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On continuation of regular solutions of partial differential equations with constant coefficients

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Introduction.

This paper deals with the problem of continuation of real analytic solutions of partial differential equations with constant coefficients. In [3], [4] we have considered the following case: Let K and U be compact convex and open convex subsets of $\boldsymbol{R}^n$ such that $K \subset U \subset \boldsymbol{R}^n$. Let $\mathcal{A}_p$ denote the real analytic solutions of the partial differential equation $p(D)u=0$ with constant coefficients. Then the quotient space $\mathcal{A}_p(U \backslash K)/\mathcal{A}_p(U)$ does not depend on U and represents the obstruction of extensibility of real analytic solutions defined outside the exceptional set K to a neighborhood of K. A satisfactory result was given there: For the single operator p, it says that $\mathcal{A}_p(U \backslash K)/\mathcal{A}_p(U)=0$ *if and only if the characteristic polynomial* $p(\zeta)$ *has no elliptic irreducible component*. (As for systems see [4].) In this paper we consider a case somewhat generalizing the preceding one: Let H be an open half space in $\boldsymbol{R}^n$; $K_1 = K \cap H$, where K is compact and convex as above; U_1 be an open convex neighborhood of K_1 in H. We discuss conditions for $\mathcal{A}_p(U_1 \backslash K_1)/\mathcal{A}_p(U_1)=0$, and give some sufficient conditions (Theorems 2.6, 2.7, and 2.12). In case $K \subset H$ this problem reduces to the preceding one.

We adopt the method employed by Grušin [1], who studied the isolated singularities of infinitely differentiable solutions. Since we treat here the sets "with boundary", we need a new (*relative*) type of *Phragmén-Lindelöf theorem* (Lemma 2.9) which plays an essential role in our method.

In §1 we consider the same problem for hyperfunction solutions and obtain a necessary and sufficient condition for the extensibility. The obtained result is used in the proof of theorem 2.6 for real analytic solutions. Though we can consider similar problems for other classes of regular solutions, we mainly concern ourselves with real analytic solutions of single operators here. Some of the remaining cases will be treated in future.

A part of these results was announced in [6]. Some of them has been

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improved on revision. For example, we can show $\mathcal{A}_p(U_1\backslash K_1)/\mathcal{A}_p(U_1)=0$ for the heat equation: $p(D)=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_{n-1}^2-\partial/\partial x_n$ and for any compact convex set K, where we employ $H=\{x\in \boldsymbol{R}^n;\ x_n<0\}$ (Corollary 2.14).

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§ 1. Preliminaries. The case of hyperfunction solutions.

Let $x=(x_1,\cdots,x_n)$ be a system of coordinates on $\boldsymbol{R}^n$. Without loss of generality we can assume that $H=\{x\in\boldsymbol{R}^n;\ x_n<0\}$. We put $x'=(x_1,\cdots,x_{n-1})$, and sometimes write $x=(x',x_n)$. Thus x' is a system of coordinates of the hyperplane $\partial H=\{x\in\boldsymbol{R}^n;\ x_n=0\}$. For saving notations we write K in place of K_1 and U in place of U_1. Since we make the discussions only for such parts in the sequel, there is no confusion. We put $L=\bar{K}$, where the closure is taken in $\boldsymbol{R}^n$. Thus K is a locally closed bounded subset of $\boldsymbol{R}^n$, L is compact and convex, $L\backslash K$ is also compact, convex and $L\backslash K\subset\partial H$. Though we need not omit the trivial case $L\backslash K=\emptyset$ (i. e. the case $L=K\Subset H$) from the logical standpoint, we clarify the matter assuming $L\backslash K\neq\emptyset$ in the sequel.

Let $p(D)$ be a partial differential operator with constant coefficients corresponding to the characteristic polynomial $p(\zeta)$, where $D=(D_1,\cdots,D_n)$, $D_j=\sqrt{-1}\,\partial/\partial x_j$. We always exclude the trivial case $p=$constant. We denote by $\mathcal{A}$ and $\mathcal{B}$ the sheaf of germs of real analytic functions and that of hyperfunctions on $\boldsymbol{R}^n$ respectively. We denote by $\mathcal{A}_p$ and $\mathcal{B}_p$ the sheaf of germs of real analytic solutions and that of hyperfunction solutions of $p(D)u=0$ respectively. As a preliminary work, we are going to seek the condition for $\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)=0$ below.

Now we write down the fundamental exact sequence of cohomology groups which is often used in [4].

$$(1.1)\qquad 0\longrightarrow H_K^0(U,\mathcal{B}_p)\longrightarrow H^0(U,\mathcal{B}_p)\xrightarrow{\lambda} H^0(U\backslash K,\mathcal{B}_p)$$
$$\longrightarrow H_K^1(U,\mathcal{B}_p)\longrightarrow H^1(U,\mathcal{B}_p)\,.$$

The last term vanishes due to the existence theorem of Komatsu (see [4], p. 416, (1.7)). Thus, with the usual notations $\mathcal{B}_p(U)=H^0(U,\mathcal{B}_p)$, $\mathcal{B}_p(U\backslash K)=H^0(U\backslash K,\mathcal{B}_p)$, we have

$$(1.2)\qquad H_K^1(U,\mathcal{B}_p)\cong\mathcal{B}_p(U\backslash K)/\lambda\mathcal{B}_p(U)\,.$$

On the other hand, since we have a flabby resolution

$$(1.3)\qquad 0\longrightarrow\mathcal{B}_p\longrightarrow\mathcal{B}\xrightarrow{p}\mathcal{B}\longrightarrow 0\,,$$

(see [8], Theorem 5), we can calculate $H_K^1(U, \mathscr{B}_p)$ from the following cochain complex

$$(1.4) \qquad 0 \longrightarrow H_K^0(U, \mathscr{B}) \xrightarrow{p} H_K^0(U, \mathscr{B}) \longrightarrow 0\,.$$

Thus we have

$$(1.5) \qquad H_K^1(U, \mathscr{B}_p) = H_K^0(U, \mathscr{B})/pH_K^0(U, \mathscr{B})\,.$$

Now we claim that the mappings λ and p in (1.2) and (1.5) are injective. In fact we have

LEMMA 1.1 (Kawai [7 bis], Schapira [14]). $H_K^0(U, \mathscr{B}_p) = 0$.

This is a variant of the so called *Holmgren's uniqueness theorem.*

PROOF. Take $u \in H_K^0(U, \mathscr{B}_p)$ arbitrarily. Let $u_1 \in H_L^0(\boldsymbol{R}^n, \mathscr{B})$ be an extension of u. By the assumption $v = p(D)u_1$ belongs to $H_{L\backslash K}^0(\boldsymbol{R}^n, \mathscr{B})$. Since $H_L^0(\boldsymbol{R}^n, \mathscr{B})$ and $H_{L\backslash K}^0(\boldsymbol{R}^n, \mathscr{B})$ are spaces of hyperfunctions with compact supports, we can apply the Fourier (-Laplace) transform $\tilde{v} = \langle \exp \sqrt{-1}\langle x, \zeta\rangle, v\rangle$ and obtain the identity $p(\zeta)\tilde{u}_1 = \tilde{v}$ for entire functions. By the inequality of division (see, e. g., [2], Lemma 3.1.7) we see that $\tilde{u}_1$ has the same growth order as $\tilde{v}$, namely, the estimate of the following type: given any $\varepsilon > 0$ we have, with some $C_\varepsilon > 0$,

$$|\tilde{u}_1(\zeta)| \leqq C_\varepsilon \exp(\varepsilon|\zeta| + H_{L\backslash K}(\zeta))\,.$$

Here $H_{L\backslash K}(\zeta) = \sup_{x \in L\backslash K} \mathrm{Re}\langle x, \sqrt{-1}\zeta\rangle$ is the supporting function of $L\backslash K$. Therefore by the Paley-Wiener-Ehrenpreis-Martineau theorem we conclude that $u_1 \in H_{L\backslash K}^0(\boldsymbol{R}^n, \mathscr{B})$, namely, $u = 0$ in U. q. e. d.

On account of the above lemma we omit the symbol λ hereafter. Now we employ a more complicated form of the fundamental exact sequence of relative cohomology groups: For a triple of open sets $X \supset Y \supset Z$, we have the exact sequence

$$(1.6) \qquad \begin{aligned} 0 &\longrightarrow H_{X\backslash Y}^0(X, \mathscr{F}) \longrightarrow H_{X\backslash Z}^0(X, \mathscr{F}) \longrightarrow H_{Y\backslash Z}^0(Y, \mathscr{F}) \\ &\longrightarrow H_{X\backslash Y}^1(X, \mathscr{F}) \longrightarrow H_{X\backslash Z}^1(X, \mathscr{F}) \longrightarrow H_{Y\backslash Z}^1(Y, \mathscr{F}) \\ &\longrightarrow H_{X\backslash Y}^2(X, \mathscr{F}) \longrightarrow \cdots\,. \end{aligned}$$

We apply this sequence to the sets $X = \boldsymbol{R}^n$, $Y = \boldsymbol{R}^n\backslash(L\backslash K)$, $Z = \boldsymbol{R}^n\backslash L$, and to the sheaf $\mathscr{F} = \mathscr{B}_p$. We have $H_{Y\backslash Z}^0(Y, \mathscr{B}_p) = H_K^0(Y, \mathscr{B}_p) = H_K^0(U, \mathscr{B}_p) = 0$ by the excision theorem and by Lemma 1.1; $H_{X\backslash Y}^2(X, \mathscr{B}_p) = 0$ because $\mathscr{B}_p$ is of flabby dimension $\leqq 1$ by (1.3); $H_{Y\backslash Z}^1(Y, \mathscr{B}_p) = H_K^1(U, \mathscr{B}_p)$ by the excision theorem. Thus we have the exact sequence

$$(1.7) \qquad 0 \longrightarrow H_{L\backslash K}^1(\boldsymbol{R}^n, \mathscr{B}_p) \longrightarrow H_L^1(\boldsymbol{R}^n, \mathscr{B}_p) \longrightarrow H_K^1(U, \mathscr{B}_p) \longrightarrow 0\,.$$

Thus we have proved

THEOREM 1.2.

$$\begin{aligned}\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U) &\cong H^1_K(U, \mathcal{B}_p)\\ &\cong H^0_K(U, \mathcal{B})/pH^0_K(U, \mathcal{B})\\ &\cong H^1_L(\boldsymbol{R}^n, \mathcal{B}_p)/H^1_{L\backslash K}(\boldsymbol{R}^n, \mathcal{B}_p)\,.\end{aligned}$$

REMARK 1. From this theorem we see especially that the factor space $\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)$ does not depend on the particular choice of U.

REMARK 2. Theorem 1.2 holds for single operators even if the convexity of K is not assumed. In fact, we have the global existence theorem of Harvey-Komatsu for any open set $U\subset \boldsymbol{R}^n$: $p(D)\mathcal{B}(U)=\mathcal{B}(U)$. On the other hand we have obviously $H^0_K(U, \mathcal{B}_p)\subset H^0_{\Gamma K}(H, \mathcal{B}_p)$, where ΓK is the convex hull of K. Hence $H^0_K(U, \mathcal{B}_p)=0$ by Lemma 1.1. The remaining reasoning is the same as above.

Since L and $L\backslash K$ are compact convex subsets of $\boldsymbol{R}^n$, we have the following representations for the spaces $H^1_L(\boldsymbol{R}^n, \mathcal{B}_p)$, $H^1_{L\backslash K}(\boldsymbol{R}^n, \mathcal{B}_p)$ by vectors of holomorphic functions on the variety $N(p)=\{\zeta\in \boldsymbol{C}^n;\ p(\zeta)=0\}$ of roots of p:

$$(1.8)\qquad \begin{aligned} H^1_{L\backslash K}(\boldsymbol{R}^n, \mathcal{B}_p) &= \widetilde{\mathcal{B}[L\backslash K]}/p\widetilde{\mathcal{B}[L\backslash K]}=\widetilde{\mathcal{B}[L\backslash K]}\{p, d\}\,,\\ H^1_L(\boldsymbol{R}^n, \mathcal{B}_p) &= \widetilde{\mathcal{B}[L]}/p\widetilde{\mathcal{B}[L]}=\widetilde{\mathcal{B}[L]}\{p, d\}\,,\end{aligned}$$

and the exact sequences:

$$(1.9)\qquad \begin{aligned} &0 \longrightarrow \widetilde{\mathcal{B}[L\backslash K]} \xrightarrow{p(\zeta)} \widetilde{\mathcal{B}[L\backslash K]} \xrightarrow{d} \widetilde{\mathcal{B}[L\backslash K]}\{p, d\} \longrightarrow 0\,,\\ &0 \longrightarrow \widetilde{\mathcal{B}[L]} \xrightarrow{p(\zeta)} \widetilde{\mathcal{B}[L]} \xrightarrow{d} \widetilde{\mathcal{B}[L]}\{p, d\} \longrightarrow 0\,.\end{aligned}$$

These are the so called *Fundamental Principle* for $\mathcal{A}(L)$ etc.; see [**4**], Theorem 3.8. We briefly explain the notations.

We employ the notation $\mathcal{B}[L]=H^0_L(\boldsymbol{R}^n, \mathcal{B})$ and $\mathcal{B}[L\backslash K]=H^0_{L\backslash K}(\boldsymbol{R}^n, \mathcal{B})$ as in our earlier papers. $\widetilde{\mathcal{B}[L]}$ and $\widetilde{\mathcal{B}[L\backslash K]}$ denote the Fourier transform of $\mathcal{B}[L]$ and $\mathcal{B}[L\backslash K]$ respectively. Due to the Paley-Wiener-Ehrenpreis-Martineau theorem, $\widetilde{\mathcal{B}[L]}$ is the space of entire functions $F(\zeta)$ with the following growth condition: for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|F(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|+H_L(\zeta))\,.$$

Here $H_L(\zeta)=\sup\limits_{x\in L}\operatorname{Re}\langle x, \sqrt{-1}\,\zeta\rangle$ is the *supporting function* of L. Similarly $\widetilde{\mathcal{B}[L\backslash K]}$ is characterized in the same way with $H_L(\zeta)$ replaced by $H_{L\backslash K}(\zeta)$, the supporting function of $L\backslash K$. $\Big($Note that we employ the Fourier transform $\tilde{u}(\zeta)=\int \exp(\sqrt{-1}\langle x, \zeta\rangle)u(x)dx.\Big)$

The symbol d denotes a *noetherian operator* corresponding to $p(\zeta)$. In general, it is a vector-valued differential operator with polynomial coefficients composed with the restriction map to the variety $N(p)$, and characterized by the following condition: a polynomial $F(\zeta)$ can be divided by $p(\zeta)$ if and only if $dF=0$. In our present case we can assume that each irreducible component of the algebraic variety $N(p)$ is normally placed with respect to ζ_1. Hence if $p=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ is the irreducible decomposition of p, we can employ $d=\{d_1, \cdots, d_k\}$ with

$$d_\lambda F(\zeta)=\Big[{}^t\Big(1, \frac{\partial}{\partial\zeta_1}, \cdots, \frac{\partial^{\alpha_\lambda-1}}{\partial\zeta_1^{\alpha_\lambda-1}}\Big)F(\zeta)\Big]\Big|_{N_\lambda}, \qquad \text{for} \quad \zeta\in N_\lambda,\ \lambda=1, \cdots, k,$$

where $N_\lambda=\{\zeta\in \boldsymbol{C}^n;\ p_\lambda(\zeta)=0\}$ is the λ-th irreducible component of $N(p)$. For definiteness we employ this noetherian operator in the sequel.

A vector of holomorphic functions $\{F_\lambda(\zeta);\ \lambda=1, \cdots, k\}$ on $\{N_\lambda;\ \lambda=1, \cdots, k\}$ is called a *holomorphic p-function* if it is locally in the image of the noetherian operator, namely, if for any point $\zeta_0\in N(p)$ there exist a neighborhood V of ζ_0 in $\boldsymbol{C}^n$ and a holomorphic function $F(\zeta)$ on V such that $F_\lambda(\zeta)=d_\lambda F(\zeta)$ for $\zeta\in N_\lambda\cap V$.

Lastly, $\widetilde{\mathscr{B}[L]}\{p, d\}$ and $\widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$ denote the spaces of holomorphic p-functions which satisfy the same growth condition as that of $\widetilde{\mathscr{B}[L]}$ described above and that of $\widetilde{\mathscr{B}[L\backslash K]}$ respectively. For fuller details of the terminology see [**13**].

DEFINITION 1.3 (Notation). We denote by $\widetilde{\mathscr{B}[L]/\mathscr{B}[L\backslash K]}\{p, d\}$ the quotient space $\widetilde{\mathscr{B}[L]}\{p, d\}/\widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$.

Combining this with Theorem 1.2 we have

PROPOSITION 1.4. *$\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)\underset{\tilde{d}}{\cong}\widetilde{\mathscr{B}[L]/\mathscr{B}[L\backslash K]}\{p, d\}$. The correspondence is given in the following way: For $u\in\mathscr{B}_p(U\backslash K)$, let $[u]\in\mathscr{B}(U)$ be an extension of u and let $[[p(D)[u]]]\in\mathscr{B}[L]$ be an extension of $p(D)[u]\in H_K^0(U, \mathscr{B})$. Then, $\tilde{d}\cdot u=d\cdot\widetilde{[[p(D)[u]]]}\in\widetilde{\mathscr{B}[L]}\{p, d\}$* mod *$\widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$ is the corresponding element in the right hand side.*

REMARK 3. The notation in Definition 1.3 is natural since we have the following commutative exact diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{\mathscr{B}[L\backslash K]} \arrow[r, "p(\zeta)"] \arrow[d] & \widetilde{\mathscr{B}[L\backslash K]} \arrow[r, "d"] \arrow[d] & \widetilde{\mathscr{B}[L\backslash K]}\{p, d\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\mathscr{B}[L]} \arrow[r, "p(\zeta)"] \arrow[d] & \widetilde{\mathscr{B}[L]} \arrow[r, "d"] \arrow[d] & \widetilde{\mathscr{B}[L]}\{p, d\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{\mathscr{B}[L]/\mathscr{B}[L\backslash K]} \arrow[r, "p(\zeta)"] \arrow[d] & \widetilde{\mathscr{B}[L]/\mathscr{B}[L\backslash K]} \arrow[r, "d"] \arrow[d] & \widetilde{\mathscr{B}[L]/\mathscr{B}[L\backslash K]}\{p, d\} \arrow[r] \arrow[d] & 0. \\
& 0 & 0 & 0 &
\end{tikzcd}$$

REMARK 4. In the case of single operator p, we can deduce the isomorphism in Proposition 1.4 directly by the explicit definition of the mapping $\tilde{d}$ given there, as in the proof of [3], Lemma 4. We only need the fundamental principle (1.9), thus avoiding tedious sequences of arrows. But the above way of argument is so general as to be applied to the systems of operators. (See Remark 5 after the proof of Theorem 1.5.)

THEOREM 1.5. *$\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)=0$ if and only if for any $\varepsilon>0$, there exists some $C_\varepsilon>0$ such that the following inequality holds:*

$$H_L(\zeta) \leqq \varepsilon|\zeta| + H_{L\backslash K}(\zeta) + C_\varepsilon, \qquad \zeta \in N(p). \tag{1.10}$$

PROOF. The sufficiency follows directly from Proposition 1.4. In fact, assuming the above inequality we have the inclusion $\widetilde{\mathscr{B}[L]}\{p, d\} \subset \widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$, hence the right hand side of the identity in the proposition vanishes.

Now we prove the necessity. Let $a\in L$ be an arbitrary point. Let $E\in\mathscr{B}(\boldsymbol{R}^n)$ be a translation of a fundamental solution: $p(D)E=\delta(x-a)$, where δ is the *Dirac delta function*. Clearly E belongs to $\mathscr{B}_p(U\backslash K)$. Therefore $\widetilde{\mathscr{B}[L]}\{p, d\}$ contains a vector function $d[[\widetilde{p(D)E}]] = d\cdot\exp(\sqrt{-1}\langle a, \zeta\rangle)$, which contains the function $\exp(\sqrt{-1}\langle a, \zeta\rangle)$ in its components. Now suppose that $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)=0$. Then by Proposition 1.4 we have $\widetilde{\mathscr{B}[L]}\{p, d\} \subset \widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$, so that the following estimate must hold for the function $\exp(\sqrt{-1}\langle a, \zeta\rangle)$: for any $\varepsilon>0$ there exists some $C_\varepsilon>0$ such that

$$|\exp(\sqrt{-1}\langle a, \zeta\rangle)| \leqq C_\varepsilon \exp(\varepsilon|\zeta| + H_{L\backslash K}(\zeta)). \tag{1.11}$$

The desired inequality (1.10) follows from this. In fact, suppose that (1.10) does not hold. Then, there are an $\varepsilon>0$ and a sequence $\{\zeta^{(k)}\}_{k=1}^\infty \subset N(p)$ such that

$$H_L(\zeta^{(k)}) \geqq \varepsilon|\zeta^{(k)}| + H_{L\setminus K}(\zeta^{(k)}) + k\,. \tag{1.12}$$

Since L is compact the supremum can be replaced by the maximum: We have $H_L(\zeta) = \max_{x\in L} \mathrm{Re}\langle x, \sqrt{-1}\,\zeta\rangle$, hence we can find a sequence $\{a_k\} \subset L$ such that $\mathrm{Re}\langle a_k, \sqrt{-1}\,\zeta^{(k)}\rangle = H_L(\zeta^{(k)})$. Taking a subsequence if necessary, we can assume that a_k converge to some point $a_0 \in L$ when k tends to infinity. Since a is arbitrary we can take $a = a_0$. Now by (1.12) we have

$$\mathrm{Re}\langle a_k, \sqrt{-1}\,\zeta^{(k)}\rangle \geqq \varepsilon|\zeta^{(k)}| + H_{L\setminus K}(\zeta^{(k)}) + k\,.$$

On the other hand, by (1.11) we have

$$\mathrm{Re}\langle a_0, \sqrt{-1}\,\zeta^{(k)}\rangle \leqq (\varepsilon/2)|\zeta^{(k)}| + H_{L\setminus K}(\zeta^{(k)}) + \log C_{\varepsilon/2}\,.$$

Combining these we obtain

$$\begin{aligned}(\varepsilon/2)|\zeta^{(k)}| &\leqq \mathrm{Re}\langle a_0 - a_k, \sqrt{-1}\,\zeta^{(k)}\rangle - k + \log C_{\varepsilon/2} \\ &\leqq |a_0 - a_k|\cdot|\zeta^{(k)}| - k + \log C_{\varepsilon/2}\,.\end{aligned}$$

Hence

$$((\varepsilon/2) - |a_0 - a_k|)|\zeta^{(k)}| \leqq -k + \log C_{\varepsilon/2}\,.$$

Since $|a_0 - a_k| \to 0$ by the assumption, and $|\zeta^{(k)}| \to \infty$ by (1.12), we have a contradiction when we let $k \to \infty$. q. e. d.

REMARK 5. Let $p(D)$ be a general system of operators corresponding to the matrix $p(\zeta): \mathcal{P}^s \to \mathcal{P}^t$, where $\mathcal{P}$ denotes the ring of all the polynomials of ζ. Let $p_1(D)$ be a *compatibility system* of p. Put $M = \mathrm{Coker}\, p'$, where p' is the transposed matrix of p. Then a similar argument combined with the results of [4] gives the following isomorphisms:

$$\begin{aligned}\mathcal{B}_p(U\setminus K)/\lambda\mathcal{B}_p(U) &= H^1_K(U, \mathcal{B}_p) \\ &= H^0_K(U, \mathcal{B}_{p_1})/p[H^0_K(U, \mathcal{B})]^s \\ &= H^1_L(\boldsymbol{R}^n, \mathcal{B}_p)/H^1_{L\setminus K}(\boldsymbol{R}^n, \mathcal{B}_p) \\ &= \widetilde{\mathcal{B}[L]}\{\mathrm{Ext}^1(M, \mathcal{P}), d'_p\}/\widetilde{\mathcal{B}[L\setminus K]}\{\mathrm{Ext}^1(M, \mathcal{P}), d'_p\}\,.\end{aligned}$$

Here $\widetilde{\mathcal{B}[L]}\{\mathrm{Ext}^1(M, \mathcal{P}), d'_p\}$ denotes the space of vectors of holomorphic functions on the family of algebraic varieties $N(\mathrm{Ext}^1(M, \mathcal{P}))$ which have the same growth order as elements of $\widetilde{\mathcal{B}[L]}$ and which are locally in the image of a certain noetherian operator d'_p. $\widetilde{\mathcal{B}[L\setminus K]}\{\mathrm{Ext}^1(M, \mathcal{P}), d'_p\}$ has a similar meaning with $\widetilde{\mathcal{B}[L]}$ replaced by $\widetilde{\mathcal{B}[L\setminus K]}$. (For the details see [4], p. 421.) Thus the assertion of Thorem 1.5 can be modified to the case of systems in the following way: $\mathcal{B}_p(U\setminus K)/\lambda\mathcal{B}_p(U) = 0$ if and only if the inequality (1.10) is satisfied for $\zeta \in N(\mathrm{Ext}^1(M, \mathcal{P}))$, where λ denotes the restriction map. If, and only if p is determined, we can omit the symbol λ, that is, we have the unique way of extension of solutions.

Now, let us say for the present that a closed convex cone C with the origin as its vertex is the *propagation cone of* p, if its dual cone

$$C^0=\{\eta\in \boldsymbol{R}^n;\ \langle x, \eta\rangle \geqq 0 \text{ for any } x\in C\}$$

is the smallest of the closed cones which contain every cone Γ satisfying the following condition: For any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|\mathrm{Im}\,\zeta|\leqq \varepsilon|\zeta|+C_\varepsilon, \qquad \text{when} \quad \zeta\in N(p) \quad \text{and} \quad \mathrm{Im}\,\zeta\in\Gamma. \tag{1.13}$$

(Of course such a cone as Γ may not exist. Then we must put $C=\boldsymbol{R}^n$.) We can paraphrase the condition (1.10) as follows.

LEMMA 1.6. (1.10) *is equivalent to the following condition:*

$$(a+C)\cap H\subset K \qquad \textit{for any} \quad a\in K. \tag{1.14}$$

Here $a+C$ *denotes the set* $\{x\in \boldsymbol{R}^n;\ x=a+y,\ y\in C\}$.

PROOF. First note that (1.14) implies in particular that C is properly contained in the upper half space $\{x\in \boldsymbol{R}^n;\ x_n\geqq 0\}$. (Given two cones Γ_1, Γ_2, in the upper half space with their vertices at the origin, we will say that Γ_1 is properly contained in Γ_2 if $\Gamma_1\cap\{x\in \boldsymbol{R}^n;\ x_n=1\}\Subset \Gamma_2\cap\{x\in \boldsymbol{R}^n;\ x_n=1\}$.) Since (1.14) is a relation concerning two convex sets, it is obviously equivalent to the following:

$$(a+C)\cap \partial H\subset L\backslash K. \tag{1.15}$$

Further, it can be rewritten in terms of supporting functions:

$$H_{(a+C)\cap\partial H}(\zeta)\leqq H_{L\backslash K}(\zeta). \tag{1.16}$$

Put $a=(a', a_n)$. We have obviously

$$\begin{aligned} &H_{(a+C)\cap\partial H}(\zeta) \\ &\quad=\sup_x\,[\mathrm{Re}\,\langle x, \sqrt{-1}\,\zeta\rangle;\ x\in C\cap\{x\in \boldsymbol{R}^n;\ x_n=-a_n\}]-\langle a, \mathrm{Im}\,\zeta\rangle \\ &\quad=\sup_x\,[-a_n\,\mathrm{Re}\,\langle x, \sqrt{-1}\,\zeta\rangle;\ x\in C\cap\{x\in \boldsymbol{R}^n;\ x_n=1\}]-\langle a, \mathrm{Im}\,\zeta\rangle \\ &\quad=a_n\inf_x\,[\langle x, \mathrm{Im}\,\zeta\rangle;\ x\in C\cap\{x\in \boldsymbol{R}^n;\ x_n=1\}]-\langle a, \mathrm{Im}\,\zeta\rangle. \end{aligned} \tag{1.17}$$

Now assuming (1.10) let us prove (1.14). Let C' be the convex proper cone in the upper half space, with its vertex at the origin, which satisfies

$$(a+C')\cap\partial H=L\backslash K \tag{1.18}$$

for some point $a\in K$. (That is, let C' be the translation to the origin of the convex proper cone generated by the half lines drawn from the point a to the points of $L\backslash K$.) In the following we will show that the estimate (1.13) is satisfied for any cone properly contained in C'^0, the dual cone of C'. That will show, in particular, that $C^0 \supset C'^0$, namely, $C \subset C'$. Since $a \in K$ is arbitrary, we will conclude that C must satisfy (1.15) for any $a \in K$. Thus (1.14) will be proved.

Apply the calculation in (1.17) to C'. Let Γ be a cone properly contained in C'^0. Then there exists $\delta > 0$ such that

$$\inf_x [\langle x, \operatorname{Im} \zeta \rangle ; \ x \in C' \cap \{x \in \boldsymbol{R}^n ; \ x_n = 1\}] \geqq \delta |\operatorname{Im} \zeta| ,$$

when $\operatorname{Im} \zeta \in \Gamma$. Thus (1.18) and (1.17) imply

$$H_{L \setminus K}(\zeta) = H_{(a+C') \cap \partial H}(\zeta) \leqq a_n \delta |\operatorname{Im} \zeta| - \langle a, \operatorname{Im} \zeta \rangle , \qquad \text{when} \quad \operatorname{Im} \zeta \in \Gamma .$$

On the other hand, (1.10) implies, for any $\varepsilon > 0$,

$$\begin{aligned} a_n \delta |\operatorname{Im} \zeta| - \langle a, \operatorname{Im} \zeta \rangle &\leqq a_n \delta |\operatorname{Im} \zeta| + H_L(\zeta) \\ &\leqq a_n \delta |\operatorname{Im} \zeta| + H_{L \setminus K}(\zeta) + \varepsilon |\zeta| + C_\varepsilon , \end{aligned}$$

with some $C_\varepsilon > 0$, when $\zeta \in N(p)$. Thus we conclude

$$(-a_n)\delta |\operatorname{Im} \zeta| \leqq \varepsilon |\zeta| + C_\varepsilon , \qquad \text{when} \quad \operatorname{Im} \zeta \in \Gamma \text{ and } \zeta \in N(p) .$$

Since $(-a_n)\delta$ is a positive constant independent of ε, we have proved (1.13), hence (1.14), as remarked at the beginning.

Conversely, assume (1.14). Then, by (1.16) and by the calculation of (1.17) we have

$$a_n \inf_x [\langle x, \operatorname{Im} \zeta \rangle ; \ x \in C \cap \{x \in \boldsymbol{R}^n ; \ x_n = 1\}] - \langle a, \operatorname{Im} \zeta \rangle \leqq H_{L \setminus K}(\zeta) .$$

Assume first that $\operatorname{Im} \zeta$ does not belong to the interior of C^0. Then, we have

$$\inf_x [\langle x, \operatorname{Im} \zeta \rangle ; \ x \in C \cap \{x \in \boldsymbol{R}^n ; \ x_n = 1\}] \leqq 0 .$$

Since $a_n < 0$, we can thereby omit the first term and obtain

$$(1.19) \qquad -\langle a, \operatorname{Im} \zeta \rangle \leqq H_{L \setminus K}(\zeta) .$$

Now consider the set

$$\Gamma_\varepsilon = \{\eta ; \ \langle x, \eta \rangle \geqq \varepsilon |\eta| \ \text{ for any } x \in C \cap \{x \in \boldsymbol{R}^n ; \ x_n = 1\}\} .$$

This is a closed cone properly contained in C^0. Thus by the assumption on C we have, when $\operatorname{Im} \zeta \in \Gamma_\varepsilon$ and $\zeta \in N(p)$,

$$|\operatorname{Im} \zeta| \leqq \varepsilon |\zeta| + C_\varepsilon ,$$

with some $C_\varepsilon > 0$. Hence, for such ζ we have obviously

$$(1.20) \qquad H_L(\zeta) \leqq H_{L \setminus K}(\zeta) + A\varepsilon |\zeta| + C_\varepsilon ,$$

where A is a constant depending only on the size of L.

On the other hand, assume $\operatorname{Im}\zeta \notin \Gamma_\varepsilon$. Let $x(\operatorname{Im}\zeta)$ be one of the points of $C \cap \{x \in \boldsymbol{R}^n;\ x_n = 1\}$ at which $\langle x, \operatorname{Im}\zeta\rangle$ attains its minimum. (Note that $C \cap \{x \in \boldsymbol{R}^n;\ x_n = 1\}$ is a compact set.) We have

$$\inf_x \Big[\Big\langle x, \operatorname{Im}\zeta - \frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big\rangle;\ x \in C \cap \{x \in \boldsymbol{R}^n;\ x_n = 1\}\Big]$$

$$\leqq \Big\langle x(\operatorname{Im}\zeta), \operatorname{Im}\zeta - \frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big\rangle$$

$$\leqq \varepsilon|\operatorname{Im}\zeta| - \varepsilon|\operatorname{Im}\zeta| = 0.$$

Hence we have from (1.19)

$$-\langle a, \operatorname{Im}\zeta\rangle \leqq -\Big\langle a, \operatorname{Im}\zeta - \frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big\rangle - \Big\langle a, \frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big\rangle$$

$$\leqq H_{L\setminus K}\Big(\zeta - \sqrt{-1}\frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big) - \Big\langle a, \frac{\varepsilon|\operatorname{Im}\zeta| x(\operatorname{Im}\zeta)}{\langle x(\operatorname{Im}\zeta), x(\operatorname{Im}\zeta)\rangle}\Big\rangle$$

$$\leqq H_{L\setminus K}(\zeta) + B\varepsilon|\zeta|,$$

where B is a constant depending only on the sets L and C. Thus taking the supremum with respect to $a \in K$ in the left hand side, we have

$$H_L(\zeta) \leqq H_{L\setminus K}(\zeta) + B\varepsilon|\zeta|. \tag{1.21}$$

Combining (1.20) with (1.21) we obtain

$$H_L(\zeta) \leqq H_{L\setminus K}(\zeta) + (A+B)\varepsilon|\zeta| + C_\varepsilon,$$

for $\zeta \in N(p)$. Since $\varepsilon > 0$ is arbitrary, we have obtained (1.10). q. e. d.

We shall say that p is *hyperbolic with respect to the direction* $(0, \cdots, 0, 1)$ (*in the sense of hyperfunctions*) if the propagation cone of p is properly contained in the upper half space $\{x \in \boldsymbol{R}^n;\ x_n \geqq 0\}$. Thus, we have

COROLLARY 1.7. *$\mathscr{B}_p(U\setminus K)/\mathscr{B}_p(U) = 0$ if and only if p is hyperbolic with respect to $(0, \cdots, 0, 1)$ and its propagation cone satisfies* (1.14).

In [7] Kawai has proved that we can construct a fundamental solution E of $p(D)E = \delta$ whose support is contained in a cone properly contained in the upper half space if and only if p is hyperbolic with respect to $(0, \cdots, 0, 1)$. (In fact, we can immediately make his proof so precise as to see that the convex hull of the support of the fundamental solution agrees with the propagation cone C.)

It is known (cf. [7]) that p is hyperbolic with respect to $(0, \cdots, 0, 1)$ if and only if $p_m(0, \cdots, 0, 1) \neq 0$ and the equation $p_m(\zeta', \zeta_n) = 0$ in ζ_n has only real roots when ζ' is real, where p_m is the principal part of p and $\zeta = (\zeta', \zeta_n)$ is the corresponding partition of the variables. The propagation cone of p is readily seen to be the following:

$C=$ the dual cone of the connected component of the set
$\{\eta\in \boldsymbol{R}^n;\ p_m(\eta)\neq 0\}$ containing $(0,\cdots,0,1)$.

Note that Corollary 1.7 has a meaning even in case p is a general system (see Remark 5), though the significance of the propagation cone is not clear in that case.

REMARK 6. If we take the propagation cone of a hyperbolic operator to be the convex hull of the support of the fundamental solution specified above, then we can prove Corollary 1.7 more directly. In fact, assume that $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)=0$. Let E be a solution of $p(D)E=\delta$, where δ is the Dirac delta function. For $a\in K$, the function $u(x)=E(x-a)|_{U\backslash K}$ belongs to $\mathscr{B}_p(U\backslash K)$. Hence by the assumption u can be extended to an element $v\in\mathscr{B}_p(U)$. The function $w(x)=E(x-a)-v(x)$ on U satisfies $p(D)w=\delta$, and $\operatorname{supp} w\subset K$. By the usual argument we can conclude from these relations that p is hyperbolic with respect to $(0,\cdots,0,1)$. Therefore, let E be the fundamental solution for which the convex hull of its support gives the propagation cone C. Now assume that (1.14) dose not hold for some point $a\in K$. Then the function $u(x)=E(x-a)|_{U\backslash K}$ cannot be extended to an element of $\mathscr{B}_p(U)$, which shows the necessity of (1.14). In fact, if there would exist an extension $v\in\mathscr{B}_p(U)$, then applying Lemma 1.1 to the set $(a+C)\cap H$ we would have $v\equiv 0$. (We can assume without loss of generality that $(a+C)\cap H\subset U$, U being sufficiently large.) Hence $E(x-a)=0$ in $U\backslash K$. This is a contradiction.

Conversely, let p be hyperbolic with respect to $(0,\cdots,0,1)$ and assume that (1.14) is satisfied. Then we can little by little solve the Cauchy problem and construct the solution on the whole U. Condition (1.14) guarantees the coherency of the local solutions given in individual steps. We omit the details.

REMARK 7. We give an example which shows that in order to extend hyperfunction solutions the hyperbolicity of p is not sufficient in general and the condition on the shape of K is really necessary. Assume $n=2$ for simplicity. For the set $K=\{(s,-t);\ -t\leqq s\leqq t,\ 0<t\leqq 1\}\subset\boldsymbol{R}^2$, the condition (1.14) is not satisfied for any operator p. Thus we have $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)\neq 0$ for any p. Note that this has been the case when $K\Subset U$. Let $p(D)=D_1$, for example. The following function really gives a nontrivial element of $\mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U)$.

$$u(x_1,x_n)=\begin{cases}1-x_1 & \text{for}\quad x_1>-x_n,\ 0<x_1\leqq 1\,,\\ 1+x_1 & \text{for}\quad x_1<x_n,\ -1\leqq x_1<0\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

It is obvious that we can also give an example in infinitely differentiable solutions.

I do not know the exact literature concerning this problem. Only we point out the work of Malgrange [**10**]. In that work he proved that infinitely differentiable solutions of an overdetermined system defined in $U\backslash K$ can be continued to K, and remarked that there exist single operators having the same property for fixed K. (See also Palamodov [**13**] for distribution solutions.) The result of this section can be easily translated to other classes of solutions of non-quasianalytic type. We only have to change the meaning of hyperbolicity according to the classes of solutions. (Cf. the argument of Remark 6.) Essential difference arises when we treat quasianalytic solutions or when we treat regular solutions outside a set K with no interior points.

§ 2. Continuation of real analytic solutions.

Let $\mathcal{A}$ denote the sheaf of germs of real analytic functions, $\mathcal{A}_p$ the sheaf of germs of real analytic solutions of $p(D)u=0$. We employ the usual notations $\mathcal{A}(U)=H^0(U, \mathcal{A})$, $\mathcal{A}_p(U)=H^0(U, \mathcal{A}_p)$. Now we are going to discuss the conditions on p on which $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$ holds. For this purpose, we first quote a result on propagation of regularities.

THEOREM 2.1 (T. Kawai). $\mathcal{B}_p(U)\cap\mathcal{A}(U\backslash K)\subset\mathcal{A}_p(U)$.

The proof is carried out in a way similar to Lemma 1.1 but with a more delicate argument using the Fourier hyperfunctions and the sheaf of rapidly decreasing real analytic functions. See [7], Theorem 5.1.1.

REMARK 8. From this theorem we see that the natural map

$$\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)\longrightarrow\mathcal{B}_p(U\backslash K)/\mathcal{B}_p(U)$$

is injective. Hence, in showing that a solution $u\in\mathcal{A}_p(U\backslash K)$ can be extended to an element of $\mathcal{A}_p(U)$, we only have to show that it can be extended to an element of $\mathcal{B}_p(U)$.

We must make a detailed study of the image of the real analytic solutions under the map $\tilde{d}$ in the isomorphism in Proposition 1.4. For this purpose, we are going to seek another expression of the map $\tilde{d}$ for real analytic solutions. Let $\chi\in C^\infty(U)$ be such that $\chi=1$ on a neighborhood of K, and $\overline{\operatorname{supp}\chi}\cap\partial U\subset L\backslash K$, where the closure or the boundary are taken in $\boldsymbol{R}^n$. Take $u\in\mathcal{A}_p(U\backslash K)$ arbitrarily. Then $\operatorname{supp}p(D)(\chi u)\cap K=\emptyset$, so that, we can extend $p(D)(\chi u)$ to K by zero and obtain an element of $H^0_{\operatorname{supp}\chi}(U, C^\infty)$. Let $[[p(D)(\chi u)]]_0$ be one of its extension to an element of $H^0_{\overline{\operatorname{supp}\chi}}(\boldsymbol{R}^n, \mathcal{B})$.

LEMMA 2.2.

$$\tilde{d}\cdot u=-d\cdot\widetilde{[[p(D)(\chi u)]]_0}\mod\widetilde{\mathcal{B}[L\backslash K]}\{p, d\}\,.$$

PROOF. Let $[u]\in\mathcal{B}(U)$ be an extension of u and let $[[\chi[u]]]\in H^0_{\overline{\operatorname{supp}\chi}}(\boldsymbol{R}^n, \mathcal{B})$ be an extention of $\chi[u]$. Then we have obviously

$$p(D)[[\chi[u]]]\equiv[[p(D)(\chi u)]]_0+[[p(D)[u]]] \mod \mathscr{B}[L\backslash K].$$

Hence, the definition of the noetherian operator implies

$$0=d\cdot p(\zeta)\widetilde{[[\chi[u]]]}\equiv d\widetilde{[[p(D)(\chi u)]]_0}+d\widetilde{[[p(D)[u]]]} \mod \widetilde{\mathscr{B}[L\backslash K]}\{p,d\}.$$

Thus we have proved

$$\begin{aligned}\tilde{d}\cdot u&=d\widetilde{[[p(D)[u]]]} \mod \widetilde{\mathscr{B}[L\backslash K]}\{p,d\}\\ &=-d\widetilde{[[p(D)(\chi u)]]_0} \mod \widetilde{\mathscr{B}[L\backslash K]}\{p,d\}.\end{aligned}$$

q. e. d.

PROPOSITION 2.3. *Let $\{F_\lambda(\zeta);\ \lambda=1,\cdots,k\}$ be a holomorphic p-function which represents the residue class $\tilde{d}\cdot u$ corresponding to $u\in\mathscr{A}_p(U\backslash K)$. Then, for every λ, any component $F(\zeta)$ of the vector $F_\lambda(\zeta)$ has the following property: Given any $\varepsilon>0$ and any infra-exponential entire function $J(\zeta)$, we can find holomorphic functions $f_{J,\varepsilon}$ and $g_{J,\varepsilon}$ on the corresponding component N_λ such that they give the decomposition $J(\zeta)F(\zeta)=f_{J,\varepsilon}(\zeta)+g_{J,\varepsilon}(\zeta)$ and they satisfy the following estimates. $f_{J,\varepsilon}$ satisfies, for any $\eta>0$ and for some $C_{J,\varepsilon,\eta}>0$ depending on η,*

$$(2.1)\qquad |f_{J,\varepsilon}(\zeta)|\leqq C_{J,\varepsilon,\eta}\exp(\eta|\zeta|+\varepsilon|\mathrm{Im}\,\zeta|+H_{L\backslash K}(\zeta)),\quad \text{when}\quad \zeta\in N_\lambda.$$

$g_{J,\varepsilon}$ satisfies, for any $k\geqq 0$ and for some $C_{J,\varepsilon,k}>0$ depending on k,

$$(2.2)\qquad |g_{J,\varepsilon}(\zeta)|\leqq C_{J,\varepsilon,k}(1+|\zeta|)^{-k}\exp\Big(\varepsilon|\mathrm{Im}\,\zeta'|+\frac{\varepsilon}{2}\mathrm{Im}\,\zeta_n+H_L(\zeta)\Big),\quad \text{when}\ \zeta\in N_\lambda.$$

PROOF. We employ here the local operator with constant coefficients. Local operators are a kind of differential operators of infinite order appearing in the theory of hyperfunctions. We employ here only those with constant coefficients. By the Fourier transform they correspond to the operators of multiplication by the infra-exponential entire functions (or, entire functions of minimal type of order one). For the details see [5], § 1.

Now let $J(D)$ be a local operator with constant coefficients. First remark that $J(D)u$ also belongs to $\mathscr{A}_p(U\backslash K)$. Thus by Lemma 2.2 we have

$$\tilde{d}\cdot J(D)u=-d\widetilde{[[p(D)(\chi J(D)u)]]_0} \mod \widetilde{\mathscr{B}[L\backslash K]}\{p,d\}.$$

By a smooth cut-off function we decompose $[[p(D)(\chi J(D)u)]]_0$ into $v+w$ such that $\mathrm{supp}\,v\subset\{x_n>-\varepsilon\}\cap L_\varepsilon$, $\mathrm{supp}\,w\subset\Big\{x_n<-\frac{\varepsilon}{2}\Big\}\cap L_\varepsilon$ and $w\in C_0^\infty(\boldsymbol{R}^n)$, where L_ε is the ε-neighborhood of L. Thus

$$\tilde{d}\cdot J(D)u=-d\cdot\tilde{v}-d\cdot\tilde{w} \mod \widetilde{\mathscr{B}[L\backslash K]}\{p,d\}.$$

By the Paley-Wiener theorem the two terms on the right hand side satisfy the desired estimates. Adjusting by a suitable element of $\widetilde{\mathscr{B}[L\backslash K]}\{p,d\}$, we have obtained the decomposition for any representative $F(\zeta)$ of $\tilde{d}\cdot J(D)u$.

In order to obtain the decomposition for $J(\zeta)F(\zeta)$, recall that $\vec{d}\cdot J(D)u$ has the following form

$$
\begin{aligned}
(2.3)\qquad & \vec{d}\cdot J(D)u \\
&= d[[\widetilde{p(D)[J(D)u]}]] \bmod \widetilde{\mathscr{B}[L\backslash K]}\{p, d\} \\
&= d[[\widetilde{J(D)p(D)[u]}]] + d[[\widetilde{p(D)([J(D)u]-J(D)[u])}]] \bmod \widetilde{\mathscr{B}[L\backslash K]}\{p, d\} \\
&= d\cdot J(D)[[\widetilde{p(D)[u]}]] + d\cdot p(D)[[\widetilde{[J(D)u]-J(D)[u]}]] \bmod \widetilde{\mathscr{B}[L\backslash K]}\{p, d\} \\
&= d\cdot J(\zeta)[[\widetilde{p(D)[u]}]] + d\cdot p(\zeta)[[\widetilde{[J(D)u]-J(D)[u]}]] \quad \bmod \widetilde{\mathscr{B}[L\backslash K]}\{p, d\}\,.
\end{aligned}
$$

Here $J(\zeta)$ is the infra-exponential entire function corresponding to $J(D)$. The last term vanishes by the definition of the noetherian operator. Thus we have

$$\vec{d}\cdot J(D)u = d\cdot J(\zeta)[[\widetilde{p(D)[u]}]] \quad \bmod \widetilde{\mathscr{B}[L\backslash K]}\{p, d\}\,.$$

Further the component $d_\lambda J(\zeta)[[\widetilde{p(D)[u]}]]$ of $d\cdot J(\zeta)[[\widetilde{p(D)[u]}]]$ corresponding to the irreducible component N_λ of $N(p)$ has the following form

$$
d_\lambda J(\zeta)[[p(D)[u]]] = \begin{bmatrix} J(\zeta)[[\widetilde{p(D)[u]}]]|_{N_\lambda} \\ \vdots \\ \dfrac{\partial^{\alpha_\lambda-1}}{\partial\zeta_1^{\alpha_\lambda-1}} J(\zeta)[[\widetilde{p(D)[u]}]]|_{N_\lambda} \end{bmatrix}
$$

$$
= \begin{bmatrix} J(\zeta) & & 0 \\ \vdots & \ddots & \\ \dfrac{\partial^{\alpha_\lambda-1}}{\partial\zeta_1^{\alpha_\lambda-1}} J(\zeta) & \cdots & J(\zeta) \end{bmatrix} \times \begin{bmatrix} [[\widetilde{p(D)[u]}]]|_{N_\lambda} \\ \vdots \\ \dfrac{\partial^{\alpha_\lambda-1}}{\partial\zeta_1^{\alpha_\lambda-1}} [[\widetilde{p(D)[u]}]]|_{N_\lambda} \end{bmatrix}.
$$

(Here we have assumed that we can adopt the noetherian operator $d=\{d_\lambda\}$, $d_\lambda = {}^t(1, \cdots, \partial^{\alpha_\lambda-1}/\partial\zeta_1^{\alpha_\lambda-1})|_{N_\lambda}$ indicated in the explanations of (1.9) in §1. Therefore the above equality is a direct consequence of the Leibniz formula.) Note that the derivatives of $J(\zeta)$ are also infra-exponential entire functions. Thus, from this formula we see that each component of $F_\lambda(\zeta) = {}^t([[\widetilde{p(D)[u]}]]|_{N_\lambda}, \cdots, \partial^{\alpha_\lambda-1}/\partial\zeta_1^{\alpha_\lambda-1}[[\widetilde{p(D)[u]}]]|_{N_\lambda})$ also admits a decomposition stated in our proposition, step by step from the earlier ones. q. e. d.

So far we have fixed a convex set K to which we intended to extend the solutions. Now we need to consider a family $\mathscr{K}$ of such sets.

LEMMA 2.4. *Let $\mathscr{K}$ be a family of sets K's which satisfies: if $K \in \mathscr{K}$, then also $[K+(0, \cdots, 0, \delta)] \cap \{x_n < 0\} \in \mathscr{K}$ for any $\delta > 0$. Assume that for each irreducible component p_λ of p we have $\mathscr{A}_{p_\lambda}(U\backslash K)/\mathscr{A}_{p_\lambda}(U) = 0$ for any $K \in \mathscr{K}$ and for any open convex neighborhood U. Then we have $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U) = 0$*

for any $K \in \mathcal{K}$ *and for any such* U.

PROOF. Let $p = p_1 \cdots p_k$ be the irreducible decomposition of p. (Some of p_j may agree.) We employ the induction on k. Assume that the lemma is proved for any p with $k-1$ irreducible components. Take $u \in \mathcal{A}_p(U \backslash K)$. Then $v_1 = p_2(D) \cdots p_k(D)u \in \mathcal{A}_{p_1}(U \backslash K)$. Therefore by the assumption we have $v_1 \in \mathcal{A}_{p_1}(U)$. The equation

$$p_2(D) \cdots p_k(D)u_1 = v_1$$

has an analytic solution u_1 in a convex neighborhood U_1 of $K_1 = K \cap \{x_n < -\varepsilon\}$ which is relatively compact in U (see [**9**], Theorem 3.1). Put $u_2 = u - u_1$. Then $u_2 \in \mathcal{A}_{p_2 \cdots p_k}(U_1 \backslash K_1)$. Thus, by the assumption on $\mathcal{K}$ we can apply the induction hypothesis to the operator $p_2 \cdots p_k$ and to the set K_1, since our operator is invariant under translation along x_n-axis. Thus we conclude that $u_2 \in \mathcal{A}(U_1)$, hence $u = u_1 + u_2 \in \mathcal{A}(U_1)$. Due to the property of unique continuation of analytic functions, we conclude that $u \in \mathcal{A}(U \cap \{x_n < -\varepsilon\})$. Since ε is arbitrary, we finally conclude that $u \in \mathcal{A}(U)$, which obviously implies $u \in \mathcal{A}_p(U)$. q. e. d.

REMARK 9. The converse implication of the lemma trivially holds with no restriction on the family $\mathcal{K}$. In fact, we have $\mathcal{A}_{p_\lambda}(U \backslash K) / \mathcal{A}_{p_\lambda}(U) \subset \mathcal{A}_p(U \backslash K) / \mathcal{A}_p(U)$ for each λ.

Next we need to change the convex neighborhood U of K. For this reason we discuss to what extent U affects the possibility of continuation of real analytic solutions.

LEMMA 2.5. *Let* $\mathcal{K}$ *be a family of* K*'s satisfying the condition of Lemma 2.4. Let* C_0 *be either a fixed open convex cone with its vertex at the origin and properly contained in the upper half space* $\{x \in \boldsymbol{R}^n;\ x_n > 0\}$, *or a fixed cylindrical domain of the form* $C_0 = \{x \in \boldsymbol{R}^n;\ x_n > 0,\ |x_j| < c,\ j = 1, \cdots, n-1\}$. *Assume that we have* $\mathcal{A}_p(V \backslash K) / \mathcal{A}_p(V) = 0$ *for any* $K \in \mathcal{K}$ *and for any* $V \supset K$ *of the form* $V = (a + C_0) \cap H$ *with* $a = (0, \cdots, 0, a_n) \in H$. *(In case* C_0 *is cylindrical, we assume that for every* $K \in \mathcal{K}$ *there exists at least one such* V *containing* K.*) Then we have* $\mathcal{A}_p(U \backslash K) / \mathcal{A}_p(U) = 0$ *for any pair* $U \supset K$ *with* $K \in \mathcal{K}$.

PROOF. First assume that the given $U \supset K$ is convex, and contained in some V of the form $(a + C_0) \cap H$. Put

$$U_{\varepsilon,\delta} = \{x \in U;\ \mathrm{dis}(x, \partial U) > \delta\} \cap \{x_n < -\varepsilon\}\ ,$$

$$V_\varepsilon = V \cap \{x_n < -\varepsilon\}\ ,$$

and

$$K_\varepsilon = K \cap \{x_n \leqq -\varepsilon\}\ .$$

Let $\bar{U}_{\varepsilon,\delta}$, $\bar{V}_\varepsilon$ be their closure taken in $\boldsymbol{R}^n$. We choose δ so small that $\bar{U}_{\varepsilon,\delta} \supset K_\varepsilon$. Now we employ the exact sequence:

$$
(2.4) \qquad
\begin{array}{c}
\begin{array}{ccccccc}
 & & \overset{\displaystyle 0}{\|} & & & & \\
0 \longrightarrow & H^0_{K_\varepsilon}(\bar U_{\varepsilon,\delta}, \mathcal{A}_p) & \longrightarrow & H^0(\bar U_{\varepsilon,\delta}, \mathcal{A}_p) & \longrightarrow & H^0(\bar U_{\varepsilon,\delta}\backslash K_\varepsilon, \mathcal{A}_p) & \\
\longrightarrow & H^1_{K_\varepsilon}(\bar U_{\varepsilon,\delta}, \mathcal{A}_p) & \longrightarrow & H^1(\bar U_{\varepsilon,\delta}, \mathcal{A}_p)\,. & & & \\
 & & & \underset{\displaystyle 0}{\|} & & &
\end{array}
\end{array}
$$

(We here sketch the proof of $H^1(\bar U_{\varepsilon,\delta}, \mathcal{A}_p)=0$. (C. f. [**9**].) We have the exact sequence:

$$
\begin{aligned}
0 \longrightarrow H^0(\bar U_{\varepsilon,\delta}, \mathcal{A}_p) \longrightarrow H^0(\bar U_{\varepsilon,\delta}, \mathcal{A}) \overset{p}{\longrightarrow} H^0(\bar U_{\varepsilon,\delta}, \mathcal{A}) \\
\longrightarrow H^1(\bar U_{\varepsilon,\delta}, \mathcal{A}_p) \longrightarrow H^1(\bar U_{\varepsilon,\delta}, \mathcal{A})\,.
\end{aligned}
$$

Here $H^1(\bar U_{\varepsilon,\delta}, \mathcal{A})=0$ due to Malgrange's theorem. Since p is surjective in the space $\mathcal{A}(\bar U_{\varepsilon,\delta})$ for a compact convex set $\bar U_{\varepsilon,\delta}$, we have thus $H^1(\bar U_{\varepsilon,\delta}, \mathcal{A}_p)=0$.) Thus we have from (2.4),

$$
(2.5) \qquad H^1_{K_\varepsilon}(\bar U_{\varepsilon,\delta}, \mathcal{A}_p)=\mathcal{A}_p(\bar U_{\varepsilon,\delta}\backslash K_\varepsilon)/\mathcal{A}_p(\bar U_{\varepsilon,\delta})\,,
$$

and similarly

$$
(2.6) \qquad H^1_{K_\varepsilon}(\bar V_\varepsilon, \mathcal{A}_p)=\mathcal{A}_p(\bar V_\varepsilon\backslash K_\varepsilon)/\mathcal{A}_p(\bar V_\varepsilon)\,.
$$

On the other hand, the excision theorem shows that the left hand sides of (2.5) and (2.6) agree. Hence we have

$$
(2.7) \qquad \mathcal{A}_p(\bar V_\varepsilon\backslash K_\varepsilon)/\mathcal{A}_p(\bar V_\varepsilon)\cong \mathcal{A}_p(\bar U_{\varepsilon,\delta}\backslash K_\varepsilon)/\mathcal{A}_p(\bar U_{\varepsilon,\delta})\,,
$$

where the isomorphism is obviously given by the restriction map from $\mathcal{A}_p(\bar V_\varepsilon\backslash K_\varepsilon)$ to $\mathcal{A}_p(\bar U_{\varepsilon,\delta}\backslash K_\varepsilon)$. Now take $u\in\mathcal{A}_p(U\backslash K)$ arbitrarily. Restricting u to $\bar U_{\varepsilon,\delta}\backslash K_\varepsilon$, we take it as an element of $\mathcal{A}_p(\bar U_{\varepsilon,\delta}\backslash K_\varepsilon)$. Then (2.7) shows that there exist $v\in\mathcal{A}_p(\bar V_\varepsilon\backslash K_\varepsilon)$ and $w\in\mathcal{A}_p(\bar U_{\varepsilon,\delta})$ such that

$$
(2.8) \qquad u=v+w\,, \qquad \text{on} \quad \bar U_{\varepsilon,\delta}\backslash K\,.
$$

By the assumption of the lemma every element $v\in\mathcal{A}_p(\bar V_\varepsilon\backslash K_\varepsilon)$ can be continued analytically to V_ε. Thus the right hand side of (2.8) can be continued analytically to $U_{\varepsilon,\delta}$, hence to U_ε. Since $\varepsilon>0$ is arbitrary, we conclude that $u\in\mathcal{A}_p(U)$.

Finally assume that U is not convex (and not small). Since K itself is convex, we can find a convex neighborhood U_1 of K such that $U\supset U_1\supset K$ and $V\supset U_1$ for some V. We have obviously $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)\subset\mathcal{A}_p(U_1\backslash K)/\mathcal{A}_p(U_1)$, where the injection is induced from the restriction map. Since we have proved that the latter equals zero, we conclude that $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.

q. e. d.

Now we present our main results. The first one is a direct corollary from the result of § 1, and does not rely on Proposition 2.3.

THEOREM 2.6. *Assume that each irreducible component p_λ of p satisfies the following condition: There exists a sequence of directions $\vartheta_\lambda^{(k)}$, $k=1,2,\cdots$ converging to $(0,\cdots,0,1)$ such that p_λ is hyperbolic with respect to every $\vartheta_\lambda^{(k)}$. Then $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.*

PROOF. Since K is arbitrary now, we can apply Lemma 2.4. Therefore we only have to prove our theorem for each irreducible component. First assume that p_λ is hyperbolic with respect to $(0,\cdots,0,1)$. Let C be the propagation cone of p_λ. Let K' be the closure in H of the following set

$$\bigcup_{a\in K}(a+C)\cap H.$$

Assume that U is of the form $U=(a+C_0)\cap H$ for some $a=(0,\cdots,0,a_n)\in H$, where C_0 is an open convex cone with its vertex at the origin, containing $C\backslash\{0\}$, and properly contained in the upper half space. We have obviously $U\supset K'$. Thus by Corollary 1.7 we have $\mathcal{B}_{p_\lambda}(U\backslash K')/\mathcal{B}_{p_\lambda}(U)=0$. Hence by Remark 8 we have $\mathcal{A}_{p_\lambda}(U\backslash K')/\mathcal{A}_{p_\lambda}(U)=0$. By the uniqueness of analytic continuation we have the obvious inclusion $\mathcal{A}_{p_\lambda}(U\backslash K)/\mathcal{A}_{p_\lambda}(U)\subset\mathcal{A}_{p_\lambda}(U\backslash K')/\mathcal{A}_{p_\lambda}(U)$ induced from the restriction mapping. Thus we conclude that $\mathcal{A}_{p_\lambda}(U\backslash K)/\mathcal{A}_{p_\lambda}(U)=0$. Due to Lemma 2.5, this holds for any U.

Next we consider the general case. Put $H_k=\{x\in\boldsymbol{R}^n;\ \langle\vartheta_\lambda^{(k)},x\rangle<-\delta_k\}$, where δ_k is a suitable sequence of positive numbers tending to zero such that H_k does not contain any point of $L\backslash K$. We apply the above result to each set $K_k=K\cap H_k$ and $U_k=U\cap H_k$, taking a suitable direction as the x_n-axis. Thus we obtain $\mathcal{A}_{p_\lambda}(U_k\backslash K_k)/\mathcal{A}_{p_\lambda}(U_k)=0$. Therefore if $u\in\mathcal{A}_{p_\lambda}(U\backslash K)$, then we have $u\in\mathcal{A}_{p_\lambda}(U_k)$ for any k. If we let $k\to\infty$, K_k approaches K and U_k approaches U. Thus by the uniqueness of analytic continuation we finally conclude that $u\in\mathcal{A}_{p_\lambda}(U)$. q. e. d.

REMARK 10. The condition of Theorem 2.5 depends on the lower order terms. In fact the polynomial $(\zeta_1^2-\zeta_2\zeta_n)(\zeta_1^2+\zeta_2\zeta_n)$ $(n=3)$ satisfies it, but the perturbed one

$$(\zeta_1^2-\zeta_2\zeta_n)(\zeta_1^2+\zeta_2\zeta_n)+1$$

does not. For it is irreducible, hence it does not contain any hyperbolic factor.

Now we give results really depending on the analyticity of the solutions, i. e., employing Proposition 2.3.

THEOREM 2.7. *Assume that each irreducible component p_λ of p satisfies either of the following two conditions.*

1) *The same condition as Theorem 2.6.*

2) *There exists a non-characteristic direction $(\vartheta_\lambda,0)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}$ such that $K\subset\{(x',x_n);\ \langle\vartheta_\lambda,x'\rangle=0\}$ and that the roots τ of $p(\zeta'+\tau\vartheta_\lambda,\zeta_n)=0$ for fixed*

$\zeta' \in \boldsymbol{C}^{n-1}$ *satisfy the following estimate for* $\mathrm{Im}\,\zeta_n \geqq 0$: *There exists a constant* b *(possibly depending on* ζ'*) such that given any* $\varepsilon > 0$ *we have*

$$|\mathrm{Im}\,\tau| \leqq \varepsilon|\zeta_n| + b|\mathrm{Im}\,\zeta_n| + C_{\zeta',\varepsilon}, \tag{2.9}$$

with some constant $C_{\zeta',\varepsilon} > 0$ *depending on* ζ' *and* ε. *Then* $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U) = 0$.

PROOF. Put $\mathcal{K} = \{[K+(0, \cdots, 0, \delta)] \cap H;\ \delta \geqq 0\}$. Then $\mathcal{K}$ satisfies the condition of Lemma 2.4, and each element of $\mathcal{K}$ also satisfies the condition in our theorem. Therefore, again we can prove the theorem separately for each irreducible component. For the components satisfying the condition 1), the proof is already given. Now let p_λ satisfy the second condition. Hereafter we simply write p for this component. By a suitable change of the coordinate system in the x'-space we can assume that $(\vartheta_\lambda, 0) = (1, 0, \cdots, 0)$. Then, we can assume without loss of generality that

$$K = \{(0, x'', x_n);\ -c_n \leqq x_n < 0,\ |x_j| \leqq c,\ j = 2, \cdots, n-1\},$$

where $x'' = (x_2, \cdots, x_{n-1})$. (This reduction is just the same as that made in the proof of Theorem 2.6. This time we employ a cylinder as C_0 in Lemma 2.5.) This means that

$$H_L(\zeta) = c|\mathrm{Im}\,\zeta''| + \max\{c_n\,\mathrm{Im}\,\zeta_n, 0\},$$

where $\zeta = (\zeta_1, \zeta'', \zeta_n)$ and $|\mathrm{Im}\,\zeta''| = \sum_{j=2}^{n-1}|\mathrm{Im}\,\zeta_j|$. Thus for each ζ'' fixed. the algebraic equation in ζ_1:

$$p(\zeta_1, \zeta'', \zeta_n) = 0$$

has m roots $\zeta_1 = \tau_j(\zeta'', \zeta_n)$, $j = 1, \cdots, m$, each satisfying (2.9) with $C_{\zeta',\varepsilon}$ replaced by $C_{\zeta'',\varepsilon}$, where m is the order of p. Let $\Delta(\zeta'', \zeta_n)$ be the discriminant of this equation. Since p is irreducible, Δ is not identically equal to zero. Hence, when ζ'' is fixed, either Δ is identically equal to zero as a polynomial in ζ_n, or Δ is different from zero for $|\zeta_n| \geqq \delta(\zeta'')$. Here, $\delta(\zeta'')$ is the largest modulus of the roots of the equation $\Delta(\zeta'', \zeta_n) = 0$ in ζ_n. As for the case $\Delta \equiv 0$, we can factorize p and make the same argument to the irreducible components. Thus, in any case, there exists $\delta(\zeta'')$ such that τ_j are holomorphic in ζ_n on $\mathrm{Im}\,\zeta_n \geqq \delta(\zeta'')$ or on $\mathrm{Im}\,\zeta_n \leqq -\delta(\zeta'')$. The variety $N(p)$ is covered in the following way:

$$N(p) = [\{|\mathrm{Im}\,\zeta_n| \leqq \delta(\zeta'')\} \cap N(p)] \cup N_+^{(1)} \cup N_-^{(1)} \cup \cdots \cup N_+^{(m)} \cup N_-^{(m)}, \tag{2.10}$$

where

$$N_\pm^{(j)} = \{\zeta_1 = \tau_j(\zeta'', \zeta_n);\ \pm\mathrm{Im}\,\zeta_n \geqq \delta(\zeta'')\}.$$

Let $u \in \mathcal{A}_p(U\backslash K)$ and let $F(\zeta)$ be a component of a representative of $\tilde{\partial}\cdot u$. $F(\zeta)$ satisfies the following estimate corresponding to the space $\widetilde{\mathcal{B}[L]}\{p, d\}$: given any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

(2.11) $$|F(\zeta)| \leqq C_\varepsilon \exp(\varepsilon|\zeta| + H_L(\zeta)).$$

On the other hand, $F(\zeta)$ satisfies the condition in Proposition 2.3. By this fact we are going to prove that $F(\zeta)$ satisfies the following estimate corresponding to the space $\widetilde{\mathscr{B}[L\backslash K]}\{p, d\}$: given any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

(2.12) $$|F(\zeta)| \leqq C_\varepsilon \exp(\varepsilon|\zeta| + H_{L\backslash K}(\zeta)).$$

Due to Proposition 1.4 and Remark 8, that will complete the proof. When $\operatorname{Im}\zeta_n \leqq 0$, (2.11) clearly implies (2.12), since in our case

$$H_L(\zeta) = c|\operatorname{Im}\zeta''| + \max\{c_n \operatorname{Im}\zeta_n, 0\} \quad \text{and} \quad H_{L\backslash K}(\zeta) = c|\operatorname{Im}\zeta''|.$$

Therefore no difficulty occurs on $N_-^{(j)}$. From now on we consider a fixed $N_+^{(j)}$ and simply write τ for τ_j. Thus for ζ'' fixed, the holomorphic function $G(\zeta_n) = F(\tau(\zeta'', \zeta_n), \zeta'', \zeta_n)$ of one variable ζ_n satisfies the following condition: for any infra-exponential entire function $J(\zeta_n)$ and for any $\varepsilon > 0$, we have the decomposition $J(\zeta_n)G(\zeta_n) = f_{J,\varepsilon}(\zeta_n) + g_{J,\varepsilon}(\zeta_n)$, where $f_{J,\varepsilon}$, $g_{J,\varepsilon}$ are holomorphic in $\operatorname{Im}\zeta_n \geqq \delta(\zeta'')$ and satisfy

(2.13) $$|f_{J,\varepsilon}(\zeta_n)| \leqq C_{J,\varepsilon} \exp(\varepsilon|\zeta_n| + \varepsilon|\zeta''| + \varepsilon|\tau(\zeta'', \zeta_n)| + \varepsilon|\operatorname{Im}\tau(\zeta'', \zeta_n)| + (c+\varepsilon)|\operatorname{Im}\zeta''|),$$

(2.14) $$|g_{J,\varepsilon}(\zeta_n)| \leqq C_{J,\varepsilon} \exp\Big(\varepsilon|\operatorname{Im}\tau(\zeta'', \zeta_n)| + \Big(c_n + \frac{\varepsilon}{2}\Big)\operatorname{Im}\zeta_n + (c+\varepsilon)|\operatorname{Im}\zeta''|\Big).$$

(We put $\eta = \varepsilon$ in (2.1) and $k = 0$ in (2.2).)

Now we prepare a few lemmas.

LEMMA 2.8. *Let b be a fixed constant. Assume that the function $u(t)$ of one variable $t \geqq 0$ satisfies the following condition: given any function $\varphi(t) > 0$, monotone increasing to infinity when t tends to infinity, there exist a positive constant C_φ depending on φ such that*

$$|u(t)| \leqq C_\varphi \exp\Big(bt - \frac{t}{\varphi(t)}\Big).$$

Then, there exist a constant $b' < b$ and a constant $C > 0$ such that

$$|u(t)| \leqq C \exp(b't).$$

PROOF. Assume the contrary. Then, for any positive N we can find another positive number $\psi(N)$ so that

$$\sup_{0 \leqq t \leqq \psi(N)} |u(t)| \exp\Big(-bt + \frac{1}{N}t\Big) \geqq N.$$

Clearly we can assume that $\psi(N)$ is monotone increasing to infinity when N

tends to infinity. Therefore the inverse function $s=\varphi(t)$ of $t=\psi(s)$ is also monotone increasing to infinity, and we have

$$\begin{aligned}\sup_{0\leqq t\leqq N}|u(t)|\exp\left(-bt+\frac{t}{\varphi(t)}\right)&\geqq\sup_{0\leqq t\leqq N}|u(t)|\exp\left(-bt+\frac{t}{\varphi(t)}\right)\\&\geqq\sup_{0\leqq t\leqq\psi(M)}|u(t)|\exp\left(-bt+\frac{1}{M}t\right)\\&\geqq M.\end{aligned}$$

Here we put $M=\varphi(N)$. M increases infinitely when N does. This contradicts the assumption. q. e. d.

LEMMA 2.9. *Assume that the holomorphic function $G(z)$ of one variable for $\operatorname{Im} z\geqq 0$ satisfies the following condition: There exist positive constants a, b, ε, and $q<1$ such that for any infra-exponential entire function $J(z)$ we have a decomposition $JG=f_J+g_J$, where f_J, g_J are holomorphic in $\operatorname{Im} z\geqq 0$ and satisfy*

$$(2.15)\qquad \begin{aligned}|f_J(z)|&\leqq C_J\exp(\varepsilon|z|+a\operatorname{Re}(-\sqrt{-1}\,z)^q),\\|g_J(z)|&\leqq C_J'\exp(a\operatorname{Re}(-\sqrt{-1}\,z)^q+b|\operatorname{Im} z|),\end{aligned}$$

with some constants C_J and C_J'. Here $(\ )^q$ denotes the principal branch of the power function. Then G satisfies

$$|G(z)|\leqq C_1\exp(\varepsilon|z|+a\operatorname{Re}(-\sqrt{-1}\,z)^q)+C_1'\exp(a\operatorname{Re}(-\sqrt{-1}\,z)^q+\varepsilon|\operatorname{Im} z|),$$

where C_1 (C_1') is the constant C_J (C_J') in (2.15) corresponding to $J=1$.

PROOF. Put $z=x+\sqrt{-1}\,y$. We have for $y>0$

$$(2.16)\qquad |J(\sqrt{-1}\,y)G(\sqrt{-1}\,y)|\leqq C_Je^{\varepsilon y+ay^q}+C_J'e^{by+ay^q}.$$

Now assume that $\varepsilon<b$. Given a function φ which is monotone increasing to infinity, choose J so that

$$|J(\sqrt{-1}\,y)|\geqq C\cdot\exp\left(\frac{y}{\varphi(y)}+ay^q\right)\qquad\text{for}\quad y\geqq 0.$$

(For the construction of such J see [5], Lemma 1.2.) Dividing the both sides of (2.16) by J we have

$$|G(\sqrt{-1}\,y)|\leqq C_\varphi\exp\left(by-\frac{y}{\varphi(y)}\right)\qquad\text{for}\quad y\geqq 0.$$

Thus by Lemma 2.8 we have, with some $b'<b$

$$|G(\sqrt{-1}\,y)|\leqq C\exp(b'y).$$

Therefore, for any J, the function g_J appearing in the decomposition $JG=f_J+g_J$ has the following two estimates:

$$|g_J(z)|\leqq C_J'\exp(a\operatorname{Re}(-\sqrt{-1}\,z)^q+b|\operatorname{Im} z|);$$

and for any $\eta > 0$,

$$|g_J(\sqrt{-1}\,y)| \leqq |J(\sqrt{-1}\,y)G(\sqrt{-1}\,y)| + |f_J(\sqrt{-1}\,y)|$$
$$\leqq C_\eta \exp(\max(\varepsilon, b')y + \eta y),$$

with some $C_\eta > 0$. Now consider the function

$$h(z) = g_J(z)\exp(\max(\varepsilon, b')\sqrt{-1}\,z - a(-\sqrt{-1}\,z)^q).$$

$h(z)$ is bounded on the real axis by the constant C_J'; of exponential growth on $\mathrm{Im}\, z \geqq 0$; and on the imaginary axis satisfies, for any $\eta > 0$,

$$|h(\sqrt{-1}\,y)| \leqq C_\eta \exp(\eta y),$$

with some $C_\eta > 0$. Therefore, by the Phragmén-Lindelöf theorem $h(z)$ is bounded on $\mathrm{Im}\, z \geqq 0$ by the same constant C_J'. Thus $g_J(z)$ satisfies the new estimate

$$|g_J(z)| \leqq C_J' \exp(a\,\mathrm{Re}(-\sqrt{-1}\,z)^q + \max(\varepsilon, b')|\mathrm{Im}\, z|),$$

with $b' < b$. When $\varepsilon < b'$, we can apply the same argument and replace b' by another smaller one. Repeating this process we can finally replace b by ε. For, assume the contrary. Then we have the following situation: There exists some $b_0 \geqq \varepsilon$ such that g_J has the following estimate for any $\eta > 0$,

$$|g_J(z)| \leqq C_{J,\eta} \exp(a\,\mathrm{Re}(-\sqrt{-1}\,z)^q + (b_0 + \eta)|\mathrm{Im}\, z|),$$

with some $C_\eta > 0$, but g_J does not satisfy the following one

$$|g_J(z)| \leqq C_J' \exp(a\,\mathrm{Re}(-\sqrt{-1}\,z)^q + b_0|\mathrm{Im}\, z|).$$

Applying, however, the Phragmén-Lindelöf theorem to

$$g_J(z)\exp(b_0\sqrt{-1}\,z - a(-\sqrt{-1}\,z)^q)$$

we can show that the first inequality implies the second one. This is absurd, hence we have proved that g_J satisfies

$$|g_J(z)| \leqq C_J' \exp(a\,\mathrm{Re}(-\sqrt{-1}\,z)^q + \varepsilon|\mathrm{Im}\, z|).$$

Thus, choosing $J=1$, we have proved that $G(z) = f_1(z) + g_1(z)$ satisfies the desired estimate. q. e. d.

REMARK 11. Let $\varphi(r)$ be a function of $r \geqq 0$ monotone increasing to infinity. Assume that $\sum_{m=1}^{\infty} 1/m\varphi(m) < \infty$. Then we have the following variant of Lemma 2.9:

Assume that the holomorphic function $G(z)$ of one variable for $\mathrm{Im}\, z \geqq 0$ satisfies the following condition: There exist positive constants b, ε such that for any infra-exponential entire function $J(z)$ we have a decomposition $JG = f_J + g_J$, where f_J, g_J are holomorphic in $\mathrm{Im}\, z \geqq 0$ and satisfy

$$|f_J(z)| \leqq C_J \exp\left(\varepsilon|z|+\frac{|\mathrm{Re}\, z|}{\varphi(|\mathrm{Re}\, z|)}\right),$$

$$|g_J(z)| \leqq C'_J \exp\left(\frac{|\mathrm{Re}\, z|}{\varphi(|\mathrm{Re}\, z|)}+b|\mathrm{Im}\, z|\right),$$

with some C_J and C'_J. Then G satisfies

$$|G(z)| \leqq C_1 \exp\left(\varepsilon|z|+\frac{|z|}{\psi(|z|)}\right)+C'_1 \exp\left(\frac{|z|}{\psi(|z|)}+\varepsilon|\mathrm{Im}\, z|\right),$$

where $\psi(r)$ is another function of $r \geqq 0$ monotone increasing to infinity.

In fact we can find an infra-exponential entire function $a(z)$ which does not vanish in $\mathrm{Im}\, z \geqq 0$ and satisfies $\mathrm{Re}[\log a(x)] \geqq |x|/\varphi(|x|)$ for real x. (Cf. [**5**], Proof of Theorem 2.6.) Therefore we can apply the proof of Lemma 2.9 replacing the power function $(-\sqrt{-1}\, z)^q$ by $\log a(z)$. Finally we can find ψ such that $\mathrm{Re}[\log a(z)] \leqq |z|/\psi(|z|)$ (cf. [**5**], Lemma 1.1).

LEMMA 2.10. *Assume that an entire function $F(z, w)$ of the variables $z \in \boldsymbol{C}$, $w \in \boldsymbol{C}^l$ satisfies the following two estimates: For any $\varepsilon>0$ and for any fixed $w \in \boldsymbol{C}^l$, we have, with some $C_{\varepsilon,w}>0$,*

$$|F(z, w)| \leqq C_{\varepsilon,w} \exp(\varepsilon|z|). \tag{2.17}$$

There exist fixed constants $a>0$, $b>0$ such that for any $\varepsilon>0$ we have, with some $C_\varepsilon>0$,

$$|F(z, w)| \leqq C_\varepsilon \exp(\varepsilon(|z|+|w|)+a|\mathrm{Im}\, z|+b|\mathrm{Im}\, w|). \tag{2.18}$$

Then F satisfies another estimate: For any $\varepsilon>0$ we have, with some $C_\varepsilon>0$,

$$|F(z, w)| \leqq C_\varepsilon \exp(\varepsilon(|z|+|w|)+b|\mathrm{Im}\, w|).$$

Note that we employ the notation $|w|=\sum_{j=1}^l |w_j|$ and $|\mathrm{Im}\, w|=\sum_{j=1}^l |\mathrm{Im}\, w_j|$.

PROOF. Estimate (2.18) shows that F is the Fourier transform of a hyperfunction $v(s, t)$ whose support is contained in the real compact set

$$L_{a,b}=\{(s, t) \in \boldsymbol{R}\times\boldsymbol{R}^l;\ |s| \leqq a,\ |t_j| \leqq b,\ j=1, \cdots, l\}.$$

On the other hand, Proposition 1.13′ of Martineau [**12**], Chapitre II shows that the two estimates (2.17), (2.18) imply, for any $\varepsilon>0$,

$$|F(z, w)| \leqq C_\varepsilon \exp(\varepsilon|z|+K_\varepsilon|w|),$$

with some $C_\varepsilon>0$ and $K_\varepsilon>0$. This shows that the function $v(s, t)$, considered as an analytic functional, has its porter in the polydisk

$$D_\varepsilon=\{(\sigma, \tau) \in \boldsymbol{C}\times\boldsymbol{C}^l;\ |\sigma| \leqq \varepsilon,\ |\tau_j| \leqq K_\varepsilon,\ j=1, \cdots, l\}.$$

Combining these two informations with Théorème 3.3 b) in [**12**], Chapitre I, we conclude that v has its support in $L_{\varepsilon,b}=L_{a,b}\cap D_\varepsilon$. Since ε is arbitrary, we finally conclude that $\mathrm{supp}\, v \subset L_{0,b}$, by the uniqueness of the supports of real analytic functionals. This obviously implies the desired estimate.

q. e. d.

END OF PROOF OF THEOREM 2.7. From our assumption that $(1, 0, \cdots, 0)$ is non-characteristic with respect to p, we have

$$|\tau(\zeta'', \zeta_n)| \leqq M(|\zeta_n| + |\zeta''|), \tag{2.19}$$

with a constant $M > 0$. On the other hand, from (2.9) we have, due to Seidenberg's theorem, for $\operatorname{Im} \zeta_n \geqq \delta(\zeta'')$,

$$|\operatorname{Im} \tau(\zeta'', \zeta_n)| \leqq a \operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q + b|\operatorname{Im} \zeta_n| + C_{\zeta''},$$

where a, b, q are constants possibly depending on ζ'', and $q < 1$. (Note that for $\operatorname{Im} z \geqq 0$ we have $|z|^q \leqq \mu \operatorname{Re}(-\sqrt{-1}\,z)^q$ with $\mu = 1/\cos(q/2)\pi$.) Putting these estimates into (2.13), (2.14), we obtain for $\operatorname{Im} \zeta_n \geqq \delta(\zeta'')$

$$\begin{aligned} &|f_{J,\varepsilon}(\zeta_n)| \\ &\quad \leqq C_{J,\varepsilon} \exp\{\varepsilon(M+1)|\zeta_n| + \varepsilon a \operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q + \varepsilon b|\operatorname{Im} \zeta_n| + C_{\zeta''}\}, \\ &|g_{J,\varepsilon}(\zeta_n)| \\ &\quad \leqq C_{J,\varepsilon} \exp\left\{\varepsilon a \operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q + \varepsilon b|\operatorname{Im} \zeta_n| + \left(c_n + \frac{\varepsilon}{2}\right)|\operatorname{Im} \zeta_n| + C_{\zeta''}\right\}. \end{aligned} \tag{2.20}$$

Thus we can apply Lemma 2.9 to the function $G(\zeta_n) \exp(\sqrt{-1}\,\varepsilon b \zeta_n)$ for the region $\operatorname{Im} \zeta_n \geqq \delta(\zeta'')$ and conclude that

$$\begin{aligned} |F(\zeta)| &= |G(\zeta_n)| \\ &\leqq C_{\zeta'',\varepsilon} \exp(\varepsilon(M+1)|\zeta_n| + a \operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q + \varepsilon b|\operatorname{Im} \zeta_n|) \\ &\leqq C'_{\zeta'',\varepsilon'} \exp(\varepsilon'|\zeta_n|), \end{aligned} \tag{2.21}$$

for $\zeta \in \bigcup_{j=1}^{n} N_+^{(j)}$, where ε' is another arbitrary positive number and $C'_{\zeta'',\varepsilon'}$ is a constant depending on ζ'' and ε'. Since this type of estimate trivially holds on $\{|\operatorname{Im} \zeta_n| \leqq \delta(\zeta'')\} \cap N(p)$ and on $\bigcup_{j=1}^{n} N_-^{(j)}$, we conclude that given any $\varepsilon > 0$,

$$|F(\zeta)| \leqq C_{\zeta'',\varepsilon} \exp(\varepsilon|\zeta_n|) \qquad \text{for} \quad \zeta \in N(p). \tag{2.22}$$

On the other hand, we have, directly from (2.11) and (2.9), (2.19), for any $\varepsilon > 0$,

$$|F(\zeta)| \leqq C_\varepsilon \exp(\varepsilon(|\zeta''| + |\zeta_n|) + c|\operatorname{Im} \zeta''| + c_n|\operatorname{Im} \zeta_n|). \tag{2.23}$$

(Here we used the assumption $K \subset \{x_1 = 0\}$. Otherwise, the right hand side would have contained the variable $|\operatorname{Im} \zeta_1|$ to a more great extent.) Now consider the functions

$$F_j(\zeta'', \zeta_n) = \sigma_j(F(\tau_1(\zeta'', \zeta_n), \zeta'', \zeta_n), \cdots, F(\tau_m(\zeta'', \zeta_n), \zeta'', \zeta_n)),$$

where σ_j is the j-th fundamental symmetric polynomial and τ_k, $k=1, \cdots, m$ are the roots of equation $p(\zeta_1, \zeta'', \zeta_n)=0$ in ζ_1. F_j are well defined as single-valued functions and entire on C^{n-1}, since the set $\{\Delta=0\}$ is a removable singularity for them. F_j satisfy the estimates similar to (2.22), (2.23) with the constants c and c_n replaced by jc and jc_n. Thus we can apply Lemma 2.10 to F_j and conclude that

$$|F_j(\zeta'', \zeta_n)| \leqq C_\varepsilon \exp(\varepsilon(|\zeta''|+|\zeta_n|)+jc|\mathrm{Im}\,\zeta''|).$$

As the roots of

$$\lambda^m - F_1(\zeta'', \zeta_n)\lambda^{m-1} + \cdots + (-1)^m F_m(\zeta'', \zeta_n) = 0,$$

the original function F satisfies the similar estimate

$$|F(\zeta)| \leqq C_\varepsilon \exp(\varepsilon(|\zeta''|+|\zeta_n|)+c|\mathrm{Im}\,\zeta''|) \tag{2.24}$$

(see, e.g., Malgrange [**11**], Lemma 2.3). This is nothing but the desired estimate (2.12). Thus, on account of Proposition 1.4, u can be extended as a hyperfunction solution to the whole U. Due to Remark 8 after Theorem 2.1, the extended solution is real analytic. q. e. d.

REMARK 12. As the above proof shows (see, especially, (2.21)), it is probable that we can replace b in the assumption (2.9) by b_ε satisfying $\varepsilon b_\varepsilon \to 0$ when $\varepsilon \to 0$.

REMARK 13. The condition $K \subset \{\langle \vartheta, x' \rangle = 0\}$ is essential. In fact, we have the following example. Consider the wave operator $p(D) = \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_n^2$ $(n=3)$. Put $\vartheta = (1, 0)$. Then the roots τ of the equation $p(\tau+\zeta_1, \zeta_2, \zeta_n) = 0$ obviously satisfy the condition (2.9). Thus we can apply Theorem 2.7 if $K \subset \{x_1 = 0\}$. On the other hand, we have the following solution u of $p(D)u = 0$.

$$u(x_1, x_2, x_n) = \frac{1}{\sqrt{v}} \log\left\{\left(\frac{x_2}{v}\right)^2 + \frac{1}{1-k^2}\left(\frac{x_n + kx_1 - \frac{1-k^2}{2k}}{v} + k\right)^2\right\},$$

where,

$$v = v(x_1, x_2, x_n) = -\left(x_1 + \frac{1}{2}\right)^2 + x_2^2 + \left(x_n - \frac{1}{2k}\right)^2,$$

and k is a constant satisfying $0 < k < 1$. The singularity of u agrees with the following hyperbola near its vertex $(0, 0, \sqrt{1-k^2}/2k)$,

$$\begin{cases} x_2 = 0 \\ x_n^2 - x_1^2 = \dfrac{1-k^2}{4k^2}. \end{cases}$$

Therefore if $L \backslash K$ has an interior in ∂H, we can construct a non-trivial element of $\mathcal{A}_p(U \backslash K)/\mathcal{A}_p(U)$ modifying this solution. This example is obtained from the solution $\log(x_2^2+x_n^2)$ by means of the *Lorenz transformation* and the *conformal transformation* combined with suitable translations. We can give similar example also for *ultrahyperbolic equations*. For example, for $p(D)=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2-\partial^2/\partial x_3^2-\partial^2/\partial x_n^2$ $(n=4)$, we can also apply Theorem 2.7 if $K\subset\{x_1=0\}$. But we have the following solution

$$u(x_1, x_2, x_3, x_n)=\frac{1}{v}\log\left\{\left(\frac{x_3}{v}+\frac{1}{2}\right)^2+\frac{1}{1-k^2}\left(\frac{x_n+kx_1}{v}+k\right)^2\right\},$$

where $v=x_1^2+x_2^2-x_3^2-x_n^2$ and $0<k<1$. The singularity of u contains the following two dimensional variety defined by the equations

$$\begin{cases} 2x_3+(x_1^2+x_2^2-x_3^2-x_n^2)=0 \\ \dfrac{1}{k}x_n+x_1+(x_1^2+x_2^2-x_3^2-x_n^2)=0\,. \end{cases}$$

After an elementary calculation we can see that it has the following " minimal point ".

$$\left(\frac{7a}{3k}-\frac{2}{3},\ 0,\ \frac{2a}{3k}-\frac{1}{3},\ a\right)$$

where

$$a=\frac{k(2+\sqrt{3(1-k^2)})}{1+3k^2}\,.$$

I thank Professor K. Aomoto for his advice on these subjects.

We give a simple consequence of Theorem 2.7.

COROLLARY 2.11. *Assume that the principal part of p does not contain ζ_n and let $K=\{(0,\cdots,0,x_n);\ -c_n\leqq x_n<0\}$. Then we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.*

PROOF. Obviously, every irreducible component of p has the same property, and the condition on K is compatible with that in Lemma 2.4. Thus we can take as $(\vartheta_\lambda, 0)$ any direction which is non-characteristic with respect to the component to apply Theorem 2.7. In fact we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$ if only K is contained in a hyperplane perpendicular to a non-characteristic direction $(\vartheta, 0)$ of p. Thus we have proved a stronger assertion. q. e. d.

REMARK 14. In spite of the above examples in Remark 13, we expect that we can generalize Corollary 2.11 to those K which may have interior points. For some class of operators we can really do it employing Theorem 2.12 below.

To avoid unnecessary complication, we give the result in the fixed coordinate system and for an irreducible p. Recall that we put $\zeta=(\zeta', \zeta_n)=(\zeta_1, \zeta'', \zeta_n)$.

THEOREM 2.12. *Let p be an irreducible polynomial. Assume that $(1, 0, \cdots, 0)$ is a non-characteristic direction of $p(D)$. Let $I\subset\{2,\cdots,n-1\}$ be a subset of indices. We write $|\mathrm{Re}\,\zeta^I|=\sum_{i\in I}|\mathrm{Re}\,\zeta_i|$. Assume that there exist positive* *constants A, b and B such that we have the following estimates for $\zeta\in N(p)$: Given any $\varepsilon>0$ there exists some constant $C_\varepsilon>0$ such that*

$$|\operatorname{Im}\zeta_1|\leqq\varepsilon|\zeta''|+\varepsilon|\zeta_n|+A|\operatorname{Re}\zeta^I|+b|\operatorname{Im}\zeta''|+C_\varepsilon, \tag{2.25}$$

$$|\operatorname{Re}\zeta^I|\leqq\varepsilon|\zeta|+B|\operatorname{Im}\zeta'|+C_\varepsilon. \tag{2.26}$$

Then we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.

PROOF. Let us employ the notations in the proof of Theorem 2.7. This time without loss of generality we can assume that

$$K=\{(x_1,\cdots,x_{n-1},x_n)\in\boldsymbol{R}^n;\ -c_n\leqq x_n<0,\ |x_j|\leqq c,\ j=1,\cdots,n-1\}.$$

Due to Seidenberg's theorem and (2.25), we have, for the roots $\tau=\zeta_1$ of $p(\zeta)=0$,

$$|\operatorname{Im}\tau(\zeta'',\zeta_n)|\leqq a\operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q+C_{\zeta''}\qquad\text{if}\quad\operatorname{Im}\zeta_n\geqq\delta(\zeta''),$$

with some positive constants a and q satisfying $q<1$ and possibly depending on ζ''. (Recall that for $\operatorname{Im}z\geqq 0$ we have $|z|^q\leqq\mu\operatorname{Re}(-\sqrt{-1}\,z)^q$ with $\mu=1/\cos(q/2)\pi$.) Thus in place of (2.20) we obtain

$$|f_{J,\varepsilon}(\zeta_n)|\leqq C_{J,\varepsilon}\exp(\varepsilon(M+1)|\zeta_n|+(c+\varepsilon)a\operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q+C_{\zeta''}),$$

$$|g_{J,\varepsilon}(\zeta_n)|\leqq C_{J,\varepsilon}\exp\Big((c+\varepsilon)a\operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q+\Big(c_n+\frac{\varepsilon}{2}\Big)|\operatorname{Im}\zeta_n|+C_{\zeta''}\Big).$$

Thus we can apply Lemma 2.9 and conclude that for any $\varepsilon>0$,

$$|F(\zeta)|\leqq C_{\zeta'',\varepsilon}\exp(\varepsilon(M+1)|\zeta_n|+(c+\varepsilon)a\operatorname{Re}(-\sqrt{-1}\,\zeta_n)^q),$$

with some constant $C_{\zeta'',\varepsilon}>0$. Thus we again obtain (2.22). On the other hand, instead of (2.23) we have, for any $\varepsilon>0$,

$$|F(\zeta)|\leqq C_\varepsilon\exp(\varepsilon(|\zeta''|+|\zeta_n|)+cA|\operatorname{Re}\zeta^I|+c(b+1)|\operatorname{Im}\zeta''|+c_n|\operatorname{Im}\zeta_n|), \tag{2.27}$$

with some $C_\varepsilon>0$. Since the right hand side of (2.27) contains $\operatorname{Re}\zeta^I$, we cannot apply Lemma 2.10. Therefore we prepare another lemma.

LEMMA 2.13. *Assume that an entire function $F(z,w,v)$ of the variables $z\in\boldsymbol{C}$, $w\in\boldsymbol{C}^l$, $v\in\boldsymbol{C}^k$, satisfies the following two estimates: For any $\varepsilon>0$, and for any fixed $w\in\boldsymbol{C}^l$, $v\in\boldsymbol{C}^k$, we have, with some $C_{\varepsilon,w,v}>0$,*

$$|F(z,w,v)|\leqq C_{\varepsilon,w,v}\exp(\varepsilon|z|). \tag{2.28}$$

There exist positive constants a, b and A such that for any $\varepsilon>0$, we have, with some $C_\varepsilon>0$,

$$|F(z,w,v)|\leqq C_\varepsilon\exp(\varepsilon(|z|+|w|+|v|)+a|\operatorname{Im}z|+b|\operatorname{Im}w|+A|v|). \tag{2.29}$$

Then F satisfies another estimate: For any $\varepsilon>0$, we have, with some $C_\varepsilon>0$, (2.30) $$|F(z, w, v)| \leqq C_\varepsilon \exp(\varepsilon(|z|+|w|+|v|)+b|\operatorname{Im} w|+A|v|).$$

PROOF. (2.29) shows that F is the Fourier-Laplace transform of an analytic functional $\mu(Z, W, V)$ which admits the following set as a porter

$$M=\{(Z, W, V)\in \boldsymbol{C}\times\boldsymbol{C}^l\times\boldsymbol{C}^k;\ \operatorname{Im} Z=0,\ |\operatorname{Re} Z|\leqq a,$$
$$\operatorname{Im} W_j=0,\ |\operatorname{Re} W_j|\leqq b,\ j=1, \cdots, l,\ |V_i|\leqq A,\ i=1, \cdots, k\}.$$

On the other hand, Proposition 1.13′ of [12], Chapitre II shows that (2.28) and (2.29) imply for any $\varepsilon>0$,

$$|F(z, w, v)|\leqq C_\varepsilon \exp(\varepsilon(|z|+|w|+|v|)+K_\varepsilon(|w|+|v|)),$$

with some $C_\varepsilon>0$ and $K_\varepsilon>0$. This means that μ also admits the following sets as porters

$$M_\varepsilon=\{(Z, W, V)\in \boldsymbol{C}\times\boldsymbol{C}^l\times\boldsymbol{C}^k;$$
$$|Z|\leqq\varepsilon,\ |W_j|\leqq K_\varepsilon,\ j=1, \cdots, l,\ |V_i|\leqq K,\ i=1, \cdots, k\}.$$

Without loss of generality we can assume that $\varepsilon\leqq a$, $K_\varepsilon\geqq b$, and $K_\varepsilon\geqq A$. In the following we will show that for any fixed $\varepsilon>0$, $M\cup M_\varepsilon$ is polynomially convex. Assuming this for a moment, we can prove the lemma in the following way. By Théorème 2.2 of [12], Chapitre I, we conclude that μ has its porter in the intersection $M\cap M_\varepsilon$:

$$M\cap M_\varepsilon=\{(Z, W, V)\in \boldsymbol{C}\times\boldsymbol{C}^l\times\boldsymbol{C}^k;\ \operatorname{Im} Z=0,\ |\operatorname{Re} Z|\leqq\varepsilon,$$
$$\operatorname{Im} W_j=0,\ |\operatorname{Re} W_j|\leqq b,\ j=1, \cdots, l,\ |V_i|\leqq A,\ i=1, \cdots, k\}.$$

This means that $F=\tilde{\mu}$ satisfies the estimate (2.30) with ε replaced, e.g., by 2ε. Since $\varepsilon>0$ is arbitrary, the proof will be completed.

Now it remains to show that $M\cup M_\varepsilon$ is polynomially convex. Let $P=(Z^0, W^0, V^0)\notin M\cup M_\varepsilon$. We give a polynomial which vanishes at P, but does not on $M\cup M_\varepsilon$. Put

$$N=\{(Z, W)\in \boldsymbol{C}\times\boldsymbol{C}^l;\ \operatorname{Im} Z=0,\ |\operatorname{Re} Z|\leqq a,$$
$$\operatorname{Im} W_j=0,\ |\operatorname{Re} W_j|\leqq b,\ j=1, \cdots, l\},$$
$$N_\varepsilon=\{(Z, W)\in \boldsymbol{C}\times\boldsymbol{C}^l;\ |Z|\leqq\varepsilon,\ |W_j|\leqq K_\varepsilon,\ j=1, \cdots, l\}.$$

Then, the complement $C(M\cup M_\varepsilon)$ is covered in the following way:

$$C(M\cup M_\varepsilon)=CM\cap CM_\varepsilon=S_0\cup S_1\cup S_2,$$

where

$$S_0=\{(Z, W, V);\ (Z, W)\notin N\cup N_\varepsilon\},$$
$$S_1=\{(Z, W, V);\ |V_i|>K_\varepsilon \text{ for some } i\},$$
$$S_2=\{(Z, W, V);\ (Z, W)\notin N_\varepsilon,\ |V_i|>A \text{ for some } i\}.$$

First, let $P\in S_1$. Assume, e.g., that $|V_i^0|>K_\varepsilon$. Then the linear function $V_i-V_i^0$ obviously fits our purpose. Next assume that $P\in S_0$. This time the problem reduces to that in the space $\boldsymbol{C}\times\boldsymbol{C}^l$ of the variables (Z, W). In fact, we will prove below that $N\cup N_\varepsilon$ is polynomially convex in $\boldsymbol{C}\times\boldsymbol{C}^l$. Assuming this for a moment, we have a polynomial $\phi(Z, W)$ vanishing at (Z^0, W^0) but not on $N\cup N_\varepsilon$. This ϕ is the desired one for our P. Now let $P\in S_2$. We can assume at the same time that $P\notin S_0$, hence $P\in N$. Thus we have especially

$$\operatorname{Im} Z^0=0 \quad \text{and} \quad |Z^0|>\varepsilon; \qquad |V_i^0|>A \text{ for some } i.$$

Without loss of generality we can assume that $Z^0=|Z^0|$. We employ the polynomial

$$\phi(Z, V)=\Big(\frac{Z}{|Z^0|}\Big)^p\Big(\frac{\alpha V_i-A}{|V_i^0|-A}\Big)-1,$$

where $\alpha=\bar{V}_i^0/|V_i^0|$ and p is a positive integer specified later. Clearly $\phi(Z^0, V^0)=0$. Assume that $\phi(Z, V)=0$ for some point $(Z, W, V)\in M\cup M_\varepsilon$. If $|Z|\leqq\varepsilon$, then we have

$$1<\Big(\frac{\varepsilon}{|Z^0|}\Big)^p\Big|\frac{\alpha V_i-A}{|V_i^0|-A}\Big|.$$

Therefore if we choose p so large that

$$\Big(\frac{|Z^0|}{\varepsilon}\Big)^p(|V_i^0|-A)>K_\varepsilon+A,$$

(which is possible because $|Z^0|/\varepsilon>1$), then we have

$$|V_i|>K_\varepsilon.$$

Thus ϕ does not vanish at such points. If $|Z|>\varepsilon$, then the assumption $(Z, W, V)\in M\cup M_\varepsilon$ implies $(Z, W, V)\in M$. Hence Z is real and $|V_i|<A$. Since we can assume that p is even, the identity $\phi(Z, W, V)=0$ implies

$$\alpha V_i-A>0, \qquad \text{hence} \quad |V_i|>A.$$

This is a contradiction. Thus ϕ does not vanish on $M\cup M_\varepsilon$.

Lastly we show that $N\cup N_\varepsilon$ is polynomially convex. The complement $C(N\cup N_\varepsilon)$ is covered in the following way:

$$C(N\cup N_\varepsilon)=CN\cap CN_\varepsilon=S_{00}\cup S_{01}\cup S_{02}\cup S_{03},$$

where

$$S_{00}=\{(Z, W);\ |Z|>\varepsilon,\ \operatorname{Im} W_j\neq 0 \text{ for some } j\},$$

$$S_{01}=\{(Z, W);\ |W_j|>K_\varepsilon \text{ for some } j\},$$

$$S_{02}=\{(Z, W);\ |Z|>\varepsilon,\ \operatorname{Im} W_j=0,\ j=1,\cdots,l,\ |\operatorname{Re} W_j|>b \text{ for some } j\},$$

$$S_{03}=\{(Z, W);\ |Z|>\varepsilon,\ \operatorname{Im} Z\neq 0\}\cup\{(Z, W);\ |\operatorname{Re} Z|>a\}.$$

The sets S_{01} and S_{03} are treated in the same way as S_1 employing linear functions. The set S_{02} is treated in the same way as S_2. In fact, we can assume at the same time that $P=(Z^0, W^0)\notin S_{03}$, hence that $Z^0=|Z^0|$ and $W_j^0=\mathrm{Re}\, W_j^0>b$. Thus we can employ

$$\phi(Z, W)=\Big(\frac{Z}{|Z^0|}\Big)^p\Big(\frac{W_j-b}{|W_j^0|-b}\Big)-1\,,$$

where p is a positive even integer such that

$$\Big(\frac{|Z^0|}{\varepsilon}\Big)^p(|W_j^0|-b)>K_\varepsilon+b\,.$$

Finally assume that $P\in S_{00}$. Assume $\mathrm{Im}\, W_j^0\neq 0$. Then we can employ

$$\phi(Z, W)=\Big(\frac{Z}{Z^0}\Big)^p\frac{W_j}{W_j^0}-1\,.$$

ϕ does not vanish on N if we choose p so that

$$p\arg Z^0+\arg W_j^0\not\equiv 0 \mod \pi\,.$$

Since $\arg W_j^0\not\equiv 0 \bmod \pi$, there exist infinitely many such p's. On the other hand, if $\phi(Z, W)=0$ for some $(Z, W)\in N_\varepsilon$, then we have

$$|W_j|=|W_j^0|\Big(\frac{|Z^0|}{|Z|}\Big)^p\geqq |W_j^0|\Big(\frac{|Z^0|}{\varepsilon}\Big)^p\,.$$

Therefore if we choose p so large that

$$|W_j^0|\Big(\frac{|Z^0|}{\varepsilon}\Big)^p>K_\varepsilon\,,$$

(which is possible because $|Z^0|/\varepsilon>1$), then ϕ also does not vanish on N_ε. Thus the proof of Lemma 2.13 is completed.

END OF PROOF OF THEOREM 2.12. We again employ the symmetric polynomials and define F_j in the same way. On account of (2.22) and (2.27) we can apply Lemma 2.13 to F_j. Thus we Obtain: for any $\varepsilon>0$,

$$|F(\zeta)|\leqq C_\varepsilon \exp(\varepsilon(|\zeta''|+|\zeta_n|)+c(A+b+1)|\zeta^I|+c(b+1)|\mathrm{Im}\,\zeta''|)\,,$$

with some $C_\varepsilon>0$. Substituting the right hand side of the assumption (2.26), we finally obtain: for any $\varepsilon>0$,

$$|F(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|+c(A+b+1)(B+1)|\mathrm{Im}\,\zeta'|)\,,\qquad \text{for}\quad \zeta\in N(p)\,, \tag{2.31}$$

with another $C_\varepsilon>0$. Now put

$$K'=\{(x_1,\cdots, x_{n-1}, x_n)\in \boldsymbol{R}^n\,;$$
$$-c_n\leqq x_n<0,\ |x_j|\leqq c(A+b+1)(B+1),\ j=1,\cdots, n-1\}\,.$$

Due to Lemma 2.5, we can assume that U is a rectangular parallelepiped containing K'. Then, on account of Proposition 1.4, the obtained inequality (2.31) shows that every solution $u \in \mathcal{A}_p(U\backslash K)$ can be extended as a hyperfunction solution to the whole U, if u is modified on $K'\backslash K$. By Remark 8 after Theorem 2.1 and by the uniqueness of analytic continuation, we conclude that $u \in \mathcal{A}_p(U)$. q. e. d.

REMARK 15. The condition (2.25) obviously implies that the highest term of p does not contain ζ_n. But the latter condition does not imply (2.25) and (2.26) in general. In fact $p(\zeta)=(\zeta_1+\zeta_2)(\zeta_1+\sqrt{-1}\,\zeta_2)+1$ $(n=3)$ is a counter-example. This example also shows that the condition of Theorem 2.12 posed on every irreducible component is not stable under perturbation by lower order terms. (Cf. Remark 10 after Theorem 2.6. I do not know whether the perturbation by lower order terms really destroys the identity $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$ or not. In the case $K \Subset U$ the necessary and sufficient condition for $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$ really depends on the lower order terms of p. Therefore we must be careful.)

We give a typical example satisfying (2.25) and (2.26).

COROLLARY 2.14. *Consider the operator* $p(D)=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_{n-1}^2-\partial/\partial x_n$ *corresponding to the heat equation. Then we have* $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$.

PROOF. We only have to verify the conditions (2.25), (2.26) on the polynomial $p(\zeta)$. We choose $I=\{2,\cdots,n-1\}$. Then (2.25) is easily verified. We are going to show that there exists a constant B such that for any $\varepsilon>0$ we have with some $C_\varepsilon>0$,

$$|\mathrm{Re}\,\zeta_j|\leqq\varepsilon|\zeta|+B|\mathrm{Im}\,\zeta'|+C_\varepsilon,\quad\text{if}\quad \zeta_1^2+\cdots+\zeta_{n-1}^2-\sqrt{-1}\,\zeta_n=0, \tag{2.32}$$

for $j=2,\cdots,n-1$. For the convenience of the notation we put $j=n-1$. We have

$$\zeta_{n-1}=\pm\sqrt{-\zeta_1^2-\cdots-\zeta_{n-2}^2+\sqrt{-1}\,\zeta_n}\,.$$

Put $\zeta_j=\xi_j+\sqrt{-1}\,\eta_j$, $j=1,\cdots,n$. First assume that

$$\xi_1^2+\cdots+\xi_{n-2}^2\leqq 3(\eta_1^2+\cdots+\eta_{n-2}^2)+2(|\xi_n|+|\eta_n|)\,.$$

Then we have directly

$$\begin{aligned}|\mathrm{Re}\,\zeta_{n-1}|&\leqq|\zeta_{n-1}|\\&\leqq\sqrt{\sum_{j=1}^{n-2}|\zeta_j|^2}+\sqrt{|\zeta_n|}\\&\leqq\sqrt{4\sum_{j=1}^{n-2}|\eta_j|^2+2\sqrt{2}\,|\zeta_n|}+\sqrt{|\zeta_n|}\\&\leqq 2\sum_{j=1}^{n-2}|\eta_j|+(\sqrt{2\sqrt{2}}+1)\sqrt{|\zeta_n|}\,.\end{aligned}$$

Thus (2.32) is satisfied with $B\geqq 2$.

Conversely assume that

$$\xi_1^2+\cdots+\xi_{n-2}^2\geqq 3(\eta_1^2+\cdots+\eta_{n-2}^2)+2(|\xi_n|+|\eta_n|).$$

Let a be a positive real number and b be a real number. Then we have obviously

$$|\mathrm{Im}\sqrt{a+\sqrt{-1}\,b}\,|=\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}=\frac{|b|}{\sqrt{2(\sqrt{a^2+b^2}+a)}}\leqq\frac{|b|}{2\sqrt{a}}.$$

We apply this inequality, employing

$$\begin{aligned}a&=\xi_1^2+\cdots+\xi_{n-2}^2-\eta_1^2-\cdots-\eta_{n-2}^2+\eta_n\\&\geqq\frac{1}{2}(\xi_1^2+\cdots+\xi_{n-2}^2+\eta_1^2+\cdots+\eta_{n-2}^2)+|\xi_n|+|\eta_n|+\eta_n\\&>0.\end{aligned}$$

Thus we have

$$\begin{aligned}&|\mathrm{Re}\,\zeta_{n-1}|\\&=|\mathrm{Im}\sqrt{\zeta_1^2+\cdots+\zeta_{n-2}^2-\sqrt{-1}\,\zeta_n}\,|\\&=\left|\mathrm{Im}\sqrt{(\xi_1^2+\cdots+\xi_{n-2}^2-\eta_1^2-\cdots-\eta_{n-2}^2+\eta_n)+2\sqrt{-1}\left(\xi_1\eta_1+\cdots+\xi_{n-2}\eta_{n-2}-\frac{1}{2}\xi_n\right)}\right|\\&\leqq\frac{\left|\xi_1\eta_1+\cdots+\xi_{n-2}\eta_{n-2}-\frac{1}{2}\xi_n\right|}{\sqrt{\xi_1^2+\cdots+\xi_{n-2}^2-\eta_1^2-\cdots-\eta_{n-2}^2+\eta_n}}\\&\leqq\sqrt{2}\,\frac{\sqrt{\xi_1^2+\cdots+\xi_{n-2}^2}\sqrt{\eta_1^2+\cdots+\eta_{n-2}^2}+\frac{1}{2}|\xi_n|}{\sqrt{\xi_1^2+\cdots+\xi_{n-2}^2+\eta_1^2+\cdots+\eta_{n-2}^2+2(|\xi_n|+|\eta_n|+\eta_n)}}\\&\leqq\sqrt{2}\sqrt{\eta_1^2+\cdots+\eta_{n-2}^2}+\frac{\frac{1}{2}|\xi_n|}{\sqrt{|\xi_n|+|\eta_n|+\eta_n}}\\&\leqq\sqrt{2}\sum_{j=1}^{n-2}|\eta_j|+\frac{1}{2}\sqrt{|\xi_n|}.\end{aligned}$$

Thus (2.32) is also satisfied if $B\geqq 2$. q. e. d.

Summing up, we have proved the following result: Assume that each irreducible component of p satisfies one of

I) the assumption of Theorem 2.6;

II) that of Theorem 2.7, 2); and

III) that of Theorem 2.12 (for a suitable coordinate system in x'-space).

Then we have $\mathcal{A}_p(U\backslash K)/\mathcal{A}_p(U)=0$. At this time, the more the number of the components corresponding to II), the thinner the set K must be.

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