, h). Let $A^e \sim B^e$ in (D^e, h^e) and $A^e, B^e \in V^e$. Then $A^e | J^k \sim B^e | J^k$ in (V^k, h^k). Hence $L^e + A^e \in V^e$ if and only if $L^e + B^e \in V^e$ and we have $A^e \sim B^e$ in (V^e, h^e). Therefore (D^e, h^e) is a separation of (V, h). If $V \subset D_0 \subset D$, then clearly $\lambda_{Vk}(D^e, h^e) \supset \lambda_{Vk}(D_0^e, h^e)$. The reverse inclusion follows from $\lambda_{Vk}(D^e, h^e) \subset \lambda_V(D^e, h^e) \subset \lambda_{D_0}(D^e, h^e) = (D_0^e, h^e)$ and $\lambda_{Vk}(D^e, h^e) = \lambda_{Vk}(\lambda_{Vk}(D^e, h^e)) \subset \lambda_{Vk}(D_0^e, h^e)$.

In order to prove the if part of Theorem 1.2, we need show that there is a distributive extension (D^e, h^e) of (D^k, h^k) such that $\lambda_V(D^e, h^e)$ is a digital extension of (V, h).

Lemma 1.6 says that the class of digital extension of (V, h) obtained for a given $D \supset V$ in such a manner is included in that class constructed for D_0 if $V \subset D_0 \subset D$.

Now we will show that in a family of all distributive subset $\{D \mid V \subset D\}$ there exists the minimum distributive subset D_0, and (D_0, h) is finite if so is (V, h).

LEMMA 1.7. *Let* V *be a semi-modular subset of* W^J. *Then for any* $[\varphi, j]$ *of* $\sigma(V)$, *the number of changes* $[\theta, i] \in \sigma(V)$ *such that* $[\theta, i] \le [\varphi, j]$ *is finite, and hence* $\sigma(V)$ *is a change diagram.* (For the definitions of change diagram and $\sigma(V)$, see 7.1 and 9.3 of [6].)

PROOF. It is known that $\sigma(V)$ is partially ordered set, and if $[\theta, i]$ is in $\sigma(V)$ and $\theta > 1$, then $[\theta - 1, i]$ is in $\sigma(V)$ and $[\theta - 1, i] < [\theta, i]$ (see 7.4 and 7.5 of [6]).

Assume that there are infinite number of $[\theta, i]$ in $\sigma(V)$ such that $[\theta, i] \le [\varphi, j]$. Since J is a finite set, there exists an i_0 of J such that for infinitely many η, it follows $[\eta, i_0] \le [\varphi, j]$, that is, $[\theta, i_0] \le [\varphi, j]$ for all $\theta > 0$. Now let M be an arbitrary element of V. Then $M_j < \varphi$, because $[M_{i_0} + 1, i_0] < [\varphi, j]$. Thus from the definition of $\sigma(V)$, $[\varphi, j]$ cannot be in $\sigma(V)$, contradicting the assumption. Therefore the number of $[\theta, i] \in \sigma(V)$ such that $[\theta, i] \le [\varphi, j]$ must be finite for every $[\varphi, j]$ of $\sigma(V)$, and then $\sigma(V)$ is a change diagram.

LEMMA 1.8. *Let* V *be a semi-modular subset of* W^J *and let* D *be a distributive subset of* W^J *such that* $D \supset V$. *If* $[\theta, i]$ *and* $[\varphi, j]$ *are changes of* $\sigma(V)$ *and* $[\theta, i] \le [\varphi, j]$ *in* $\sigma(D)$, *then* $[\theta, i] \le [\varphi, j]$ *in* $\sigma(V)$.

PROOF. Let $[\theta, i]$ and $[\varphi, j]$ be changes satisfying the above condition. Then from the semi-modularity of V, there exists an A of V such that $A_i \geq \theta$ and $A_j \geq \varphi$.

Now we consider an arbitrary covering sequence in V from 0^J to A. Then this sequence is also a covering sequence in D, and the corresponding sequence of changes contains changes $[\theta, i]$ and $[\varphi, j]$ in the given order by 7.6 of [6]. Therefore by 7.6 of [6], $[\theta, i] \leq [\varphi, j]$ in $\sigma(V)$, completing the proof.

LEMMA 1.9. *Let V be a semi-modular subset of W^J and recursively define a subset $V(i+1)$ by $V(1) = V$ and $V(i+1) = V(i) \wedge V = \{A \wedge B \mid A \in V(i), B \in V\}$. Then $V(n)$, $n = |J|$, is the minimum distributive subset including V, and $V(m) = V(n)$ if $n \leq m$.*

PROOF. Let P be an element of $V(m)$. Then P may be written as $P = P(1) \wedge P(2) \wedge \cdots \wedge P(m)$ where each $P(i)$, $1 \leq i \leq m$, is in V and $m \geq n$. Then for each $k \in J$ there exists $P(i_k)$ in $\{P(i), 1 \leq i \leq m\}$ such that $P_k = P(i_k)_k$. Hence $P(1) \wedge \cdots \wedge P(m) \leq P(i_1) \wedge \cdots \wedge P(i_n) \leq P$, and this implies $P = P(i_1) \wedge P(i_2) \wedge \cdots \wedge P(i_n) \in V(n)$. Since $V(i) \subset V(j)$ if $i \leq j$, we have $V(m) = V(n)$ for $m \geq n = |J|$. Now let $M = \bigwedge_i M(i)$ and $N = \bigwedge_i N(i)$ be elements of $V(n)$. Then $M \wedge N \in V(2n) = V(n)$, $M \vee N = \bigwedge_{i,j} (M(i) \vee M(j)) \in V(n^2) = V(n)$ and it is clear that $0^J \in V(n)$. Thus, in order to prove that $V(n)$ is distributive, it suffices to show that if $M \ll N$ (N covers M) in $V(n)$, then $N = M + \delta^p$ for some $p \in J$. In this case, however, we may assume without loss of generality $M_i = M(i)_i$ and $N_i = N(i)_i$. Then $N = \bigwedge_{1 \leq i \leq n} (M(i) \vee N(i))$ because $(M(i) \vee N(i))_i = N_i$ and $N_i = (\bigwedge_i N(i))_i \leq (\bigwedge_i (M(i) \vee N(i)))_i \leq N_i$. Now consider a covering sequence $M(i) = C(i, 0), C(i, 1), \cdots, C(i, r_i) = M(i) \vee N(i)$ in V for each pair $M(i)$ and $M(i) \vee N(i)$. Since $\bigwedge_i C(i, 0) = M$ and $\bigwedge_i C(i, i_r) = N$, there exist $C(i, i_1)$ for each $i \in J$ and $C(k, k_1+1)$ such that $M = \bigwedge_i C(i, i_1) < (\bigwedge_{i \neq k} C(i, i_1)) \wedge C(k, k_1+1) \leq N$, where $0 \leq i_1 \leq r_i$, $i \in J$ and $k_1 + 1 \leq r_k$. Since $C(k, k_1+1)$ is written as $C(k, k_1) + \delta^p$ for some $p \in J$, $(\bigwedge_{i \neq k} C(i, i_1)) \wedge C(k, k_1+1)$ must be $M + \delta^p$. This implies that $N = M + \delta^p$ because N covers M in $V(n)$. Hence $V(n)$ is a distributive subset. Moreover, if D is a distributive subset of W^J and $V \subset D$, then $V(n) \subset D$, that is, $V(n)$ is the minimum distributive subset including V, completing the proof of the lemma.

LEMMA 1.10. *Let M be an element of a semi-modular subset V of W^J. If M covers only $M(1), M(2), \cdots, M(m)$, $m \geq 2$, then for each $M(i)$ there exists at least an $M(i_r)$, $1 \leq i_r \leq m$, such that $M(i) \wedge M(i_r) \in V$ and $M(i) \neq M(i_r)$. In particular, if $m \leq 2$ for every element M of V, then V is distributive.*

PROOF. Since g.l.b. $(M(i), M(j)) = M(i) \frown M(j)$ is in V and $m \leq |J|$, for

given $M(i)$ we have an $M(i_r)$ such that $M(i) \frown M(i_r)$ is maximal in a set $\{M(i) \frown M(j),\ 1 \leqq j \leqq m\}$. Now we consider two covering sequences in V; $M(i) \frown M(i_r) = A(0),\ A(1),\ \cdots,\ A(p) = M(i)$, and $M(i) \frown M(i_r) = B(0),\ B(1),\ \cdots,$ $B(q) = M(i_r)$. Then $A(1) \neq B(1)$. From the definition of $M(i_r)$, we have $A(1) \lor B(1) \not\leqq M(j)$ for every $M(j)$, $1 \leqq j \leqq m$. If this is not the case there exists an $M(j)$ $(1 \leqq j \leqq m)$ such that $M(i) \frown M(j) \geqq A(1) > M(i) \frown M(i_r)$, contradicting the assumption. Since $A(1) \lor B(1) \leqq M$, we have $A(1) \lor B(1) = M$. This implies $M = M(i) \frown M(i_r) + \delta^s + \delta^t$, where $A(1) = A(0) + \delta^s$ and $B(1) = B(0) + \delta^t$ for some distinct s and t of J. Therefore, $M(i) = A(1)$, $M(i_r) = B(1)$ and $M(i) \frown M(i_r)$ $= M(i) \land M(i_r)$, which completes the first part of the lemma. The second part of the lemma is the direct consequence of the first part.

Now let us show that $\mu(\sigma(V))$ is the desired minimum distributive subset including V.

THEOREM 1.11. *If V is a semi-modular subset of W^J then $\mu(\sigma(V))$ is the minimum distributive subset including V. That is, $\mu(\sigma(V)) = V(n)$, where $V(n)$ is the subset defined in 1.8. In particular, if every element of V covers at most two states of V then V is distributive and $\mu(\sigma(V)) = V$.*

PROOF. By 1.7, $\sigma(V)$ is a change diagram. Hence $\mu(\sigma(V))$ is well defined and $\mu(\sigma(V)) \supset V$. Thus by 1.9, to prove the first part of the theorem we have only to show that $\mu(\sigma(V)) \subset V(n)$, $n = |J|$. Let $A \in \mu(\sigma(V))$, and $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V(n))$ and $A_i < \theta$. If we have $A_j < \varphi$, then $A \in \mu(\sigma(V(n)))$, that is, $\mu(\sigma(V))$ $\subset V(n)$. Since $[\eta, k] \in \sigma(V)$ if and only if $[\eta, k] \in \sigma(V(n))$, $[\theta, i]$ and $[\varphi, j]$ are changes of $\sigma(V)$. Therefore by 1.8, we have $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V)$, and from the assumption that $A \in \mu(\sigma(V))$, we have $A_j \leqq \varphi$.

The latter part of the theorem follows directly from the former part of the theorem and 1.10.

COROLLARY 1.12. *If A is an element of a semi-modular subset V, then $\mu(\sigma(V_A)) = \mu(\sigma(V))_A$.*

PROOF. Since V_A is a semi-modular subset, by 1.11,

$$\mu(\sigma(V_A)) = V_A(n)$$
$$= \{M(1) \land M(2) \land \cdots \land M(n) \mid M(i) \in V_A\}$$
$$= \{N(1) \land N(2) \land \cdots \land N(n) - A \mid N(i) \in V,\ N(i) \geqq A\}$$
$$= \{M - A \mid M \in V(n),\ M \geqq A\}$$
$$= \mu(\sigma(V))_A .$$

COROLLARY 1.13. *Let V be a semi-modular subset and let A and B be elements of V. If $V_A = V_B$, then $\mu(\sigma(V))_A = \mu(\sigma(V))_B$. In particular, if $A \sim B$ in a semi-modular state chart (V, h) then $A \sim B$ in $(\mu(\sigma(V)), h)$.*

PROOF. By 1.12, if $V_A = V_B$ then

$$\mu(\sigma(V))_A = \mu(\sigma(V_A)) = \mu(\sigma(V_B)) = \mu(\sigma(V))_B \ .$$

In particular, if $A \sim B$ in (V, h), then $h_A = h_B$ in V_A $(= V_B)$. Moreover, from the definition of state charts, the domain of h is extended to $V(n) = \mu(\sigma(V))$ by taking $h(N)_j = h(M)_j$, for any $N \in \mu(\sigma(V)) - V$ and $j \in J$ and for some $M \in V$ such that $N_j = M_j$. Hence $(\mu(\sigma(V)), h)$ is a distributive state chart and $h_A = h_B$ in $\mu(\sigma(V))_A$ $(= \mu(\sigma(V))_B)$. Therefore $A \sim B$ in $(\mu(\sigma(V)), h)$, completing the proof.

DEFINITION 1.14. Let V be a semi-modular subset of W^J. Define a relation $\mathfrak{P}$ on V by $M \mathfrak{P} N$ if and only if $V_M = V_N$. Then $\mathfrak{P}$ is an equivalence relation on V and a family of equivalence classes $V/\mathfrak{P}$ is well defined. V is said to be *periodic* whenever $V/\mathfrak{P}$ is a finite family of equivalence classes. It is to be noted that when our discussions are focussed on binary state charts, (V, h) is finite if and only if V is periodic, and $M \sim N$ if and only if $M \mathfrak{P} N$ and $h(M) = h(N)$.

Now we will show $\mu(\sigma(V))$ is periodic if so is V, and for this purpose, next two lemmas are prepared.

LEMMA 1.15. *Let V be a semi-modular subset of W^J. Then for any subset J_p of J, $\sigma(V)|J_p = \sigma(V|J_p)$ and $\mu(\sigma(V|J_p)) = \mu(\sigma(V))|J_p$.*

PROOF. By 1.7, $\sigma(V)$ and $\sigma(V|J_p)$ are change diagrams, and $\mu(\sigma(V))$ and $\mu(\sigma(V|J_p))$ are well defined. If $[\theta, i] \in \sigma(V)|J_p$, then $[\theta, i] \in \sigma(V)$ and there is an M of V such that $M_i = \theta = (M|J_p)_i$. This means $[\theta, i] \in \sigma(V|J_p)$. Conversely if $[\theta, i] \in \sigma(V|J_p)$, then there exists an M of V satisfying $(M|J)_i = M_i = \theta$, and hence $[\theta, i] \in \sigma(V)|J_p$.

Now suppose $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V)|J_p$. If $M^p \in V|J_p$ then we have an M of V such that $M|J_p = M^p$, where if $M_i^p = M_i < \theta$ then $M_j^p = M_j < \varphi$. Therefore $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V|J_p)$.

Secondly let us assume $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V|J_p)$, $M \in V$ and $M_i < \theta$. In this case, however, $M|J_p \in V|J_p$. Since $(M|J_p)_i < \theta$, we have $(M|J_p)_j = M_j < \varphi$, and so $[\theta, i] \leqq [\varphi, j]$ in $\sigma(V)|J_p$. Hence $\sigma(V|J_p) = \sigma(V)|J_p$. Let us show the remaining part of the lemma. By 9.3 of [7] and 9.6 of [6], $\sigma(\mu(\sigma(V)|J_p)) = \sigma(\mu(\sigma(V)))|J_p = \sigma(V)|J_p$. Therefore by 9.2 of [7],

$$\mu(\sigma(V)|J_p) = \mu(\sigma(\mu(\sigma(V)|J_p)))$$

$$= \mu(\sigma(\mu(\sigma(V)))|J_p)$$

$$= \mu(\sigma(V))|J_p \ .$$

Hence from the first part of the lemma,

$$\mu(\sigma(V|J_p)) = \mu(\sigma(V)|J_p) = \mu(\sigma(V))|J_p \ ,$$

completing the proof.

LEMMA 1.16. *Let V be a distributive subset of W^J and $P = \{J_p\}$ be a finite*

family of subsets of J satisfying that for any pair $\{i, j\} \subset J$ there exists J_p of P such that $\{i, j\} \subset J_p$. Then $A\mathfrak{P}B$ in V if and only if $(A|J_p)\mathfrak{P}(B|J_p)$ for every $J_p \subset J$. Furthermore, V is periodic if and only if $V|J_p$ is periodic for every J_p of P.

PROOF. The only if part of the lemma is a direct consequence of 5.2 of [7], that is, $(V|J_p)_{A|J_p} = V_A|J_p$. Let us show the if part of the lemma.

Let $L \in V_A$. Suppose that $[\theta, i] \leq [\varphi, j]$ in $\sigma(V)$ and $(B+L)_i < \theta$. Then, by the assumption, there exists a J_p of P such that $\{i, j\} \subset J_p$, and $(A|J_p)\mathfrak{P}(B|J_p)$ in $V|J_p$. On the other hand, we have $\sigma(V)|J_p = \sigma(V|J_p)$ by 1.15 and $[\theta, i] \leq [\varphi, j]$ in $\sigma(V|J_p)$. Since $(V|J_p)_{A|J_p} = (V|J_p)_{B|J_p}$ and $(A+L)|J_p \in V|J_p$, we obtain $(B+L)|J_p \in V|J_p$. Hence $((B+L)|J_p)_j = (B+L)_j < \varphi$. This implies $L \in V_B$. The same reasoning shows that if $L \in V_B$ then $L \in V_A$ and hence we have $V_A = V_B$, that is, $A\mathfrak{P}B$. Next, suppose $A\mathfrak{P}B$. Then $(A|J_p)\mathfrak{P}(B|J_p)$ in $V|J_p$, and therefore $|V/\mathfrak{P}|$, the number of elements of $V/\mathfrak{P}$, is not greater than $\prod_{J_p \in P} |(V|J_p)/\mathfrak{P}|$. However, each $V|J_p$ is periodic and $\prod_{J_p \in P} |(V|J_p)/\mathfrak{P}|$ must be finite. This completes the proof of the lemma.

THEOREM 1.17. *Let V be a semi-modular subset of W^J. If V is periodic, then $\mu(\sigma(V))$ is periodic.*

PROOF. If $|J| \leq 2$, then $\mu(\sigma(V)) = V$ by 1.11, and the assertion of 1.17 holds.

Suppose 1.17 is true for $|J| \leq n$. Now let $|J| = n+1$. For each $p \in J$ we define J_p by taking $J_p = J - p$. Then the family $P = \{J_p, p \in J\}$ satisfies the condition of 1.16. Moreover, if V is periodic then $V|J_p$ is periodic and so is $\mu(\sigma(V|J_p))$ by the assumption. Therefore $\mu(\sigma(V))|J_p$ is periodic by 1.15, and $\mu(\sigma(V))$ is periodic by 1.16. The proof is thus completed by the induction on the number of elements of J.

PROOF OF THEOREM 1.2. Instead of proving 1.2, we shall show the next theorem. The proof of Theorem 1.2 is all the same except that of the last part of Theorem 1.2'.

THEOREM 1.2'. *A binary and semi-modular state chart (V, h) has its binary and digital extension if and only if it has a finite, binary and distributive separation (D^k, h^k) for $D = \mu(\sigma(V))$. In fact, if (D^e, h^e) is a binary and digital extension of (D^k, h^k) and (V^k, h^k) is simple in (D^k, h^k), then $\lambda_{VP}(D^e, h^e)$ is a digital extension of (V, h), where $J \subset J^p \subset J^k$ and $V^p = V^k|J^p$. Furthermore, if a digital extension (V^e, h^e) is induced by a distributive extension (D^e, h^e) of a given distributive state chart (D, h), then (V^e, h^e) is induced by a finite separation $(\mu(\sigma(V))^e, h^e)$ of (V, h).*

PROOF. Let $D = \mu(\sigma(V))$ and (D^k, h^k) be a finite, binary distributive separation of (V, h). Then there exists a binary, distributive and digital extension (D^e, h^e) of (D^k, h^k) [8]. (D^e, h^e) is clearly an extension of (D, h), and we define $(V^e, h^e) = \lambda_{VP}(D^e, h^e)$. By 1.3, (V^e, h^e) is a binary and semi-modular extension

of (V, h).

Let A^e and B^e be elements of V^e and $h^e(A^e) = h^e(B^e)$. Since (D^e, h^e) is digital, $A^e \sim B^e$ in (D^e, h^e). Then $A^e | J^k \sim B^e | J^k$ in (D^k, h^k) and $A^e | J^k \sim B^e | J^k$ in (V^k, h^k) because (D^k, h^k) is a separation. Hence $A^e | J^p \sim B^e | J^p$ for an arbitrary J^p such that $J \subset J^p \subset J^k$. Now suppose $L^e \in V^e_{Ae}$. Then $L^e \in D^e_{Ae} = D^e_{Be}$ and $L^e + B^e \in D^e$ because $A^e \sim B^e$ in (D^e, h^e). However, since $L^e | J^p \in V^p_{Ae | J^p} = V^p_{Be | J^p}$, we have $(L^e + B^e) | J^p \in V^p$. Therefore we have $L^e + B^e \in V^e$ and $L^e \in V^e_{Be}$. A similar argument shows that if $L^e \in V^e_{Be}$ then $L^e \in V^e_{Ae}$. Therefore $V^e_{Ae} = V^e_{Be}$, and by the assumption $h^e(A^e) = h^e(B^e)$, we obtain $A^e \sim B^e$ in (V^e, h^e), completing the proof of the if part of the theorem.

Now let (V^e, h^e) be a semi-modular and digital extension of (V, h), and let $(D^e, h^e) = (\mu(\sigma(V^e)), h^e)$. Then by 1.15,

$$\mu(\sigma(V^e)) | J = \mu(\sigma(V^e | J)) = \mu(\sigma(V)),$$

that is, (D^e, h^e) is a distributive extension of $(\mu(\sigma(V)), h)$, and is finite by 1.17. Suppose that A^e and B^e are elements of V^e and $A^e \sim B^e$ in (D^e, h^e). Since $h^e(A^e) = h^e(B^e)$, $A^e \sim B^e$ in (V^e, h^e). Therefore (D^e, h^e) is a separation of (V, h). This completes the proof of the first part of 1.2′.

Now let (V^e, h^e) be a digital extension of (V, h), and $(V^e, h^e) = \lambda_{Vp}(D^e, h^e)$, where (D^e, h^e) is an arbitrary distributive extension of a given state chart (D, h). Then by 1.17, $(\mu(\sigma(V^e)), h^e)$ is a finite separation of (V, h), and $V^e \subset \mu(\sigma(V^e)) \subset D^e$ by 1.11. Therefore, $\lambda_{Vp}(\mu(\sigma(V^e)), h^e) = (V^e, h^e)$, completing the proof of 1.2′.

§ 2. Some elementary properties of the induced synthetic classes and v-similarity classes.

The synthetic relation and induced synthetic relation $\underset{\sim}{\mathcal{L}}$ were defined in [8]. That is $A \underset{\sim}{\mathcal{L}} B$ in V if there exists a digital semi-modular extension (V^e, h^e) such that $A^e \sim B^e$ in V^e for some extensions A^e and B^e of A and B, respectively. We at first notice the following remark without proof, because we can prove it by the same argument as 1.4 and 1.5 of [8], where it was assumed that (V, h) was distributive.

REMARK 2.1. Let (V, h) be a semi-modular state chart. Then there exists a digital extension (V^e, h^e) if and only if (V, h) admits a synthetic relation. Furthermore, the induced synthetic relation $\underset{\sim}{\mathcal{L}}$ is a congruence on V and we have $\underset{\sim}{\mathcal{L}} \leq \underset{\sim}{v} \leq \sim$, where $\underset{\sim}{v}$ is the v-similarity relation of (V, h) (see 4.4 of [7]), and $\alpha \leq \beta$ implies that if $A\beta B$ then $A\alpha B$ for binary relations α and β.

Remark 2.1 indicates that to synthesize a digital state chart (V^e, h^e) it is required to investigate properties of v-similarity classes rather than that of similarity classes when we are concerned with semi-modular state charts (it

is to be noticed that if (V, h) is distributive then $\cong\, =\, \sim$).

Let (V, h) be a finite and semi-modular state chart with nodes J. By $[A]$, $[A]_v$ and $[A]_e$ we denote the similarity class, v-similarity class and the induced synthetic class each containing a state A of V, respectively. As in the case of similarity classes, we denote $S_1 \mathfrak{F} S_2$ for any subsets S_1 and S_2 of W^J if there are elements M of S_1 and N of S_2 such that $M \leq N$, and we write $S_1 \mathfrak{E} S_2$ if $S_1 \mathfrak{F} S_2$ and $S_2 \mathfrak{F} S_1$. Moreover, let $J_Z = \{i \mid i \in J,\ Z_i \neq 0\}$ for a cycle Z of $Z(M)$, $M \in V$, and let $J_M = \bigcup_{Z \in Z(M)} J_Z$ and $\overline{J_M} = J - J_M$. Since $A \cong B$ implies $A \sim A \vee B \sim B$ and $\cong$ is a congruence on V, $Z([M]_v) = Z(M) = Z([M])$.

LEMMA 2.2. *A semi-modular state chart (V, h) is finite if and only if the family of v-similarity classes $V/\cong$ is finite.*

PROOF. (V, h) is finite if $V/\cong$ is finite because $\cong\, \leq\, \sim$. Now let (V, h) be finite and S be one of similarity classes of V. Since $\cong$ is restricted to be a congruence on S, $S/\cong$ is well defined and we have

$$V/\cong\, = \bigcup_{S \in V/\sim} S/\cong\,.$$

Let $m(S) = \{M(i) \mid M(i) \in S,\ M(i) \text{ is minimal in } S\}$. Then $m(S)$ is a finite subset of S, by 3.2 of $[6]$. Moreover if M is in S, then there is an $M(i)$ of $m(S)$ such that $M(i) \leq M$ and $M(i) \sim M = M(i) \vee M$, that is, $[M]_v = [M(i)]_v$. Therefore, we have $|S/\cong| \leq |m(S)|$ and

$$|V/\cong| \leq |V/\sim| \cdot \max_{S \in V/\sim} |S/\cong| \leq |V/\sim| \cdot \max_{S \in V/\sim} |m(S)|\,.$$

Hence $|V/\cong|$ is finite because both $|V/\sim|$ and $\max_S |m(S)|$ is finite, completing the proof of the lemma.

LEMMA 2.3. *Let S be a v-similarity class of a finite semi-modular state chart (V, h) and $Z \in Z(S)$. Then $S|J_Z = \{M|J_Z \mid M \in S\}$ is a totally ordered set.*

PROOF. Suppose that M and N be elements of S and $N|J_Z \nleq M|J_Z$. Then $M \sim M \vee N$ and $(M \vee N - M)|J_Z \neq 0^{J_Z}$. From the orthogonality of cycles of S, we have $(M \vee N - M)|J_Z \geq Z|J_Z$ and $(Z|J_Z)_i > 0$ for all i of J_Z. Hence it follows that $M|J_Z < N|J_Z$. Therefore we have either $M|J_Z \geq N|J_Z$ or $M|J_Z < N|J_Z$. This completes the proof of the lemma.

LEMMA 2.4. *Let (V, h) be a semi-modular state chart with nodes J and let $M \sim N$ in (V, h). Then $M \cong N$ if and only if either $\overline{J_M} = \phi$ or $M|\overline{J_M} = N|\overline{J_N}$.*

PROOF. It is to be noticed that $J_M = J_N$ by 7.3 of $[7]$.

Suppose $M \cong N$ and $\overline{J_M} \neq \phi$. Since $M \sim M \vee N \sim N$, $M_i = (M \vee N)_i = N_i$ for all $i \in \overline{J_M}$. Hence $M|\overline{J_M} = N|\overline{J_N}$.

Let us prove the if part. If either $\overline{J_M} = \phi$ or both $\overline{J_M} \neq \phi$ and $M|\overline{J_M} = N|\overline{J_N}$, then there exists an A of V such that $M \vee N \leq A$ and $A \sim M \sim N$. Hence $M \sim A = M \vee A$ and $N \sim A = N \vee A$. This implies, however, $M \cong N$

because $\mathcal{L}$ is a congruence on V, completing the proof of the lemma.

THEOREM 2.5. *Let (V, h) be a finite and semi-modular state chart, M and N be elements of V such that $Z(M) = Z(N)$. Then $M \mathcal{L} N$ if and only if both $M|J_Z = N|J_Z \bmod Z|J_Z$ for all $Z \in Z(M)$ and $M|\overline{J_M} = N|\overline{J_N}$ if $\overline{J_M} \neq \phi$.*

PROOF. If $M \mathcal{L} N$ then $M|\overline{J_M} = N|\overline{J_N}$ provided $\overline{J_M} \neq \phi$, by 2.3. Furthermore, $M \sim M \vee N \sim N$ and by 2.6 of [7], $M \vee N$ can be represented as

$$(*) \qquad M \vee N = M + \sum_{Z(i) \in Z(M)} a_i Z(i)$$

$$= N + \sum_{Z(j) \in Z(N)} b_j Z(j), \qquad a_i, b_j \in W.$$

Hence from the orthogonality of cycles, we have, for every $Z \in Z(M)$,

$$(M \vee N)|J_Z = M|J_Z \bmod Z|J_Z$$

and

$$(M \vee N)|J_Z = N|J_Z \bmod Z|J_Z.$$

Hence

$$M|J_Z = N|J_Z \bmod Z|J_Z.$$

Conversely, if the sufficient condition of 2.5 is fulfilled, then $M|\overline{J_M} = (M \vee N)|\overline{J_M} = N|\overline{J_N}$ if $\overline{J_M} \neq \phi$, and $M \vee N$ can be written as $(*)$. Hence by 2.6 of [7], $M \sim M \vee N \sim N$, completing the proof of the theorem.

Now we will define cycles of the induced synthetic classes and show in 2.11 that they can be written as $k_i Z(i)$. Although this result is not directly utilized in §3, it indicates that the method constructing separations of (V, h) is reasonable.

LEMMA 2.6. *Let (V^e, h^e) be a digital extension of a semi-modular state chart (V, h) with nodes J, and $\mathcal{L}$ be the induced synthetic relation. Then the following properties are satisfied.*

(1) *If $A^e \in V^e$ then $[A^e]|J \subset [A^e|J]_e$.*

(2) *If $A^e, B^e \in V^e$, $A^e \leq B^e$ and $A^e|J = B^e|J$, then $[A^e]|J \subset [B^e]|J$.*

(3) *For any $A \in V$, there exists an extension $A^e \in V^e$ such that $[A^e]|J = [A^e|J]_e = [A]_e$.*

PROOF. Let $A^e \sim B^e$ in (V^e, h^e). Then $A^e|J \mathcal{L} B^e|J$ from the definition of $\mathcal{L}$. Hence $[A^e]|J \subset [A^e|J]_e$.

Suppose that C^e is in $[A^e]$. Then $A^e \sim C^e$ and we have $B^e - A^e + C^e \sim C^e$ and $(B^e - A^e + C^e)|J = C^e|J$. Therefore $C^e|J \subset [B^e]|J$, and this completes the proof of (2).

Let $\{[A(i)^e], 1 \leq i \leq p\}$ be the family of all similarity classes such that $A(i)^e|J = A$, $1 \leq i \leq p$. Since (V^e, h^e) is digital, p is certainly a positive integer. Let $A^e = \bigvee_{1 \leq i \leq p} A(i)^e$. Then $A(i)^e \leq A^e \in V^e$ and $A^e|J = A$. Therefore by (2) we have $[A(i)^e]|J \subset [A^e]|J$, $1 \leq i \leq p$. Now let $B \in [A]_e$. Then there exist B^e and $A(0)^e$ of V^e with the property $B^e \sim A(0)^e$, $B^e|J = B$ and $A(0)^e|J = A$. However,

$A(0)^e$ must be a member of some $[A(k)^e]$ $(1 \le k \le p)$. Hence $A \in [A(k)^e] | J$ $\subset [A^e] | J$, that is, $[A]_e \subset [A^e] | J$. Therefore, by (1) we have $[A^e] | J = [A]_e$, completing the proof of (3).

DEFINITION 2.7. Let (V, h) be a finite and semi-modular state chart, (V^e, h^e) be a digital extension of (V, h) and $\overset{e}{\sim}$ be the induced synthetic relation of (V, h). By $s(A)$, $A \in V$, we denote the set $\{s \mid 0^J < s \in W^J, A \ll A+s$ in $[A]_e\}$, where $A \ll A+s$ in $[A]_e$ means that both $A, A+s \in [A]_e$ and if $A < B < A+s$ then $B \notin [A]_e$. If $A \overset{e}{\sim} B$ then $s(A) = s(B)$ by 2.6, and $s([A]_e)$ is well defined, by taking $s([A]_e) = s(A)$. We say s $(\in s(A))$ an *induced synthetic cycle* of A (defined by (V^e, h^e) or $\overset{e}{\sim}$), and $s(A)$ (or equivalently $s([A]_e)$) is called the set of induced synthetic cycles of A (or $[A]_e$). Then from the orthogonality of cycles, there exists only one cycle s^e of $Z(A^e)$ such that $s^e | J = s$, $A^e | J = A$ and $A^e \in V^e$. Hence $s(i) \wedge s(j) = 0^J$ if $s(i)$ and $s(j)$ are distinct elements of $s(A)$.

LEMMA 2.8. *Let* (V^e, h^e) *be a digital extension of a semi-modular state chart* (V, h). *If* $[A]_e \overset{\sim}{\mathfrak{F}} [B]_e$ *and* $s \in s(A)$ *then* s *is written as* $s = \sum\limits_{1 \le i \le m} a_i s(i)$, $a_i \in W$, *where* $s(B) = \{s(i), 1 \le i \le m\}$. *Furthermore if* $[A]_e \overset{\sim}{\mathfrak{E}} [B]_e$, *then* $s(A) = s(B)$.

PROOF. By (3) of 2.6, there exist A^e and B^e of V^e with the property $A^e | J = A$, $B^e | J = B$, $[A^e] | J = [A]_e$ and $[B^e] | J = [B]_e$, where we may assume without loss of generality $A \le B$. Let $C^e = A^e \vee B^e$. Then $C^e | J = B$ and by (1) and (2) of 2.6,

$$[C^e] | J \subset [C^e | J]_e = [B]_e = [B^e] | J \subset [C^e] | J.$$

Hence $[C^e] | J = [B^e] | J$. Thus we have $s(A) = Z(A^e) | J$, $s(B) = Z(C^e) | J = Z(B^e) | J$ and $A^e \le C^e$. Therefore by 7.4 of [7] we have

$$s = \sum\limits_{1 \le j \le m} a_j s(j).$$

The latter part of the lemma is obtained by the above result (see 7.2 of [7]).

LEMMA 2.9. *Let* (V^e, h^e) *be a digital extension of a semi-modular state chart* (V, h) *and let* $Z \in Z(A)$ *for* $A \in V$. *Then, there exist positive integers* r *and* k *such that* $A(p) \overset{e}{\sim} A(q)$ *if and only if* $p, q \ge r$ *and* $p = q \bmod k$, *where* $A(j) = A+jZ$ *for an arbitrary non-negative integer* j.

PROOF. Since the number of equivalence classes defined by $\overset{e}{\sim}$ is finite, there must be $A(s)$ and $A(t)$, $s < t$, of V such that $A(s) \overset{e}{\sim} A(t)$. Denote by r the minimum number s satisfying the above condition, and by k the corresponding minimum positive integer j such that $A(r) \overset{e}{\sim} A(r+j)$, that is, $kZ \in s(A(r))$. Now let $A(p) \overset{e}{\sim} A(q)$. From the definition of r, we have $p, q \ge r$ and $[A(r)]_e \overset{\sim}{\mathfrak{E}} [A(p)]_e$, because $A(r) \overset{e}{\sim} A(r+ck)$ for any $c \in W$ and there exists an integer c such that $A(r) \le A(p) \le A(r+ck)$. Hence $s(A(r)) = s(A(p))$ by 2.8, and we obtain the result that $A(p) \overset{e}{\sim} A(q)$ if and only if $p = q \bmod k$, com-

pleting the proof of the lemma. ☛

LEMMA 2.10. *Let* (V^e, h^e) *be a digital extension of a semi-modular state chart* (V, h) *and* $Z(A) = \{Z(i), 1 \leq i \leq m\}$ *and* r_i *be the integer whose existence was ensured in* 2.9 *for each i. Then there exist integral vectors* $k, r \in W^m$ *such that if* $p, q \in W^m$ *and* $p, q \geq r$, *then* $A(p) \stackrel{s}{\sim} A(q)$ *if and only if* $p_i = q_i \bmod k_i$ *for all i, where* $A(a)$ *denotes*

$$A + \sum_{1 \leq i \leq m} a_i Z(i) \quad \text{for any } a \in W^m.$$

PROOF. Denote by a^i an element of W^m whose i-th component is a_i and j-th component is zero if $i \neq j$. Let k^i be the minimum vector such that $A(r) \stackrel{s}{\sim} A(r+k^i)$, $1 \leq i \leq m$, and $k = \sum_{1 \leq i \leq m} k^i$ (by 2.9 and the definition of r, such a k^i exists). Then $k_i Z(i) \in s(A(r))$ and $[A(r)]_e \mathfrak{C} [A(p)]_e$ if $p \geq r$ because there exists an $a \in W^m$ such that $A(r) \leq A(p) \leq A(r+a_i k^i)$ and $A(r) \stackrel{s}{\sim} A(r+a_i k^i)$. Hence by 2.8 and 2.9, $A(p) \stackrel{s}{\sim} A(q)$ if and only if $p_i = q_i \bmod k_i$, $1 \leq i \leq m$, completing the proof of the lemma.

THEOREM 2.11. *Let* (V, h) *be a semi-modular state chart which admits an induced synthetic relation* $\stackrel{s}{\sim}$. *If* s *is an induced synthetic cycle of* $A \in V$, *then* $s = k_i Z(i)$ *for some* $k_i \in W$, *where* $Z(i)$ *is a cycle of* $Z(A)$ *such that* $s \wedge Z(i) = 0^J$.

PROOF. By 2.6, s may be written as $s = \sum_{1 \leq i \leq m} a_i Z(i)$, where $a \in W^m$ and $|Z(A)| = m$. Furthermore, by 2.8 and 2.10, there exist r and k of W^m such that $k_i Z(i) \in s(A(r))$, where $A(r) = A + \sum_{1 \leq i \leq m} r_i Z(i)$ and $r_i \neq 0$ if and only if $a_i \neq 0$. Since there are p and q of W satisfying

$$A(r) \leq A + \sum_{1 \leq i \leq m} p \, a_i Z(i) = A + ps$$

$$\leq A + q \sum_{1 \leq i \leq m} k_i Z(i),$$

we have $A(r) \mathfrak{C} (A + q \sum_i k_i Z(i))$ and $A \mathfrak{C} (A + ps)$. Therefore $A \mathfrak{C} A(r)$ and by 2.8, $s = k_i Z(i)$ if $s \wedge A(i) \neq 0^J$, which is the desired result.

§3. Proof of Main Theorem.

Let (V, h) be a binary, finite and semi-modular state chart. For the proof of Main Theorem it suffices, by 1.2 or 1.2′, to show that there exists a finite, binary and distributive separation (D^k, h^k) of (V, h). At first we will define a special type of extension called a separation with respect to $V/\sim$. Secondly, some preliminary lemmas will be prepared to show that there exists a separation with respect to $V/\sim$. Then finally we will complete the proof of Main Theorem in 3.7, where the existence of a separation of (V, h) will be shown.

Definition 3.1. Let (V, h) be a finite*and semi-modular state chart with nodes J, and let $S = \{S_i\}$ be a finite family of mutually disjoint subsets of V. Moreover, let (D, h) be a distributive state chart such that $(V, h) \subset (D, h)$. An extension (D^k, h^k) of (D, h) is called an S-*separation* or a *separation with respect to* S whenever it satisfies the followings.

(1) (D^k, h^k) is finite.

(2) If $A^k, B^k \in D^k$, $h^k(A^k) = h^k(B^k)$ and $A^k|J$, $B^k|J \in \bigcup_i S_i$ then $A^k|J$, $B^k|J \in S_i$ for some S_i of S.

Lemma 3.2. *If* (D^k, h^k) *is a separation with respect to* $V/\sim$, *then* (D^k, h^k) *is a finite separation of* (V, h). *In fact,* $\lambda_V(D^k, h^k)$ *is simple in* (D^k, h^k).

Proof. Let $(V^k, h^k) = \lambda_V(D^k, h^k)$, and let $A^k \sim B^k$ in (D^k, h^k). Then $A^k|J \sim B^k|J$ by the assumption.

Now suppose $L^k + A^k \in V^k$, viz., $L^k + A^k \in D^k$ and $(L^k + A^k)|J \in V$. Then $L^k + B^k \in D^k$ and $(L^k + B^k)|J = L^k|J + B^k|J \in V$. This implies $L^k + B^k \in V^k$. By the same argument, we have $L^k + A^k \in V^k$ if $L^k + B^k \in V^k$. Therefore $V^k_{A^k} = V^k_{B^k}$. Since $h^k_{A^k} = h^k_{B^k}$ on $D^k_{A^k}$, we obtain $h^k_{A^k} = h^k_{B^k}$ on $V^k_{A^k}$. Hence $A^k \sim B^k$ in (V^k, h^k), completing the proof of the lemma.

Lemma 3.3. *Let* (V, h) *be a finite and semi-modular state chart and* $S = \{S_i\}$ *be a finite family of mutually disjoint subsets of* V. *Then the following properties are satisfied.*

(1) *There is an* S-*separation of* (V, h) *if and only if there is an* (S_p, S_q)-*separation for every distinct pair* (S_p, S_q).

(2) *There exists an* (S_p, S_q)-*separation if and only if there exists an* (S_{p_i}, S_{q_j})-*separation for all* $1 \leq i \leq r$ *and* $1 \leq j \leq t$, *where* $\{S_{p_i}, 1 \leq i \leq r\}$ *and* $\{S_{q_j}, 1 \leq j \leq t\}$ *are partitions of* S_p *and* S_q, *respectively.*

Proof. If (V^k, h^k) is an (S_1, S_2)-separation, then it is also an (S_3, S_4)-separation if $S_3 \subset S_1$ and $S_4 \subset S_2$. Hence the only if parts of (1) and (2) are satisfied. Now let (V^k, h^k) be an (S_p, S_q)-separation of (V, h) with nodes J^k, where k corresponds to a non-ordered pair (S_p, S_q). In this case, however, we may assume without loss of generality that system (V^k, h^k), $k \in \{(p, q)\}$, is co-intersectional.

Now let us show that $(V^e, h^e) = \bigotimes_k (V^k, h^k)$ is an S-separation. Since each (V^k, h^k) is finite, (V^e, h^e) is finite by 6.7 of [7], and the assertion (1) of 3.1 holds. Suppose A^e and B^e are states of V^e such that $A^e|J \in S_p$ and $B^e|J \in S_q$. Then there exists an (S_p, S_q)-separation (V^k, h^k) (with nodes J^k) such that $h^k(A^e|J^k) \neq h^k(B^e|J^k)$. Hence we have the relation $h^e(A^e) \neq h^e(B^e)$, and (V^e, h^e) is an (S_p, S_q)-separation.

The if part of (2) is verified similarly.

Definition 3.4. Let $[\Sigma, H]$ be a binary change chart whose corresponding state chart (D, h) is finite. It is assumed that the set J of nodes of the chart

does not contain 0. We define an extension $[\Sigma^k, H^k]$, called the *type β extension with period* $\gamma \geq 4$, by taking $\Sigma^k = \Sigma \cup [\eta, 0]$, $H[\eta, 0] = \eta \bmod 2$, $\eta > 0$, and with the ordering for each non-negative integer c,

(1) $[\theta_0 + c\gamma + 1, r] \ll [2c+1, 0] \ll [\theta_0 + c\gamma + 2, r]$ if $[\theta_0 + c\gamma + 2, r]$

exists where $[\theta_0 + 1, r] < [\varphi_0, r] < [\theta_0 + \gamma, r]$, and

(2) $[\varphi_0 + c\gamma + 1, r] \ll [2c+2, 0] \ll [\varphi_0 + c\gamma + 1, r]$ if $[\varphi_0 + c\gamma + 1, r]$

exists, and $[\theta_0 + 1, r] < [\varphi_0, r] < [\theta_0 + \gamma, r]$.

In particular when $\Sigma^k = \Sigma \cup [1, 0]$ with the ordering

(3) $$[\theta_0 + 1, r] \ll [1, 0] \ll [\theta_0 + 2, r],$$

$[\Sigma^k, H^k]$ is said to be the *type α extension*.

The corresponding state chart (D^k, h^k) is called an extension of type α or of type β according as $[\Sigma^k, H^k]$ is an extension of type α or of type β.

Then it is easily verified that $A^k \in D^k$ if and only if $A^k | J \in D$ and

(4) $$A_0^k = \begin{cases} 0 & , & 0 \leq A_r^k \leq \theta_0 \\ 2c \text{ or } 2c+1 & , & A_r^k = \theta_0 + c\gamma + 1 \\ 2c+1 & , & \theta_0 + c\gamma + 1 < A_r^k \leq \varphi_0 + c\gamma \\ 2c+1 \text{ or } 2c+2, & A_r^k = \varphi_0 + c\gamma + 1 \\ 2c+2 & , & \varphi_0 + c\gamma + 1 < A_r^k \leq \theta_0 + (c+1)\gamma \end{cases}$$

when (D^k, h^k) is of type β, and

(5) $$A_0^k = \begin{cases} 0 & , & 0 \leq A_r^k \leq \theta_0 \\ 0 \text{ or } 1, & A_r^k = \theta_0 + 1 \\ 1 & , & A_r^k > \theta_0 + 1 \end{cases}$$

when (D^k, h^k) is of type α.

LEMMA 3.5. *Let (D, h) be a finite, binary distributive state chart, and S_1 and S_2 be subsets of D with the property that if $M \in S_1$, then $M_r = \theta_0 \bmod \gamma$ and if $N \in S_2$ then $N_r = \varphi_0 \bmod \gamma$, where $\theta_0 < \varphi_0 < \theta_0 + \gamma$ and γ is a positive even integer ≥ 4. Then the extension (D^k, h^k) of type β defined in 3.4 is an (S_1, S_2)-separation. In particular, if $M_r = \theta_0$ for all $M \in S_1$, then extension (D^k, h^k) of type α defined in 3.4 is also an (S_1, S_2)-separation.*

PROOF. By (4) and (5) of 3.4, it is clear that $h^k(M^k)_0 = h_0^k(\theta_0) = 0$, $h^k(N^k)_0 = h_0^k(\varphi_0) = 1$ and $h^k(M^k) \neq h^k(N^k)$, if M^k and N^k are extensions of $M \in S_1$ and $N \in S_2$, respectively. Hence, to prove the first part of the lemma, it is sufficient to show that (D^k, h^k) is finite.

At first, let $A \sim B$ in D, A^k and B^k be extensions of A and B, respectively,

and $A_r = B_r < \theta_0$. Then by 3.4, $A_0^k = B_0^k = 0$. If $L^k \in D_{A^k}^k$ and $L^k | J = L$, then $A+L \in D$ and we have $B+L \in D$. Since $(A+L)_r = (B+L)_r$, there is a P^k of D^k such that $P^k | J = B+L$ and $P_0^k = (A^k+L^k)_0$ from the definition of D^k. Since $(P^k-B^k)_0 = L_0^k$ and $(P^k-B^k) | J = L$, we have $P^k-B^k = L^k \in D_{B^k}^k$. Conversely, if $L^k \in D_{B^k}^k$ then $L^k \in D_{A^k}^k$ by a similar argument. Hence we have $D_{A^k}^k = D_{B^k}^k$ and $A^k \sim B^k$ because (D^k, h^k) is binary.

Now suppose $h^k(A^k) = h^k(B^k)$, $A^k | J = A \sim B = B^k | J$ in D, and $B_r = A_r + c\gamma$, $c \in W$ and $A_r \geqq \theta_0$. Then $B_0^k = A_0^k + 2c$ by (4) of 3.4, and if $L^k + A^k \in D^k$ then $L+A$ and $L+B$ are in D, where $(L+A)_r + c\gamma = (L+B)_r$. Therefore by 3.4, there is a P^k of D^k such that $P^k | J = B+L$ and $P_0^k = (L^k+A^k)_0 + 2c$. Hence $(P^k-B^k) | J = L$ and $(P^k-B^k)_0 = L_0^k$, that is, $L^k \in D_{B^k}^k$. The same reasoning shows that $L^k \in D_{A^k}^k$ if $L^k \in D_{B^k}^k$. Hence $A^k \sim B^k$. Combining the relation with the above result, we have the result that any similarity class of (D, h) is split into at most finite similarity classes of (D^k, h^k), and therefore (D^k, h^k) is finite.

The latter part of the lemma is verified by a similar argument.

LEMMA 3.6. *Let S_A and S_B be subsets of W^J such that for any M of S_A and N of S_B,*

(1) $M = A \bmod Z$, $N = B \bmod Z$, *and*

(2) $M \neq N \bmod Z$,

where A, B and Z are elements of W^J and $Z_i \neq 0$ for all $i \in J$. Then there exists a positive integer k such that S_A, S_B are partitioned into k classes (some of them may be empty) $S_A(i)$, $S_B(i)$, $0 \leqq i \leqq k-1$ respectively, with properties that for any M, P of $S_A(i)$ and N, Q of $S_B(j)$,

(3) $M = P \bmod kZ$, $N = Q \bmod kZ$, *and*

(4) *there exists an $r \in J$, depending on $S_A(i)$ and $S_B(j)$, such that $M_r \neq N_r$* $\bmod kZ_r$.

PROOF. With no loss of generality we may assume that A and B are the minimum points of W^J satisfying the condition (1). Let

$$S(A) = \{M \mid M \in W^J, \; M = A \bmod Z\}$$

and $\qquad\qquad S(B) = \{N \mid N \in W^J, \; N = B \bmod Z\}$.

Then $S_A \subset S(A)$ and $S_B \subset S(B)$, and if there exist partitions of $S(A)$ and $S(B)$, satisfying (3) and (4) of the lemma, then they induce the desired partitions of S_A and S_B. Hence we may assume that $S_A = S(A)$ and $S_B = S(B)$. Let $B(b) = B+bZ$ be the minimum element of S_B such that $A < B+bZ$, and let us define an integer k by the minimum positive integer with the property that there exists an r of J such that $A_r \neq B(b)_r \bmod kZ_r$.

Now we define $S_A(i)$ and $S_B(j)$, $0 \leqq i, j \leqq k-1$ by

$$S_A(i) = \{A(p) \mid A(p) = A+pZ, \; p = i \bmod k, \; p \geqq 0\} \,,$$

and $\qquad\qquad S_B(j) = \{B(q) \mid B(q) = B+qZ, \; q = j \bmod k, \; q \geqq 0\}$.

Then it is evident that $S_A(i)$ and $S_B(j)$ satisfy (3) of the lemma. Let us show that they satisfy also (4) of the lemma.

If $k=1$ then $S_A(i)=S_A$, $S_B(j)=S_B$, and (4) holds as shown by the definition of r and k. Therefore, we may assume $k>1$. In this case, however, $A_t = B(b)_t \bmod Z_t$ for all $t \in J$. Hence we have a $t \in J$ such that $A_t = B(b)_t$ because otherwise we have $A < B(b)-Z \in B(b)$, contradicting the assumption.

Now let $A(p) \in S_A(i)$ and $B(q) \in S_B(j)$. Then $A(p)_t = B(b+p)_t$ and there exists $B(q')$ of S_B such that $b+p \leq q' < b+p+k$ and $q' = q \bmod k$. Therefore, if $q' \neq b+p$, then $A(p)_t < B(q')_t < B(b+p+k)_t = A(p+k)_t$ and $A(p)_t \neq B(q)_t \bmod kZ_t$. On the other hand, since $A(p)_r \neq B(b+p)_r \bmod kZ_r$, we have $A(p)_r \neq B(q')_r \bmod kZ_r$ if $q' = b+p$. This completes the proof of the lemma.

THEOREM 3.7. *Let (V, h) be a binary, finite and semi-modular state chart. Then there exists a finite separation (D^k, h^k) of (V, h). In fact, there exists a separation (D^k, h^k) with respect to $V/\sim$.*

PROOF. By 1.17, $(\mu(\sigma(V)), h)$ (or (W^J, h)) is binary, finite and distributive state chart including (V, h). Now let (D, h) be an arbitrary finite, binary and distributive state chart including (V, h), and let $[M]_v$, $[N]_v \in V/\overset{v}{\sim}$, $[M]_v \neq [N]_v$ and $h(M)=h(N)$. Suppose, at first, that $[M]_v \mathfrak{F} [N]_v$ is not satisfied. Then $\overline{J_N} \neq \phi$ and by 2.5, there exists an r of $\overline{J_N}$ such that for any $A \in [M]_v$ and $B \in [N]_v$, $A_r > B_r = N_r$. Hence by 3.5, we have an $([M]_v, [N]_v)$-separation of type α.

Next let $[M]_v \mathfrak{F} [N]_v$, where without loss of generality we may assume $[N]_v \mathfrak{F} [M]_v$, that is, $[M]_v \mathfrak{E} [N]_v$. Then $Z(M)=Z(N)$, and by 2.5, there exists a $Z \in Z(M)$ such that if $P \in [M]_v$ and $Q \in [N]_v$, then

$$M|J_Z = P|J_Z \bmod Z|J_Z, \qquad N|J_Z = Q|J_Z \bmod Z|J_Z,$$

and $$M|J_Z \neq N|J_Z \bmod Z|J_Z.$$

Therefore by 3.6, $[M]_v$ and $[N]_v$ are partitioned into $\{S_1(i), 0 \leq i \leq k-1\}$ and $\{S_2(j), 0 \leq j \leq k-1\}$, respectively, each satisfying (3) and (4) of 3.6. Therefore by 3.5, there exists an $(S_1(i), S_2(j))$-separation of type β for each pair $(S_1(i), S_2(j))$. Hence by (2) of 3.3, we have an $([M]_v, [N]_v)$-separation. Thus we obtain, by 2.2 and (1) of 3.3, a separation (D^k, h^k) with respect to $V/\overset{v}{\sim}$. Since $\overset{v}{\sim} \leq \sim$, (D^k, h^k) is a separation with respect to $V/\sim$. Therefore, by 3.2 there exists a finite separation of (V, h), completing the proof of Theorem 3.7.

COROLLARY 3.8. *If (D^k, h^k) is distributive and both (V, h) and (D^k, h^k) are finite then $\lambda_V(D^k, h^k)$ is finite.*

PROOF. In order to prove the corollary, it is sufficient to show that $\lambda_V(D^k)$ is periodic. Thus we have only to show that the corollary holds when (D^k, h^k) is binary. Let (D^p, h^p) be a separation with respect to $V/\sim$, where it may be assumed that (D^k, h^k) and (D^p, h^p) is co-intersectional, viz., the intersection

of (D^k, h^k) and (D^p, h^p) is equal to (V, h). Let $(D^e, h^e) = (D^k, h^k) \otimes (D^p, h^p)$. (D^e, h^e) is clearly a separation with respect to $V/\sim$ by 1.6. Hence by 3.2, $\lambda_V(D^e, h^e)$ is simple and finite, and $\lambda_V(D^e)$ is periodic. By 1.5, however, we have $\lambda_V(D^e) | J^k = \lambda_V(D^e | J^k) = \lambda_V(D^k)$, where J^k is nodes of D^k. Therefore $\lambda_V(D^k)$ is periodic, completing the proof.

REMARK. Let $\{V, h\}$ and $\{D, h\}$ be sets of semi-modular extensions $\{(V^k, h^k)\}$ and distributive extensions $\{(D^k, h^k)\}$ of (V, h) and (D, h), respectively, where $(V, h) \subset (D, h)$ and (D, h) is distributive. We have shown that there exists a mapping $\lambda : \{D, h\} \to \{V, h\}$ with the following properties.

(1) If (D^k, h^k) is finite then $\lambda(D^k, h^k)$ is finite.

(2) If $D_1 \subset D_2$ then $\lambda\{D_1, h\} \subset \lambda\{D_2, h\}$.

(3) For any $(D^k, h^k) \in \{D, h\}$ there exists a $(D_0^k, h^k) \in \{D_0, h\}$ such that $\lambda(D^k, h^k) = \lambda(D_0^k, h^k)$ where $D_0 = \mu(\sigma(V))$.

(4) If (D, h) is binary and finite, then there exists a $(D^e, h^e) \in \{D, h\}$ such that $\lambda(D^e, h^e)$ is digital.

However, we have not confirmed as yet whether for any (V^k, h^k) there exists a (D^k, h^k) (may be not distributive) such that $\lambda(D^k, h^k) = (V^k, h^k)$ or not, even in the case (V^k, h^k) is binary and digital.

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