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# Homology operations on ring spectrum of $H^{\infty}$ type and their applications

By Akihiro TSUCHIYA*

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## § 0. Introduction and statement of results.

Homology operations on iterated loop spaces were defined by Araki-Kudo [2] and Dyer-Lashof [5]. These operations are essential for studying the homology of the spaces such as $QX$, $F$, $BF$, $BPL$, etc.

On the other hand when we consider the spaces such as $QS^0$, they are not only infinite loop spaces but they have reduced join products which are infinitely homotopy commutative.

The purpose of this paper is to study the relations between these two products and relations between two Dyer-Lashof operations $Q^i$ and $\bar{Q}^j$ defined by loop products and reduced join products respectively.

In § 1, we formulate the ring spectrum of $H^{\infty}$ type. It is roughly speaking a ring spectrum $(X, \mu)$ with unit, whose product is parametrized by $W\Sigma_q$, so that there exist $\Sigma_q$ equivariant maps

$$\bar{\theta}_q: W\Sigma_q \times X^{(q)} \longrightarrow X. \tag{0-1}$$

In § 2, in the first part, we reconstruct the parametrization of the loop product on iterated loop spaces by $W\Sigma_q$ due to Araki-Kudo, and Dyer-Lashof in the following way. Define

$$c: QS^0(q) \times (QX)^q \longrightarrow QX \tag{0-2}$$

by composition, and then we approximate $QS^0(q)$ by $W\Sigma_q$ equivariantly by taking a $\Sigma_q$ equivariant map $\varphi_q$: $W\Sigma_q \to QS^0(q)$, then we get

$$\theta_q: W\Sigma_q \times (QX)^q \longrightarrow QX. \tag{0-3}$$

If $X$ is a ring spectrum of $H^{\infty}$ type using (0-1) we get

$$\bar{\theta}_q: W\Sigma_q \times (QX)^q \longrightarrow QX. \tag{0-4}$$

The main point of this section is the following.

PROPOSITION 2-8. *The general distributive law. The following diagram is*

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*equivariantly homotopy commutative over* $\Sigma_p \int \Sigma_q \xrightarrow{\tilde{\theta}_\otimes} \Sigma_p \times \Sigma_{qp} \xrightarrow{\pi_2} \Sigma_{qp}$.

$$\begin{array}{ccc}
W\Sigma_p \times (W\Sigma_q \times (QX)^q)^p & \xrightarrow{id \times (\theta_q)^p} & W\Sigma_p \times (QX)^p \\
\downarrow & & \downarrow \tilde{\theta}_p \\
W\Sigma_p \times (W\Sigma_q)^p \times ((QX)^q)^p & & \\
\downarrow \Delta_{qp+1} \times id \times \square & & QX \\
(W\Sigma_p)^{qp+1} \times (W\Sigma_q)^p \times \prod_{J \in S} (QX)^p_J & & \uparrow \theta_{qp} \\
\downarrow & & \\
W\Sigma_p \times (W\Sigma_q)^p \times \prod_{J \in S} (W\Sigma_p \times (QX)^p)_J & \xrightarrow{\theta_\otimes \times \prod(\tilde{\theta}_p)_J} & W\Sigma_{qp} \times \prod_{J \in S} (QX)_J .
\end{array}$$

In § 3, using (0-3) and (0-4) we define Dyer-Lashof operations $Q^i$ and $\bar{Q}^j$. The first proposition is the following Mixed Cartan formula.

PROPOSITION 3-7.

$$\bar{Q}^j(x * y) = \sum_{j_0 + \cdots + j_p = j} \Sigma \bar{Q}_0^{j_0}(x_0 \otimes y_0) * \cdots * \bar{Q}_p^{j_p}(x_p \otimes y_p)$$

*where* $\Delta_{p+1}(x \otimes y) = \Sigma(x_0 \otimes y_0) \otimes \cdots \otimes (x_p \otimes y_p)$. *And we have*

$$\bar{Q}_0^j(x \otimes y) = \bar{Q}^j(\varepsilon(x) \cdot y), \quad \bar{Q}_p^j(x \otimes y) = \bar{Q}^j(x \cdot \varepsilon(y)) .$$

*For* $0 < i < p$ *put* $m_i = \frac{1}{p}\binom{p}{i}$ *then*

$$\bar{Q}_i^j(x \otimes y) = [m_i] \circ (\Sigma Q^j(x_1 \circ \cdots \circ x_i \circ y_1 \circ \cdots \circ y_{p-i})$$

*where* $\Delta_i x = \Sigma x_1 \otimes \cdots \otimes x_i$, *and* $\Delta_{p-i} y = \Sigma y_1 \otimes \cdots \otimes y_{p-i}$.

Then we study relations between $Q^i$ and $\bar{Q}^j$, for example, we get

PROPOSITION 3-14. *When* $p=2$, *in* $H_*(QX : Z_2)$

$$\bar{Q}^{t-k} Q^k(x) = \sum_{j, c_1, c_2, \Delta x = \Sigma x' \otimes x''} \binom{t-k-1-j-2c_1}{k-j-c_1+c_2}$$
$$Q^{t-j-c_1-c_2} \bar{Q}^{c_1}(x') * Q^j \bar{Q}^{c_2}(x'') .$$

In § 4, the preceding results are used to determine the Dyer-Lashof operations on $H_*(SF)$, $H_*(BSF)$, $H_*(SPL)$ and $H_*(BSPL)$ when $p$ is an odd prime number.

PROPOSITION 4-3. *The Pontrjagin ring* $H_*(SF)$ *is the free commutative algebra generated by*

i) $\beta^\varepsilon Q^j[1] * [1-p]$.

ii) $Q^I[1] * [1-p^2]$, $|I| = 2$, $e(I) + \varepsilon_1 > 0$, $I$: *admissible.*

iii) $\bar{Q}^J(Q^I[1]*[1-p^2])$, $|I|\geqq 2$, $|J|\geqq 1$, $e(J, I)+\varepsilon_1>0$,
$(J, I)$: *admissible*, $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$.

THEOREM 4-4. *The Pontrjagin ring $H_*(BSF)$ is the free commutative ring generated by*

i) $\tilde{z}_i$, $\sigma(Q^j[1]*[1-p])$, $i, j=1, 2, \cdots$.

ii) $\sigma(Q^I[1]*[1-p^2])$, $|I|=2$, $e(I)+\varepsilon_1\geqq 1$, $I$: *admissible.*

iii) $\bar{Q}^J(\sigma(Q^I[1]*[1-p^2]))$, $|I|=2$, $|J|\geqq 1$, $e(J, I)+\varepsilon_1>1$,
$(J, I)$: *admissible*, $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$.

THEOREM 4-22. *The Pontrjagin ring $H_*(BSPL)$ is the free commutative ring generated by*

1) $\bar{b}_j$ $j=1, 2, \cdots$.

2) $\sigma(\bar{x}_I)$; $I=(\delta, r, \varepsilon, s)$, $e(I)+\delta\geqq 1$, $I$: *admissible.*

3) $\bar{Q}^J(\sigma(\bar{x}_I))$, $I=(\delta, r, \varepsilon, s)$, $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r\geqq 1$,
$e(J, I)+\varepsilon_1>1$, $(J, I)$: *admissible.*

*And the map $j_*$: $H_*(BSPL)\to H_*(BSF)$ is as follows.*

$$1)\quad j_*(\bar{b}_j)=\begin{cases} 0 & \text{if } j\not\equiv 0 \ \frac{p-1}{2} \\ c_j\tilde{z}_k+\text{dec.} & \text{if } j=\frac{k-1}{2}k, \quad c_j\neq 0. \end{cases}$$

$$2)\quad j_*(\sigma(\bar{x}_I))=\begin{cases} \sigma(\beta^\delta Q^r\beta^\varepsilon Q^s[1]*[1-p^2]), & \text{if } (\delta, \varepsilon)\neq(0, 0) \\ \sigma(Q^rQ^s[1]*[1-p^2])+\binom{r-1}{s(p-1)}\sigma(Q^{r+s}[1]*[1-p]) & \text{if } (\delta, \varepsilon)=(0, 0). \end{cases}$$

## §1. Multiplicative ring spectra of $H^\infty$ type.

**1-1.** At first we shall review the Boardman's $CW$ spectra after Vogt [**16**].

We denote by $\mathscr{F}$, the category of pointed finite $CW$ complexes and base point preserving continuous maps and by $I\mathscr{F}$, the subcategory of $\mathscr{F}$ with same objects of $\mathscr{F}$, and whose morphisms are inclusions as subcomplexes.

Let $R^\infty$ be the prehilbert space with orthonormal basis $e_i=(0, \cdots, 0, 1, 0, \cdots)$, $i=1, 2, \cdots$. Let $V$ be a prehilbert space isomorphic to $R^\infty$. For each finite dimensional subspace $A$ of $V$, take a copy $\mathscr{F}(A)$ of $\mathscr{F}$, and for each inclusion map $\alpha$: $A\subset B$, define a function $S_\alpha$: $\mathscr{F}(A)\to\mathscr{F}(B)$ by $S_\alpha(X)=(\alpha(A)^{cB}\cup\infty)\wedge X$, $S_\alpha f=(\alpha(A)^{cB}\cup\infty)\wedge f$, where $(\alpha A)^{cB}$ is the orthonormal complement of $\alpha(A)$ in $B$ and $D\cup\infty$ is the one point compactification, and $\wedge$ is the smash product. If we take $I\mathscr{F}(A)$ instead of $\mathscr{F}(A)$, the inclusion $\alpha$: $A\subset B$ induces a functor $S_\alpha$: $I\mathscr{F}(A)\to I\mathscr{F}(B)$.

DEFINITION 1-1. The category of finite $CW$ spectra associated to $V$, denoted by $\mathscr{F}(V)$, is the *limit category of the* 2-*diagrams* $\{\mathscr{F}(A)\}$ *associated to*

$V$. So that the objects of $\mathcal{F}(V)$ are all pairs $(X, A)$ where $X \in \mathcal{F}$, and $A \subset V$ finite dimensional subspaces. And the set of morphisms are defined by $\mathcal{F}(V)((X, A), (Y, B)) = \lim_{C} \mathcal{F}(S_\alpha A, S_\beta B)$, where $\alpha: A \subset C$ and $\beta: B \subset C$.

The subcategory $I\mathcal{F}(V)$ of $\mathcal{F}(V)$ is defined to be *the limit of the same 2-diagrams with* $\mathcal{F}$ *replaced by* $I\mathcal{F}$.

For each $A \subset V$, there is a canonical inclusion functor $J(A): \mathcal{F}(A) \to \mathcal{F}(V)$ and $I\mathcal{F}(A) \to I\mathcal{F}(V)$ defined by $J(A)(X) = (X, A)$, $J(A)(f) = \lim_{C} S_\alpha f$, where $\alpha: A \subset C$.

DEFINITION 1-2. The category $\mathcal{S}(V)$ of the $CW$ spectra associated to $V$ is the category whose set of objects are all the sets of the directed non-empty diagrams in $I\mathcal{F}(V)$, and whose sets of morphisms are defined by $\mathcal{S}(X, Y) = \underleftarrow{\lim}_{\alpha} \underrightarrow{\lim}_{\beta} \mathcal{F}(V)(X_\alpha, Y_\beta)$, where $X = \{X_\alpha, \alpha \in I\}$, $Y = \{Y_\beta, \beta \in J\}$.

The subcategory $I\mathcal{S}(V)$ of $\mathcal{S}(V)$ is the subcategory of $\mathcal{S}(V)$ with same objects, and whose sets of morphisms are defined by $I\mathcal{S}(X, Y) = \underleftarrow{\lim}_{\alpha} \underrightarrow{\lim}_{\beta} I\mathcal{F}(V)(X_\alpha, Y_\beta)$.

REMARK 1-3. The category $\mathcal{F}(V)$ and $I\mathcal{F}(V)$ are considered as full subcategories of $\mathcal{S}(V)$ and $I\mathcal{S}(V)$ respectively.

For $X, Y \in \mathrm{ob}\, \mathcal{F}$, the set of morphisms $\mathcal{F}(X, Y)$ is *topologized by compact open topology*. And for $X, Y \in \mathcal{F}(V)$, the set of morphisms $\mathcal{F}(V)(X, Y) = \underrightarrow{\lim}_{C} (S_\alpha X, S_\beta Y)$ is *topologized by taking direct limit of topological spaces*. And then for $X, Y \in \mathcal{S}(V)$, the set $\mathcal{S}(X, Y) = \underleftarrow{\lim}_{\alpha} \underrightarrow{\lim}_{\beta} \mathcal{F}(V)(X_\alpha, Y_\beta)$ is *topologized by taking at first* $\underrightarrow{\lim}$ *topology and then taking* $\underleftarrow{\lim}$ *topology in the category of compactly generated spaces.*

Let $V$ and $W$ be prehilbert spaces isomorphic to $R^\infty$. Give $V$ and $W$ the fine topology, i.e., a subset $U$ of $V$ is open if and only if $U \cap A$ open in $A$ for all finite dimensional subspaces. Let $LIE(V, W)$ be *the space of all linear isometric maps from* $V$ *to* $W$ with function space topology in the category of compactly generated spaces.

LEMMA 1-4. *$LIE(V, W)$ is contractible.*

If $f$ is an element of $LIE(V, W)$ then $f$ induces functors $f_*: \mathcal{F}(V) \to \mathcal{F}(W)$ and $\mathcal{S}(V) \to \mathcal{S}(W)$, and for any two $f, g$, the functors $f_*$ and $g_*$ are coherently naturally equivalent.

For prehilbert spaces $V$ and $W$, isomorphic to $R^\infty$, we define an external smash product

$$\wedge: \mathcal{F}(V) \times \mathcal{F}(W) \longrightarrow \mathcal{F}(V \oplus W), \tag{1-1}$$

$$\wedge: \mathcal{S}(V) \times \mathcal{S}(W) \longrightarrow \mathcal{S}(V \oplus W)$$

as follows. For finite dimensional subspaces $A \subset V$, $B \subset W$, $K_{A,B}$ denote the following composite functor. $K_{A,B}: \mathcal{F}(A)\times\mathcal{F}(B)\xrightarrow{\wedge}\mathcal{F}(A\oplus B)\xrightarrow{J(A\oplus B)}\mathcal{F}(V\oplus W)$, then this commutes with suspension functor $S_\alpha\times S_\beta: \mathcal{F}(A)\times\mathcal{F}(B)\to\mathcal{F}(A')\times\mathcal{F}(B')$, where $\alpha: A\subset A'$, $\beta: B\to B'$, so that we can take the limit and get the functor $\wedge: \mathcal{F}(V)\times\mathcal{F}(W)\to\mathcal{F}(V\oplus W)$. Considering a directed non-empty diagram in $I\mathcal{F}(V)$ and $I\mathcal{F}(W)$, we get the functor $\wedge: \mathcal{S}(V)\times\mathcal{S}(W)\to\mathcal{S}(V\oplus W)$.

**1.2.** For any positive integer $q$, denote by $\Sigma_q$, the symmetric group of $q$-elements, and by $\pi: W\Sigma_q\to B\Sigma_q$, the classifying bundle of principal $\Sigma_q$ bundle. So that $W\Sigma_q$ is a contractible space on which $\Sigma_q$ acts freely (to the right). And we can assume $W\Sigma_q$ has a structure of $CW$ complex on which $\Sigma_q$ acts cell wisely.

Let $V^q=V\oplus\cdots\oplus V$ be the direct sum of $q$ copies of $V$, where $V$ is a prehilbert space isomorphic to $R^\infty$. And $\Sigma_q$ acts on $V^q$ *by permutation of factors*, i.e., $\sigma(x_1,\cdots,x_q)=(x_{\sigma^{-1}(1)},\cdots,x_{\sigma^{-1}(q)})$, $\sigma\in\Sigma_q$, $x_i\in V$. Then $\Sigma_q$ acts on $LIE(V^q, W)$ by $\sigma(f)=f\circ\sigma$, $f\in LIE(V, W)$, $\sigma\in\Sigma_q$.

LEMMA 1-5. *There exists a continuous $\Sigma_q$ equivariant map $\varphi_q: W\Sigma_q\to LIE(V^q, W)$, i.e., $\varphi_q(w\sigma^{-1})=\sigma\varphi_q(w)$, $\sigma\in\Sigma_q$, $w\in W\Sigma_q$, and any two such maps are $\Sigma_q$ equivariantly homotopic.*

PROOF. By Bredon, the obstruction for the existence for $\varphi_q$ are in $H^i_{\Sigma_q}(W\Sigma_q, \pi_i(LIE(V^q, W)))$, where $H^i_{\Sigma_q}(:)$ is the $\Sigma_q$ equivariant cohomology in the sense of Steenrod [**13**]. But by Lemma 1-4, $\pi_i(LIE(V^q, W))=0$. And the obstruction to make a homotopy lies in $H^i_{\Sigma_q}(W\Sigma_q, \pi_i(LIE(V^q, W)))=0$. Now we fix a $\Sigma_q$ equivariant map $\varphi_q: W\Sigma_q\to LIE(V^q, W)$.

DEFINITION 1-6. *A ($\Sigma_q$ equivariant) carrier $\chi$ dominating $\varphi_q$* associates with each finite dimensional subspace $A\subset V^q$, a finite dimensional subspace $\chi A\subset W$ and positive integer $m(A)$ with the following properties.

1) $A\subset B$ implies $\chi A\subset\chi B$ and $m(A)\leqq m(B)$, (and $\dim(A)\to\infty$ implies $\dim\chi A\to\infty$ and $m(A)\to\infty$).

2) $\Sigma_q$ equivariant, i.e., for any $\sigma\in\Sigma_q$, $\chi(\sigma A)=\sigma\chi A$.

3) For all $w\in W\Sigma_q^{(m(A))}$, $\varphi_q(w, A)\subseteqq\chi A$, where $W\Sigma_q^{(m(A))}$ means the $m(A)$ skeleton of $W\Sigma_q$.

REMARK 1-7. A carrier $\chi$ dominating $\varphi_q$ always exists.

For each carrier $\chi$ we associate a functor

$$EP_q: \mathcal{S}(V)\longrightarrow\mathcal{S}(W) \tag{1-2}$$

by the following way. For each finite dimensional subspace $A\subset V$, let $EA=\{(w, v)\in W\Sigma_q^{(m(A))}\times\chi A^q,\ v\in(\varphi_q(w, A^q))^{c\chi A^q}\}$, with the subspace topology. Then the projection onto the first coordinate $\xi_A: EA\to W\Sigma_q^{(m(A^q))}$ defines a vector bundle. On $EA$, the symmetric group $\Sigma_q$ acts by the formula $\sigma((w, v))=(w\sigma^{-1}, v)$, and this is well defined by (1-3), 2). This makes $\xi_A$ a $\Sigma_q$ equivariant vector bundle. Define a functor $EP_q(A): \mathcal{F}(A)\to\mathcal{F}(\chi A)$ by the formula

$EP_q(X)=T(\xi_A)\ltimes X\wedge\cdots\wedge X=T(\xi_A)\ltimes X^{(q)}$, where $T(\xi_A)$ is the Thom complex of $\xi_A$ and $X\ltimes Y=X\times Y/X\times *$, and $X^{(q)}=X\wedge\cdots\wedge X$ is the $q$-th smash product of $X$. Then $\Sigma_q$ acts on $EP_q(X)$ by $\sigma(e, x_1, \cdots, x_q)=(\sigma e, x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(q)})$, $e\in T(\xi_A)$ and $x_i\in X$, and $\sigma\in\Sigma_q$, then the functor $EP_q(A)$ is compatible with the suspension functor up to natural equivalence with coherence condition. So we get a functor $EP_q\colon \mathcal{F}(V)\to\mathcal{S}(W)$, and taking a directed non-empty diagram we get the functor $EP_q\colon\mathcal{S}(V)\to\mathcal{S}(W)$.

REMARK 1-8. For each $X\in\mathcal{S}(V)$, the symmetric group $\Sigma_q$ acts on $EP_q(X)$, and for each map $f\colon X\to Y$, $EP_q(f)\colon EP_q(X)\to EP_q(Y)$ is $\Sigma_q$ equivariant map.

REMARK 1-9. If we take another $\Sigma_q$ equivariant map $\varphi'_q$, and carrier $\chi'$ dominating $\varphi'_q$, then these define the functor $EP'_q\colon\mathcal{S}(V)\to\mathcal{S}(W)$. Then we can construct a $\Sigma_q$ equivariant map $\tilde{\varphi}_q\colon W\tilde{\Sigma}_q=W\Sigma_q\times I\to LIE(V^q, W)$, and carrier $\tilde{\chi}$ dominating $\tilde{\varphi}_q$ with the following properties,

(1) $\tilde{\varphi}_q|W\Sigma_q\times 0=\varphi_q$, $\tilde{\varphi}_q|W\Sigma_q\times 1=\varphi'_q$. (2) $\chi\subset\tilde{\chi}|W\Sigma_q\times 0$, $\chi'\subset\tilde{\chi}|W\Sigma_q\times 1$. Then the inclusion map $W\Sigma_q=W\Sigma_q\times 0\subset W\tilde{\Sigma}_q=W\Sigma_q\times I$, and the $\Sigma_q$ equivariant deformation retract $\pi\colon W\tilde{\Sigma}_q=W\Sigma_q\times I\to W\Sigma_q\times 1=W\Sigma_q$ define $\Sigma_q$ equivariant natural transformation $\Phi\colon EP_q\to E\tilde{P}_q$, and $\Phi'\colon E\tilde{P}_q\to EP'_q$. And composing these we get a natural transformation $\Psi=\Phi'\circ\Phi\colon EP_q\to EP'_q$, which is $\Sigma_q$ equivariant and homotopy equivalent, i. e., for any $X\in\mathcal{S}(V)$, $\Psi(X)\colon EP_q(X)\to EP'_q(X)$ is $\Sigma_q$ equivariant and homotopy equivalent.

**1-3.** For any positive integers $p$ and $q$, we consider $\Sigma_p\times\Sigma_q$ as a subgroup of $\Sigma_{p+q}$ by considering that $\Sigma_p$ acts on first $p$ elements and $\Sigma_q$ acts on last $q$ elements. Let $\Sigma_p\int\Sigma_q$ be the wreath product of $\Sigma_q$ by $\Sigma_p$. So that $\Sigma_p\int\Sigma_q$ is considered as a subgroup of $\Sigma_{pq}$ as follows, at first the set of $pq$ elements $\{1, 2, \cdots, pq\}$ is divided in $p$ blocks of $q$ elements by $\{1, \cdots, q\}, \cdots, \{(p-1)q+1, \cdots, pq\}$. And $(\sigma, \tau_1, \cdots, \tau_p)\in\Sigma_p\int\Sigma_q$, $\sigma\in\Sigma_p$, $\tau_i\in\Sigma_q$, operates on $\{1, \cdots, pq\}$ by the formula, at first $\tau_i$ operates on the $i$-th block by permutation for $i=1, \cdots, p$, and then $\sigma$ operates on blocks by permutation.

If $G$ is any subgroup of $\Sigma_q$, by the same method of 1-2, we can take a $G$ equivariant map $\varphi_G\colon WG\to LIE(V^q, W)$ and carrier $\chi_G=\chi$ dominating $\varphi_G$. And we can define a functor $EP_G\colon\mathcal{S}(V)\to\mathcal{S}(W)$. And if we change $\varphi_G$ by $\varphi'_G$ and $\chi_G$ by $\chi'_G$, then we get a functor $EP'_G\colon\mathcal{S}(V)\to\mathcal{S}(W)$. And by the same method of Remark 1-9 we get a natural transformation $\Psi_G\colon EP_G\to EP'_G$, which is $G$ equivariant and homotopy equivalent.

Now we fix positive integers $p$ and $q$. And fix an element $f\in LIE((R^\infty)^2, R^\infty)$. Then define functor $EP_p\wedge EP_q\colon\mathcal{S}(R^\infty)\to\mathcal{S}(R^\infty)$ by the formula,

$$EP_p\wedge EP_q\colon\mathcal{S}(R^\infty)\xrightarrow{EP_p\times EP_q}\mathcal{S}(R^\infty)\times\mathcal{S}(R^\infty)\xrightarrow{\wedge}\mathcal{S}(R^\infty\oplus R^\infty)\xrightarrow{f_*}\mathcal{S}(R^\infty). \tag{1-3}$$

And denote by $EP_p(EP_q)$ the composite functor,

(1-4) $$EP_pEP_q: \mathcal{S}(R^{\infty}) \xrightarrow{EP_q} \mathcal{S}(R^{\infty}) \xrightarrow{EP_p} \mathcal{S}(R^{\infty}).$$

Then for any $X$, $\Sigma_p \times \Sigma_q$ acts on $(EP_p \wedge EP_q)(X)$, $(\Sigma_p \int \Sigma_q$ acts on $EP_pEP_q(X)$,) and for any $f: X \to Y$, $(EP_p \wedge EP_q)(f): (EP_p \wedge EP_q)(X) \to (EP_p \wedge EP_q)(Y)$ is $\Sigma_p \times \Sigma_q$ equivariant, $(EP_pEP_q(f): EP_pEP_q(X) \to EP_pEP_q(Y)$ is $\Sigma_p \int \Sigma_q$ equivariant).

We define functors,

(1-5) $$I_{p,q} = I: EP_p \wedge EP_q \longrightarrow EP_{p+q},$$
$$J_{p,q} = J: EP_pEP_q \longrightarrow EP_{pq}$$

by the following way. At first we take $G = \Sigma_p \times \Sigma_q \subset \Sigma_{p+q}$, and define $\varphi_G$ by the composite $\varphi_G: W(\Sigma_p \times \Sigma_q) = W\Sigma_p \times W\Sigma_q \xrightarrow{\varphi_p \times \varphi_q} LIE((R^{\infty})^p, R^{\infty}) \times LIE((R^{\infty})^q, R^{\infty}) \xrightarrow{\times} LIE((R^{\infty})^{p+q}, (R^{\infty})^2) \xrightarrow{f_*} LIE((R^{\infty})^{p+q}, R^{\infty})$, and carrier $\chi_G$ by definition, for $A \subset (R^{\infty})^p$, $B \subset (R^{\infty})^q$, $\chi_G(A \times B) = f \circ (\chi_p(A) \times \chi_q(B))$, and $m(A \times B) = \min(m(A), m(B))$, and extending these data. Then the functor $EP_{\Sigma_p \times \Sigma_q}$ associating these data $\varphi_G$ and $\chi_G$ is naturally equivalent to the functor $EP_p \wedge EP_q$. On the second, we define another $\varphi'_G$ and $\chi'_G$ by the formula, $\varphi'_G: W(\Sigma_p \times \Sigma_q) \xrightarrow{i} W\Sigma_{p+q} \xrightarrow{\varphi_{p+q}} LIE((R^{\infty})^{p+q}, R^{\infty})$, and $\chi_G(A) = \chi_{\Sigma_{p+q}}(A)$, $m'_G(A) = m_{\Sigma_{p+q}}(A)$. By these $\varphi'_G$ and $\chi'_G$ we get the functor $EP'_G: \mathcal{S}(R^{\infty}) \to \mathcal{S}(R^{\infty})$. And by definition we get the natural transformation $\Phi: EP'_G \to EP_{p+q}$. On the other hand by preceding remark we get natural transformation $\Phi_G: EP_G \cong EP_p \wedge EP_q \to EP'_G$. Composing $\Phi$ and $\Phi_G$ we get the natural transformation $I_{p,q}$.

By the same method we get the natural transformation $J_{p,q}$.

REMARK 1-10. $I_{p,q}$ (respectively $J_{p,q}$) is $\Sigma_p \times \Sigma_q$ (respectively $\Sigma_p \int \Sigma_q$) equivariant natural transformation.

**1-4.** Consider *the sphere spectrum* $S^0 \in \mathcal{T}(R^{\infty})$. This is considered as follows. At first consider $S^0 \in \mathcal{T}(R^0)$, the two points with distinguished point, then the sphere spectrum $S^0$ is by definition $J(R^0)(S^0)$, where $J(R^0): \mathcal{T}(R^0) \to \mathcal{T}(R^{\infty})$ is the canonical inclusion. Then $EP_q(S^0)$ is considered as the diagram

$$\{T(\xi_A) \ltimes ((A \cup \infty) \wedge \cdots \wedge (A \cup \infty)),\ A \subset R^{\infty}\}.$$

Define $\Sigma_q$ equivariant map

(1-6) $$\bar{\theta}_q: EP_q(S^0) \longrightarrow S^0,$$

in the following way. At first we consider the following $\Sigma_q$ equivariant homeomorphism

$$(1\text{-}7)\qquad \begin{aligned} T(\xi_A)\ltimes((A\cup\infty)\wedge\cdots\wedge(A\cup\infty)) &\cong T(E_A\times(A\oplus\cdots\oplus A)) \\ &\cong T(W\Sigma_q^{(m(A^q))}\times\chi A^q) \\ &\cong W\Sigma_q^{(m(A^q))}\ltimes(\chi(A^q)\cup\infty), \end{aligned}$$

where $\Sigma_q$ operates on $T(W\Sigma_q^{(m(A^q))}\times(\chi A^q))$ by $\sigma(w, e)=(w\sigma^{-1}, e)$, $\sigma\in\Sigma_q$, $w\in W\Sigma_q^{(m(A^q))}$, $e\in\chi(A^q)$. Then define $\bar{\theta}_q(A)$ by composite: $T(\xi_A)\ltimes(A\cup\infty)^q\to W\Sigma_q^{(m(A^q))}\ltimes(\chi A^q\cup\infty)\xrightarrow{\pi_2}(\chi A^q\cup\infty)$. Going to the limit we get $\bar{\theta}_q$.

PROPOSITION 1-11. *The maps $\bar{\theta}_q(S^0)=\bar{\theta}_q$, $q=1, 2, \cdots$, satisfy the following properties:*

0) $\bar{\theta}_1: WP_1(S^0)\cong S^0\to S^0$ *is homotopic to identity.*

1) *The following diagram is $\Sigma_p\times\Sigma_q$ equivariantly homotopy commutative.*

$$(1\text{-}8)\qquad \begin{array}{ccc} EP_p(S^0)\wedge EP_q(S^0)\cong(EP_p\wedge EP_q)(S^0) & \xrightarrow{I_{p,q}} & EP_{p+q}(S^0) \\ \downarrow{\scriptstyle \bar{\theta}_p\wedge\bar{\theta}_q} & & \downarrow{\scriptstyle \bar{\theta}_{p+q}} \\ S^0\wedge S^0 & \xrightarrow[f_*]{} & S^0 \end{array}$$

*where $f\in LIE((R^\infty)^2, R^\infty)$.*

2) *The following diagram is $\Sigma_p\int\Sigma_q$ equivariantly homotopy commutative.*

$$(1\text{-}9)\qquad \begin{array}{ccc} EP_pEP_q(S^0) & \xrightarrow{EP_p(\bar{\theta}_q)} & EP_p(S^0) \\ \downarrow{\scriptstyle J_{p,q}} & & \downarrow{\scriptstyle \bar{\theta}_p} \\ EP_{pq}(S^0) & \xrightarrow{\bar{\theta}_{pq}} & S^0. \end{array}$$

**1-5.** At first we fix an element $f\in LIE((R^\infty)^2, R^\infty)$. Consider a spectrum $X\in\mathcal{S}(R^\infty)$, we assume $X$ is $-1$ connected, i.e., $\pi_i(X)=0$, $i<0$. We write $X\wedge X$ for $f_*(X\wedge X)$ if there is no confusion.

DEFINITION 1-12. A $-1$ connected spectrum with a map $i: S^0\to X$ and $\mu: X\wedge X=f_*(X\wedge X)\to X$, is said to be a *ring spectrum* if:

(1) $\mu$ is homotopy associative.

(2) $i$ gives homotopy unit, i.e., the following diagram is homotopy commutative:

$$\begin{array}{ccccc} S^0\wedge X & \xrightarrow{i\wedge id} & X\wedge X & \xleftarrow{id\wedge i} & X\wedge S^0 \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\cong} & \\ & & X & & \end{array}$$

DEFINITION 1-13. A ring spectrum $(X, i, \mu)$ is said to be $H^\infty$ *type* if, for each positive integer $q$, given a $\Sigma_q$ equivariant map $\bar{\theta}_q: EP_q(X)\to X$ with the following properties.

0) $\bar{\theta}_1: EP_1X\cong X\to X$ is homotopic to identity.

1) The following diagram is $\Sigma_p\times\Sigma_q$ homotopy commutative.

$$(1\text{-}10)\qquad \begin{array}{ccc} EP_p(X)\wedge EP_q(X)\cong(EP_p\wedge!EP_q)(X) & \xrightarrow{I_{p,q}} & EP_{p+q}(X) \\ \downarrow{\scriptstyle \bar{\theta}_p\wedge\bar{\theta}_q} & & \downarrow{\scriptstyle \bar{\theta}_{p+q}} \\ X\wedge X\cong f_*(X\wedge X) & \xrightarrow[\mu]{} & X. \end{array}$$

2) The following diagram is $\Sigma_p\int\Sigma_q$ homotopy commutative.

$$(1\text{-}11)\qquad \begin{array}{ccc} EP_pEP_q(X) & \xrightarrow{EP_p(\bar{\theta}_q)} & EP_pX \\ \downarrow{\scriptstyle J_{p,q}} & & \downarrow{\scriptstyle \bar{\theta}_p} \\ EP_{pq}(X) & \xrightarrow{\bar{\theta}_{pq}} & X. \end{array}$$

3) The following diagram is $\Sigma_q$ homotopy commutative.

$$(1\text{-}12)\qquad \begin{array}{ccc} EP_q(S^0) & \xrightarrow{EP_q(i)} & EP_q(X) \\ \downarrow{\scriptstyle \bar{\theta}_q} & & \downarrow{\scriptstyle \bar{\theta}_q} \\ S^0 & \xrightarrow{i} & X. \end{array}$$

DEFINITION 1-14. Let $X$, $Y$ be two ring spectra of type $H^\infty$, a map $f$: $X\to Y$ is said to be *a map of type* $H^\infty$ if:

(1) The following diagram is homotopy commutative.

$$\begin{array}{ccc} S^0 & \xrightarrow{i} & X \\ & \searrow_{i} & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

(2) For each positive integer $q$, the following diagram is $\Sigma_q$ equivariantly homotopy commutative:

$$\begin{array}{ccc} EP_q(X) & \xrightarrow{EP_q(f)} & EP_q(Y) \\ \downarrow{\scriptstyle \bar{\theta}_q} & & \downarrow{\scriptstyle \bar{\theta}_q} \\ X & \xrightarrow{f} & Y. \end{array}$$

## § 2. The associated infinite loop space $Q(X)$ of $X$.

**2-1.** At first we fix an identification $(S^1, *)=(I/\partial I, \partial I/\partial I)\xrightarrow{\cong}(R^1\cup\infty, \infty)$

which preserve orientation. So that we get $(S^n, *)=(I^n/\partial I^n, \partial I^n/\partial I^n) \xrightarrow{\cong} (R^n \cup \infty, \infty)$.

For pointed topological space $X$, we denote *the n-th iterated loop space* over $X$ by $\Omega^n X$, so that $\Omega^n X = F((I^n, \partial I^n), (X, *)) \cong F((S^n, *), (X, *)) \cong F((R^n \cup \infty, \infty), (X, *))$, where $F((X, A), (Y, B))$ denotes the space of all continuous maps $f: X \to Y$ with $f(A) \subset B$.

Let $\underline{\underline{\mathrm{CO}}}$ denote the category of compactly generated topological space. We define a functor

$$Q: \mathcal{S}(R^\infty) \longrightarrow \underline{\underline{\mathrm{CO}}} \tag{2-1}$$

in the following way. For any $X \in \mathcal{T}(A)$, where $A$ is a finite dimensional subspace of $R^\infty$, we define $Q(X)$ as

$$\begin{aligned} Q(X) &= \varinjlim_n \mathcal{T}(R^n \cup \infty, S_\alpha X) \\ &= \varinjlim_n \mathcal{T}(S^n, S_\alpha X) = \varinjlim_n \Omega^n S_\alpha X , \end{aligned}$$

where $\alpha: A \subset R^n$, and $R^n$ denotes the subspace of $R^\infty$ spanned by $e_1, e_2, \cdots, e_n$. Then this is compatible with the suspension functor $S_\alpha: \mathcal{T}(A) \to \mathcal{T}(B)$, where $\alpha: A \subset B$. So that taking the limit of the 2-diagram we get the functor $Q: \mathcal{T}(R^\infty) \to \underline{\underline{\mathrm{CO}}}$. And taking the directed non-empty diagram we get the desired functor $Q: \mathcal{S}(R^\infty) \to \underline{\underline{\mathrm{CO}}}$.

Now consider the functor $S^n \wedge \mathcal{T} \to \mathcal{T}$ defined by $(S^n \wedge)(X) = S^n \wedge X$. This defines the functor $S^n \wedge : \mathcal{T}(A) \to \mathcal{T}(A)$, for $A \subset R^\infty$. And taking the limit process we get the functor

$$S^n \wedge : \mathcal{S}(R^\infty) \longrightarrow \mathcal{S}(R^\infty) . \tag{2-2}$$

On the other hand, for any pointed topological space $X$, we can define a canonical map $g_n: S^n \wedge \Omega^n X \to \Omega^n S^n X$, defined by $(g_n(x, l))(y) = x \wedge l(y)$, where $x \in S^n$, $l \in \Omega^n X$, $y \in X$. This defines the natural transformation $G_n: (S^n \wedge) \circ Q \to Q \circ (S^n \wedge)$ between the two functors from $\mathcal{S}(R^\infty)$ to $\underline{\underline{\mathrm{CO}}}$.

And taking the adjoint map, we get the natural transformation

$$F_n: Q \longrightarrow \Omega^n \circ Q \circ (S^n \wedge) . \tag{2-3}$$

PROPOSITION 2-1. *For any* $\boldsymbol{X} \in \mathcal{S}(R^\infty)$, *the map* $F_n(\boldsymbol{X}): Q(\boldsymbol{X}) \to \Omega^n Q(S^n \wedge \boldsymbol{X})$ *is homotopy equivalence.*

**2-2.** For each positive integer, $n$ and $q$, we denote by $S^n(q) = S^n \vee \cdots \vee S^n$, the one point union of $q$ spheres of dimension $n$. The symmetric group $\Sigma_q$ operates on $S^n(q)$ by permutations of factors, i.e., $\sigma((x, i)) = (x, \sigma^{-1}(i))$ where $\sigma \in \Sigma_q$, $x \in S^n$, $i = 1, \cdots, q$. Let $(S^n)^q$ denote the product of $q$ spheres of dimension $n$. The group $\Sigma_q$ operates on $(S^n)^q$ by the permutations of factors. Let $j_n: S^n(q) \to (S^n)^q$ denote the canonical inclusion.

LEMMA 2-2. *Let $G$ be a subgroup of $\Sigma_q$, and $K$ be a CW complex on which $G$ acts freely and cell wisely, and assume* $\dim K \leqq n-2$. *Then the map $(j_n)_*$: $[K, \Omega^n S^n(q)]_G \to [K, \Omega^n(S^n)^q]_G$ is one to one onto, where $[\ ,\ ]_G$ denotes the set of $G$ equivariant homotopy classes of continuous $G$ maps.*

PROOF. Since $j_{n*}: \pi_i(\Omega^n S^n(q)) \to \pi_i(\Omega^n(S^n)^q)$ are isomorphic for $i \leqq n-2$, this follows from Bredon's obstruction theory [4].

Now define the inclusion map $i: \Omega^n S^n \to \Omega^{n+1} S^{n+1}$ by $i(l) = l \wedge id_1$, where $l \in \Omega^n S^n$, $id_1: S^1 \to S^1$, identity map. Define $i_q = i: \Omega^n(S^n)^q \to \Omega^{n+1}(S^{n+1})^q$ by the formula $i_q = i \times \cdots \times i: \Omega^n(S^n)^q \cong \Omega^n S^n \times \cdots \times \Omega^n S^n \xrightarrow{i \times \cdots \times i} \Omega^{n+1}S^{n+1} \times \cdots \times \Omega^{n+1}S^{n+1} = \Omega^{n+1}(S^{n+1})^q$. And also define $i_q = i: \Omega^n S^n(q) \to \Omega^{n+1}S^{n+1}(q)$ by $i(l) = l \wedge id_1$ where $l \in \Omega^n S^n$, and we identify $S^n(q) \wedge S^1 \cong S^{n+1}(q)$. Then it is easy to show that the following diagram is $\Sigma_q$ equivariantly commutative.

$$\begin{array}{ccc} \Omega^n S^n(q) & \longrightarrow & \Omega^n(S^n)^q \\ \downarrow & & \downarrow \\ \Omega^{n+1}S^{n+1}(q) & \longrightarrow & \Omega^{n+1}(S^{n+1})^q . \end{array} \tag{2-4}$$

Denote by $QS^0(q)$, the limit space $\varinjlim_n \Omega^n S^n(q)$ and by $Q(S^0)^q$, the limit space $\varinjlim_n \Omega^{n+1}(S^{n+1})^q$.

PROPOSITION 2-3. *For any subgroup $G$ of $\Sigma_q$, there exists a $G$ equivariant map $\theta_G: WG \to QS^0(q)$ which has the following properties.*

1) $\theta_G(WG^{(n-2)}) \subseteqq \Omega^n S^n(q)$.

2) $j_n \circ \theta_G|_{WG^{(n-2)}}: WG^{(n-2)} \to \Omega^n(S^n)^q$ *is $G$ equivariantly homotopic to the $G$ map whose adjoint map $WG^{(n-2)} \ltimes S^n \to (S^n)^q$ is $\Delta_q \circ \pi_2$, where $\pi_2: WG^{(n-2)} \times S^n \to S^n$, $\Delta_q: S^n \to (S^n)^q$.*

*And $G$ equivariant map $WG \to QS^0(q)$ which has the properties* 1) *and* 2), *(replaced $(n-2)$ by any number $k(n) \leqq n-2$, which has the property $k(n) \to \infty$ when $n \to \infty$) is unique up to $G$ equivariant homotopy.*

PROOF. This is the direct consequence of Lemma 2-2.

2-3. For each spectrum $X \in \mathcal{S}(R^\infty)$ we define a map

$$\theta_q:\ W\Sigma_q \times Q(X)^q \longrightarrow Q(X)$$

by the composition $W\Sigma_q \times Q(X)^q \xrightarrow{\theta_{\Sigma_q} \times id} QS^0(q) \times Q(X)^q \xrightarrow{c_q} Q(X)$ where the map $c_q$ is defined by the following way. Let $w \in QS^0(q)$, and $l_1, \cdots, l_q \in Q(X)$, then there is a large integer $N$, such that $w \in \Omega^N S^N(q)$, $l_i \in F(S^N, S_\alpha X(A))$ for some $\alpha: A \subset R^N \subset R^\infty$. Then $c_q(w, l_1, \cdots, l_q) \in F(S^N, S_\alpha X(A))$ represents the following map.

$$c_q(w\,;\,l_1,\cdots,l_q):\ S^N \xrightarrow{\ w\ } S^N\vee\cdots\vee S^N \xrightarrow{\ l_1\vee\cdots\vee l_q\ } S_\alpha X(A)\,.$$

This definition is compatible to the suspension map, and $c_q$ is well defined.

PROPOSITION 2-4. *The map $\theta_q$ has the following properties.*

1) *$\Sigma_q$ equivariant, i. e., $\theta_q(w, l_1, \cdots, l_q)=\theta_q(w\sigma, l_{\sigma^{-1}(1)}, \cdots, l_{\sigma^{-1}(q)})$ where $\sigma\in\Sigma_q$, $w\in W\Sigma_q$, $l_i\in Q(X)$.*

2) *Functorial for map $f: X\to Y$ in $\mathcal{S}(R^\infty)$, i. e., the following diagram is commutative.*

$$\text{(2-5)}\qquad \begin{array}{ccc} W\Sigma_q\times Q(X)^q & \xrightarrow{\ id\times Q(f)^q\ } & W\Sigma_q\times Q(Y)^q \\ \downarrow{\scriptstyle\theta_q} & & \downarrow{\scriptstyle\theta_q} \\ Q(X) & \xrightarrow{\ \ Q(f)\ \ } & Q(Y)\,. \end{array}$$

3) *For any $w\in W\Sigma_q$, $\theta_q(w\,;\ ): Q(X)^q\to Q(X)$ is homotopic to the q-th iterated loop product $\mu_q: Q(X)^q\to Q(X)$.*

4) *The following diagram is $\Sigma_p\times\Sigma_q$ equivariantly homotopy commutative, where $\mu_\vee: Q(X)^2\to Q(X)$ is a loop product.*

$$\text{(2-6)}\qquad \begin{array}{ccc} W\Sigma_p\times Q(X)^p\times W\Sigma_q\times Q(X)^q & \longrightarrow & W\Sigma_p\times W\Sigma_p\times Q(X)^p\times Q(X)^q \\ & & \downarrow \\ \Big\downarrow{\scriptstyle\theta_p\times\theta_q} & & W\Sigma_{p+q}\times Q(X)^{p+q} \\ & & \downarrow{\scriptstyle\theta_{p+q}} \\ Q(X)\times Q(X) & \xrightarrow{\ \ \mu_\vee\ \ } & Q(X)\,. \end{array}$$

5) *The following diagram is $\Sigma_p\int\Sigma_q$ equivariantly homotopy commutative.*

$$\text{(2-7)}\qquad \begin{array}{ccccc} W\Sigma_p\times(W\Sigma_q\times Q(X)^q)^p & & \xrightarrow{\ id\times(\theta_q)^p\ } & & W\Sigma_p\times Q(X)^p \\ \downarrow & & & & \downarrow{\scriptstyle\theta_p} \\ W\Sigma_p\times(W\Sigma_q)^p\times Q(X)^{pq} & \longrightarrow & W\Sigma_{pq}\times Q(X)^{pq} & \xrightarrow{\ \theta_{pq}\ } & Q(X)\,. \end{array}$$

PROOF. 1) follows from the following commutative diagram.

$$\begin{array}{ccccc} S^N & \xrightarrow{\ w\ } & S^N\vee\cdots\vee S^N & \xrightarrow{\ \ l_1\vee\cdots\vee l_q\ \ } & S_\alpha X(A) \\ & \searrow^{w\sigma} & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle id} \\ & & S^N\vee\cdots\vee S^N & \xrightarrow{\ \ l_\sigma^{-1}(1)\cdots l_\sigma^{-1}(q)\ \ } & S_\alpha X(A)\,. \end{array}$$

2) and 3) follow directly from the definition. Now fix an element $\mu_2\in QS^0(2)$ such that $\mu_2$ is the image of the element $\Omega^1S^1(2)$ defined by

$$\mu_2(t)=\begin{cases}(2t,1)\in S^2\vee S^1 & \text{if } 0\leqq t\leqq 1/2\\ (t-1,2)\in S^2\vee S^1 & \text{if } 1/2\leqq t\leqq 1\,.\end{cases}$$

And define $\theta_{\Sigma_p\times\Sigma_q}\colon W\Sigma_p\times W\Sigma_q\to QS^0(p+q)$ by the formula, $\theta_{\Sigma_p\times\Sigma_q}\colon W\Sigma_p\times W\Sigma_q \xrightarrow{\theta_{\Sigma_p}\times\theta_{\Sigma_q}} Q(S^0(p))\times Q(S^0(q)) \xrightarrow{Q(i_1)\times Q(i_2)} QS^0(p+q)\times QS^0(p+q) \xrightarrow{\mu_2\times id} QS^0(2)\times (QS^0(p+q))^2 \xrightarrow{c_2} QS^0(p+q)$. Then $\theta_{\Sigma_p\times\Sigma_q}$ is a $\Sigma_p\times\Sigma_q$ equivariant map: and define $\theta_{p,q}\colon (W\Sigma_p\times W\Sigma_q)\times(QX)^{p+q}\to Q(X)$ by the formula $(W\Sigma_p\times W\Sigma_q)\times Q(X)^{p+q} \xrightarrow{\theta_{\Sigma_p\times\Sigma_q}\times id} QS^0(p+q)\times Q(X)^{p+q} \xrightarrow{c_{p+q}} QX$. Then $\theta_{p,q}$ coincides to the map $\mu\circ(\theta_p\times\theta_q)$ if we define $\mu$ by means of $\mu_2$. On the other hand, we define $\Sigma_p\times\Sigma_q$ equivariant map $\theta'_{\Sigma_p\times\Sigma_q}\colon W\Sigma_p\times W\Sigma_q\to QS^0(p+q)$ by the composition map $W\Sigma_p\times W\Sigma_q\to W\Sigma_{p+q} \xrightarrow{\theta_{\Sigma_p\times\Sigma_q}} QS^0(p+q)$. And define $\theta'_{p,q}\colon W\Sigma_p\times W\Sigma_q\times Q(X)^{p+q}\to Q(X)$ by this $\theta'_{\Sigma_p\times\Sigma_q}$, then this map $\theta'_{p,q}$ coincides with another direction map of diagram (2-6). On the other hand by Proposition 2-3, $\theta_{p,q}$ and $\theta'_{p,q}$ are $\Sigma_p\times\Sigma_q$ equivariantly homotopic. So we get the $\Sigma_p\times\Sigma_q$ homotopy commutativity of diagram (2-6).

Proof of 5) is very similar to that of 4), so that we omit the proof.

For any $X$ and $Y\in\mathcal{S}(R^\infty)$, we define a map

(2-8) $$\mu\wedge\colon Q(X)\times Q(Y)\longrightarrow Q(X\wedge Y)$$

in the following way, where we fix an element $f\in LIE((R^\infty)^2, R^\infty)$, and $X\wedge Y=f_*(X\wedge Y)$. Let $l_1\in Q(X)$ and $l_2\in Q(Y)$, then they are represented by $l_1\in\mathcal{F}(S^{N_1}, S_\alpha X(A))$ and $l_2\in\mathcal{F}(S^{N_2}, S_\beta Y(B))$, where $\alpha\colon A\subset R^{N_1}$, $\beta\colon B\subset R^{N_2}$. Then $\mu\wedge(l_1,l_2)$ is represented by $l_1\wedge l_2\in\mathcal{F}(S^{N_1}\wedge S^{N_2}, (S_\alpha X\wedge S_\beta Y)(f(A\oplus B)))$. Then going to limit we get the map $\mu\wedge$.

REMARK. $\mu\wedge$ is functorial for $X$ and $Y$.

Let $id\in QS^0$ denote the element represented by identity map $id\colon S^N\to S^N$. And we define $h_q\colon B\Sigma_q\to QS^0$ by $W\Sigma_q/\Sigma_q=B\Sigma_p\to W\Sigma_q\times_{\Sigma_q}(id)^q\to W\Sigma_q\times Q(S^0)^q \xrightarrow{\theta_q} QS^0$.

PROPOSITION 2-5. *The following diagram is $\Sigma_q$ equivariantly homotopy commutative.*

(2-9) $$\begin{array}{ccc} W\Sigma_q\times Q(X) \xrightarrow{id\times\Delta_q} W\Sigma_q\times Q(X)^q & \xrightarrow{\theta_q} & Q(X)\\ \downarrow & & \uparrow\cong\\ B\Sigma_q\times Q(X) \xrightarrow[h_q\times id]{} QS^0\times QX & \xrightarrow[\mu\wedge]{} & Q(S^0\wedge X)\,.\end{array}$$

**2-4.** Let $X$ be a ring spectrum of $H^\infty$ type. For each positive integer $q$, we define a $\Sigma_q$ equivariant map

(2-10) $$\bar{\theta}_q\colon W\Sigma_q\times Q(X)^q\longrightarrow Q(X)\,.$$

At first we define a $\Sigma_q$ equivariant map

(2-11) $$\tilde{\theta}_q\colon W\Sigma_q\times Q(X)^q\longrightarrow Q(EP_q(X))\,.$$

Then the map $\bar{\theta}_q$ is by definition the composite map $Q(\mu_q)\circ\tilde{\theta}_q\colon W\Sigma_q\times Q(X)^q\to Q(EP_q(X))\to Q(X)$. Let $(w\,;l_1,\cdots,l_q)\in W\Sigma_q\times Q(X)^q$, we can assume $w\in W\Sigma_q^{(m)}$, $l_1,\cdots,l_q\in\mathcal{F}(S^N,S_\alpha)((A))$, where $\alpha\colon A\subset R^N$, and $m\leqq m(A^q)$. Then the element $\tilde{\theta}_q(w,l_1,\cdots,l_q)\in\mathcal{F}(\chi((R^N)^q)\cup\infty, T\xi_{R^N}|\!\!\times(S_\alpha X)\wedge\cdots\wedge(S_\alpha X))$ represents the following map.

$$\begin{array}{c}\chi((R^N)^q)\cup\infty\\ \big\downarrow w\times id\\ W\Sigma_q^{(m((R^N)^q))}|\!\!\times(\chi((R^N)^q\cup\infty))\\ \big\downarrow\cong\text{ by (1-7)}\\ T\xi_{R^N}|\!\!\times(S^N\wedge\cdots\wedge S^N)\\ \big\downarrow id|\!\!\times(l_1\wedge\cdots\wedge l_q)\\ T\xi_{R^N}|\!\!\times(S_\alpha X\wedge\cdots\wedge S_\alpha X)\,.\end{array}$$

And taking the limit we get the map $\tilde{\theta}_q$.

PROPOSITION 2-6. *The maps $\bar{\theta}_q$ have the following properties.*

1) *$\Sigma_q$ equivariant, i.e., $\theta_q(w\sigma\,;l_{\sigma^{-1}(1)},\cdots,l_{\sigma^{-1}(q)})=l(w\,;l_1,\cdots,l_q)$ for $w\in W\Sigma_q$, $l_i\in Q(X)$, $\sigma\in\Sigma_q$.*

2) *Functorial for a map $f\colon X\to Y$ of $H^\infty$ type, i.e., the following diagram is $\Sigma_q$ equivariantly homotopy commutative.*

(2-12) $$\begin{array}{ccc}W\Sigma_q\times Q(X)^q & \xrightarrow{id\times Q(f)^q} & W\Sigma_q\times Q(Y)^q\\ \big\downarrow\bar{\theta}_q & & \big\downarrow\bar{\theta}_q\\ Q(X) & \xrightarrow{Q(f)} & Q(Y)\,.\end{array}$$

3) *For any $w\in W\Sigma_q$, $\bar{\theta}_q(w\colon\ )\colon Q(X)^q\to Q(X)$ is homotopic to the map $Q(\mu_q)\cdot\mu_\wedge^q\colon Q(X)^q\to Q(X\wedge\cdots\wedge X)\to Q(X)$.*

4) *The following diagram is $\Sigma_p\times\Sigma_q$ equivariantly homotopy commutative.*

(2-13) $$\begin{array}{ccc}W\Sigma_p\times Q(X)^p\times W\Sigma_q\times Q(X)^q & \longrightarrow & W\Sigma_p\times W\Sigma_q\times Q(X)^{p+q}\\ \big\downarrow\bar{\theta}_p\times\bar{\theta}_q & & \big\downarrow\\ Q(X)\times Q(X) & & W\Sigma_{p+q}\times Q(X)^{p+q}\\ \big\downarrow\mu_\wedge & & \big\downarrow\bar{\theta}_{p+q}\\ Q(X\wedge X) & \xrightarrow{Q(\mu)} & Q(X)\,.\end{array}$$

5) *The following diagram is $\Sigma_p\textstyle\int\Sigma_q$ equivariantly homotopy commutative.*

(2-14)
$$\begin{array}{ccc}
W\Sigma_p\times(W\Sigma_q\times Q(X)^q)^p & \xrightarrow{\ id\times(\bar{\theta}_q)^p\ } & W\Sigma_p\times Q(X)^p \\
\downarrow & & \downarrow \bar{\theta}_p \\
W\Sigma_p\times(W\Sigma_q)^p\times Q(X)^{pq} \longrightarrow W\Sigma_{pq}\times Q(X)^{pq} & \xrightarrow{\ \bar{\theta}_{pq}\ } & Q(X).
\end{array}$$

6) *The following diagram is homotopy commutative.*

(2-15)
$$\begin{array}{ccccc}
QS^0\times QX & \xrightarrow{\ Q(i)\times id\ } & QX\times QX & \xrightarrow{\ \mu_\wedge\ } & Q(X\wedge X) \\
\downarrow \mu_\wedge & & \cong & & \downarrow Q(\mu) \\
Q(S^0\wedge X) & & \longrightarrow & & QX.
\end{array}$$

PROPOSITION 2-7. *The following diagram is $\Sigma_q$ equivariantly homotopy commutative.*

(2-16)
$$\begin{array}{ccc}
W\Sigma_q\times(QX)^q\times QY & \xrightarrow{\ id\times id\times\Delta_q\ } & W\Sigma_q\times(QX\times QY)^q \\
 & & \downarrow id\times(\mu_\wedge)^q \\
\downarrow \theta_q\times id & & W\Sigma_q\times Q(X\wedge Y)^q \\
 & & \downarrow \theta_q \\
QX\times QY & \xrightarrow{\ \mu_\wedge\ } & Q(X\wedge Y).
\end{array}$$

**2-5.** At the last of this section, we describe the relation between $\theta_p$ and $\bar{\theta}_q$. This relation is in some sense the distributive law. For each positive integer $p$ and $q$, a group homomorphism

(2-17)
$$\theta_\otimes :\ \Sigma_p\textstyle\int\Sigma_q \longrightarrow \Sigma_{q^p}$$

is defined as follows. Denote by $S_q$, *the set of $q$ elements*, $S_q=\{1, 2, \cdots, q : \bmod q\}$. Consider the set $S=S_q\times\cdots\times S_q$, the $p$-th direct product of $S_q$. Then the set $S$ consists of $q^p$ elements. We denote an element of $S$ by $J=(j_1, \cdots, j_p)$ where $j_i\in S_q$, $i=1, \cdots, p$. We consider $\Sigma_{q^p}$ as the permutation group of the set $S$. Now the group $\Sigma_p\int\Sigma_q$ operates on the set $S$ by the following.

(2-18)
$$\tilde{\sigma}((j_1, \cdots, j_p))=(\sigma_{\sigma^{-1}(1)}(j_{\sigma^{-1}(1)}), \cdots, \sigma_{\sigma^{-1}(p)}(j_{\sigma^{-1}(p)}))$$

where $\tilde{\sigma}=(\sigma\,;\sigma_1, \cdots, \sigma_p)\in\Sigma_p\int\Sigma_q$. This defines a homomorphism $\theta_\otimes :\Sigma_p\int\Sigma_q\to\Sigma_{q^p}$.

Now for each $J\in S$, take a copy $(QX)^p_J$ of $(QX)^p$ and consider the direct product $\prod_{J\in S}(QX)^p_J$. Then the group $\Sigma_p\times\Sigma_{q^p}$ operates on $\prod_{J\in S}(QX)^p_J$ by the following rule. $\Sigma_p$ operates on each $(QX)^p_J$ by the permutation of the factor, and $\Sigma_{q^p}$ operates on $\prod_{J\in S}(QX)^p_J$ by permuting the coordinates indexed by $J\in S$.

Consider the following homomorphism

$$\tilde{\theta}_{\otimes}: \Sigma_p \int \Sigma_q \longrightarrow \Sigma_p \times \Sigma_{qp} \tag{2-19}$$

defined by $\tilde{\theta}_{\otimes}((\sigma, \sigma_1, \cdots, \sigma_p)) = (\sigma, \theta_{\otimes}((\sigma, \sigma_1, \cdots, \sigma_p)))$.

We define *an equivariant map* $\square = \square_{p,q}$ over $\tilde{\theta}_{\otimes}: \Sigma_p \int \Sigma_q \to \Sigma_p \times \Sigma_{qp}$ by

$$\square : ((QX)^q)^p = \prod_{(i,j) \in S_p \times S_q} (QX)_{(i,j)} \longrightarrow \prod_{J \in S} (QX)_J^p \tag{2-20}$$

where $S_p = \{1, \cdots, p: \bmod p\}$, $S_q = \{1, \cdots, q \bmod q\}$, and for $X = \prod_{(i,j)} x_{(i,j)} \in \prod_{(i,j)} (QX)_{(i,j)}$ the $J$-th component $\square(X)_J$ of $\square(X)$ is $x_{(1,j_1)} \times \cdots \times x_{(p,j_p)}$ where $J = (j_1, \cdots, j_p)$.

And a map $\theta_{\otimes}: W\Sigma_p \times (W\Sigma_q)^p \to W\Sigma_{qp}$ is defined to be an equivariant map over $\theta_{\otimes}: \Sigma_p \int \Sigma_q \to \Sigma_{qp}$.

PROPOSITION 2-8. (The general distributive law.) *The following diagram is equivariantly homotopy commutative over*

$$\Sigma_p \int \Sigma_q \xrightarrow{\tilde{\theta}_{\otimes}} \Sigma_p \times \Sigma_{qp} \xrightarrow{\pi_2} \Sigma_{qp}.$$

$$\begin{array}{ccc} W\Sigma_p \times (W\Sigma_q \times (QX)^q)^p & \xrightarrow{id \times (\theta_q)^p} & W\Sigma_p \times (QX)^p \\ \downarrow & & \downarrow \tilde{\theta}_p \\ W\Sigma_p \times (W\Sigma_q)^p \times ((QX)^q)^p & & QX \\ \downarrow \Delta_{qp+1} \times id \times \square & & \uparrow \\ (W\Sigma_p)^{qp+1} \times (W\Sigma_q)^p \times \prod_{J \in S} (QX)_J^p & & \theta_{qp} \\ \downarrow & & | \\ W\Sigma_p \times (W\Sigma_q)^p \times \prod_{J \in S} (W\Sigma_p \times (QX)^p)_J & \xrightarrow{\theta_{\otimes} \times \prod(\tilde{\theta}_p)_J} & W\Sigma_{qp} \times \prod_{J \in S} (\theta X)_J . \end{array} \tag{2-21}$$

To prove this proposition, we prepare some more facts. At first we define

$$\varphi_S: W\Sigma_p \times (W\Sigma_q)^p \longrightarrow QS^0(S) \tag{2-22}$$

in the following way, where $S^0(S)$ is the one point union $\bigvee_{J \in S} S_J^0$, of the sphere spectra $S^0$ indexed by $S$. Let $(w; w_1, \cdots, w_p) \in W\Sigma_p \times (W\Sigma_q)^p$, then we can assume $w \in W\Sigma_p^{(n)}$, and $w_i \in W\Sigma_q^{(m)}$, $i = 1, \cdots, p$, for some $n$ and $m$. Then $\varphi_S$ $(w: w_1, \cdots, w_p)$ represents the following composite map in $\mathcal{F}(\chi((R^N)^p) \cup \infty, \bigvee_{J \in S} (\chi((R^N)^p) \cup \infty)_J)$.

$$
\begin{array}{l}
\chi((R^N)^p)\cup\infty \\
\quad\big\downarrow w\times id \\
W\Sigma_p^{(m((R^N)^p))}\ltimes(\chi((R^N)^p)\cup\infty) \\
\quad\big\downarrow \cong \text{ by (1-7)} \\
T(\xi_{R^N})\ltimes((R^N\cup\infty)\wedge\cdots\wedge(R^N\cup\infty))\cong T(\xi_{R^N})\ltimes(S^N\wedge\cdots\wedge S^N) \\
\quad\big\downarrow id\times(\varphi_q(w_1)\wedge\cdots\wedge\varphi_q(w_p)) \\
T(\xi_{R^N})\ltimes(S^N(q)\wedge\cdots\wedge S^N(q)) \\
\quad\big\downarrow \cong \\
T(\xi_{R^N})\ltimes(\bigvee_{J\in S} S_J^{pN}) \\
\quad\big\downarrow \cong \\
\bigvee_{J\in S} T(\xi_{R^N})\ltimes S_J^{pN} \\
\quad\big\downarrow \cong \text{ by (1-7)} \\
\bigvee_{J\in S}(W\Sigma_p^{(m((R^N)^p))}\ltimes(\chi((R^N)^p)\cup\infty))_J \\
\quad\big\downarrow \vee\pi_2 \\
\bigvee_J(\chi((R^N)^\infty)\cup\infty)\,.
\end{array}
$$

LEMMA 2-9. *The map $\varphi_S$ is equivariant map over $\theta_\otimes\colon \Sigma_p\int\Sigma_q\to\Sigma_{qp}$.*

Now using this $\varphi_S$ we define an equivariant map over $\theta_\otimes$

(2-24) $$\theta_S\colon W\Sigma_p\times(W\Sigma_q)^p\times\prod_{J\in S}Q(X)_J\longrightarrow Q(X)$$

by composition $W\Sigma_p\times(W\Sigma_q)^p\times\prod_{J\in S}Q(X)_J\xrightarrow{\varphi_S\times id}QS^0(S)\times\prod_{J\in S}Q(X)_J\xrightarrow{c_S}Q(X)$.

LEMMA 2-10. *The diagram (2-21) is equivariantly commutative over $\theta_\otimes$ if we replace the right hand side map $\theta_{qp}$ by $\theta_S$.*

PROOF. This is the direct consequence of definitions.

Now define an equivariant map $\varphi'_S$ over $\theta_\otimes$ by

(2-25) $$\varphi'_S\colon W\Sigma_p\times(W\Sigma_q)^p\xrightarrow{\theta_\otimes}W\Sigma_{qp}\xrightarrow{\theta_{qp}}QS^0(q^p)=QS^0(S)\,.$$

Now using this $\varphi'_S$ we define an equivariant map

(2-26) $$\theta'_S\colon W\Sigma_p\times(W\Sigma_q)^p\times\prod_{J\in S}Q(X)_J\longrightarrow Q(X)$$

by composition $W\Sigma_p\times(W\Sigma_q)^p\times\prod_{J\in S}Q(X)_J\xrightarrow{\varphi'_S\times id}QS^0(S)\times\prod_{J\in S}Q(X)_J\xrightarrow{c_S}Q(X)$.

LEMMA 2-11. *The maps $\varphi_S$ and $\varphi'_S$ are equivariantly homotopic over $\theta_\otimes\colon \Sigma_p\times(\Sigma_q)^p\to\Sigma_{qp}$.*

Proof of this lemma is given in (2-6).

PROOF OF PROPOSITION 2-8. By definition of $\varphi'_S$ and $\theta'_S$ the following diagram is equivariantly commutative over $\theta_{\otimes}$.

$$\text{(2-27)}\qquad \begin{array}{ccc} W\Sigma_p\times(W\Sigma_q)^p\times\prod_{J\in S}Q(X)_J & \xrightarrow{\theta'_S} & Q(X) \\ \downarrow \theta_{\otimes}\times id & & \uparrow \theta_{qp} \\ W\Sigma_{qp}\times\prod_{J\in S}Q(X)_J & \xrightarrow{\cong} & W\Sigma_{qp}\times(Q(X))^{qp}. \end{array}$$

On the other hand, by Lemma 2-11, $\theta_S$ and $\theta'_S$ are equivariantly homotopic over $\theta_{\otimes}$. By Lemma 2-10, Proposition holds if we replace $\theta_{qp}$ by $\theta_S$. So that Proposition holds by (2-27).

**2-6.** PROOF OF LEMMA 2-11. We prove this lemma by applying Proposition 2-3 considering $G=\Sigma_p\int\Sigma_q$ as a subgroup of $\Sigma_{qp}$. So that we must show that $\varphi_S$ and $\varphi'_S$ satisfy the weaker conditions 1) and 2). But $\varphi'_S$ satisfies 1) and 2) since $\theta_{qp}$ satisfies these. And $\varphi_S$ satisfies 1). So we show $\varphi_S$ satisfies 2). We take for each positive integer $N$, a pair of integers $(l_1, l_2)$ satisfying $m(\chi_p((R^{l_1})^p))\leqq N$, $l_2\leqq l_1-2$, and $l_1\to\infty$, $l_2\to\infty$ when $N\to\infty$. For each $N$, consider $\varphi_q\colon W\Sigma_q^{(l_2)}\to\mathcal{F}(S^N, S^N(q))$ defined in Proposition 2-3. And take a $\Sigma_q$ equivariant homotopy $\Phi_q\colon I\times W\Sigma_q^{(l_2)}\to\mathcal{F}(S^N, (S^N)^q)$ combining $j_N\circ\theta_q$ and constant map on $\Delta_q\colon S^N\to(S^N)^q$. Consider the following diagram,

$$\text{(2-28)}\qquad \begin{array}{ccc} S^N(q)\wedge\cdots\wedge S^N(q) & \xrightarrow{j_N\wedge\cdots\wedge j_N} & (S^N)^q\wedge\cdots\wedge(S^N)^q \\ \downarrow\cong & & \downarrow k \\ \bigvee_{J\in S}S_J^{pN} & \xrightarrow[j]{} & \prod_{J\in S}S_J^{pN} \end{array}$$

where $k$ is defined by the formula: the $J$-th coordinate of $k((x_{11}\times\cdots\times x_{1q})\wedge\cdots\wedge(x_{p1}\times\cdots\times x_{pq}))$ is $(x_{(1j_1)}\wedge\cdots\wedge x_{(pj_p)})$ where $J=(j_1,\cdots,j_p)\in S$. Then the above diagram is a map over $\theta_{\otimes}$. Next consider the following diagram from (2-23),

(2-29)
$$
\begin{array}{ccc}
\chi((R^N)^p)\cup_\infty & & \\
\Big\downarrow {\scriptstyle (\cong)\ (w\times id)} & & \\
T(\xi_{R^N})\ltimes(S^N\wedge\cdots\wedge S^N) & \xrightarrow{\quad id\times(\Delta_q\wedge\cdots\wedge\Delta_q)\quad} & T(\xi_{R^N})\ltimes((S^N)^q\wedge\cdots\wedge(S^N)^q) \\
\Big\downarrow {\scriptstyle id\times(\theta_q(w_1)\wedge\cdots\wedge\theta_q(w_p))} & & \\
T(\xi_{R^N})\ltimes(S^N(q)\wedge\cdots\wedge S^N(q)) & \xrightarrow{id\times(j_N\wedge\cdots\wedge j_N)} & T(\xi_{R^N})\ltimes((S^N)^q\wedge\cdots\wedge(S^N)^q) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle id\times R} \\
T(\xi_{R^N})\ltimes(\bigvee_{J\in S} S_J^{p\,N}) & \xrightarrow{id\times j_N} & T(\xi_{R^N})\times(\prod_{J\in S} S_J^{p\,N}) \\
\Big\downarrow {\scriptstyle \cong} & & \Big\downarrow \\
\bigvee_{J\in S}(T(\xi_{R^N})\ltimes S^{p\,N})_J & \xrightarrow{\Delta_S\times j_N} & \prod_{J\in S}(T(\xi_{R^N})\times S^{p\,N})_J \\
\Big\downarrow {\scriptstyle \cong\ \text{by (1-7)}} & & \Big\downarrow {\scriptstyle \cong\ \text{by (1-7)}} \\
\bigvee_{J\in S}(W\Sigma_p^{(m((R^N)^p))}\ltimes(\chi((R^N)^p)\cup_\infty)_J & \longrightarrow & \prod_{J\in S}(W\Sigma_p^{(m((R^N)^p))}\times(\chi((R^N)^p\cup_\infty))_J \\
\Big\downarrow {\scriptstyle \vee\pi_2} & & \Big\downarrow {\scriptstyle \Pi\pi_2} \\
\bigvee_{J\in S}(\chi((R^N)^p)\cup_\infty) & \xrightarrow[j]{} & \prod_{J\in S}(\chi((R^N)^p)\cup_\infty)_J .
\end{array}
$$

And right hand composite map also defines a map $\Phi: W\Sigma_p^{(l_1)}\times(W\Sigma_q^{(l_2)})^p\rightarrow \mathcal{F}((\chi((R^N)^p)\cup\infty), \prod_{J\in S}(\chi((R^N)^p)\cup\infty)_J)$, which is equivariant over $\theta_\otimes$ and homotopy $\Phi_q$ defines a $\Sigma_p\int\Sigma_q$ equivariant homotopy between $j\circ\varphi_S$ and $\phi$. On the other hand it is easy to see that $\phi$ is a constant map to the diagonal $\Delta_S: \chi((R^N)^p)\cup_\infty\rightarrow\prod_{J\in S}(\chi((R^N)^p)\cup_\infty)_J$. This proves the lemma.

## § 3. The Dyer-Lashof Operations defined by $\theta_q$ and $\bar\theta_q$.

**3-1.** For iterated loop spaces, Kudo-Araki [2], and Dyer-Lashof [5] defined homology operations and their properties are investigated by various authors.

We define Dyer-Lashof operations on $Q(X)$, and describe their properties without proof, since proofs are essentially the same as that of previous authors.

Let $p$ denote a prime number and $Z_p=Z/pZ$. And $\pi_p$ denote the cyclic group of order $p$ and we consider $\pi_p$ as a subgroup of $\Sigma_p$.

For $x\in H_n(QX: Z_p)$, we define $Q_i(x)=\theta_{p*}\circ(i\times id)_*(e_i\underset{\pi}{\otimes}x^p)$ where $e_i\underset{\pi}{\otimes}x^p \in H_{pn+i}(W\pi_p\times_{\pi_p}(QX)^p: Z_p)$, $i=0, 1, 2, \cdots$. Then if $\deg x=n$ is even, $Q_i(x)=0$ unless $i=0$ or $-1 \bmod 2(p-1)$, and if $\deg x=n$ is odd, $Q_i(x)=0$ unless $i\equiv p-1$ or $p-2 \bmod 2(p-1)$. And $Q_{2i(p-1)-1}(x)=\beta Q_{2i(p-1)}(x)$ if $\deg x=n$ is even and $Q_{(2i+1)(p-1)-1}(x)=\beta Q_{(2i+1)(p-1)}(x)$ if $\deg x=n$ is odd.

We define

(3-1) $$Q^j(x)=(-1)^{j+m(n^2-n)/2}(m\,!)^n Q_{(2j-n)(p-1)}(x)$$

if $p$ is odd prime, where $m=\dfrac{p-1}{2}$, $\deg x=n$,

$$Q^j(x)=Q_{j-n}(x)$$

if $p$ is 2; where $n=\deg x$.

We denote Pontrjagin product on $H_*(QX; Z_p)$ by $*$. And define the pairing

(3-2) $$H_*(QS^0: Z_p)\otimes H_*(QX: Z_p)\longrightarrow H_*(QX: Z_p)$$

by $H_*(QS^0: Z_p)\otimes H_*(QX: Z_p)\xrightarrow{\cong} H_*(QS^0\times QX: Z_p)\xrightarrow{\mu_\wedge} H_*(Q(S^0\wedge X: Z_p))\cong H_*(QX: Z_p)$. And we denote this pairing by $b\circ x$ for $b\in H_*(QS^0: Z_p)$ and $x\in H_*(QX: Z_p)$.

PROPOSITION 3-1. *The following properties hold.*

1) *$Q^i$ is natural for a map $Q(f)$ where $f: X\to Y$.*

2) *$Q^i$ is an abelian group homomorphism.*

3) *If $p\geqq 3$ then*

$$Q^i(x)=0 \text{ if } \deg x>2i, \text{ and } Q^i(x)=x^{[p]} \text{ if } \deg x=2i.$$

*If $p=2$ then*

$$Q^i(x)=0 \text{ if } \deg x>i, \text{ and } Q^i(x)=x^{[2]} \text{ if } \deg x=i,$$

*where $x^{[j]}=x*\cdots*x$, the j-th Pontrjagin product.*

4) *Cartan formula.*

$$Q^i(x*y)=\sum_{i_1+i_2=i} Q^{i_1}(x)*Q^{i_2}(y),$$

$$\Delta Q^i(x)=\sum_{i_1+i_2=i}\sum Q^{i_1}(x')\otimes Q^{i_2}(x'') \quad \text{where } \Delta x=\Sigma x'\otimes x''.$$

5) *Adem relations.*

*If $j>pi$*

$$Q^jQ^i=\sum_k(-1)^{j+k}\binom{(k-i)(p-1)-1}{pk-j}Q^{j+i-k}Q^k,$$

*if $p\geqq 3$ and $j\geqq pi$*

$$Q^j\beta Q^i=\sum_k(-1)^{j+k}\binom{(k-i)(p-1)}{pk-j}\beta Q^{j+i-k}Q^k$$

$$-\sum_k(-1)^{j+k}\binom{(k-i)(p-1)-1}{pk-j-1}Q^{j+i-k}\beta Q^k.$$

6) *Nishida relations. Let $P_*^k$ denote the dual of Steenrod reduced power $P^k$, (if $p=2$ we put $P_*^k=S_{q*}^k$).*

$$P_*^sQ^r=\sum_i(-1)^{s+i}\binom{(r-s)(p-1)}{s-p^i}Q^{r-s+i}P_*^i,$$

*and if $p\geqq 3$*

$$P_*^s\beta Q^r=\sum_i(-1)^{s+i}\binom{(r-s)(p-1)-1}{s-p^i}\beta Q^{r-s+i}P_*^i$$

$$-\sum_i(-1)^{s+i}\binom{(r-s)(p-1)-1}{s-p^i-1}Q^{r-s+i}\beta P_*^i.$$

Now consider the pairing $H_*(QS^0: Z_p)\otimes H_*(QX: Z_p)\rightarrow H_*(QX: Z_p)$.

PROPOSITION 3-2. *The following distributive law holds.*

1) $(b_1*b_2)\circ x=\sum(-1)^{\deg b_2\deg x'}(b_1\circ x')*(b_2\circ x'')$.

*If, $b_1$, $b_2\in H_*(WS^0: Z_p)$, $x\in H_*(QX: Z_p)$ and $\Delta x=\sum x'\otimes x''$.*

2) $b\circ(x_1*x_2)=\sum(-1)^{\deg b''\deg x_1}(b'\circ x_1)*(b''\circ x_2)$.

*If $b\in H_*(QS^0: Z_p)$, $x_1$, $x_2\in H_*(QX: Z_p)$ and $\Delta b=\sum b'\otimes b''$.*

Let $Q_iS^0$ denote the connected component of $QS^0$ which consists of degree $i$ maps, and $[i]\in H_0(Q_iS^0: Z_p)$ denote the identity element.

PROPOSITION 3-3. *The following relations hold for $x\in H_*(QX: Z_p)$.*

1) $Q^i[1]\circ x=\sum_l Q^{k+l}(P_*^l(x))$,

2) $Q^k(x)=\sum_l Q^{k+l}[1]\circ(c(P^l)_*(x))$,

*where $c(P^l)$ denotes the conjugate of $P^l$ in the Steenrod algebra $A_p$. (If $p=2$, $P^l=S_q^l$.)*

PROOF. By Proposition 2-6, the following diagram is commutative.

$$\begin{array}{ccccc} H_*(W\pi_p/\pi_p\times QX: Z_p) & \xrightarrow{id\times\Delta_p} & H_*(W\pi_p\times_{\pi_p}(QX)^p: Z_p) & \longrightarrow & H_*(W\Sigma_p\times_{\Sigma_p}(QX)^p: Z_p) \\ \downarrow h_p\times id & & & & \downarrow \theta_p \\ H_*(QS^0\times QX: Z_p) & & \xrightarrow{\quad\mu_\wedge\quad} & & H_*(QX: Z_p). \end{array}$$

On the other hand $h_p(e_{2k(p-1)})=(-1)^kQ^k([1])$, and

$$(id\times\Delta_p)(e_{2k(p-1)}\otimes x)=\nu(q)\sum_l(-1)^le_{(2k+2pl-q)(p-1)}\otimes_\pi(P_*^l(x))^p$$

$$-\nu(q-1)\sum_l(-1)^le_{(2k+2pl-q)(p-1)+p}\otimes_\pi(P_*^l\beta x)^p.$$

If $p=2$, $(id\times\Delta_2)(e_k\otimes x)=\sum_l e_{(k+2l-q)}\otimes(P_*^l(x))^2$, where $\nu(q)=(m\,!)^q(-1)^{m(q^2-q)/2}$, and $m=(p-1)/2$, $q=\deg x$. By definition of $Q^j(x)$, we get the formula 1). And $\sum_l Q^{k+l}[1]\circ c(P^l)_*(x)=\sum_l\sum_m Q^{k+l+m}(P_*^m c(P^l)_*(x))=Q^k(x)$. So we get formula 2).

**3-2.** Let $X$ be a ring spectrum of $H^\infty$ type. Then $H_*(QX\colon Z_p)$ has product other than Pontrjagin product. Define product on $H_*(QX\colon Z_p)$ by

$$(3\text{-}3)\qquad H_*(QX\colon Z_p)\otimes H_*(QX\colon Z_p)\longrightarrow H_*(QX\colon Z_p)\colon x\otimes y\longrightarrow x\circ y$$

by $H_*(QX\colon Z_p)\otimes H_*(QX\colon Z_p)\to H_*(QX\times QX\colon Z_p)\xrightarrow{Q(\mu)} H_*(QX\colon Z_p)$.

PROPOSITION 3-4. *The following distributive law holds.*

1) $(x_1*x_2)\circ y=\sum(-1)^{\deg x_2\cdot\deg y'}(x_1\circ y')*(x_2\circ y'')$

*where* $x_1, x_2, y\in H_*(QX\colon Z_p)$, $\Delta y=\sum y'\otimes y''$.

2) $x\circ(y_1*y_2)=\sum(-1)^{\deg y_1\cdot\deg x''}(x'\circ y_1)*(x''\circ y_2)$

*where* $x, y_1, y_2\in H_*(QX\colon Z_p)$, $\Delta x=\sum x'\otimes x''$.

Now we define the operations on $H_*(QX\colon Z_p)$ by

$$(3\text{-}4)\qquad \bar{Q}^j\colon H_*(QX\colon Z_p)\longrightarrow H_{*+2j(p-1)}(QX\colon Z_p)\,,\qquad j=0,1,2,\cdots$$

by $\bar{Q}^j(x)=(-1)^{j+m(q^2-q)/2}(m\,!)^q\bar{\theta}_{p*}(i\times id)_*(e_{(2j-q)(p-1)}\otimes x^p)$, where $e_{(2j-q)(p-1)}\otimes x^p\in H_*(W\pi_p\times_{\pi_p}(QX)^p\colon Z_p)$, $\deg x=q$, $m=(p-1)/2$, for $p>2$, and $\bar{Q}^j(x)=\bar{\theta}_{2*}(e_{j-q}\otimes x^2)$ for $p=2$.

PROPOSITION 3-5. *The following relations hold.*

1) $\bar{Q}^j$ *is natural for* $Q(f)$ *where* $f\colon X\to Y$ *is a map of ring spectra of* $H^\infty$ *type.*

2) $\bar{Q}$ *is an abelian group homomorphism.*

3) *If* $p\geqq 3$ *then*

$$\bar{Q}^i(x)=0 \text{ if } \deg x>2i, \text{ and } \bar{Q}^i(x)=x\circ\cdots\circ x=x^p\,,$$

*the p-th power in join product, if* $\deg x=2i$.

*If* $p=2$ *then*

$$\bar{Q}^i(x)=0 \text{ if } \deg x>i \text{ and } \bar{Q}^i(x)=x\circ x=x^2, \text{ if } \deg x=i\,.$$

4) *Cartan formula.*

$$\bar{Q}^i(x\circ y)=\sum_{i_1+i_2=i}\bar{Q}^{i_1}(x)\circ\bar{Q}^{i_2}(y)\,.$$

$$\Delta\bar{Q}^i(x)=\sum_{i_1+i_2=i}\sum\bar{Q}^{i_1}(x')\otimes\bar{Q}^{i_2}(x'')\qquad\text{where } \Delta x=\sum x'\otimes x''\,.$$

5) *Adem relations.*

*If* $j>pi$, *then*

$$\bar{Q}^j\bar{Q}^i=\sum_k(-1)^{j+k}\binom{(k-i)(p-1)-1}{pk-j}\bar{Q}^{j+i-k}\bar{Q}^k\,.$$

*If $p\geqq 3$ and $j\geqq pi$ then*

$$\bar{Q}^j\beta\bar{Q}^i=\sum_k(-1)^{j+k}\binom{(k-i)(p-1)}{pk-j}\beta\bar{Q}^{j+i-k}\bar{Q}^k$$

$$-\sum_k(-1)^{j+k}\binom{(k-i)(p-1)-1}{pk-j-1}\bar{Q}^{j+i-k}\beta\bar{Q}^k.$$

6) *Nishida relations.*

$$P_*^s\bar{Q}^r=\sum_j(-1)^{i+s}\binom{(r-s)(p-1)}{s-p^i}\bar{Q}^{r-s+i}P_*^i.$$

*If $p\geqq 3$*

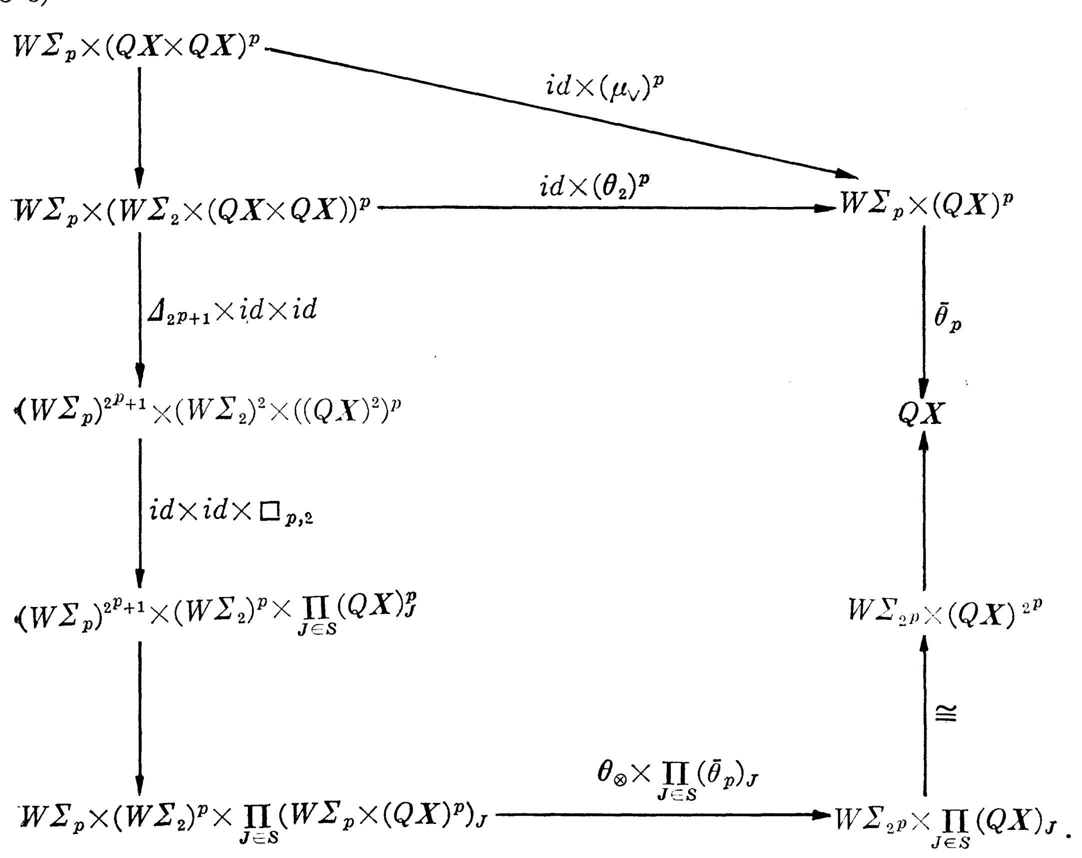
$$P_*^s\beta\bar{Q}^r=\sum_i(-1)^{i+s}\binom{(r-s)(p-1)}{s-p^i}\beta\bar{Q}^{r-s+i}P_*^i$$

$$-\sum_i(-1)^{i+s}\binom{(r-s)(p-1)-1}{s-p^i-1}\bar{Q}^{r-s+i}\beta P_*^i.$$

**3-3.** Now we shall investigate the relation between loop product $*$ and operations $\bar{Q}^j$. In § 5 of [14], I considered the primitive form of this section, and later Ib. Madsen [7], and P. May [8] got the mixed Cartan formula relating these. Our formulas are slight extensions of their results.

Consider the diagram (2-21) in the case of $q=2$.

(3-5)



Consider the set $S_{2p}=(S_2)^p=S_2\times\cdots\times S_2$, $S_2=\{1 \text{ or } 2 \bmod 2\}$, and for each integer $i$, $0\leqq i\leqq p$, consider the subset $S[i]$ of $S_{2p}$ defined by $S[i]=\{J=(j_1,\cdots,j_p)\in(S_2)^p\,;\ \#\{j_i\,;\, j_i=1\}=i\}$. Then the symmetric group $\Sigma_p$ operates on $S[i]$ by permuting the coordinates.

Now define operations $\bar{Q}_i^j$, $i=0,\cdots,p\,;\ j=0,1,2,\cdots$ by

(3-6) $$\bar{Q}_i^j\colon H_{q_1}(QX\colon Z_p)\otimes H_{q_2}(QX\colon Z_p)\longrightarrow H_{q_1+q_2+2j(p-1)}(QX\colon Z_p)$$

by the formula. $x\in H_{q_1}(QX\colon Z_p)$, $y\in H_{q_2}(QX\colon Z_p)$, $q=q_1+q_2$.

(3-7) $$\bar{Q}_i^j(x\otimes Y)=(-1)^{j+(1/2)m(q^2-q)}(m\,!)^q\theta_{\otimes,i}\circ(i\times(id)^p)\circ(e_{(2j-q)(p-1)}\otimes(x\otimes y)^p)$$

if $p\geqq 3$,

$$\bar{Q}_i^j(x\otimes y)=\theta_{\otimes,i}\circ(e_{(j-q)(p-1)}\otimes(x\otimes y)^2)$$

if $p=2$, where $\theta_{\otimes,i}\colon W\Sigma_p\times(QX\times QX)^p\to QX$ is defined by

$$\begin{array}{ccc}
W\Sigma_p\times(QX\times QX)^p & \xrightarrow{\qquad\theta_{\otimes,i}\qquad} & QX\\
\Big\downarrow{\scriptstyle \Delta_{m_i+1}\times\square_i} & & \Big\uparrow{\scriptstyle \theta_{\binom{p}{i}}}\\
W\Sigma_p^{m_i+1}\times\prod_{J\in S[i]}(QX)_J^p & & \\
\Big\downarrow & & \\
W\Sigma_p\times\prod_{J\in S[i]}(W\Sigma_p\times(QX)^p)_J & \xrightarrow{\theta_{\otimes,i}\times id} & W\Sigma_{\binom{p}{i}}\times\prod_{J\in S[i]}(QX)_J
\end{array}$$

where $\square_i\colon(QX\times QX)^p\to\prod_{J\in S[i]}(QX)_J^p$ is defined by

$$\square_i((x_{1,1},x_{1,2})\times\cdots\times(x_{p,1},x_{p,2}))=\prod_{J\in S[i]}(x_{(1,j_1)}\times\cdots\times x_{(p,j_p)})_J=(j_1,\cdots,j_p)\,.$$

PROPOSITION 3-7. *Mixed Cartan formula.*

$$\bar{Q}^j(x*y)=\sum_{j_0+\cdots+j_p=j}\sum\bar{Q}_0^{j_0}(x_0\otimes y_0)*\cdots*\bar{Q}_p^{j_p}(x_p\otimes y_p)$$

*where* $\Delta_{p+1}(x\otimes y)=\sum(x_0\otimes y_0)\otimes\cdots\otimes(x_p\otimes y_p)$.

PROOF. This follows from the following $\Sigma_p$ equivariant homotopy commutative diagram.

$$\begin{array}{ccc}
W\Sigma_p\times(QX\times QX)^p & \xrightarrow{\quad id\times(\mu_\vee)^p\quad} & W\Sigma_p\times(QX)^p\\
\Big\downarrow{\scriptstyle \Delta_{p+1}\times\Delta_{p+1}} & & \Big\downarrow{\scriptstyle \bar{\theta}_p}\\
(W\Sigma_p)^{p+1}\times(QX\times QX)^{p+1} & & QX\\
\Big\downarrow & & \Big\uparrow{\scriptstyle \mu_\vee}\\
(W\Sigma_p\times(QX\times QX))^{p+1} & \xrightarrow{\quad\prod_{j=0}^{p}(\theta_{\otimes,i})\quad} & (QX)^{p+1}\,.
\end{array}$$

And homotopy commutativity follows easily from the homotopy commutativity of the diagram (3-5).

PROPOSITION 3-8.

1) $\bar{Q}_0^j(x\otimes y)=\bar{Q}^j(\varepsilon(x)\cdot y)$

2) $\bar{Q}_p^j(x\otimes y)=\bar{Q}^j(x\cdot\varepsilon(y))$

*where $\varepsilon: H_*(QX: Z_p)\to Z_p$ denotes the augmentation.*

PROOF. This follows easily from definition.

PROPOSITION 3-9. *For $0<i<p$, $m_i=\frac{1}{p}\binom{p}{i}$ then*

$$\bar{Q}_i^j(x\otimes y)=[m_i]\circ(\sum Q^j(x_1\circ\cdots\circ x_i\circ y_1\circ\cdots\circ y_{p-i}))$$

*where $\Delta_i x=\sum x_1\otimes\cdots\otimes x_i$, $\Delta_{p-i}y=\sum y_1\otimes\cdots\otimes y_{p-i}$.*

PROOF. Let $t\in\Sigma_p$ denote the element $\begin{pmatrix}1, & \cdots, & p-1, & p\\ 2, & \cdots, & p, & 2\end{pmatrix}$, and $\pi_p$ denote the cyclic group generated by $t$, and $S_0[i]\subset S[i]$ denote a fixed complete representative set of orbit set $S^{[i]}/\pi_p$. At first consider the following $\Sigma_p$ equivariant map $F$.

$$\begin{array}{ccccc}
W\Sigma_p\times(QX\times QX) & \xrightarrow{id\times\Delta_p} & W\Sigma_p\times(QX\times QX)^p & \xrightarrow{id\times\square_t} & W\Sigma_p\times\prod\limits_{J\in S[i]}((QX)^p)_J\\
 & \searrow^{F} & & & \downarrow\\
 & W\Sigma_p\times\prod\limits_{J\in S[i]}(QX)_J & \xleftarrow{id\times\prod\bar{\theta}_p} & & W\Sigma_p\times\prod\limits_{J\in S[i]}(W\Sigma_p\times(QX)^p)_J .
\end{array}$$

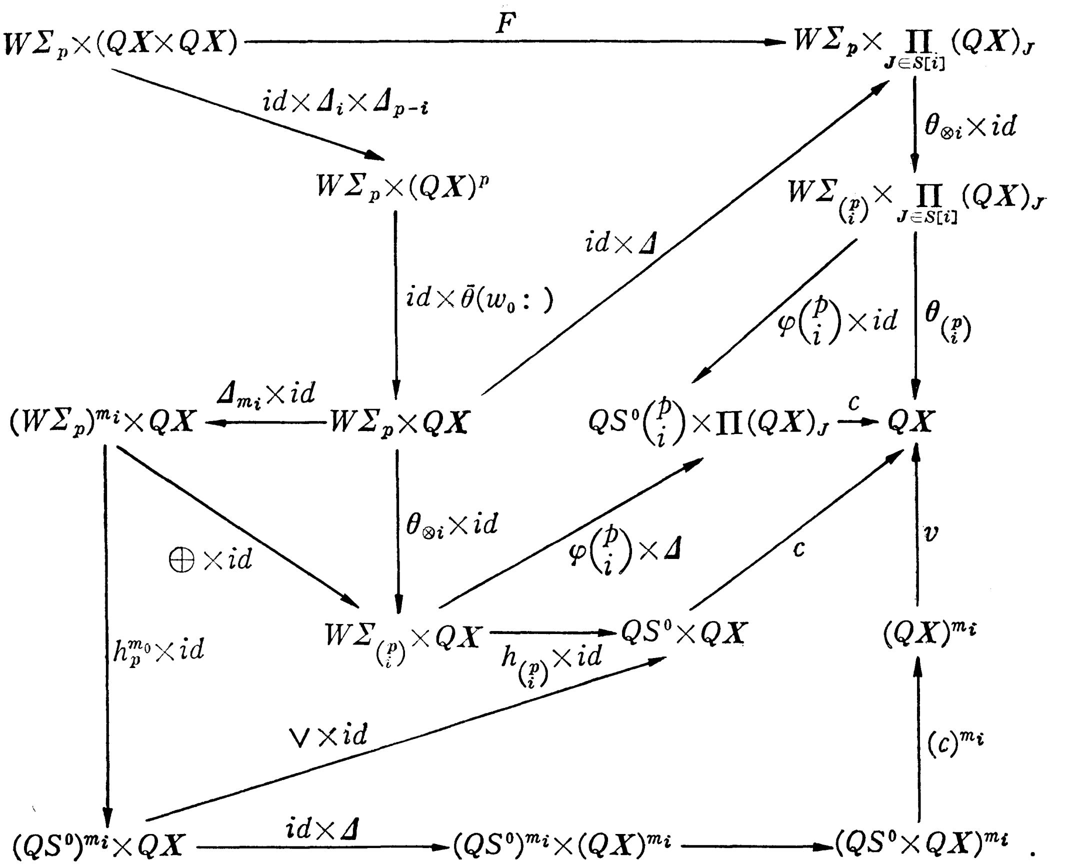
For each $J\in S_0[i]$, fix an element $\sigma(J)\in\Sigma_p$ with $\sigma(J)^{-1}((1,\cdots,1,2,\cdots,2))=(j_1,\cdots,j_p)=J$. Then we have

$$\begin{aligned}F(w,(x,y))&=(w;\prod_{J\in S_0[i]}\prod_{j=0}^{p-1}[\bar{\theta}_p(w,(t^{-j}\sigma(J))((x,\cdots,x,y,\cdots,y)))]_{(t^jJ)})\\
&=(w,\prod_{J\in S_0[i]}\prod_{j=0}^{p-1}[\bar{\theta}_p(wt^{-j}\sigma(J),(x,\cdots,x,y,\cdots,y))]_{(t^jJ)}).\end{aligned}$$

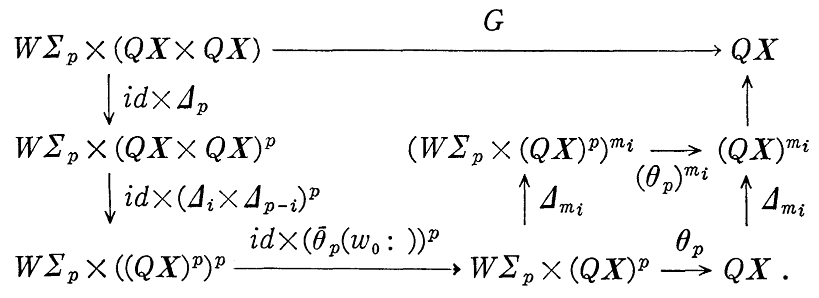
Now consider the set $A=W\Sigma_p\times\prod_{J\in S_0[i]}\prod_{j=0}^{p-1}(W\Sigma_p)_{(t^jJ)}$, and $\pi_p$ operates on this set by $t(w,\prod_J\prod_{j=0}^{p-1}w_{(t^jJ)})=(wt,\prod_J\prod_{j=0}^{p-1}(w_{(t^{j-1}J)})_{(t^jJ)})$. Then the map $W\Sigma_p\to A$ defined by $w\mapsto(w,\prod_J\prod_{j=0}^{p-1}(wt^{-j}\sigma(J))_{(t^jJ)})$ is $\pi_p$ equivariant map, and this map is $\pi_p$ equivariantly homotopic to the $\pi_p$ map $W\Sigma_p\to A$ defined by $w\mapsto(w,\prod_J\prod_{j=0}^{p-1}(w_0)_{(t^jJ)})$, where $w_0\in W\Sigma_p$ is a fixed element. So that $F$ is $\pi_p$ equivariantly homotopic to the following $\pi_p$ map $F_0$.

$$\begin{array}{ccc}
W\Sigma_p\times(QX\times QX) & \xrightarrow{id\times(\Delta_i\times\Delta_{p-i})} & W\Sigma_p\times(QX)^i\times(QX)^{p-i}\\
\downarrow F_0 & & \downarrow id\times\bar{\theta}(w_0:\ )\\
W\Sigma_p\times\prod\limits_{J\in S[i]}(QX)_J & \xleftarrow{id\times\Delta_{\binom{p}{i}}} & W\Sigma_p\times QX.
\end{array}$$

Then consider the following $\pi_p$ equivariantly homotopy commutative diagram




Denote by $G$, the left hand side map in the above diagram, $G\colon W\Sigma_p\times(QX\times QX)\to QX$. Then the following diagrams are $\Sigma_p$ equivariantly homotopy commutative.




This shows the Proposition.

**3-4.** Now we shall investigate the relations between $\beta^\varepsilon Q^i$ and $\beta^{\varepsilon'}\bar{Q}^j$.

By Proposition 3-3 and Cartan formula

(3-8) a) $\bar{Q}^iQ^j(x)=\bar{Q}^i(\sum_l Q^{j+l}[1]\circ c(P^l)_*(x))$

$$=\sum_{i_1+i_2=i}\sum_l \bar{Q}^{i_1}(Q^{j+l}[1])\circ\bar{Q}^{i_2}(c(P^l)_*(x))$$

and for $p \geqq 3$

$$\text{b)}\quad \bar{Q}^i\beta Q^j(x)=\bar{Q}^i(\sum_l (Q^{j+l}[1]\circ(\beta c(P^l)(x))+\beta Q^{j+l}[1]\circ c(P^l(x))))$$

$$=\sum_{i_1+i_2=i}\sum_l (\bar{Q}^{i_1}Q^{j+l}[1]\circ\bar{Q}^{i_2}\beta c(P^l)(x)+\bar{Q}^{i_1}\beta Q^{j+l}[1]\circ\bar{Q}^{i_2}c(P^l)(x)).$$

Now define an operation $\bar{Q}^j[1]\circ$, $j=0, 1, 2, \cdots$

(3-9) $$\bar{Q}^j[1]\circ:\ H_*(QX: Z_p)\longrightarrow H_{*+2j(p-1)}(QX: Z_p)$$

if $p\geqq 3$

$$\bar{Q}^j[1]\circ:\ H_*(QX: Z_2)\longrightarrow H_{*+j}(QX: Z_2)$$

if $p=2$ by

$$\bar{Q}^j[1]\circ x=\sum_l \bar{Q}^{j+l}(P^l_*(x))\qquad (p=2,\ P^l=S^l_q).$$

PROPOSITION 3-10.

$$\bar{Q}^j(x)=\sum_l \bar{Q}^{j+l}[1]\circ c(P^l)_*(x).$$

PROOF.

$$\sum_l \bar{Q}^{j+l}[1]\circ c(P^l)(x)=\sum_{l,k}\bar{Q}^{j+l+k}(P^k c(P^l)(x))$$

$$=\bar{Q}^j(x).$$

PROPOSITION 3-11. *In $H_*(QS^0: Z_p)$ we have*

1) $$\bar{Q}^l[1]\circ Q^n[1]=\sum_{\substack{l_1+\cdots+l_p+l_{p+1}(p-1)=l(p-1)\\ n_1+\cdots+n_{p+1}=l+n}} 0^{l_1}\cdots(p-1)^{l_p}\binom{n_1(p-1)}{l_1}\cdots\binom{n_p(p-1)}{l_p}$$

$$Q^{n_1}[1]*\cdots*Q^{n_p}[1]*([p^2-1]\circ Q^{l_{p+1}}[1]\circ Q^{n_{p+1}-l_{p+1}}[1]).$$

2) $\bar{Q}^l[1]\circ\beta Q^n[1]$

$$=\sum_{\substack{l_1+\cdots+l_p+l_{p+1}(p-1)=l(p-1)\\ n_1+\cdots+n_{p+1}=l+n\\ \varepsilon_1+\cdots+\varepsilon_{p+1}=1}} 0^{l_1}\cdots(p-1)^{l_p}\binom{n_1(p-1)-\varepsilon_1}{l_1}\cdots\binom{n_p(p-1)-\varepsilon_p}{l_p}$$

$$\beta^{\varepsilon_1}Q^{n_1}[1]*\cdots*\beta^{\varepsilon_p}Q^{n_p}[1]*([p^2-1]\circ Q^{l_{p+1}}[1]\circ\beta^{\varepsilon_{p+1}}Q^{n_{p+1}-l_{p+1}}[1]).$$

*For $p\geqq 3$.*

PROOF. Consider the following diagram

$$\begin{array}{ccc} H_*(W\pi_p/\pi_p\times\cdots\times W\pi_p/\pi_p: Z_p) & \xrightarrow{id\times\Delta_p} & H_*(W\pi_p\times_{\pi_p}(W\pi_p/\pi_p)^p: Z_p)\\ \downarrow G & & \downarrow i\times(h_p)^p\\ H_*(QS^0: Z_p) & \xleftarrow{\bar{\theta}_p} & H_*(W\Sigma_p\times_{\Sigma_p}(QS^0)^p: Z_p).\end{array}$$

Then by the same method as the proof of Proposition 3-3, it is easily proved that

$$\bar{Q}^i[1]\circ Q^j[1]=(-1)^{i+j}G(e_{2i(p-1)}\otimes e_{2j(p-1)}).$$

And for $p \geqq 3$

$$\bar{Q}^i[1] \circ \beta Q^j[1] = (-1)^{i+j} G(e_{2i(p-1)} \otimes e_{2j(p-1)-1}).$$

But by Proposition 2-8 the following diagram is commutative.

$$\begin{array}{ccc}
H_*(W\pi_p \times_{\pi_p} (W\pi_p/\pi_p)^p : Z_p) & \xrightarrow{id \times h_p} & H_*(W\Sigma_p \times_{\Sigma_p} (QS^0)^p : Z_p) \\
\downarrow & & \downarrow \bar{\theta}_p \\
H_*(W\Sigma_p \times_{\Sigma_p} (W\Sigma_p/\Sigma_p)^p : Z_p) & & H_*(QS^0 : Z_p) \\
\downarrow \theta_\otimes & \nearrow h_{p^p} & \\
H_*(W\Sigma_{p^p}/\Sigma_{p^p} : Z_p) & &
\end{array}$$

On the other hand the following diagram is commutative up to inner automorphism in $\Sigma_{p^p}$.

(3-10)
$$\begin{array}{ccc}
\pi_p \times \pi_p \xrightarrow{id \times \Delta_p} \pi_p \int \pi_p \longrightarrow \Sigma_p \int \Sigma_p & \xrightarrow{\theta_\otimes} & \Sigma_{p^p} \\
\downarrow \Delta_{p^{p-2}+p-1} & & \uparrow \oplus \\
(\pi_p \times \pi_p)^{p^{p-2}+p-1} & & \\
\downarrow \text{shuffle} & & \Sigma_{p^2} \times \Sigma_{p^p-p^2} \\
(\pi_p \times \pi_p)^p \times (\pi_p \times \pi_p)^{p^{p-2}-1} & & \uparrow \oplus \times id \\
\downarrow \prod_{j=0}^{p-1} ((\ )^j \times id) \times (\otimes)^p & & \\
(\pi_p \times \pi_p)^p \times (\Sigma_{p^2})^{p^{p-2}-1} & \xrightarrow{(\times)^p \times \oplus} & \pi_p^p \times \Sigma_{p^p-p^2}.
\end{array}$$

This proves the Proposition.

LEMMA 3-12. *The following relations hold.*

1)
$$P_*^l(Q^m[1]) = (-1)^l \binom{(m-l)(p-1)}{l} Q^{m-l}[1],$$

$$c(P^l)_*(Q^m[1]) = \binom{m(p-1)+l}{l} Q^{m-l}[1].$$

2) *For* $p \geqq 3$

$$P_*^l(\beta Q^m[1]) = (-1)^l \binom{(m-l)(p-1)-1}{l} \beta Q^{m-l}[1],$$

$$c(P^l)_*(\beta Q^m[1]) = \binom{m(p-1)+l-1}{l} \beta Q^{m-l}[1].$$

PROPOSITION 3-13. *When $p=2$, in $H_*(QS^0: Z_2)$*

$$\bar{Q}^{t-k}Q^k[1]=\sum_j \binom{t-k-1-j}{k-j} Q^{t-j}[1]*Q^j[1].$$

PROPOSITION 3-14. *When $p=2$, in $H_*(QX: Z_2)$*

$$\bar{Q}^{t-k}Q^k(x)=\sum_{j,c_1,c_2}\sum_{\Delta x=\Sigma x'\otimes x''}\binom{t-k-1-j-2c_1}{k-j-c_1+c_2}$$
$$Q^{t-j-c_1-c_2}\bar{Q}^{c_1}(x')*Q^j\bar{Q}^{c_2}(x'').$$

PROPOSITION 3-15. *When $p$ is an odd prime, then*

1) $\bar{Q}^{t-k}Q^k[1]=c_t^k(p)Q^t[1]*[p^p-p]-Q^{t-k}Q^k[1]*[p^p-p^2]$
$+*decomposable$,

2) $\bar{Q}^{t-k}\beta Q^k[1]=d_t^k(p)\beta Q^t[1]*[p^p-p]-Q^{t-k}\beta Q^k[1]*[p^p-p^2]$
$+*decomposable$,

*where $c_t^k(p)$ and $d_t^k(p)$ are some constants.*

CONJECTURE 3-16[1].

1) $c_t^k=c_t^k(p)=-\binom{t-k-1}{r(p-1)} \bmod p$,

2) $d_t^k=d_t^k(p)=\binom{t-k-1}{r(p-1)-1} \bmod p$.

## § 4. Applications.

**4-1.** In this chapter, $p$ always denotes an odd prime number. And we denote $H_*(X)$ instead of $H_*(X: Z_p)$.

For any sequence $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, where $\varepsilon_i=0$ or 1 and $j_i$ are non-negative integers, we denote $Q^J$ for $\beta^{\varepsilon_1}Q^{j_1}, \cdots, \beta^{\varepsilon_r}Q^{j_r}$. And we define *degree of $I=d(I)=\sum_i(2j_i(p-1)-\varepsilon_i)$, length of $I=|I|=r$*, and *excess of $I=e(I)=$* $2j_i-\varepsilon_1-\sum_{i=2}^{r}(2j_i(p-1)-\varepsilon_i)$. And we say that $I$ is *admissible* if $pj_i-\varepsilon_i\geqq j_{i-1}$ for $i=2, \cdots, r$.

The following result is established by Dyer-Lashof [**5**].

PROPOSITION 4-1. *The Pontrjagin ring $H_*(Q_0S^0)$ is the free commutative algebra generated by $\{Q^I[1]*[-p^{|I|}];\ e(I)+\varepsilon_1\geqq 1,\ I\colon \text{admissible}\}$.*

Now we denote $SF$ for $Q_1S^0$, then this becomes an $H$-space by composition products. The following results are proved in [**8**], [**10**] and [**14**].

PROPOSITION 4-2. *The Pontrjagin ring $H_*(SF)$ is the free commutative algebra generated by $\{Q^I[1]*[1-p^{|I|}];\ e(I)+\varepsilon_1\geqq 1,\ I\colon \text{admissible}\}$.*

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1) These conjectures are proved in § 4 by different methods.

One of the main results of this chapter are the following two results. These are proved in 4-3.

PROPOSITION 4-3. *The Pontrjagin ring* $H_*(SF)$ *is the free commutative algebra generated by*

(4-1) i) $\beta^\varepsilon Q^j[1]*[1-p]$; $\varepsilon=0$ *or* 1, $j=1, 2, \cdots$,

ii) $Q^I[1]*[1-p^2]$; $|I|=2$, $e(I)+\varepsilon_1 \geqq 1$ $I$: *admissible*,

iii) $\bar{Q}^J(Q^I[1]*[1-p^2])$; $|I|\geqq 2$, $|J|\geqq 1$, $e(J, I)+\varepsilon_1>1$,

$(J, I)$ *admissible, and* $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$.

Recall that $\mathrm{Im}\, j_*: H_*(BSO)\to H_*(BSF)$ is the free commutative algebra generated by $\tilde{z}_j$, $j=1, 2, \cdots$, $\deg \tilde{z}_j=2j(p-1)$, and we can assume $\Delta\tilde{z}_j=\sum_i \tilde{z}_i\otimes\tilde{z}_{j-i}$, $\tilde{z}_0=1$, cf. § 6 in [**13**].

We denote by $\sigma: H_*(SF)\to H_{*+1}(BSF)$, the suspension homomorphism.

THEOREM 4-4. *The Pontrjagin ring* $H_*(BSF)$ *is the free commutative algebra generated by*

(4-2) i) $\tilde{z}_i$, $\sigma(Q^j[1]*[1-p])$; $i, j=1, 2, \cdots$,

ii) $\sigma(Q^I[1]*[1-p^2])$; $|I|=2$, $e(I)+\varepsilon_1\geqq 1$, $I$: *admissible*,

iii) $\bar{Q}^J(\sigma(Q^I[1]*[1-p^2]))$; $|I|\geqq 2$, $|J|\geqq 1$, $e(J, I)+\varepsilon_1>1$,

$(J, I)$: *admissible and* $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$.

**4-2.** Now we show the following two propositions.

PROPOSITION 4-5. *In* $H_*(SF)$, *if* $\deg Q^I[1]>0$, *then*

(4-3)
$$\bar{Q}^j(Q^I[1]*[1-p^{|I|}])$$
$$=Q^jQ^I[1]*[1-p^{|I|+1}]+\bar{Q}^jQ^I[1]*[1-p^{p|I|}] \text{ mod decomposables.}$$

PROPOSITION 4-6. *In* $H_*(SF)$, *the following relations hold mod decomposables.*

(4-4) 1) $\bar{Q}^j((Q^k[1])*[1-p^p])\equiv c^k_{j+k}Q^{j+k}[1]*[1-p]-Q^jQ^k[1]*[1-p^2]$.

2) $\bar{Q}^j((\beta Q^k[1])*[1-p^p])=d^k_{j+k}\beta Q^{j+k}[1]*[1-p]-Q^i\beta Q^k[1]*[1-p^2]$.

3) *If* $|J|\geqq 2$ *then*

$$\bar{Q}^j((Q^J[1])*[1-p^{p^{|J|}}])=\sum_{|K|=|J|} c_K Q^K[1]*[1-p^{|J|}]$$

*where* $c^k_{j+k}$ *and* $d^k_{j+k}$ *are constants which appeared in Proposition* 3-15, *and* $c_K$ *are some constants.*

At first we prepare some lemmas without proofs. These can be proved using the results in § 3.

LEMMA 4-7. *For* $x, y\in H_*(QS^0)$ *with* $\deg x>0$ *and* $\deg y>0$ *we have*

(4-5) $$\sum_{i+j=n} Q^i(x)\circ Q^j(x)=z^{[p]} \quad \textit{for some } z.$$

LEMMA 4-8. *For* $x\in H_*(Q_0S^0)$ *with* $\deg x>0$ *we have*

(4-6) $$x^p=z^{[p]} \quad \textit{for some } z.$$

LEMMA 4-9. *For* $x, y\in H_*(Q_0S^0)$ *with* $\deg y>0$ *and* $y=y_0^{[p]}$ *for some* $y_0$ *we have*

(4-7) $$x*y*[1] \textit{ is decomposable in } H_*(SF).$$

LEMMA 4-10. *For* $x_i=Q^{J_i}[1]*[-p^{|J_i|}]$, $i=1,\cdots,r$, *with* $\deg x_i>0$. *We have*

(4-8) $$x_1*\cdots*x_r*[1]\equiv(-1)^r(r-1)!\,(x_1\circ\cdots\circ x_r)*[1]$$
$$\textit{mod decomposables in } H_*(SF).$$

LEMMA 4-11. *For* $x_i=Q^{I_i}[1]*[-p^{|I_i|}]$, $y_j=Q^{J_j}[1]*[-p^{|J_j|}]$, $i=1,\cdots,r$, $j=1,\cdots,s$ *and* $\deg x_i>0$, $\deg y_j>0$, *then we have*

1) *If* $r>s\geqq 1$ *then*

(4-9) $$(x_1*\cdots*x_r)\circ(y_1*\cdots*y_s)*[1]\equiv 0$$
$$\textit{mod decomposables in } H_*(SF).$$

2) *If* $r=s\geqq 1$ *then*

(4-10) $$(x_1*\cdots*x_r)\circ(y_1*\cdots*y_r)*[1]=r!\,(x_1\circ\cdots\circ x_r\circ y_1\circ\cdots\circ y_r)*[1]$$
$$\textit{mod decomposable in } H_*(SF).$$

LEMMA 4-12. *For* $x=Q^I[1]$, $y=Q^J[1]$, $\deg x>0$, $\deg y>0$, *then*

(4-11) $$\sum_{i+j=n}\bar{Q}^i(x)*Q^j(y)*[1-p^{p|I|+|J|+1}]\equiv 0$$
$$\textit{mod decomposables in } H_*(SF).$$

PROOF OF 4-5. By mixed Cartan formula

$$\begin{aligned} A&\equiv\bar{Q}^j(Q^I[1]*[-p^{|I|}]*[1])\\ &=\sum_{\substack{j_0+\cdots+j_{p-1}=j\\ I_0+\cdots+I_{p-1}=I}}\bar{Q}_0^{j_0}((Q^{I_0}[1]*[-p^{|I|}])*[1])*\cdots\\ &\qquad *\bar{Q}^{j_{p-1}}_{p-1}((Q^{I_{p-1}}[1]*[-p^{|I|}])*[1])*[1].\end{aligned}$$

For $2\leqq i\leqq p-1$ if $\deg\bar{Q}_i^{j_i}((Q^{I_i}[1]*[-p^{|I|}])*[1])>0$ then by Proposition 3-9 and Lemma 4-7, this is $z^{[p]}$ for some $z$, with $\deg z>0$. So that by Lemma 4-9, the term containing this term is decomposable. So that mod decomposable

$$A \equiv \sum_{\substack{j_0+j_1=j \\ I_0+I_1=I}} \bar{Q}_0^{j_0}((Q^{I_0}[1]*[-p^{|I|}])*[1])*\dot{Q}_1^{j_1}((Q^{I_1}[1]*[-p^{|I|}])*[1])*[1]$$

$$= \sum_{\substack{j_0+j_1=j \\ I_0+I_1=I}} \bar{Q}^{j_0}(Q^{I_0}[1]*[-p^{|I|}])*Q^{j_1}(Q^{I_1}[1]*[-p^{|I|}])*[1].$$

Again using mixed Cartan formula and Cartan formula, by Lemma 4-12, if $\deg I_0 > 0$, and $\deg I_1 > 0$, then the term

$$\sum_{j_0+j_1} \bar{Q}^{j_0}(Q^{I_0}[1]*[-p^{|I|}])*Q^{j_1}(Q^{I_1}[1]*[-p^{|I|}])*[1]$$

is decomposable in $H_*(SF)$. So we get

$$A \equiv \bar{Q}^j Q^I[1]*[1-p^{|I|}]+Q^jQ^I[1]*[1-p^{|I|+1}].$$

PROOF OF PROPOSITION 4-6. At first

$$\bar{Q}^jQ^k[1]*[1-p^p] = \sum_l (\bar{Q}^{j+l}[1]\circ c(P^l)Q^k[1])*[1-p^p].$$

So that we show that $(\bar{Q}^m[1]\circ Q^n[1])*[1-p^p]$, for $m>0$, $n\geqq 0$,

$$=\binom{(m+n)(p-1)}{m(p-1)}Q^{m+n}[1]*[1-p]-Q^m[1]\circ Q^n[1]*[1-p^2]$$

mod decomposable in $H_*(SF)$. Then we get easily the formula (4-4) 1). By Proposition 3-11

$$\bar{Q}^m[1]\circ Q^n[1]*[1-p^p]$$

$$= \sum_{\substack{m_1+\cdots+m_p+m_{p+1}(p-1)=m(p-1) \\ n_1+\cdots+n_p+n_{p+1}(p-1)=n(p-1) \\ m_i+n_i=l_i(p-1) \text{ for some } l_i}} 0^{m_1}1^{m_2}\cdots(p-1)^{m_p}\binom{l_1(p-1)}{m_1}\binom{l_2(p-1)}{m_2}\cdots\binom{l_p(p-1)}{m_p}$$

$$Q^{l_1}[1]*\cdots*Q^{l_p}[1]*([p^{p-2}-1]\circ Q^{m_{p+1}}\circ Q^{n_{p+1}}[1])*[1-p^p].$$

But for $3\leqq k\leqq p$, $0\leqq i_1<\cdots<i_k\leqq(p-1)$ and $a>0$, $b\geqq 0$,

$$\sum_{\substack{a_1+\cdots+a_k=a(p-1) \\ b_1+\cdots+b_k=b(p-1) \\ a_i+b_i=l_i(p-1)}} i_1^{a_1}\cdots i_k^{a_k}\binom{l_1(p-1)}{a_1}\cdots\binom{l_k(p-1)}{a_k}Q^{l_1}[1]\circ\cdots\circ Q^{l_k}[1]$$

$$=z^{[p]} \quad \text{for some } z.$$

And for $0\leqq i_1<i_2\leqq(p-1)$ and $a>0$, $b\geqq 0$,

$$\sum_{\substack{a_1+a_2=a(p-1) \\ b_1+b_2=b(p-1) \\ a_i+b_i=l_i(p-1)}} j_1^{a_1}i_2^{a_2}\binom{l_1(p-1)}{a_1}\binom{l_2(p-1)}{a_2}Q^{l_1}[1]\circ Q^{l_2}[1]$$

$$=Q^a[1]\circ Q^b[1].$$

So by using Lemma 4-10 mod decomposable

$$\bar{Q}^m \circ Q^n[1]*[1-p^p] = \sum_{\substack{m_1+m_{p+1}=m \\ n_1+n_{p+1}=n}} \left(\sum_{i=0}^{p-1} i^{m_1(p-1)}\binom{(m_1+n_1)(p-1)}{m_1(p-1)}\right)$$

$$Q^{m_1+n_1}[1]*([p^{p-2}-1]\circ Q^{m_{p+1}}[1]\circ Q^{n_{p+1}}[1])*[1-p^p]$$

and $k \geqq 1$ and $a>0$, $b \geqq 0$, $0 \leqq i \leqq p-1$,

$$\sum_{\substack{a_1+a_2+\cdots+a_{k+1}=a \\ b_1+b_2+\cdots+b_{k+1}=b}} i^{a_1(p-1)}\binom{(a_1+b_1)(p-1)}{a_1(p-1)}$$

$$Q^{a_1+a_2}[1]\circ(Q^{a_2}[1]\circ Q^{b_2}[1])\circ\cdots\circ(Q^{a_{k+1}}[1]\circ Q^{b_{k+1}}[1]) = z^{[p]}.$$

So by using Lemma 4-10 mod decomposable

$$\bar{Q}^m[1]\circ Q^n[1]*[1-p^p] = \left(\sum_{i=0}^{p-1} i^{m(p-1)}\binom{(m+n)(p-1)}{m(p-1)}\right)$$

$$Q^{m+n}[1]*[1-p]+([p^{p-2}-1]\circ Q^m[1]\circ Q^n[1])*[1-p^{p-2}].$$

On the other hand, using Lemma 4-10

$$([p^{p-2}-1]\circ Q^m[1]\circ Q^n[1])*[1-p^{p-2}]$$

$$= -Q^m[1]\circ Q^n[1]*[1-p] \text{ mod decomposable}.$$

So we get

$$\bar{Q}^m[1]\circ Q^n[1]*[1-p^p] \text{ for } m>0$$

$$=\binom{(m+n)(p-1)}{m(p-1)}Q^{m+n}[1]*[1-p]-Q^m[1]\circ Q^n[1]*[1-p^2].$$

2) is proved in the same way. Now we prove 3). Assume $|I|=r \geqq 2$, then since $Q^I[1]=\sum_{|I|=|J|} c_J(\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1])$. And $\bar{Q}^j(Q^I[1])=\sum_l \bar{Q}^{j+l}[1]\circ(c(P^l)(Q^I[1]))$ and $c(P^l)Q^I[1]=\sum_{|I|=|J|} d_J Q^J[1]$, it is sufficient to prove that

$$(\bar{Q}^j[1]\circ((\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1]))*[1-p^{pr}]$$

$$\equiv \sum_{|K|=|J|} c_K Q^K[1] \text{ mod decomposable}.$$

Now consider the following diagram.

$$\begin{array}{ccc}
H_*(W\pi_p/\pi_p\times(W\pi_p/\pi_p)^r) & \xrightarrow{id\times\Delta_p} & H_*(W\pi_p\times_{\pi_p}((W\pi_p/\pi_p)^r)^p) \\
\Big\downarrow G & & \Big\downarrow i\times(h_p^r)^p \\
 & & H_*(W\Sigma_p\times_{\Sigma_p}((QS^0)^r)^p) \\
 & & \Big\downarrow id\times(\mu_X^{(r)})^p \\
H_*(QS^0) & \xleftarrow{\bar{\theta}_p} & H_*(W\Sigma_p\times_{\Sigma_p}(QS^0)^p).
\end{array}$$

Then by definition and by proof of Proposition 3-11, $\bar{Q}^j[1]\circ((\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1]))$ is in the image of $G$. On the other hand, by general distributive law the following diagram commutes. And up to inner automorphism in $\Sigma_{pp^r}$ the following diagram is commutative.

$$\begin{array}{ccccc}
\pi_p\times(\pi_p)^r & \xrightarrow{id\times\oplus^r} & \pi_p\times\Sigma_{p^r} & \xrightarrow{i\times\Delta_p} & \Sigma_p\int\Sigma_{p^r} \\
\downarrow \Delta_{p^r+1} & & & & \downarrow \theta_\otimes \\
(\pi_p\times(\pi_p)^r)^{p^r+1}=(\pi_p\times(\pi_p)^r)^{p^r}\times(\pi_p\times(\pi_p)^r) & & & & \Sigma_{pp^r} \\
\downarrow (\Delta_r\times id)^{p^r}\times\oplus^{r+1} & & & & \uparrow \oplus \\
((\pi_p)^r\times(\pi_p)^r)^{p^r}\times\Sigma_{p^r+1} & & & & \\
\downarrow \prod_{i_1=0\cdots i_r=0}^{p-1\cdots p-1}((\ )^{i_1}\times\cdots\times(\ )^{i_r}\times(id)^r)\times id & & & & \\
((\pi_p)^r\times(\pi_p)^r)^{p^r}\times\Sigma_{p^r+1} & & & & \Sigma_{p2r}\times\Sigma_{pp^r-2r} \\
\downarrow (\text{shuffle})^{p^r}\times id & & & & \uparrow \oplus^{p2r}\times\oplus \\
((\pi_p\times\pi_p)^r)^{p^r}\times\Sigma_{p^r+1} & & \xrightarrow{(+)^{p^r}\times\Delta_{p^{r-1}(p(p-2)r-1)}} & & ((\pi_p)^r)^{p^r}\times(\Sigma_{p^r+1})^{p^{r-1}(p(p-2)r-1)}\,.
\end{array}$$

So that we get, since $r\geqq 2$,

$$Q^j[1]\circ((\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1]))$$
$$=(-1)^*F(e_{2j(p-1)}\otimes(e_{2j_1(p-1)-\varepsilon_1}\otimes\cdots\otimes e_{2j_r(p-1)-\varepsilon_r}))*[p^{pr-2r}]+z^{[p]}$$

where $F$ is defined by

$$\begin{array}{ccccc}
H_*(\pi_p\times(\pi_p)^r) & \xrightarrow{\Delta_{p^r}} & H_*(\pi_p\times(\pi_p)^r)^{p^r} & \xrightarrow{(\Delta_r\times id)^{p^r}} & H_*((\pi_p)^r\times(\pi_p)^r)^{p^r} \\
\downarrow F & & & & \downarrow \prod_{i_1=0\cdots i_r=0}^{p-1,\cdots,p-1}((\ )^{i_1}\times\cdots\times(\ )^{i_r}\times(id)^r) \\
H_*(QS^0) & & & & H_*((\pi_p)^r\times(\pi_p)^r)^{p^r} \\
\uparrow h_{p2r} & & & & \downarrow (\text{shuffle})^r \\
H_*(\Sigma_{p2r}) & \xleftarrow{\oplus} & H_*((\pi_p)^r)^{p^r} & \xleftarrow{(+)^{p^r}} & H_*((\pi_p\times\pi_p)^r)^{p^r}
\end{array}$$

Then by the same method of the proof of 1), by using Lemma 4-10, we get

$$\bar{Q}^j[1]\circ((\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1]))\times[1-p^{p^r}]$$
$$\equiv\sum_{i_1+\cdots+i_r=i}(0^{i_1(p-1)}+\cdots+(p-1)^{i_1(p-1)})\cdots(0^{i_r(p-1)}+\cdots+(p-1)^{i_r(p-1)})$$
$$\binom{(i_1+j_1)(p-1)-\varepsilon_1}{i_1(p-1)}\cdots\binom{(i_r+j_r)(p-1)-\varepsilon_r}{i_r(p-1)}$$
$$[(\beta^{\varepsilon_1}Q^{i_1+j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{i_r+j_r}[1])]*[1-p^r]\ \text{mod decomposable}.$$

This proves the statement 3).

COROLLARY 4-13. *If $r \geqq 2$, then we have*

$$\bar{Q}^j((\beta^{\varepsilon_1}Q^{j_1}[1])\circ\cdots\circ(\beta^{\varepsilon_r}Q^{j_r}[1]))*[1-p^{pr}]$$
$$\equiv \sum_{i_1+\cdots+i_p=i} c_{i_1+j_1}^{j_1}(\varepsilon_1)\cdots c_{i_p+j_p}^{j_p}(\varepsilon_p)$$
$$(\beta^{\varepsilon_1}Q^{i_1+j_1}[1]\circ\cdots\circ(\beta^{\varepsilon_r}Q^{i_r+j_r})]*[1-p^r] \text{ mod decomposable}$$

*in $H_*(SF)$, where*

$$c_t^k(\varepsilon)=\begin{cases} c_t^k & \varepsilon=0 \\ d_t^k & \varepsilon=1. \end{cases}$$

**4-3.** At first consider $H^*(SO)\cong \Lambda_p(u_1, u_2, \cdots)$, $\deg u_j=4j-1$; we take $u_j$ such that (i) $u_j$ are primitive elements and (ii) $\langle\sigma(u_j), P_j\rangle=1$, where $P_j$ is the $j$-th Pontrjagin class.

PROPOSITION 4-14. *For $j_*: H_*(SO)\to H_*(SF)$, the following relations hold:*

1) $j_*(u_j)=0$ *if* $j\not\equiv 0\ (p-1)/2$

2) $j_*(u_{j(p-1)/2})=(-1)^j c\beta Q^j[1]*[1-p]+d_j$

*where $c$ is a non-zero constant independent to $j$ and $d_j$ are some decomposables.*

PROOF. Assume $j\not\equiv 0\ (p-1)/2$ and $j_*(u_j)\neq 0$, then $j_*(u_j)$ is indecomposable, because $u_j$ is primitive, and $\sigma: QH_*(SF)\to H_*(BSF)$ is monomorphism in this dimension. But in this dimension $H_{4j}(BSO)\to H_{4j}(BSF)$ is zero map, so this is a contradiction. By induction on $j$, we prove 2). For $j=1$, $j_*(u_{(p-1)/2})=-1c_1\beta Q^1[1]*[1-p]$ by dimensional reason and $c_1\neq 0$ since $j_*(u_{(p-1)/2})\neq 0$. Assume 2) holds for up to $j-1$. We set, mod decomposable

$$j_*(u_{j(p-1)/2})=(-1)^j c_j\beta Q^j[1]*[1-p]+\sum_{|K|\geqq 2} c_K Q^K[1]*[1-p^{|K|}] \tag{4-12}$$

where $K$ moves the generator in Proposition 4-2. At first we prove $c_K=0$ for all $K$. Let $|K_0|\geqq 2$ be such an element that appears as $c_{K_0}\neq 0$ and $|K_0|$ is maximal among those $\{K\}$ which appear $c_K=0$. Apply $\beta\bar{Q}^j$ on both terms in (4-12). Then $j_*(\beta\bar{Q}^j(u_{j(p-1)/2}))=0$. On the other hand, by Propositions 4-5 and 4-6 the term $c_{K_0}\beta Q^jQ^{K_0}[1]*[1-p^{|K_0|+1}]$ appears in the left hand side, and this is an element of basis by Proposition 4-3. This is a contradiction. So that all $c_K=0$. And since $\sigma j_*(u_{j(p-1)/2})\neq 0$ so that $c_j\neq 0$. If $j=p^r$ for some $r$ then $a>0$ such that $P^a u_{j(p-1)/2}=xu_{(j-a)(p-1)/2}$, $x\neq 0$ and in this case $P^a\beta Q^j[1]=(-1)^a x\ \beta Q^{j-a}[1]$. Then naturality of $P^a$ shows that $c_j=c_{j-a}=c_1$. Assume $j=p^r$ for some $r\geqq 1$. Then by Kochman [**6**]

$$\bar{Q}^j(u_{(p-1)/2})=(-1)^j\binom{j-1}{p-2}u_{(j+1)((p-1)/2)}.$$

On the other hand, by Propositions 3-11, 4-6

$$\bar{Q}^j(\beta Q^1[1]*[1-p])=-\binom{(j+1)(p-1)-1}{p-2}\beta Q^{j+1}[1]*[1-p]$$

mod decomposable. By naturality of $\bar{Q}^j$ we get

$$j_*(u_{(j+1)(p-1)/2}) \equiv (-1)^{j+1} c_1 \beta Q^{j+1}[1] * [1-p] \text{ mod decomposable}.$$

Applying $P^1_*$ we get

$$j_*(u_{j(p-1)/2}) = (-1)^j c_1 \beta Q^{j+1}[1] * [1-p] \text{ mod decomposable}.$$

This proves the Proposition.

COROLLARY 4-15.

$$d_t^k = \binom{t-k-1}{k(p-1)-1} \text{ mod } p.$$

PROOF. This follows from Proposition 4-14, naturality of $\bar{Q}^j$ on $j_*: H_*(SO) \to H_*(SF)$, and Kochman's results [6] on $\bar{Q}^j(u_k)$.

PROOF OF PROPOSITION 4-3. This follows directly from Proposition 4-2, Proposition 4-5 and Proposition 4-6.

PROOF OF PROPOSITION 4-4. This follows directly from Proposition 4-3 by the same method of the Proof of Theorem II in [14], so we omit it here.

**4-4.** Let $BO\langle 8N\rangle$ denote the space obtained by killing homotopy groups $\pi_i(BO)$, $i<8N$. Let $fN: S^{8N} \to BO\langle 8N\rangle$ be the canonical generator of $\pi_{8N}(BO\langle 8N\rangle) \cong Z$. Taking the iterated loop, we get the map $g_N = \Omega^{8N} f_{8N}$: $\Omega^{8N} S^{8N} \to \Omega^{8N} BO\langle 8N\rangle$. Taking limit on $N$ we get

$$g: \varinjlim \Omega^{8N} S^{8N} = QS^0 \longrightarrow \varinjlim \Omega^{8N} BO\langle 8N\rangle = Z \times BO. \tag{4-13}$$

Consider $1 \times BO = BO_{\otimes} \subset Z \times BO$, then this $BO_{\otimes}$ becomes an $H$ space by tensor product and the map $g_1: Q_1 S^0 = SF \to BO_{\otimes}$ is an $H$ space map. The following two lemmas are proved at the end of this section.

LEMMA 4-16.

$$c_t^k \equiv -\binom{t-k-1}{k(p-1)} \text{ mod } p.$$

LEMMA 4-17. *$g_{1*}(\bar{Q}^t Q^s * [1-p^p])$ is decomposable in $H_*(BO_{\otimes})$ for any non-negative integers $t$ and $s$.*

Now consider the map $k: SF \to F/PL$.

PROPOSITION 4-18. *The following relations hold for $k_*: H_*(SF) \to H_*(F/PL)$.*

1) $k_*(Q^t Q^s[1] * [1-p^2]) \equiv -\binom{t-1}{s(p-1)} k_*(Q^{t+s}[1] * [1-p])$
*mod decomposables.*

2) $k_*(\bar{Q}^t(Q^s[1] * [1-p])) \equiv -\binom{t-1}{s(p-1)} k_*(Q^{t+s}[1] * [1-p])$
*mod decomposables.*

PROOF. The following is proved in §1 in [15]. Let $P$ denote the set of all odd prime numbers, and $C_P$ is the class of abelian groups without $P$ torsions. Then there is a $C_P$ homotopy equivalence

(4-14) $\bar{\sigma}: F/PL \longrightarrow BO_{\otimes P}$ with

(1) $\bar{\sigma}$: $H$-map (2) $g_1: SF \longrightarrow BO_{\otimes} \longrightarrow BO_{\otimes P}$ and $\bar{\sigma}\circ k: SF \to F/PL \to BO_{\otimes P}$ coincide up to homotopy.

Then the proposition follows from Lemma 4-16 and Lemma 4-17, and from Propositions 4-5 and 4-6.

The following lemma follows from Proposition 2-7 of [15] and (4-14).

PROPOSITION 4-19. *The following results hold for* $k: H_*(SF) \to H_*(F/PL)$.

1) $k_*(Q^j[1]*[1-p])$ *are indecomposable.*

2) *Image of* $k_*$ *is the polynomial algebra generated by* $k_*(Q^j[1]*[1-p])$, $j=1, 2, \cdots$.

Now consider $i_*: H_*(SPL) \to H_*(SF)$. Define $v_j \in H_{4j-1}(SPL)$ by (1) $v_j = j_*(u_j)$ if $j \equiv 0(p-1)/2$ where $j_*: H_*(SO) \to H_*(SPL)$; (2) $v_j = j_*(d_j)$ if $j \not\equiv 0(p-1)/2$, where $j_*: H_*(\Omega(F/PL)) \to H_*(SPL)$, $H_*(\Omega(F/PL)) \cong \Lambda(d_1, d_2, \cdots)$, $\deg d_i = 4i-1$.

And considering Serre Spectral sequence associated to $SPL \to SF \to F/PL$ we get for $(\varepsilon_1, j_1, \varepsilon_2, j_2) = I$:

(4-15) There exists element $\bar{x}_I \in H_*(SPL)$ with the property

$$1)\quad i_*(\bar{x}_I) = \begin{cases} \beta^{\varepsilon_1}Q^{j_1}\beta^{\varepsilon_2}Q^{j_2}[1]*[1-p^2] & \text{if } (\varepsilon_1, \varepsilon_2) \neq (0,0) \\ Q^{j_1}Q^{j_2}[1]*[1-p^2] + \binom{j_1-1}{j_2(p-1)} Q^{j_1+j_2}[1]*[1-p] & \text{if } (\varepsilon_1, \varepsilon_2) = (0,0). \end{cases}$$

2) These $\bar{x}_I$ are determined uniquely mod decomposable.

PROPOSITION 4-20. *The Pontrjagin ring* $H_*(SPL)$ *is the free commutative algebra generated by*

1) $v_j$; $j=1, 2, \cdots$.

2) $\bar{x}_I$; $I=(\delta, r, \varepsilon, s)$, $e(I)+\delta \geqq 1$, $I$: *admissible.*

3) $\bar{Q}^J \bar{x}_I$; $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r \geqq 1$, $I=(\delta, r, \varepsilon, s)$, $e(J, I)+\varepsilon_1 \geqq 1$, $(J, I)$: *admissible.*

LEMMA 4-21. *Consider the map* $i: H_*(F/PL) \to H_*(BSPL)$ *and* $j_*: H_*(BSO) \to H_*(BSPL)$. *Then the Hopf algebra generated by image of* $i_*$ *and* $j_*$ *is a polynomial algebra* $Z_p[\bar{b}_1, \bar{b}_2, \cdots]$, $\deg \bar{b}_j = 4j$, $\Delta(\bar{b}_j) = \sum_i \bar{b}_i \otimes \bar{b}_{j-i}$.

Now it is easy to prove the following theorem.

THEOREM 4-22. *The Pontrjagin ring* $H_*(BSPL)$ *is the free commutative algebra generated by*

1) $\bar{b}_j$; $j=1, 2, \cdots$.

2) $\sigma(\bar{x}_I)$; $I=(\delta, r, \varepsilon, s)$, $e(I)+\delta \geqq 1$, $I$: *admissible.*

3) $\bar{Q}^J(\sigma(\bar{x}_I))$; $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r \geqq 1$, $I=(\delta, r, \varepsilon, s)$, $e(J, I)+\varepsilon_1 > 1$, $(J, I)$: *admissible.*

PROPOSITION 4-23. *The image of* $j_*: H_*(BSPL) \to H_*(BSF)$ *is the free commutative algebra generated by*

1) $\tilde{z}_j : j=1, 2, \cdots$.

2) $$\sigma(\bar{x}_I)=\begin{cases} \sigma(\beta^\delta Q^r \beta^\varepsilon Q^s[1]*[1-p^2]) : \textit{if } (\delta, \varepsilon)\neq(0, 0) \\ \sigma(Q^r Q^s[1]*[1-p^2])+\binom{r-1}{s(p-1)}\sigma(Q^{r+s}[1]*[1-p]), \end{cases}$$

*where* $I=(\delta, r, \varepsilon, s)$, $e(I)+\delta\geqq 1$, $I$: *admissible.*

3) $\bar{Q}^J(j_*(\sigma(\bar{x}_I)))$, $J=(\varepsilon_1, j_1, \cdots, \varepsilon_r, j_r)$, $r\geqq 1$, $I=(\delta, r, \varepsilon, s)$, $e(J, I)+\varepsilon_1>1$, $(J, I)$: *admissible.*

PROOF OF LEMMA 4-16. Consider infinite loop space $Z\times BU$, $H_*(Z\times BU)$ has Dyer-Lashof operations $Q^j$. On the other hand, let $\xi : F\to X$ be a complex vector bundle, then we can construct new vector bundle $P(\xi) : W\Sigma_q\times_{\Sigma_q}E^q \to W\Sigma_q\times_{\Sigma_q}X^q$. And this defines the map $P : W\Sigma_p\times_{\Sigma_q}(Z\times BU)^q\to Z\times BU$. In § 2 of [**15**], we showed that the Dyer-Lashof operations defined by $P$ coincide with that defined by infinite loop structure of $Z\times BU$ defined by Bott periodicity. Recall that $H_*(0\times BU)=Z_p[a_1, a_2, \cdots]$ where $a_i=\gamma_*(b_i)*[-1]$, where $b_i\in H_{2i}(CP^\infty)$, and $\gamma : CP^\infty\to 1\times BU$ represents the canonical line bundle. Consider the following map $F$.

$$\begin{array}{ccccc} W\pi_p/\pi_p\times CP^\infty & \xrightarrow{id\times\Delta_p} & W\pi_p\times_{\pi_p}(CP^\infty)^p & \xrightarrow{i\times(\gamma)^p} & W\Sigma_p\times_{\Sigma_p}(Z\times BU)^p \\ & & & & \downarrow P \\ & & \xrightarrow{\qquad\qquad F \qquad\qquad} & & Z\times BU \end{array}$$

On the other hand, it is easy to see that $F$ is a representative map of the bundle $N\otimes\gamma\to W\pi_p/\pi_p\times CP^\infty$, where $N\to W\pi_p/\pi_p$ is the bundle of dimension $p$ defined by regular representation $N : \pi_p\to U(p)$. So that we get

$$F(e_{2i(p-1)}\otimes b_j)=\sum_{\substack{i_1+\cdots+i_p=i(p-1)\\ j_1+\cdots+j_p=j}} 0^{i_1}\cdots(p-1)^{i_p}\binom{i_1+j_1}{i_1}\cdots\binom{i_p+j_p}{i_p}$$
$$b_{i_1+j_1}*\cdots *b_{i_p+j_p}.$$

On the other hand, by Kochman's results [**6**] and Cartan formula

$$Q^i(b_j)=(-1)^{i+j+1}\binom{i-1}{j}b_{j+i(p-1)}*[p-1]+\text{dec}.$$

But the above argument shows that

$$(-1)^i Q^i[1]\circ b_j=(0^{i(p-1)}+\cdots+(p-1)^{i(p-1)})\binom{i(p-1)+j}{i(p-1)}b_{j+i(p-1)}*[p-1]+\text{dec}.$$

On the other hand, if $c(P^l)(b_{j(p-1)})=xb_{(j-l)(p-1)}$ then $c(P^l)Q^j[1]=(-1)^j xQ^{j-l}[1]$.

And $\quad \bar{Q}^{t-k}Q^k[1]=C^k_t Q^t[1]*[p^p-p]-Q^{t-k}Q^k[1]*[p^p-p^2]+\text{dec}.$

And $\quad \bar{Q}^{t-k}Q^k[1]=\sum_l \bar{Q}^{t-k+l}[1]\circ c(P^l)Q^k[1].$

And $\quad \bar{Q}^l[1]\circ Q^m[1]=(0^{l(p-1)}+\cdots+(p-1)^{l(p-1)})\binom{(l+k)(p-1)}{l(p-1)}$

$$Q^{l+m}[1]*[p^p-p]-Q^l[1]\circ Q^m[1]*[p^p-p^2] \text{ mod dec.}$$

$$Q^{t-k}(b_{j(p-1)})=\sum_l Q^{t-k+l}[1]\circ c(P^l)b_{j(p-1)}.$$

These show that $C_t^k=-\binom{t-k-1}{k(p-1)}$.

PROOF OF LEMMA 4-17. Consider the following map $H$.

$$\begin{array}{ccccc} W\pi_p/\pi_p\times W\pi_p/\pi_p & \xrightarrow{id\times h_p} & W\pi_p/\pi_p\times QS^0 & \xrightarrow{i\times\Delta_p} & W\Sigma_p\times_{\Sigma_p}(QS^0)^p \\ \downarrow H & & & & \downarrow \bar\theta_p \\ Z\times BU & \xleftarrow[i]{} & Z\times BO & \xleftarrow[g]{} & QS^0. \end{array}$$

By general distributive law the following diagram is commutative.

$$\begin{array}{ccccc} H_*(W\pi_p/\pi_p\times W\pi_p/\pi_p) & \xrightarrow{i\times h_p} & H_*(W\Sigma_p/\Sigma_p\times QS^0) & \longrightarrow & H_*(W\Sigma_p\times_{\Sigma_p}(QS^0)^p) \\ \downarrow i\times\Delta_p & & & & \downarrow \bar\theta_p \\ H_*(W\Sigma_p\times_{\Sigma_p}(W\pi_p/\pi_p)^p) & \xrightarrow{\theta_\otimes} & H_*(W\Sigma_{pp}/\Sigma_{pp}) & \xrightarrow{h_{pp}} & H_*(QS^0). \end{array}$$

And the map $W\pi_p/\pi_p\times W\pi_p/\pi_p \xrightarrow{\theta_\otimes\circ(I\times\Delta_p)} W\Sigma_{pp}/\Sigma_{pp}$ is analysed in (3-10). So that $H\in K(W\pi_p/\pi_p\times W\pi_p/\pi_p)$ represents $\sum_{i=0}^{p-1}(\rho_1^i\otimes N_2)+(p^{p-2}-1)N_1\otimes N_2$ where $\rho_1$ is canonical line bundle defined by $\pi_p\times\pi_p\xrightarrow{\pi_1}\pi_p\to U(1)$, and $N_i$ represent the regular representation $\pi_p\times\pi_p\xrightarrow{\pi_i}\pi_p\xrightarrow{N}U(p)$, $i=1, 2$. On the other hand, $\sum_{i=0}^{p-1}\rho_1^i=N_1$ so that $H=p^{p-2}N_1\otimes N_{20}$ so that we get

$$\begin{aligned} g_*(\bar Q^tQ^k[1]) &= [p^{p-2}]\circ g_*(Q^tQ^k[1]) \\ &= g_*([p^{p-2}]\circ Q^tQ^k[1]) \\ &= g_*(Z^{[p]}). \end{aligned}$$

So we get the lemma by Lemma 4-9.

Department of Mathematics
Nagoya University
Furo-cho, Chikusa-ku
Nagoya, Japan

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