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A classification of graded extensions in a skew Laurent polynomial ring, II

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Abstract. Let V be a total valuation ring of a division ring K with an automorphism σ and let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$, the skew Laurent polynomial ring. We classify A by distinguishing three different types based on the properties of A_1 and A_{-1}, and a complete description of A_i for all $i \in \boldsymbol{Z}$ is given in the case where A_1 is not a finitely generated left $O_l(A_1)$-ideal.

Introduction.

Let K be a division ring with an automorphism σ and let V be a *total valuation ring* of K, that is, for any non-zero $k \in K$, either $k \in V$ or $k^{-1} \in V$. A graded subring $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ of $K[X, X^{-1}; \sigma]$, the skew Laurent polynomial ring, is called a *graded total valuation ring* of $K[X, X^{-1}; \sigma]$ if for any non-zero homogeneous element aX^i of $K[X, X^{-1}; \sigma]$, either $aX^i \in A$ or $(aX^i)^{-1} \in A$, where $\boldsymbol{Z}$ is the ring of integers. A graded total valuation ring A of $K[X, X^{-1}; \sigma]$ is said to be a *graded extension* of V in $K[X, X^{-1}; \sigma]$ if $A_0 = V$.

This paper is a continuation of [**10**] which is concerned with the classification of graded extensions. In order to describe the classification in detail, we introduce some notations. For any additive subgroups I and J of K, we set:

$$\begin{aligned}
&(J : I)_l = \{a \in K \mid aI \subseteq J\},\\
&(J : I)_r = \{a \in K \mid Ia \subseteq J\},\\
&I^- = \{a^{-1} \mid a \in I, a \neq 0\} \text{ and}\\
&I^{-1} = \{a \in K \mid IaI \subseteq I\}, \text{ the } \textit{inverse} \text{ of } I.
\end{aligned}$$

In particular,

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$$O_l(I) = (I : I)_l, \text{ the } \textit{left order} \text{ of } I \text{ and}$$
$$O_r(I) = (I : I)_r, \text{ the } \textit{right order} \text{ of } I.$$

A non-zero left V-submodule I of K is called a *left V-ideal* if $Ia \subseteq V$ for some non-zero $a \in K$. Similarly we define right V-ideals and V-ideals.

Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ with $W = O_l(A_1)$. In [**10**], we classified graded extensions A of V in $K[X, X^{-1}; \sigma]$ by distinguishing five different types based on the properties of A_1 and A_{-1} in the case where A_1 is a finitely generated left $O_l(A_1)$-ideal as follows:

Case 1: A_1 is a finitely generated left W-ideal.
Type (a) $A_1 = Va = a\sigma(V)$ and $A_{-1} = V\sigma^{-1}(a^{-1})$;
Type (b) $A_1 = Wa \supset a\sigma(W)$;
Type (c) $A_1 = Wa = Wa\sigma(V) \subset a\sigma(W)$;
Type (d) $A_1 = Wa = a\sigma(W)$, $A_{-1} = J(W)\sigma^{-1}(a^{-1})$ and $J(W)^2 = J(W)$, where $J(W)$ is the Jacobson radical of W;
Type (e) $A_1 = Wa = a\sigma(W)$, $A_{-1} = J(W)\sigma^{-1}(a^{-1})$ and $J(W) = Wb^{-1}$ for some $b \in K$.

In this paper, we will classify graded extensions A of V in the case where A_1 is not a finitely generated left W-ideal. For this, we introduce further notations.

For any left V-ideal I and right V-ideal J, we define

$${}^*I = \cap\{Wc \mid I \subseteq Wc, c \in K\}, \text{ where } W = O_l(I) \text{ and}$$
$$J^* = \cap\{cU \mid J \subseteq cU, c \in K\}, \text{ where } U = O_r(J).$$

If A_1 is not a finitely generated left W-ideal, then there are two cases, that is, either ${}^*A_1 \supset A_1$ or ${}^*A_1 = A_1$. In the former case, we will obtain ${}^*A_1 = Wa$, $A_1 = J(W)a$ for some $a \in K$. In the latter case, we will divide A into two types by the properties of $M_i = A_1\sigma(A_1)\cdots\sigma^{i-1}(A_1)$ for all $i \in \boldsymbol{N}$, the set of all natural numbers. Now we can classify A by distinguishing three different types based on the properties of A_1 and A_{-1} by using $*$-operation as follows:

Case 2: A_1 is not a finitely generated left W-ideal, where $W = O_l(A_1)$.
Type (f) ${}^*A_1 \supset A_1$;
Type (g) ${}^*A_1 = A_1$ and ${}^*M_i$ is not a principal left W-ideal for any $i \in \boldsymbol{N}$;
Type (h) ${}^*A_1 = A_1$ and ${}^*M_l$ is a principal left W-ideal for some $l \in \boldsymbol{N}$.

In Section 1, we will give a complete description of A_i for all $i \in \boldsymbol{Z}$ and study types (f), (g) and (h) in the following ways:

For Type (f), $A_{-1} = U\sigma^{-1}(a^{-1})$ is a principal left U-ideal, where U is an overring of V with $\sigma(U) = a^{-1}Wa$. Then we are in a similar situation as in [**10**, Theorem 1.6]. For Type (g), it will be shown that $A = \oplus_{i\in \boldsymbol{Z}} M_i X^i$, where $M_{-i} = \sigma^{-i}((V : M_i)_r)$ for any $i \in \boldsymbol{N}$ (Theorem 1.14). For Type (h), A is not uniquely determined by the properties of A_1, A_{-1}, and the structure of A is complicated (Theorem 1.20).

In Section 2, we will provide some examples of graded extensions of V in $K[X, X^{-1}; \sigma]$ to illustrate the classification. We will discuss the ideal theory in the forthcoming paper and refer the readers to [**7**] for some basic properties of non-commutative valuation rings.

1. Main results.

Throughout this paper, V is a total valuation ring of a division ring K. We start with the following lemma whose proofs are similar to ones in [**7**, Section 6].

LEMMA 1.1. *Let I be a left V-ideal with $W = O_l(I)$ and $U = O_r(I)$. Suppose that U is a total valuation ring of K. Then*

(1) *The following are equivalent*:

(a) *I is not a principal left V-ideal.*

(b) *$I(V : I)_r = J(W)$.*

(c) *$I = J(V)I$.*

(2) *$U = O_l(I^{-1})$, $W = O_r(I^{-1})$ and $I^* = I^{-1-1} = {}^*I$.*

(3) *If ${}^*I \supset I$, then ${}^*I = Wa$ and $I = J(W)a$ for some $a \in K$.*

(4) *Suppose that I is not a principal left W-ideal. Then $I^{-1} = (V : I)_r$ and ${}_vI = (V : (V : I)_r)_l \subseteq {}^*I$.*

(5) *I is not a principal left W-ideal if and only if it is not a principal right U-ideal. In this case, in particular, $J(W)^2 = J(W)$ and $J(U)^2 = J(U)$.*

PROOF.

(1) (a)$\Longrightarrow$(b): The proof is similar to one in [**6**, Lemma 1.2].

(b)$\Longrightarrow$(c): Suppose that $I \supset J(V)I$. Then there is a $b \in I \setminus J(V)I$ with $J(V)b \supseteq J(V)I$ by [**7**, Lemma 6.3]. Thus $Ib^{-1} \subseteq O_r(J(V)) = V$ by [**7**, Lemma 6.8] and so $b^{-1} \in (V : I)_r$. Hence $1 = bb^{-1} \in I(V : I)_r = J(W)$, a contradiction. Therefore $I = J(V)I$ follows.

(c)$\Longrightarrow$(a): Suppose that $I = Vc$ for some $c \in I$. Then $Vc = I = J(V)I = J(V)c$, a contradiction. Hence I is not a principal left V-ideal.

(2) and (3): These are proved in the same ways as in [**7**, Lemma 6.10 and Proposition 6.13].

(4) Since $I = J(W)I$, we easily have $(V : I)_r = (W : I)_r$ which is equal to I^{-1}.

Hence ${}_vI \subseteq {}^*I$ follows since ${}^*I = (W : (W : I)_r)_l$ by [**7**, Proposition 6.13].

(5) Suppose that I is not a principal left W-ideal. If $I = aU$ for some $a \in K$, then $I = (aUa^{-1})a = Wa$, a contradiction. Hence I is not a principal right U-ideal. The "only if" part is similar and the last statement follows from the same argument as in [**7**, Proposition 6.13.] $\square$

In the case where $J(W)^2 = J(W)$, we have the following special properties of ideals which are needed later.

Lemma 1.2. *Let I be a left W and right U-ideal, where W is an overring of V and U is a total valuation ring of K. Suppose that $J(W)^2 = J(W)$. Then*

(1) If $I \supset Wc$ for some $c \in K$, then $J(W)I \supset Wc$.

(2) If $J(W)I \supset J(W)c$ for some $c \in K$, then $J(W)I \supset Wc$.

Proof.

(1) Let $b \in I \setminus Wc$. Then $Wb \supset Wc$ and so $cb^{-1} \in J(W)$. Thus $Wc \subseteq J(W)I$. But $J(W)I$ is not a principal left W-ideal by Lemma 1.1, since $J(W)^2 = J(W)$. Hence $J(W)I \supset Wc$.

(2) We have either $J(W)I \supset Wc$ or $J(W)I \subseteq Wc$. Suppose that $J(W)I \subseteq Wc$. Then $J(W)Ic^{-1} \subseteq J(W)$ and so $Ic^{-1} \subseteq O_r(J(W)) = W$. So $I \subseteq Wc$ and thus $J(W)I \subseteq J(W)c$, a contradiction. Hence $J(W)I \supset Wc$. $\square$

Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$. Then note that A_i is a left V and right $\sigma^i(V)$-ideal for any $i \in \boldsymbol{Z}$ by [**2**, Lemma 1.1].

Lemma 1.3. *Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$. Suppose that A_i is not a principal left V-ideal for some $i \in \boldsymbol{Z}$ with $W = O_l(A_i)$. Then*

(1) $A_{-i} = \sigma^{-i}((V : A_i)_r)$.

(2) If A_i is not a principal left W-ideal, then

(a) $A_{-i} = \sigma^{-i}((W : A_i)_r) = \sigma^{-i}(A_i^{-1})$ and

(b) If $A_i = J(W)a$ for some $a \in K$, then $A_{-i} = \sigma^{-i}(a^{-1}W)$.

Proof.

(1) $V \supseteq A_i\sigma^i(A_{-i})$ implies $\sigma^i(A_{-i}) \subseteq (V : A_i)_r$. Suppose that $(V : A_i)_r \supset \sigma^i(A_{-i})$. Then for any $c \in (V : A_i)_r \setminus \sigma^i(A_{-i})$, $c^{-1} \notin \sigma^i(A_{-i}^-)$. So $c^{-1} \in A_i$ by [**10**, Lemma 1.1]. Thus $1 = c^{-1}c \in A_i(V : A_i)_r = J(W)$ by Lemma 1.1, a contradiction. Hence $\sigma^i(A_{-i}) = (V : A_i)_r$, that is, $A_{-i} = \sigma^{-i}((V : A_i)_r)$.

(2) (a) First note that $J(W)^2 = J(W)$ and $J(W)A_i = A_i$ by Lemma 1.1. So $(V : A_i)_r = (W : A_i)_r$ by the proof of Lemma 1.1 (4). Hence $A_{-i} = \sigma^{-i}((W : A_i)_r) = \sigma^{-i}(A_i^{-1})$ since $A_i^{-1} = (W : A_i)_r$.

(b) If $A_i = J(W)a$ for some $a \in K$, then $(W : A_i)_r = a^{-1}W$, because $J(W)^2 = J(W)$. Hence $A_{-i} = \sigma^{-i}(a^{-1}W)$. □

Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ with $W = O_l(A_1)$ and $\sigma(U) = O_r(A_1)$. Then it follows that $W \supseteq V$ and $U \supseteq V$, since A_1 is a left V and right $\sigma(V)$-ideal. Suppose that A_1 is not a principal left W-ideal. Then $J(W)^2 = J(W)$ and $J(U)^2 = J(U)$ by Lemma 1.1, and there are two cases, namely, either ${}^*A_1 = A_1$ or ${}^*A_1 \supset A_1$. In the latter case, we have $A_1 = J(W)a$ and ${}^*A_1 = Wa$ for some $a \in K$ by Lemma 1.1. Conversely, if $A_1 = J(W)a$ for some $a \in K$, then ${}^*A_1 = Wa \supset A_1$ by [**7**, Lemma 6.12].

First we will study Type (f), namely, ${}^*A_1 \supset A_1$.

Proposition 1.4. *Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1};\ \sigma]$ with $W = O_l(A_1)$ and $\sigma(U) = O_r(A_1)$. Suppose that A_1 is not a principal left W-ideal and that ${}^*A_1 \supset A_1$, that is, $A_1 = J(W)a$ for some $a \in K$. Then*

(1) $\sigma(U) = a^{-1}Wa$.

(2) $A_1 = J(W)a = a\sigma(J(U))$ and $A_{-1} = \sigma^{-1}(a^{-1}W) = U\sigma^{-1}(a^{-1})$.

(3) $O_l(A_{-1}) = U$ and $O_r(A_{-1}) = \sigma^{-1}(W)$.

Proof.

(1) It follows that $\sigma(U) = O_r(J(W)a) = a^{-1}Wa$ since $O_r(J(W)) = W$.

(2) Since $\sigma(J(U)) = a^{-1}J(W)a$ by (1), we have $A_1 = J(W)a = a\sigma(J(U))$ and $A_{-1} = \sigma^{-1}(a^{-1}W) = \sigma^{-1}(\sigma(U)a^{-1}) = U\sigma^{-1}(a^{-1})$ by Lemma 1.3 and (1).

(3) This easily follows from (2). □

Now as in [**10**, Section 2], for a fixed non-zero $a \in K$, we set

$$\alpha_i = a\sigma(a)\cdots\sigma^{i-1}(a), \alpha_{-i} = \sigma^{-i}(\alpha_i^{-1}) \text{ for all } i \in \boldsymbol{N} \text{ and } \alpha_0 = 1.$$

Then we have

$$\alpha_{-i} = \sigma^{-1}(a^{-1})\sigma^{-2}(a^{-1})\cdots\sigma^{-i}(a^{-1}) \text{ for all } i \in \boldsymbol{N},\ \alpha_i = \sigma^i(\alpha_{-i}^{-1})$$

and

$$\alpha_i\sigma^i(\alpha_j) = \alpha_{i+j} \text{ for all } i, j \in \boldsymbol{Z}.$$

Furthermore, by using the properties of A_{-1} in Proposition 1.4, we can consider, as in [**10**, Section 2], the following two cases:

(a) $A_{-1} = U\alpha_{-1} \supseteq \alpha_{-1}\sigma^{-1}(U)$ (equivalently, $W\alpha_{-1} \supseteq \alpha_{-1}\sigma^{-1}(W) = A_{-1}$).
(b) $A_{-1} = U\alpha_{-1} \subset \alpha_{-1}\sigma^{-1}(U)$ (equivalently, $W\alpha_{-1} \subset \alpha_{-1}\sigma^{-1}(W) = A_{-1}$).
The following proposition is clear by [**10**, Lemma 1.1].

Proposition 1.5. *Let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1};\, \sigma]$. Set $Y = X^{-1}$ and $B_i = A_{-i}$ for all $i \in \boldsymbol{Z}$. Then $B = \oplus_{i\in\boldsymbol{Z}} B_i Y^i$ is a graded extension of V in $K[Y, Y^{-1};\, \sigma^{-1}]$.*

Since $B_1 = A_{-1}$ is a principal left U-ideal for Type (f), we have the following theorem by [**10**, Theorems 2.4, 2.5 and 2.6] and Proposition 1.5.

Theorem 1.6. *Let W be an overring of V with $J(W)^2 = J(W)$ and let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$, $A_1 = J(W)a$ which is a right $\sigma(V)$-ideal for some $a \in K$, and $A_{-1} = \sigma^{-1}(a^{-1}W)$. Set $O_r(A_1) = \sigma(U)$ for some overring U of V. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if one of the following properties hold.*

(1) If $A_{-1} = U\alpha_{-1} \supseteq \alpha_{-1}\sigma^{-1}(U)$, then $A_{-i} = U\alpha_{-i}$ and $A_i = \alpha_i\sigma^i(J(U))$ for all $i \in \boldsymbol{N}$.

(2) If $A_{-1} = U\alpha_{-1} \subset \alpha_{-1}\sigma^{-1}(U)$, then $A_{-i} = \alpha_{-i}\sigma^{-i}(W)$ and $A_i = J(W)\alpha_i$ for all $i \in \boldsymbol{N}$.

Next we will study the case where ${}^*A_1 = A_1$ and it is not a principal left W-ideal. So in the remainder of this paper, *suppose that A_1 is a left V and right $\sigma(V)$-ideal with $W = O_l(A_1)$, $\sigma(U) = O_r(A_1)$ and ${}^*A_1 = A_1$ is not a principal left W-ideal.* In this case, note that $J(W)^2 = J(W)$ and $J(U)^2 = J(U)$. We will study the graded extensions by ideal theoretic methods instead of the elements α_i above as follows:

Let $A_{-1} = \sigma^{-1}((V : A_1)_r)$ and set

$$M_0 = V, M_i = A_1\sigma(A_1)\cdots\sigma^{i-1}(A_1), M_{-i} = \sigma^{-i}((V : M_i)_r) \text{ for all } i \in \boldsymbol{N}$$

and

$$N_0 = V, N_{-i} = A_{-1}\sigma^{-1}(A_{-1})\cdots\sigma^{-i+1}(A_{-1}), N_i = \sigma^i((V : N_{-i})_r) \text{ for all } i \in \boldsymbol{N}.$$

Then we have:

$$M_i\sigma^i(M_{-i}) \subseteq V, M_i\sigma^i(M_j) = M_{i+j}, N_{-i}\sigma^{-i}(N_i) \subseteq V \text{ and}$$
$$N_{-i}\sigma^{-i}(N_{-j}) = N_{-i-j} \text{ for all } i, j \in \boldsymbol{N}.$$

Note that $N_1 = A_1$, because $A_{-1} = \sigma^{-1}(A_1^{-1})$ and is not a finitely generated left U-ideal and so, by Lemma 1.1, $N_1 = \sigma((V : A_{-1})_r) = \sigma((V : \sigma^{-1}(A_1^{-1}))_r) = \sigma((\sigma^{-1}(A_1^{-1}))^{-1}) = \sigma(\sigma^{-1}(A_1^{-1-1})) = {}^*A_1 = A_1$. Furthermore, we have that ${}^*A_1 = A_1$ is not a principal left W-ideal if and only if ${}^*A_{-1} = A_{-1}$ is not a principal left U-ideal by Proposition 1.5.

Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that ${}^*A_1 = A_1$ and it is not a principal left W-ideal, where $W = O_l(A_1)$. We will show that the properties of A depend on the properties of M_i ($i \in \boldsymbol{N}$) (see Theorems 1.14 and 1.20). There are two cases, that is,

Type (g) ${}^*M_i$ is not a principal left W-ideal for any $i \in \boldsymbol{N}$;

Type (h) ${}^*M_l$ is a principal left W-ideal for some $l \in \boldsymbol{N}$.

We will study these two cases in the remainder of this section. At first, we will show that both $M = \oplus_{i\in \boldsymbol{Z}} M_i X^i$ and $N = \oplus_{i\in \boldsymbol{Z}} N_i X^i$ are graded extensions of V in $K[X, X^{-1}; \sigma]$ with $M_1 = A_1 = N_1$.

Lemma 1.7. *Let I be a left V and right $\sigma^i(V)$-ideal for some $i \in \boldsymbol{Z}$ such that I is not a principal left V-ideal as well as not a principal right $\sigma^i(V)$-ideal. Then $(V : I)_r = (\sigma^i(V) : I)_l$.*

Proof. Since both $(V : I)_r$ and $(\sigma^i(V) : I)_l$ are left $\sigma^i(V)$-ideals, we have either $(V : I)_r \supset (\sigma^i(V) : I)_l$ or $(V : I)_r \subseteq (\sigma^i(V) : I)_l$. Suppose that the first case occurs. Then for any $b \in (V : I)_r \setminus (\sigma^i(V) : I)_l$, there is a $c \in I$ with $bc \notin \sigma^i(V)$. So $c^{-1}b^{-1} \in \sigma^i(J(V))$ and thus $c^{-1} = (c^{-1}b^{-1})b \in \sigma^i(J(V))b \subseteq (V : I)_r$. Hence $1 = cc^{-1} \in I(V : I)_r \subseteq J(V)$ by Lemma 1.1 (1), a contradiction. Similarly, we have $(\sigma^i(V) : I)_l \subseteq (V : I)_r$ by the assumptions and the right hand version of Lemma 1.1 (1). Hence $(V : I)_r = (\sigma^i(V) : I)_l$ follows. □

Lemma 1.8. *Let I and J be subsets of K, $0 \in I$, $0 \in J$ and $i \in \boldsymbol{Z}$. Then*

(1) If I is a left V-ideal and $J = \sigma^{-i}((V : I)_r)$. Then $I \cup \sigma^i(J^-) = K$.

(2) $I \cup \sigma^i(J^-) = K$ if and only if $J \cup \sigma^{-i}(I^-) = K$.

Proof.

(1) For any $b \in K \setminus I$, we have $Vb \supset I$, that is, $Ib^{-1} \subset V$. So $b^{-1} \in (V : I)_r = \sigma^i(J)$ and thus $b \in \sigma^i(J^-)$. Hence $I \cup \sigma^i(J^-) = K$.

(2) Suppose that $I \cup \sigma^i(J^-) = K$ and $c \in K \setminus J$. Then $c^{-1} \notin J^-$ and so $\sigma^i(c^{-1}) \in I$. Thus $c^{-1} \in \sigma^{-i}(I)$, that is, $c \in \sigma^{-i}(I^-)$. Hence $J \cup \sigma^{-i}(I^-) = K$. Similarly, $J \cup \sigma^{-i}(I^-) = K$ implies $I \cup \sigma^i(J^-) = K$. □

Proposition 1.9. *Let A_1 be a left V and right $\sigma(V)$-ideal with $O_l(A_1) = W$ and $O_r(A_1) = \sigma(U)$. Suppose that ${}^*A_1 = A_1$ and it is not a principal left W-ideal.*

Then both $M = \oplus_{i\in \boldsymbol{Z}} M_i X^i$ *and* $N = \oplus_{i\in \boldsymbol{Z}} N_i X^i$ *are graded extensions of* V *in* $K[X, X^{-1}; \sigma]$ *with* $M_1 = A_1 = N_1$.

Proof. By Lemma 1.8, $M_i \cup \sigma^i(M_{-i}^-) = K$ for all $i \in \boldsymbol{Z}$. So it suffices to prove that $M_i\sigma^i(M_j) \subseteq M_{i+j}$ for all $i, j \in \boldsymbol{Z}$ by [**10**, Lemma 1.1]. Let $i, j \in \boldsymbol{N}$. Then $M_i\sigma^i(M_j) = M_{i+j}$ by the definition. If $i \geq j$, then

$$\begin{aligned} M_i\sigma^i(M_{-j}) &= M_{i-j}\sigma^{i-j}(M_j)\sigma^{i-j}(\sigma^j(M_{-j})) \\ &= M_{i-j}\sigma^{i-j}(M_j\sigma^j(M_{-j})) \\ &\subseteq M_{i-j}\sigma^{i-j}(V) = M_{i-j}. \end{aligned}$$

If $i < j$, then

$$V \supseteq M_j\sigma^j(M_{-j}) = M_{j-i}\sigma^{j-i}(M_i)\sigma^{j-i}(\sigma^i(M_{-j})) = M_{j-i}\ \sigma^{j-i}(M_i\sigma^i(M_{-j})).$$

So $\sigma^{j-i}(M_i\sigma^i(M_{-j})) \subseteq (V : M_{j-i})_r = \sigma^{j-i}(M_{i-j})$ and thus $M_i\sigma^i(M_{-j}) \subseteq M_{i-j}$.

Since $J(V)A_1 = A_1 = A_1\sigma(J(V))$ by Lemma 1.1 and its right version, it follows that $J(V)M_i = M_i = M_i\sigma^i(J(V))$. So M_i is not a principal left V-ideal as well as not a principal right $\sigma^i(V)$-ideal. Hence, by Lemma 1.7, we have

$$\sigma^i(M_{-i})M_i = (\sigma^i(V) : M_i)_l M_i \subseteq \sigma^i(V). \qquad (*)$$

Thus if $i \leq j$, then

$$\sigma^i(M_{-i})M_j = \sigma^i(M_{-i})M_i\sigma^i(M_{j-i}) \subseteq \sigma^i(V)\sigma^i(M_{j-i}) = \sigma^i(M_{j-i}).$$

Hence $M_{-i}\sigma^{-i}(M_j) \subseteq M_{j-i}$.

If $i > j$, then, by $(*)$, $\sigma^i(V) \supseteq \sigma^i(M_{-i})M_j\sigma^j(M_{i-j})$ and so, by Lemma 1.7,

$$\begin{aligned} \sigma^i(M_{-i})M_j &\subseteq (\sigma^i(V) : \sigma^j(M_{i-j}))_l = \sigma^j((\sigma^{i-j}(V) : M_{i-j})_l) \\ &= \sigma^j((V : M_{i-j})_r) = \sigma^j(\sigma^{i-j}(M_{-i+j})) = \sigma^i(M_{-i+j}). \end{aligned}$$

Hence $M_{-i}\sigma^{-i}(M_j) \subseteq M_{-i+j}$ follows.

Finally, since $V \supseteq M_j\sigma^j(M_{-j}) = M_j\sigma^j(V)\sigma^j(M_{-j}) \supseteq M_j\sigma^j(M_i\sigma^i(M_{-i}))$, $\sigma^j(M_{-j}) = M_{i+j}\sigma^{i+j}(M_{-i})\sigma^j(M_{-j})$, we have $\sigma^{i+j}(M_{-i})\sigma^j(M_{-j}) \subseteq (V : M_{i+j})_r$ and so $M_{-i}\sigma^{-i}(M_{-j}) \subseteq \sigma^{-i-j}((V : M_{i+j})_r) = M_{-i-j}$. Hence M is a graded extension of V in $K[X, X^{-1}; \sigma]$. Since ${}^*A_{-1} = A_{-1}$ is not a principal left U-ideal, N is a graded extension of V in $K[X, X^{-1}; \sigma]$ by the proof above and Proposition 1.5. $\square$

Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that

${}^*A_1 = A_1$ and it is not a principal left W-ideal, where $W = O_l(A_1)$. Then either WA_i is not a principal left W-ideal for any $i \in \boldsymbol{N}$ or WA_l is a principal left W-ideal for some $l \in \boldsymbol{N}$.

Lemma 1.10. *Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that ${}^*A_1 = A_1$ and it is not a principal left W-ideal, where $W = O_l(A_1)$. Set $\sigma(U) = O_r(A_1)$. Then*

(1) *$A_1\sigma(A_{-1}) = J(W)$ and $J(W)A_{i+1} = A_1\sigma(A_i)$ for all $i \in \boldsymbol{N}$.*

(2) *If WA_i is not a principal left W-ideal for some $i \in \boldsymbol{N}(i > 1)$, then $A_i = A_1\sigma(A_{i-1})$.*

(3) *$A_{-1}\sigma^{-1}(A_1) = J(U)$ and $J(U)A_{-i-1} = A_{-1}\sigma^{-1}(A_{-i})$ for all $i \in \boldsymbol{N}$.*

(4) *If UA_{-i} is not a principal left U-ideal for some $i \in \boldsymbol{N}(i > 1)$, then $A_{-i} = A_{-1}\sigma^{-1}(A_{-i+1})$.*

Proof.

(1) It follows from Lemmas 1.1 and 1.3 that $A_1\sigma(A_{-1}) = A_1(V : A_1)_r = J(W)$. $A_{-1}\sigma^{-1}(A_{i+1}) \subseteq A_i$ implies $\sigma(A_{-1})A_{i+1} \subseteq \sigma(A_i)$. Thus $J(W)A_{i+1} = A_1\sigma(A_{-1}) A_{i+1} \subseteq A_1\sigma(A_i) \subseteq A_{i+1}$. Hence $J(W)A_{i+1} = A_1\sigma(A_i)$ since $J(W)A_1 = A_1$ and $J(W)^2 = J(W)$.

(2) By the assumptions, Lemma 1.1 and (1), we have $A_i \subseteq WA_i = J(W)WA_i = J(W)A_i = A_1\sigma(A_{i-1}) \subseteq A_i$. Hence $A_i = A_1\sigma(A_{i-1})$ follows.

(3) and (4) can be got by Proposition 1.5 and (1), (2). $\square$

Lemma 1.11. *Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that $A_1 = {}^*A_1$ and it is not a principal left W-ideal, where $W = O_l(A_1)$. Suppose that WA_i is not a principal left W-ideal for any $i \in \boldsymbol{N}$. Then $A = M$.*

Proof. Suppose that WA_i is not principal left W-ideal for any $i \in \boldsymbol{N}$. Then $A_i = A_1\sigma(A_{i-1})$ for all $i \in \boldsymbol{N}$ by Lemma 1.10. Hence we have $A_i = M_i$ for all $i \in \boldsymbol{N}$ by induction on i. Furthermore, by Lemma 1.3, we have $A_{-i} = \sigma^{-i}((V : A_i)_r) = \sigma^{-i}((V : M_i)_r) = M_{-i}$. Hence $A = M$ follows. $\square$

Note that $A_1 = {}^*A_1$ which is not a principal left $O_l(A_1)$-ideal if and only if $A_{-1} = {}^*A_{-1}$ which is not a principal left $O_l(A_{-1})$-ideal. Now the following remark is clear by Proposition 1.5 and Lemma 1.11.

Remark. Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that $A_1 = {}^*A_1$ and it is not a principal left $O_l(A_1)$-ideal. Set $U = O_l(A_{-1})$. Suppose that UA_{-i} is not a principal left U-ideal for any $i \in \boldsymbol{N}$. Then $A = N$.

Lemma 1.12. *Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ such that ${}^*A_1 = A_1$ and it is not a principal left W-ideal, where $W = O_l(A_1)$. Set*

$\sigma(U) = O_r(A_1)$.

(1) *Suppose that either* $WA_l = Wc$ *or* $A_l = J(W)c$ *for some* $c \in K$ *and* $l \in \boldsymbol{N}(l > 1)$. *Then* $\sigma(A_{l-1}) = A_1^{-1}c$ *and* $O_l(A_{l-1}) = U$.

(2) *Suppose that* $WA_l = Wc$ *for some* $c \in K$ *and* $l \in \boldsymbol{N}$ (*assume that* l *is the smallest natural number for this property*). *Then* $M_l = J(W)c$ *and* $U = W$.

(3) *Suppose that either* $UA_{-l} = Uc$ *or* $A_{-l} = J(U)c$ *for some* $c \in K$ *and* $l \in \boldsymbol{N}(l > 1)$. *Then* $\sigma^{-1}(A_{-l+1}) = A_{-1}^{-1}c$ *and* $O_l(A_{-l+1}) = W$.

(4) *Suppose that* $UA_{-l} = Uc$ *for some* $c \in K$ *and* $l \in \boldsymbol{N}$ (*assume that* l *is the smallest natural number for this property*). *Then* $N_{-l} = J(U)c$ *and* $U = W$.

Proof.

(1) Suppose that $A_l = J(W)c$. Then, from $J(W)c = A_l \supseteq A_1\sigma(A_{l-1})$, we derive $\sigma(A_{l-1}) \subseteq (J(W)c : A_1)_r = ((W : A_1)_r)c = A_1^{-1}c$, because $J(W)A_1 = A_1$. On the other hand, since $A_{-1} = \sigma^{-1}(A_1^{-1})$, we have

$$A_{l-1} \supseteq A_{-1}\sigma^{-1}(A_l) = \sigma^{-1}(A_1^{-1}A_l) = \sigma^{-1}(A_1^{-1}J(W)c) = \sigma^{-1}(A_1^{-1}c),$$

because A_1^{-1} is not a principal right W-ideal. Hence $\sigma(A_{l-1}) = A_1^{-1}c$ follows. Furthermore, $\sigma(O_l(A_{l-1})) = O_l(\sigma(A_{l-1})) = O_l(A_1^{-1}c) = O_l(A_1^{-1}) = O_r(A_1) = \sigma(U)$, which shows $O_l(A_{l-1}) = U$. In the case where $WA_l = Wc$, the statements are proved similarly.

(2) For each i $(2 \le i \le l-1)$, we have $A_i = A_1\sigma(A_{i-1})$ by Lemma 1.10 and so $A_i = M_i$ inductively. Hence $M_l = M_1\sigma(M_{l-1}) = A_1\sigma(A_{l-1}) = A_1A_1^{-1}c = J(W)c$ by (1) and Lemmas 1.1, 1.3. That $U = W$ follows from $W = O_l(J(W)c) = O_l(M_l) \supseteq O_l(M_{l-1}) \supseteq O_l(M_1) = W$ and $O_l(M_{l-1}) = O_l(A_{l-1}) = U$ by (1).

(3) and (4) can be got by Proposition 1.5 and (1), (2). □

Since M and N are graded extensions of V in $K[X, X^{-1}; \sigma]$ with $M_1 = A_1 = N_1$, we have the following remark from the proofs of Lemma 1.12.

Remark. Suppose that either $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$ or $N_{-l'} = J(U)c'$ for some $c' \in K$ and $l' \in \boldsymbol{N}$. Then $U = W$.

Lemma 1.13. *Let* $A = \oplus_{i \in \boldsymbol{Z}} A_iX^i$ *be a graded extension of* V *in* $K[X, X^{-1}; \sigma]$ *such that* ${}^*A_1 = A_1$ *and it is not a principal left* W*-ideal, where* $W = O_l(A_1)$. *Set* $\sigma(U) = O_r(A_1)$. *Then*

(1) $J(W)A_i = M_i$ *for all* $i \in \boldsymbol{N}$.

(2) $J(U)A_{-i} = N_{-i}$ *for all* $i \in \boldsymbol{N}$.

Proof.

(1) If WA_i is not a principal left W-ideal for any $i \in \boldsymbol{N}$, then $A_i = M_i$ for all

$i \in \boldsymbol{N}$ by Lemma 1.11 so that $J(W)A_i = J(W)M_i = M_i$. If WA_l is a principal left W-ideal for some $l \in \boldsymbol{N}$, then $U = W$ by Lemma 1.12. We inductively suppose that $J(W)A_i = M_i$ for some $i \in \boldsymbol{N}$. Then $M_{i+1} = M_1\sigma(M_i) = A_1\sigma(J(W)A_i) = A_1\sigma(J(U)A_i) = A_1\sigma(A_i) = J(W)A_{i+1}$ by Lemma 1.10 since A_1 is not a principal right $\sigma(U)$-ideal.

(2) can be got by Proposition 1.5 and (1). □

We are now ready to prove the following theorem which characterizes Type (g):

Theorem 1.14. *Let $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$ such that A_1 is a left V and right $\sigma(V)$-ideal with ${}^*A_1 = A_1$, which is not a principal left W-ideal, where $W = O_l(A_1)$. Suppose that ${}^*M_i$ is not a principal left W-ideal for any $i \in \boldsymbol{N}$. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if $A = M = \oplus_{i\in\boldsymbol{Z}} M_iX^i$.*

Proof. Suppose that A is a graded extension of V in $K[X, X^{-1}; \sigma]$. By Lemma 1.13. $J(W)A_i = M_i$ for all $i \in \boldsymbol{N}$. Assume that $WA_j = Wc$ for some $j \in \boldsymbol{N}$ and $c \in K$. Then ${}^*M_j = {}^*(J(W)A_j) = {}^*(J(W)c) = Wc$, a contradiction. Thus WA_i is not a principal left W-ideal for any $i \in \boldsymbol{N}$. Hence $A = M = \oplus_{i\in\boldsymbol{Z}} M_iX^i$ by Lemma 1.11. Conversely, if $A = M$, then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ by Proposition 1.9. □

Since $N = \oplus_{i\in\boldsymbol{Z}} N_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ with $N_1 = A_1$, we have the following

Corollary 1.15. *Suppose that ${}^*M_i$ is not a finitely generated left W-ideal for any $i \in \boldsymbol{N}$. Then $N = M$.*

Finally we will study Type (h), that is, $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$ (see Lemma 1.1).

Lemma 1.16. *Suppose that $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$ (assume that l is the smallest natural number for this property). Then*

(1) *For any $i \in \boldsymbol{N}$, ${}^*M_i \supset M_i$ if and only if $i \in l\boldsymbol{N}$. In this case, $M_{il} = J(W)c\sigma^l(c)\cdots\sigma^{(i-1)l}(c)$ for all $i \in \boldsymbol{N}$.*

(2) *$O_l(M_i) = W$ for all $i \in \boldsymbol{N}$.*

Proof.

(1) First note that $U = W$ by the remark to Lemma 1.12. For any $i \in \boldsymbol{N}$, let $\beta_i = c\sigma^l(c)\cdots\sigma^{(i-1)l}(c)$. Suppose that $M_{il} = J(W)\beta_i$. Then $M_{(i+1)l} = M_l\sigma^l(M_{il}) = M_l\sigma^l(J(W)\beta_i) = M_l\sigma^l(J(U))\sigma^l(\beta_i) = M_l\sigma^l(\beta_i) = J(W)\beta_{i+1}$. Thus ${}^*M_i \supset M_i$ for all

$i \in l\boldsymbol{N}$. So it suffices to prove that ${}^*M_p = M_p$ if $p \notin l\boldsymbol{N}$. Before that, we have to prove the statement (2). Let $i \in \boldsymbol{N}$. Then there is a $j \in \boldsymbol{N}$ with $jl \geq i$. So $W = O_l(J(W)\beta_j) = O_l(M_{jl}) \supseteq O_l(M_i) \supseteq W$ and hence $O_l(M_i) = W$. Write $p = il + j$ $(0 < j < l)$ and suppose that ${}^*M_p \supset M_p$, that is, $M_p = J(W)b$ for some $b \in K$ by Lemma 1.1. Then $J(W)b = M_j\sigma^j(M_{il}) = M_j\sigma^j(J(W)\beta_i) = M_j\sigma^j(\beta_i)$. So $M_j = J(W)b\sigma^j(\beta_i^{-1})$, which contradicts to the choice of l. Hence ${}^*M_p = M_p$ for all $p \in \boldsymbol{N}$ with $p \notin l\boldsymbol{N}$. $\square$

Now the following Lemma is clear by Proposition 1.5 and Lemma 1.16.

LEMMA 1.17. *Suppose that $N_{-l} = J(U)d$ for some $d \in K$ and $l \in \boldsymbol{N}$ (assume that l is the smallest natural number for this property). Then*

(1) *For any $i \in \boldsymbol{N}$, ${}^*N_{-i} \supset N_{-i}$ if and only if $i \in l\boldsymbol{N}$. In this case, $N_{-il} = J(U)d\sigma^{-l}(d)\cdots\sigma^{(-i+1)l}(d)$ for all $i \in \boldsymbol{N}$.*

(2) *$O_l(N_{-i}) = U$ for all $i \in \boldsymbol{N}$.*

LEMMA 1.18. *$M_k = J(W)b$ for some $b \in K$ and $k \in \boldsymbol{N}$ if and only if $N_{-k} = J(U)\sigma^{-k}(b^{-1})$. In this situation, $\sigma^k(W) = b^{-1}Wb$.*

PROOF. Note that $U = W$ by the remark to Lemma 1.12. Suppose that $M_k = J(W)b$. Then $M_{-k} = \sigma^{-k}(b^{-1}W)$ by Lemma 1.3 and $J(W)b = J(W)b\sigma^k(W)$ since M_k is a right $\sigma^k(W)$-ideal. So $b\sigma^k(W)b^{-1} \subseteq O_r(J(W)) = W$. To prove that $\sigma^k(W) = b^{-1}Wb$, suppose that $\sigma^k(W) \subset b^{-1}Wb$. Then, applying Lemma 1.13 to $M = \oplus_{i\in\boldsymbol{Z}}M_iX^i$, we have

$$\begin{aligned}
N_{-k} &= J(W)M_{-k} = J(W)\sigma^{-k}(b^{-1}W) \\
&= J(W)\sigma^{-k}(b^{-1}Wb)\sigma^{-k}(b^{-1}) \\
&= \sigma^{-k}(b^{-1}Wb)\sigma^{-k}(b^{-1}) = \sigma^{-k}(b^{-1}W),
\end{aligned}$$

which is a contradiction, because N_{-k} is not a principal right $\sigma^{-k}(W)$-ideal. Hence $\sigma^k(W) = b^{-1}Wb$ follows. Therefore, by Lemma 1.13, $N_{-k} = J(W)\sigma^{-k}(b^{-1}W) = J(W)W\sigma^{-k}(b^{-1}) = J(W)\sigma^{-k}(b^{-1})$ as desired. Now, by Proposition 1.5, we can get that $N_{-k} = J(U)\sigma^{-k}(b^{-1})$ implies $M_k = J(W)b$. This completes the proof. $\square$

Suppose that $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$. Then $M_{-l} = \sigma^{-l}(M_l^{-1}) = \sigma^{-l}(c^{-1}W) = W\sigma^{-l}(c^{-1}) \supset J(W)\sigma^{-l}(c^{-1}) = N_{-l}$ by Lemmas 1.3 and 1.18. Thus we have the following remark.

REMARK. Suppose that $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$. Then $M \neq N$.

LEMMA 1.19. *Let $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$ (assume that l is the smallest natural number for this property) and $B = \oplus_{j\in \boldsymbol{Z}} A_{jl}X^{jl}$ is a graded extension of V in $K[X^l, X^{-l}; \sigma^l]$ with $J(W)c \subseteq A_l \subseteq Wc$. Then*

(1) $J(W)\sigma^{-l}(c^{-1}) \subseteq A_{-l} \subseteq W\sigma^{-l}(c^{-1})$.

(2) $J(W)A_{jl} = M_{jl} = A_{jl}\sigma^{jl}(J(W))$ *for all* $j \in \boldsymbol{N}$.

(3) $J(W)A_{-jl} = N_{-jl} = A_{-jl}\sigma^{-jl}(J(W))$ *for all* $j \in \boldsymbol{N}$.

(4) $N_{-i} = M_{-i}$ *for all* $i \in \boldsymbol{N} \setminus l\boldsymbol{N}$.

PROOF.

(1) First note that $U = W$ and $\sigma^l(W) = c^{-1}Wc$ by the remark to Lemma 1.12 and Lemma 1.18. Since $c^{-1}W = (V : J(W)c)_r \supseteq (V : A_l)_r \supseteq (V : Wc)_r \supseteq c^{-1}J(W)$, we have

$$W\sigma^{-l}(c^{-1}) \supseteq \sigma^{-l}((V : A_l)_r) \supseteq J(W)\sigma^{-l}(c^{-1}).$$

So if A_l is not a finitely generated left V-ideal, then the statement follows since $A_{-l} = \sigma^{-l}((V : A_l)_r)$ by Lemma 1.3. If A_l is a finitely generated left V-ideal, say, $A_l = Vb$ for some $b \in A_l$, then $J(W)c \subseteq Wb \subseteq Wc$ implies $Wb = Wc$ since $J(W)^2 = J(W)$, that is, cb^{-1} is a unit in W. It follows from the proof of [**10**, Lemma 1.4] that $\sigma^l(A_{-l}) \supseteq b^{-1}J(V)$ and so $A_{-l} \supseteq \sigma^{-l}(b^{-1}J(V)) \supseteq \sigma^{-l}(b^{-1}J(W)) = \sigma^{-l}(c^{-1}J(W)) = J(W)\sigma^{-l}(c^{-1})$. Furthermore from $V \supseteq A_l\sigma^l(A_{-l})$, we derive $\sigma^l(A_{-l}) \subseteq (V : A_l)_r = b^{-1}V$. Thus $A_{-l} \subseteq \sigma^{-l}(b^{-1}V) \subseteq \sigma^{-l}(b^{-1}W) = \sigma^{-l}(c^{-1}W) = W\sigma^{-l}(c^{-1})$. Hence in any case, we have $J(W)\sigma^{-l}(c^{-1}) \subseteq A_{-l} \subseteq W\sigma^{-l}(c^{-1})$ as desired.

(2) It follows from (1) and Lemma 1.18 that $J(W)A_{-l} = J(W)\sigma^{-l}(c^{-1}) = N_{-l}$ and $A_{-l}\sigma^{-l}(J(W)) = J(W)\sigma^{-l}(c^{-1})$, which is the proof of (3) in the case where $j = 1$. Now we prove the statement by induction on j. If $j = 1$, then $J(W)A_l = J(W)c = M_l$ and $M_l = J(W)c = c\sigma^l(J(W)) = A_l\sigma^l(J(W))$ since $c\sigma^l(J(W)) \subseteq A_l \subseteq c\sigma^l(W)$. Suppose that $J(W)A_{jl} = M_{jl}$ for some $j \in \boldsymbol{N}$. Then

$$\begin{aligned} J(W)A_{jl+l} \supseteq J(W)A_l\sigma^l(A_{jl}) &= J(W)c\sigma^l(A_{jl}) = c\sigma^l(J(W)A_{jl}) \\ &= c\sigma^l(M_{jl}) = c\sigma^l(J(W)\beta_j) = J(W)\beta_{j+1} = M_{jl+l}, \end{aligned}$$

where $\beta_j = c\sigma^l(c)\cdots\sigma^{(j-1)l}(c)$ as in Lemma 1.16. Suppose that $J(W)A_{jl+l} \supset M_{jl+l} = J(W)\beta_{j+1}$. Then, by Lemma 1.2, $J(W)A_{jl+l} \supset W\beta_{j+1}$ and so $c^{-1}J(W)$ $A_{jl+l} \supset c^{-1}W\beta_{j+1}$. Thus, operating σ^{-l} on both sides, we have

$$J(W)\sigma^{-l}(c^{-1})\sigma^{-l}(A_{jl+l}) = \sigma^{-l}(c^{-1}J(W)A_{jl+l}) \supset \sigma^{-l}(c^{-1}W\beta_{j+1}) = W\beta_j.$$

So

$$WA_{jl} \supseteq J(W)A_{-l}\sigma^{-l}(A_{jl+l}) = J(W)\sigma^{-l}(c^{-1})\ \sigma^{-l}(A_{jl+l}) \supset W\beta_j.$$

Thus, by Lemma 1.2, $J(W)A_{jl} \supset J(W)\beta_j = M_{jl}$, a contradiction. Hence $J(W)A_{jl+l} = M_{jl+l}$ follows. We can prove that $M_{jl} = A_{jl}\sigma^{jl}(J(W))$ for all $j \in \boldsymbol{N}$ by the right version.

(3) can be got by Proposition 1.5 and (2).

(4) Let $i \in \boldsymbol{N} \setminus l\boldsymbol{N}$. Then ${}^*M_i = M_i$ and $W = O_l(M_i)$ by Lemma 1.16. So $M_{-i} = \sigma^{-i}(M_i^{-1})$ by lemma 1.3. Since M_i is not a principal left W-ideal, M_i^{-1} is not a principal right W-ideal so that it is not a principal left W'-ideal, where $W' = O_r(M_i^{-1})$ and it contains $\sigma^i(W)$. In particular, M_i^{-1} is not a principal left $\sigma^i(W)$-ideal. Thus $J(W)M_{-i} = M_{-i}$ by Lemma 1.1. Hence $N_{-i} = J(W)M_{-i} = M_{-i}$ by Lemma 1.13. □

In the case where either ${}^*A_1 \supset A_1$ or ${}^*A_1 = A_1$ and ${}^*M_i$ is not a principal left W-ideal for any $i \in \boldsymbol{N}$, the graded extension $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ is uniquely determined by A_1 and A_{-1} (see Theorems 1.6 and 1.14). However, in the case where ${}^*A_1 = A_1$ and ${}^*M_l$ is a principal left W-ideal for some $l \in \boldsymbol{N}$, that is, $M_l = J(W)c$ for some $c \in K$, A is not uniquely determined by A_1 and A_{-1} (see the remark after Lemma 1.18) and we are now ready to prove the following theorem which characterizes Type (h).

Theorem 1.20. *Let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ such that $A_0 = V$, A_1 is a left V and right $\sigma(V)$-ideal with ${}^*A_1 = A_1$ which is not a principal left W-ideal, where $W = O_l(A_1)$. Suppose that $M_l = J(W)c$ for some $c \in K$ and $l \in \boldsymbol{N}$ (assume that l is the smallest natural number for this property). Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if*

(1) $A_i = M_i$ for all $i \in \boldsymbol{Z} \setminus l\boldsymbol{Z}$.

(2) $B = \oplus_{j\in\boldsymbol{Z}} A_{jl} X^{jl}$ is a graded extension of V in $K[X^l, X^{-l}; \sigma^l]$ with $J(W)c \subseteq A_l \subseteq Wc$.

Proof. Note that $U = W$ by the remark to Lemma 1.12. Suppose that A is a graded extension of V in $K[X, X^{-1}; \sigma]$. Then, by Lemma 1.13, $M_l = J(W)c = J(W)A_l \subseteq A_l$ and so $A_l c^{-1} \subseteq O_r(J(W)) = W$. Hence $J(W)c \subseteq A_l \subseteq Wc$. Thus it remains to prove that $A_i = M_i$ for all $i \in \boldsymbol{Z} \setminus l\boldsymbol{Z}$. Let $i \in \boldsymbol{N} \setminus l\boldsymbol{N}$. Then ${}^*M_i = M_i$ and $A_i \supseteq J(W)A_i = M_i$ by Lemmas 1.13 and 1.16. Suppose that $A_i \supset M_i$ and let $d \in A_i \setminus M_i$. Then $Wd \supset M_i$ and so $J(W)d \supseteq M_i$. If $J(W)d = M_i$, then $Wd = {}^*(J(W)d) = {}^*M_i = M_i$, which is a contradiction, because $J(W)M_i = M_i$. Thus $J(W)d \supset M_i$. Then $J(W)A_i \supseteq J(W)d \supset M_i$, a contradiction. Hence $A_i = M_i$

follows. Now, by Proposition 1.5, we can get that $A_{-i} = N_{-i}$ and so $A_{-i} = M_{-i}$ by Lemma 1.19.

Conversely, suppose that (1) and (2) hold. For any $i \in \boldsymbol{Z}$, we have $A_i \cup \sigma^{-i}(A_i^-) = K$ by [**10**, Lemma 1.1] and the assumptions. So it suffices to prove that $A_i\sigma^i(A_j) \subseteq A_{i+j}$ for all $i, j \in \boldsymbol{Z}$, which will be proved in the following four cases:

(i) $i \notin l\boldsymbol{Z}$ and $j \notin l\boldsymbol{Z}$. Then $A_i\sigma^i(A_j) = M_i\sigma^i(M_j) \subseteq M_{i+j} \subseteq A_{i+j}$, because $M_{i+j} = A_{i+j}$ if $i + j \notin l\boldsymbol{Z}$ and $A_{i+j} \supseteq J(W)A_{i+j} = M_{i+j}$ if $i + j \in l\boldsymbol{Z}$ by Lemma 1.19.

(ii) $i \notin l\boldsymbol{Z}$, $j \in l\boldsymbol{Z}$. Then $i + j \notin l\boldsymbol{Z}$ and $A_i\sigma^i(A_j)\sigma^{i+j}(J(W)) = A_i\sigma^i(A_j\sigma^j(J(W)) = M_i\sigma^i(M_j) \subseteq M_{i+j}$ by Lemma 1.19. So

$$\begin{aligned} A_i\sigma^i(A_j) &\subseteq \left(A_i\sigma^i(A_j)\sigma^{i+j}(W)\right)_v = \left(A_i\sigma^i(A_j)\sigma^{i+j}(J(W))\right)_v \\ &\subseteq \left(M_{i+j}\right)_v \subseteq \left(M_{i+j}\right)^* = M_{i+j} = A_{i+j} \end{aligned}$$

by Lemma 1.1, where $I_v = \left(\sigma^{i+j}(W) : (\sigma^{i+j}(W) : I)_l\right)_r$ for a right $\sigma^{i+j}(W)$-ideal I.

(iii) $i \in l\boldsymbol{Z}$, $j \notin l\boldsymbol{Z}$. In this case, it is proved in the same way as in (ii) by considering $J(W)A_i\sigma^j(A_j)$.

(iv) $i, j \in l\boldsymbol{Z}$. This case is clear by the assumption. □

2. Examples.

In this section, we will provide concrete examples of graded extensions of V in $K[X, X^{-1}; \sigma]$ for illustrating the classification.

Let W be an overring of V with $J(W)^2 = J(W)$ and $\sigma = 1$. Then the following is a trivial example satisfying Theorem 1.6.

EXAMPLE 2.1. $A = \oplus_{i \in \boldsymbol{N}} WX^{-i} \oplus V \oplus (\oplus_{i \in \boldsymbol{N}} J(W)X^i)$ is a graded extension of V in $K[X, X^{-1}]$.

Let $F_0[Y_i^r \mid i \in \boldsymbol{Z}, r \in \boldsymbol{Q}]$ be a commutative domain over a field F_0 in indeterminates Y_i with $Y_i^r \cdot Y_i^s = Y_i^{r+s}$ and let $F = F_0(Y_i^r \mid i \in \boldsymbol{Z}, r \in \boldsymbol{Q})$ be its quotient field, where $\boldsymbol{Q}$ is the field of rational numbers. We define a $\sigma \in \mathrm{Aut}\,(F)$ as follows;

$$\sigma(a) = a \text{ for all } a \in F_0 \text{ and } \sigma(Y_i^r) = Y_{i-1}^r \text{ for all } i \in \boldsymbol{Z} \text{ and } r \in \boldsymbol{Q}.$$

Furthermore, let $G = \oplus_{i \in \boldsymbol{Z}} \boldsymbol{Q}_i$, the direct sum of $\boldsymbol{Q}_i$ with $\boldsymbol{Q} = \boldsymbol{Q}_i$, which is a totally ordered abelian group by lexicographic ordering and we define a map v from F to G as follows;

$v(a) = 0$ for any $a \in F_0$ and for any homogeneous element $\alpha = Y_{i_1}^{r_1} \cdots Y_{i_n}^{r_n}$ $(i_1 < \cdots < i_n)$, $v(\alpha) = (\cdots, 0, r_1, \cdots, r_n, 0, \cdots)$, i.e., the i_j-component of $v(\alpha)$ is $r_j (1 \leq j \leq n)$ and the other components of it are all zeroes.

Furthermore, let $\beta = \beta_1 + \cdots + \beta_m$ be any element in $F_0[Y_i^r \mid i \in \boldsymbol{Z}, r \in \boldsymbol{Q}]$,where β_i are homogenous elements. Then we define $v(\beta) = \min\{v(\beta_i) \mid 1 \leq i \leq m\}$. As usual, we can extend the map v to F, which is a valuation of F. We denote by V_0 the valuation ring of F determined by v, that is, $V_0 = \{\alpha\beta^{-1} \mid v(\alpha\beta^{-1}) = v(\alpha) - v(\beta) \geq 0$, where $\alpha, \beta \in F_0[Y_i^r \mid i \in \boldsymbol{Z}, r \in \boldsymbol{Q}]$ with $\beta \neq 0\}$.

Note that $\sigma(V_0) = V_0$, since σ is just shifting and that, for any $\alpha\beta^{-1} \in F$, $V_0\alpha\beta^{-1} = V_0 Y_{i_1}^{r_1} \cdots Y_{i_n}^{r_n}$ for some homogeneous element $Y_{i_1}^{r_1} \cdots Y_{i_n}^{r_n}$ by the construction of v. We set $U_i = \cup\{V_0 Y_i^r \mid r \in \boldsymbol{Q}\}$, an overring of V_0. Then $\sigma(U_i) = U_{i-1} \supset U_i$ for all $i \in \boldsymbol{Z}$.

Let $F[t, \sigma]$ be the skew polynomial ring over F in an indeterminate t and let $K = F(t, \sigma)$ be the quotient ring of $F[t, \sigma]$ which is a division ring.

As in [**8**, Section 1], we define the map

$$\varphi : F[t, \sigma]_{(t)} \longrightarrow F$$

by $\varphi(f(t)g(t)^{-1}) = f(0)g(0)^{-1}$, where $f(t), g(t) \in F[t, \sigma]$, $g(0) \neq 0$ and $F[t, \sigma]_{(t)}$ is the localization of $F[t, \sigma]$ at the maximal ideal (t). We let

$$V = \varphi^{-1}(V_0) = V_0 + tF[t, \sigma]_{(t)} \text{ and } W_i = \varphi^{-1}(U_i) = U_i + tF[t, \sigma]_{(t)},$$

the complete inverse images of V_0 and U_i by φ respectively for any $i \in \boldsymbol{Z}$. Then V and W_i are all total valuation rings of $F(t, \sigma)$ by [**8**, Proposition 1.6]. Furthermore, we have the following properties which are easily proved by the definitions:

(1) $\sigma(V) = V$ and $\sigma(W_i) = W_{i-1} \supset W_i$ for any $i \in \boldsymbol{Z}$.

(2) $Y_j^r V = V Y_j^r$ and $Y_j^r W_i = W_i Y_j^r$ for any $i, j \in \boldsymbol{Z}$ and $r \in \boldsymbol{Q}$.

Let π be a positive real number but not a rational number and set

$$A_1 = \cup\{t^{-1} Y_0^{-r} V \mid r < \pi, r \in \boldsymbol{Q}\} = \cup\{V Y_1^{-r} t^{-1} \mid r < \pi, r \in \boldsymbol{Q}\}.$$

Then A_1 satisfies the following:

(a) $W_2 = O_l(A_1)$ and $W_1 = O_r(A_1)$.

(b) A_1 is not a principal left W_2-ideal.

(c) $A_1 = \cap\{W_2 Y_1^{-s} t^{-1} \mid s > \pi, s \in \boldsymbol{Q}\}$ so that ${}^*A_1 = A_1$.

PROOF. First note that $t^{-1}F[t,\sigma]_{(t)} \supseteq A_1 \supseteq F[t,\sigma]_{(t)}$, which are easily obtained from the construction of A_1.

(a) To prove $W_2 \subseteq O_l(A_1)$, let $r, s \in \boldsymbol{Q}$ with $r < \pi$. Then

$$F[t,\sigma]_{(t)} t \cdot t^{-1} Y_0^{-r} V = F[t,\sigma]_{(t)} V = F[t,\sigma]_{(t)} \subseteq A_1$$

and

$$Y_2^s t^{-1} Y_0^{-r} V = t^{-1} Y_1^s Y_0^{-r} V = t^{-1} Y_0^{-(r_1+r)} Y_0^{r_1} Y_1^s V \subseteq t^{-1} Y_0^{-(r_1+r)} V \subseteq A_1,$$

where $r_1 \in \boldsymbol{Q}$ with $r + r_1 < \pi$ and $r_1 > 0$. Hence $W_2 \subseteq O_l(A_1)$ follows since $W_2 = U_2 + tF[t,\sigma]_{(t)}$.

To prove the converse inclusion, let $\alpha \in O_l(A_1)$. Since $K = \cup\{t^i F[t,\sigma]_{(t)} \mid i \in \boldsymbol{Z}\}$, we can write $\alpha = t^i c$ for some $i \in \boldsymbol{Z}$ and $c \in U(F[t,\sigma]_{(t)})$, where $U(F[t,\sigma]_{(t)})$ is the set of units in $F[t,\sigma]_{(t)}$. If $i < 0$, then $\alpha t^{-1} = t^i c t^{-1} = t^{i-1}\sigma(c) \notin A_1$, since $A_1 \subseteq t^{-1}F[t,\sigma]_{(t)}$, which is impossible so that $i \geq 0$. If $i > 0$, then $\alpha \in tF[t,\sigma]_{(t)} \subseteq W_2$. So we may assume that $i = 0$, that is, $\alpha \in U(F[t,\sigma]_{(t)})$. Since $F[t,\sigma]_{(t)} = F + tF[t,\sigma]_{(t)}$, we can write $\alpha = b + td$, where $b \in F$, and $d \in F[t,\sigma]_{(t)}$. Suppose that $\alpha \notin W_2$. Then $b \notin U_2$ and $b = Y_{i_1}^{l_1} \cdots Y_{i_n}^{l_n} u$ for some $l_i \in \boldsymbol{Q}, i_1 < \cdots < i_n$ and $u \in U(V_0)$ as it is noticed before. If either $i_1 \geq 2$ or $l_1 > 0$, then $b \in U_2$. So we may assume that $i_1 < 2$ and $l_1 < 0$. If $i_1 < 1$, then $A_1 \ni \alpha t^{-1} = t^{-1}\sigma(\alpha) = t^{-1} Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} \sigma(u) + \sigma(d)$, which implies $t^{-1} Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} \in A_1$ and so $t^{-1} Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} = t^{-1} Y_0^{-r}(v_0 + te)$ for some $r \in \boldsymbol{Q}$ with $r < \pi$, $v_0 \in V_0$, $e \in F[t,\sigma]_{(t)}$. Thus $Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} - Y_0^{-r} v_0 = Y_0^{-r} te \in tF[t,\sigma]_{(t)} = J(F[t,\sigma]_{(t)})$ and $Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} - Y_0^{-r} v_0$ is non-zero and is a unit in $F[t,\sigma]$, a contradiction, because $i_1 - 1 < 0$ and $l_1 < 0$. Hence $i_1 = 1$ and $l_1 < 0$. In this case, there is an $r \in \boldsymbol{Q}$ with $r < \pi$ and $l_1 - r < -\pi$. Then $A_1 \ni \alpha t^{-1} Y_0^{-r} = t^{-1}\sigma(\alpha) Y_0^{-r} = t^{-1}[Y_{i_1-1}^{l_1} \cdots Y_{i_n-1}^{l_n} \sigma(u) + t\sigma(d)]$ $Y_0^{-r} = t^{-1} Y_0^{-s} u_1$ for some $s \in \boldsymbol{Q}$ with $s < \pi$ and $u_1 = u_0 + td_1 \in V$, where $u_0 \in V_0$ and $d_1 \in F[t,\sigma]_{(t)}$. Hence, as before, we have $Y_0^{l_1 - r} Y_{i_2-1}^{l_2} \cdots Y_{i_n-1}^{l_n} \sigma(u) = Y_0^{-s} u_0$, a contradiction, because $-s > -\pi > l_1 - r$. Thus $\alpha \in W_2$ and hence $W_2 = O_l(A_1)$. Similarly, we can prove that $W_1 = O_r(A_1)$.

(b) It follows that $A_1 = \cup\{W_2 Y_1^{-r} t^{-1} \mid r < \pi, r \in \boldsymbol{Q}\}$ by (a) and that $W_2 Y_1^{-r} t^{-1} \supset W_2 Y_1^{-s} t^{-1}$ if $r > s$. Hence A_1 is not a principal left W_2-ideal.

(c) Let s and $r \in \boldsymbol{Q}$ with $s > \pi > r$. Then $W_2 Y_1^{-s} t^{-1} \supset W_2 Y_1^{-r} t^{-1}$ and so $A_1 \subseteq \cap\{W_2 Y_1^{-s} t^{-1} \mid s > \pi, s \in \boldsymbol{Q}\}$. To prove the converse inclusion, let $\alpha = ct^i$ for some $c \in U(F[t,\sigma]_{(t)})$ and $i \in \boldsymbol{Z}$ with $\alpha \in \cap\{W_2 Y_1^{-s} t^{-1} \mid s > \pi, s \in \boldsymbol{Q}\}$. Suppose that $\alpha \notin A_1$. If $i \geq 0$, then $\alpha \in F[t,\sigma]_{(t)} \subseteq A_1$, a contradiction. If $i \leq -2$, then $ct^i \in W_2 Y_1^{-s} t^{-1}$ implies $c \in W_2 Y_1^{-s} t \subseteq J(F[t,\sigma]_{(t)})$, a contradiction. Hence we may assume that $i = -1$. As before, let $c = b + td$, where $b \in F$ and $d \in F[t,\sigma]_{(t)}$ and let $b = Y_{i_1}^{l_1} \cdots Y_{i_n}^{l_n} u$, where $i_1 < \cdots < i_n$, $l_i \in \boldsymbol{Q}$, $l_1 < 0$ (since α is not in A_1) and

$u \in U(V_0)$. Then for any $s > \pi > r$, we have $W_2Y_1^{-s}t^{-1} \supseteq W_2\alpha = W_2ct^{-1} \supset W_2Y_1^{-r}t^{-1}$, which implies $U_2Y_1^{-s} \supseteq U_2Y_{i_1}^{l_1} \cdots Y_{i_n}^{l_n}u \supset U_2Y_1^{-r}$. It follows that $i_1 = 1$ and $-s \leq l_1 < -r$ for any $s, r \in \boldsymbol{Q}$ with $s > \pi > r > 0$. Hence $l_1 = -s$ for some $s \in \boldsymbol{Q}$ with $s > \pi$. This implies $W_2\alpha \supset W_2Y_1^{-s_1}t^{-1}$ for any $s_1 \in \boldsymbol{Q}$ with $\pi < s_1 < -l_1$, a contradiction. Hence $A_1 = \cap\{W_2Y_1^{-s}t^{-1} \mid s > \pi, s \in \boldsymbol{Q}\}$ follows. In particular, ${}^*A_1 = A_1$. □

We set $M_i = A_1\sigma(A_1)\cdots\sigma^{i-1}(A_1)$. Then we have

(d) $M_i = \cup\{VY_1^{-r}t^{-i} \mid r < i\pi, r \in \boldsymbol{Q}\} = \cup\{W_2Y_1^{-r}t^{-i} \mid r < i\pi, r \in \boldsymbol{Q}\}$ for all $i \in \boldsymbol{N}$ and they are not finitely generated left W_2-ideals.

(e) $M_i = \cap\{W_2Y_1^{-s}t^{-i} \mid s > i\pi, s \in \boldsymbol{Q}\}$ so that ${}^*M_i = M_i$.

(f) $(V : M_i)_r = \cup\{t^iY_1^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$ so that $M_{-i} = \sigma^{-i}((V : M_i)_r) = \cup\{t^iY_{1+i}^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$.

Proof.

(d) The first statement follows by induction on i and the second statement is clear from the proof of (b).

(e) This is clear from (c)(we use $\pi' = i\pi$ instead of π in (c)).

(f) It is clear from (d) that $(V : M_i)_r \supseteq \cup\{t^iY_1^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$. To prove the converse inclusion, let $\alpha = t^jc \in (V : M_i)_r$ for some $c \in U(F[t,\sigma]_{(t)})$ and $j \in \boldsymbol{Z}$. We may suppose that $j \geq i$, because $VY_1^{-r}t^{-i}t^jc \subseteq V \subseteq F[t,\sigma]_{(t)}$ for any $r \in \boldsymbol{Q}$ with $r < i\pi$. If $j > i$, then $\alpha = t^it^{j-i}c \in t^itF[t,\sigma]_{(t)} \subseteq t^iY_1^sV$ for any $s \in \boldsymbol{Q}$ with $s > i\pi$. If $j = i$, then, as before, let $c = b + td$ for some $b \in F$ and $d \in F[t,\sigma]_{(t)}$ and write $b = Y_{i_1}^{l_1} \cdots Y_{i_n}^{l_n}u$ for some $u \in U(V_0)$, $i_1 < \cdots < i_n$ and $l_i \in \boldsymbol{Q}$. Since $V \ni Y_1^{-r}t^{-i}\alpha$ for any $r < i\pi, r \in \boldsymbol{Q}$, we have $Y_1^{-r}b \in V_0$, that is, $b \in Y_1^rV_0$. This implies $l_1 > 0$ and $i_1 \leq 1$. If $i_1 < 1$, then $\alpha = t^i(b + td) \in t^i(Y_1^sV_0 + Y_1^stF[t,\sigma]_{(t)}) \subseteq t^iY_1^sV$ for any $s > i\pi, s \in \boldsymbol{Q}$, because $v(b) > v(Y_1^s)$. If $i_1 = 1$ and $l_1 < i\pi$, then there is an $r \in \boldsymbol{Q}$ with $l_1 < r < i\pi$ and $b \notin Y_1^rV_0$, a contradiction. If $i_1 = 1$ and $l_1 > i\pi$, then there is an $s \in \boldsymbol{Q}$ with $l_1 > s > i\pi$ and $bV_0 \subseteq Y_1^sV_0$. So we have

$$\alpha = t^ic = t^i(b + td) \in t^i(Y_1^sV_0 + Y_1^stF[t,\sigma]_{(t)}) = t^iY_1^sV.$$

Hence $(V : M_i)_r = \cup\{t^iY_1^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$ follows. In particular, $M_{-i} = \cup\{t^iY_{1+i}^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$. □

Thus we have the following example of a graded extension A of V in $K[X, X^{-1};\sigma]$ satisfying all conditions in Theorem 1.14.

Example 2.2. Under the notations and assumptions as above, let $A_i = \cup\{VY_1^{-r}t^{-i} \mid r < i\pi, r \in \boldsymbol{Q}\}$ and $A_{-i} = \cup\{t^iY_{1+i}^sV \mid s > i\pi, s \in \boldsymbol{Q}\}$. Then

$A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$.

In order to obtain more concrete examples of Theorem 1.6, let $A_1 = J(W_2)t^j$, which is a left W_2 and right $t^{-j}W_2t^j (= \sigma^{-j}(W_2) = W_{2+j})$-ideal. So, by using the notation in Section 1, $W = W_2$, $\sigma(U) = W_{2+j}$, $\alpha_i = t^{ij}$, $\alpha_{-i} = t^{-ij}$ for all $i \in \boldsymbol{N}$ and $A_{-1} = U\alpha_{-1} = W_{3+j}t^{-j}$. Thus we have the following:

(1) $A_{-1} = U\alpha_{-1} = \alpha_{-1}\sigma^{-1}(U)$ if and only if $j = -1$.
(2) $A_{-1} = U\alpha_{-1} \supset \alpha_{-1}\sigma^{-1}(U)$ if and only if $j > -1$.
(3) $A_{-1} = U\alpha_{-1} \subset \alpha_{-1}\sigma^{-1}(U)$ if and only if $j < -1$.

Hence we have the following example illustrating Theorem 1.6.

Example 2.3. Under the notations and assumptions as above, if $j \geq -1$, then $A = \oplus_{i\in \boldsymbol{N}} W_{3+j}t^{-ij}X^{-i} \oplus V \oplus (\oplus_{i\in \boldsymbol{N}} t^{ij} J(W_{3-i+j})X^i)$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ and if $j < -1$, then $A = \oplus_{i\in \boldsymbol{N}} t^{-ij}W_{2+i}X^{-i} \oplus V \oplus (\oplus_{i\in \boldsymbol{N}} J(W_2)t^{ij}X^i)$ is a graded extension of V in $K[X, X^{-1}; \sigma]$.

Finally we will provide examples satisfying all conditions in Theorem 1.20. Let V be a total valuation ring of K with rank one and suppose that $J(V) \supset (0)$ is an exceptional prime segment with C, the non-Goldie prime ideal. Then ${}^*C = C$ such that $O_l(C) = V = O_r(C)$ and it is not a finitely generated left V-ideal (cf. [**3**]). Let l be a natural number with ${}^*(C^l) = Vc = cV$ for some $c \in K$ (assume that l is the smallest natural number for this property and $l > 1$)(cf. [**1**, p. 3173]). Then $C^l = J(V)c$. Thus we have the following example (in the case $\sigma = 1$):

Example 2.4. Under the notations and assumptions above, let $A_1 = C$. Then

$$\begin{aligned} A = {} & \oplus_{i\in \boldsymbol{N}\setminus l\boldsymbol{N}} (C^i)^{-1}X^{-i} \oplus (\oplus_{j\in \boldsymbol{N}} Vc^{-j}X^{-jl}) \\ & \oplus V \oplus (\oplus_{i\in \boldsymbol{N}\setminus l\boldsymbol{N}} C^iX^i) \oplus (\oplus_{j\in \boldsymbol{N}} Vc^jX^{jl}) \end{aligned}$$

and

$$\begin{aligned} B = {} & \oplus_{i\in \boldsymbol{N}\setminus l\boldsymbol{N}} (C^i)^{-1}X^{-i} \oplus (\oplus_{j\in \boldsymbol{N}} Vc^{-j}X^{-jl}) \\ & \oplus V \oplus (\oplus_{i\in \boldsymbol{N}\setminus l\boldsymbol{N}} C^iX^i) \oplus (\oplus_{j\in \boldsymbol{N}} J(V)c^jX^{jl}) \end{aligned}$$

are graded extensions of V in $K[X, X^{-1}]$.

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