J. Math. Soc. Japan
Vol. 26, No. 4, 1974

Quasi-permutation modules over finite groups, II

By Shizuo ENDO and Takehiko MIYATA

(Received July 17, 1973)

In this paper we continue the investigation of quasi-permutation modules over finite groups, begun in [4] and [5]. The notation and terminology are the same as those in [5].

Let Π be a finite group and denote the projective class group of the integral group algebra $Z\Pi$ by $C(Z\Pi)$. Let $\Omega_{Z\Pi}$ be a maximal order in $Q\Pi$ containing $Z\Pi$. As in [5] we put $\tilde{C}(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}$ is a projective ideal of $Z\Pi$ such that $\Omega_{Z\Pi}\mathfrak{A}\oplus\Omega_{Z\Pi}\cong\Omega_{Z\Pi}\oplus\Omega_{Z\Pi}\}$ and $C^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}$ is a quasi-permutation projective ideal of $Z\Pi\}$. We further define $\tilde{C}^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}$ is a projective ideal of $Z\Pi$ such that $\mathfrak{A}\oplus S\cong Z\Pi\oplus S$ for a permutation Π-module $S\}$.

In [5] we raised the following basic problem on quasi-permutation projective modules:

'*For a finite group* Π $\tilde{C}(Z\Pi)=C^q(Z\Pi)$?'

It was proved in [5] that if Π is an abelian group or a p-group where p is an odd prime, then the answer to the problem is affirmative.

This study is mainly centered on this problem. We will show that, for a fairly extensive class of finite groups, the answer to the problem is affirmative. But we will also give some examples of finite groups Π such that $C^q(Z\Pi)\subsetneqq\tilde{C}(Z\Pi)$.

First we will give the following:

[I] *The induction theorems hold for the functors* $\tilde{C}(Z\cdot)$, $C^q(Z\cdot)$ *and* $\tilde{C}^q(Z\cdot)$.

A finite group Π is said to be of split type over Q if any simple component of $Q\Pi$ is isomorphic to a full matrix algebra over its center.

As an application of [I] the following result can be shown.

[II] *Let* Π *be one of the following groups:*

(1) *a nilpotent group whose* 2*-Sylow subgroup is of split type over* Q;

(2) *an extension of a* p*-group whose subgroups are of split type over* Q *by a cyclic group of order prime to* p.

Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

Next, using the Rosen's theorem ([14]) and the Artin's theorem ([1]), we prove the following:

[III] *Let* Π *be one of the following groups:*

(1) *a semidirect product of a cyclic normal subgroup of order* n *and a cyclic* p*-subgroup such that* $(p, n)=1$ *where* p *is an odd prime;*

(2) *a dihedral group* D_n *of order* $2n$.

Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

Furthermore, applying [I] and [III], we get the following:

[IV] *Let* Π *be one of the following groups:*

(1) *the projective special linear group* $PSL(2, p^f)$ *where* p *is a prime and* $f \geqq 0$;

(2) *the Janko simple group* J_1;

(3) *the symmetric group* S_n, $n \leqq 7$.

Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

On the other hand, the following result can be deduced from the Artin's theorem and the Mackey's subgroup theorem.

[V] *Let* Π *be one of the following groups:*

(1) *the semidirect product of the cyclic normal subgroup* $C=\langle\sigma\rangle$ *of order* 15 *and the cyclic subgroup* $P=\langle\tau\rangle$ *of order* 4 *such that* $\tau^{-1}\sigma\tau=\sigma^2$;

(2) *the alternating group* A_n, $n=8, 9$.

Then $C^q(Z\Pi) \subsetneqq \tilde{C}(Z\Pi)$.

§1. The induction theorems.

Let F be a Frobenius functor and let M be a Frobenius F-module (for the definitions see [**11**]). Let $\mathfrak{M}$ be a class of finite groups. For any finite group Π we define $F_{\mathfrak{M}}(\Pi)$ (resp. $M_{\mathfrak{M}}(\Pi)$) to be the sum of the images of the maps $i_*: F(\Pi')\rightarrow F(\Pi)$ (resp. $M(\Pi')\rightarrow M(\Pi)$) for all $i: \Pi' \subseteqq \Pi$ with $\Pi' \in \mathfrak{M}$. The following result is the most important one in the theory of Frobenius modules.

(A) ([**11**], (3.4).) *Suppose that* $e\cdot F(\Pi) \subseteqq F_{\mathfrak{M}}(\Pi)$ *for some positive integer* e. *Then* $e\cdot M(\Pi) \subseteqq M_{\mathfrak{M}}(\Pi)$.

Let R be a Dedekind domain and let Π be a finite group. We will denote the Grothendieck ring of $R\Pi$ by $G(R\Pi)$ ([**16**]). The functor $G(R\cdot)$ is the most typical Frobenius functor.

From now we will assume that K is an algebraic number field and that R is the ring of all algebraic integers in K. $\mathfrak{C}$ will denote the class of all cyclic groups. $\mathfrak{E}_K$ will denote the class of all K-elementary groups and, especially, $\mathfrak{H}$ will denote the class of all hyperelementary groups. Now the well-known induction theorem can be stated as follows:

(B) ([**16**].) *Let* Π *be a finite group. Then:*

(1) $|\Pi|\cdot G(Q\Pi) \subseteqq G_{\mathfrak{C}}(Q\Pi)$ (Artin).

(2) $G_{\mathfrak{E}_K}(K\Pi)=G(K\Pi)$, *and especially* $G_{\mathfrak{H}}(Q\Pi)=G(Q\Pi)$ (Brauer-Witt-Berman).

Let Π be a finite group and let $\Omega_{R\Pi}$ be a maximal R-order in $K\Pi$ containing $R\Pi$. We denote by $C(R\Pi)$ and $C(\Omega_{R\Pi})$ the (reduced) projective class group of $R\Pi$ and $\Omega_{R\Pi}$, respectively. Then we have the natural epimorphism $\nu: C(R\Pi)\to C(\Omega_{R\Pi})$ (e.g. [**5**]). Put $\tilde{C}(R\Pi)=\operatorname{Ker}\nu$. Then we easily see that $\tilde{C}(R\Pi)=\{[\mathfrak{A}]-[R\Pi]\in C(R\Pi)\mid \mathfrak{A}$ is a projective ideal of $R\Pi$ such that $\Omega_{R\Pi}\mathfrak{A}\oplus\Omega_{R\Pi}\cong\Omega_{R\Pi}\oplus\Omega_{R\Pi}\}=\{[\mathfrak{A}]-[R\Pi]\in C(R\Pi)\mid\mathfrak{A}$ is a projective ideal of $R\Pi$ such that $\mathfrak{A}\oplus X\cong R\Pi\oplus X$ for some finitely generated $R\Pi$-module $X\}$ (e.g. [**5**], (2.4)). R. G. Swan proved in [**16**], § 9 that the functor $C(R\cdot)$ is a Frobenius $G(K\cdot)$-module so that by (A) and (B) the induction theorem holds for $C(R\cdot)$.

We first give

THEOREM 1.1. *The functor $\tilde{C}(R\cdot)$ is a Frobenius $G(K\cdot)$-submodule of $C(R\cdot)$. Let Π be a finite group. Then:*

(1) $|\Pi|\cdot\tilde{C}(Z\Pi)\subseteq\tilde{C}_{\mathfrak{C}}(Z\Pi)$.

(2) $\tilde{C}_{\mathfrak{C}_K}(R\Pi)=\tilde{C}(R\Pi)$ *and especially* $\tilde{C}_{\mathfrak{H}}(Z\Pi)=\tilde{C}(Z\Pi)$.

PROOF. The second part of the theorem is an immediate consequence of (A), (B) and the first part. Hence we only need to prove the first part. Let Π be a finite group, let Π' be a subgroup of Π and let $i:\Pi'\to\Pi$ be the inclusion map. In [**16**], § 9 the following maps have been defined: (i) $i_*: C(R\Pi')\to C(R\Pi)$; (ii) $i^*: C(R\Pi)\to C(R\Pi')$; (iii) $\mu: G(K\Pi)\times C(R\Pi)\to C(R\Pi)$. In fact Swan proved that these maps make $C(R\cdot)$ a Frobenius $G(K\cdot)$-module. Accordingly it suffices to check a) $i_*(\tilde{C}(R\Pi'))\subseteq\tilde{C}(R\Pi)$, b) $i^*(\tilde{C}(R\Pi))\subseteq\tilde{C}(R\Pi')$ and $\mu(G(K\Pi)\times\tilde{C}(R\Pi))\subseteq\tilde{C}(R\Pi)$.

a) Let $[\mathfrak{A}']-[R\Pi']$ be an element of $\tilde{C}(R\Pi')$. Then there is a finitely generated $R\Pi'$-module X' such that $\mathfrak{A}'\oplus X'\cong R\Pi'\oplus X'$. Tensoring this with $R\Pi$ over $R\Pi'$, we get $(R\Pi\otimes_{R\Pi'}\mathfrak{A}')\oplus(R\Pi\otimes_{R\Pi'}X')\cong R\Pi\oplus(R\Pi\otimes_{R\Pi'}X')$. This implies that $i_*([\mathfrak{A}']-[R\Pi'])=[R\Pi\otimes_{R\Pi'}\mathfrak{A}']-[R\Pi]\in\tilde{C}(R\Pi)$.

b) This is evident.

c) By the definition of μ it suffices to show that $\mu(G(R\Pi)\times\tilde{C}(R\Pi))\subseteq\tilde{C}(R\Pi)$. Let $[\mathfrak{A}]-[R\Pi]\in\tilde{C}(R\Pi)$ and $[M]\in G(R\Pi)$ where M is a finitely generated R-projective $R\Pi$-module. Then we have $\mathfrak{A}\oplus X\cong R\Pi\oplus X$ for some finitely generated $R\Pi$-module X. Tensoring this with M over R, we get $(M\otimes_R\mathfrak{A})\oplus(M\otimes_R X)\cong(M\otimes_R R\Pi)\oplus(M\otimes_R X)$. Since M is R-projective, both $M\otimes_R\mathfrak{A}$ and $M\otimes_R R\Pi$ are $R\Pi$-projective ([**16**], Prop. 5.1). Therefore $\mu([M]\times([\mathfrak{A}]-[R\Pi]))=[M\otimes_R\mathfrak{A}]-[M\otimes_R R\Pi]\in\tilde{C}(R\Pi)$.

COROLLARY 1.2. *$C(\Omega_{R\cdot})$ is a Frobenius $G(K\cdot)$-module.*

PROOF. By the definition of $\tilde{C}(R\cdot)$ we have $C(\Omega_{R\cdot})=C(R\cdot)/\tilde{C}(R\cdot)$. Since $\tilde{C}(R\cdot)$ is a Frobenius $G(K\cdot)$-submodule of $C(R\cdot)$ by (1.1), $C(\Omega_{R\cdot})$ is a Frobenius $G(K\cdot)$-module.

COROLLARY 1.3 (Reiner-Ullom [**12**]). *Let Π be a finite p-group. Then*

$\tilde{C}(Z\Pi)$ *is a p-group.*

PROOF. In case Π is cyclic this can easily be proved ([**2**], p. 604, (5.9), etc.). Therefore in the general case this follows from (1.1), (1).

Let R be a commutative ring with unit element. Let Π be a finite group. We define $B(R\Pi)$ to be the abelian group given by generators $[\bigoplus_{i=1}^{t} R\Pi/\Pi'_i]$ where $\Pi'_1, \Pi'_2, \cdots, \Pi'_t$ are subgroups of Π with relations

$$[(\bigoplus_{i=1}^{t} R\Pi/\Pi'_i)\oplus(\bigoplus_{j=1}^{s} R\Pi/\Pi''_j)]=[\bigoplus_{i=1}^{t} R\Pi/\Pi'_i]+[\bigoplus_{j=1}^{s} R\Pi/\Pi''_j].$$

Then it is clear that $B(R\cdot)$ is a Frobenius functor.

Now we consider the case where $R=Z$. Here we have the commutative diagram of Frobenius functors:

$$\begin{array}{ccc} B(Z\cdot) & \xrightarrow{\alpha_Z} & G(Z\cdot) \\ \downarrow \beta & & \downarrow \gamma \\ B(Q\cdot) & \xrightarrow{\alpha_Q} & G(Q\cdot) \end{array}$$

where β and γ are epimorphisms while α_Q is a monomorphism. It is easily seen (e.g. [**16**], Prop. 4.1) that $|\Pi|\cdot B(Q\Pi)\subseteqq B_{\mathfrak{C}}(Q\Pi)$. Further we have

PROPOSITION 1.4. *Let Π be a finite group. Then $B_{\mathfrak{S}}(Q\Pi)=B(Q\Pi)$.*

PROOF. This can be seen, for example, in the Swan's proof of the Witt-Berman induction theorem ([**16**], § 4). In fact, let ρ be a finite group with a cyclic normal subgroup σ such that the extension $1\to\sigma\to\rho\to\rho/\sigma\to 1$ splits. As in the proof of [**16**], Lemma 4.4 we can make $Q\sigma$ a $Q\rho$-module. Then, for any subgroup σ' of σ, we can find a subgroup ρ' of ρ such that $Q\sigma/\sigma'\cong Q\rho/\rho'$ as $Q\rho$-modules. Further it is easily seen that $B(Q\sigma)=G(Q\sigma)$. Hence a function f_p in [**16**], Lemma 4.5 can be chosen in $B_{\mathfrak{S}}(Q\Pi)$. Therefore, along the same line as in the Swan's proof ([**16**], p. 564), we can prove that $B_{\mathfrak{S}}(Q\Pi)=B(Q\Pi)$.

REMARK 1.5. The monomorphism $\alpha_{Q\Pi}: B(Q\Pi)\to G(Q\Pi)$ is not always an isomorphism. In fact, J.-P. Serre noted in [**15**], p. 120, Ex. 4 that if Π is the direct product of a cyclic group of order 3 and a quaternion group of order 8, then $B(Q\Pi)\subsetneqq G(Q\Pi)$. Recently J. Ritter proved in [**13**] that if Π is a finite p-group then $B(Q\Pi)=G(Q\Pi)$. For further informations, see (3.3), (5.3) and [**17**].

As in [**4**] we define $C^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{A}$ is a quasi-permutation projective ideal of $Z\Pi\}$. Further we define $\tilde{C}^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\,|\,\mathfrak{A}$ is a projective ideal of $Z\Pi$ such that $\mathfrak{A}\oplus S\cong Z\Pi\oplus S$ for some permutation Π-module $S\}$. Then both $C^q(Z\Pi)$ and $\tilde{C}^q(Z\Pi)$ are submodules of $C(Z\Pi)$ and $\tilde{C}^q(Z\Pi)\subseteqq\tilde{C}(Z\Pi)\cap C^q(Z\Pi)$.

THEOREM 1.6. *The functors $C^q(Z\cdot)$ and $\tilde{C}^q(Z\cdot)$ are Frobenius $B(Q\cdot)$-sub-*

modules of $C(Z\cdot)$. *In particular, for any finite group* Π, $C^q_{\mathfrak{P}}(Z\Pi)=C^q(Z\Pi)$ *and* $\tilde{C}^q_{\mathfrak{P}}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. Both $C^q(Z\cdot)$ and $\tilde{C}^q(Z\cdot)$ are clearly Frobenius $B(Z\cdot)$-submodules of $C(Z\cdot)$. Since $C(Z\cdot)$ is a Frobenius $B(Q\cdot)$-module, we have $\ker\beta\cdot C(Z\cdot)=0$. Therefore both $C^q(Z\cdot)$ and $\tilde{C}^q(Z\cdot)$ are Frobenius $B(Q\cdot)$-submodules of $C(Z\cdot)$. The second part of the theorem follows immediately from the first part, (1.4) and (A).

§2. Restatements of the problem.

Let Π be a finite group. Let A_Π be the set of all subgroups of Π and let B_Π be the set of all subgroups Π' of Π such that $Z\Pi/\Pi'$ satisfies the Eichler's condition (ε) ([**5**]). We put $T_\Pi=(\bigoplus_{\Pi'\in B_\Pi} Z\Pi/\Pi')\oplus(\bigoplus_{\Pi''\in A_\Pi-B_\Pi}[Z\Pi/\Pi'']^{(2)})$.

Let $\boldsymbol{C}_\Pi$ be the class of all (finitely generated Z-free) Π-modules. Let M, $M'\in\boldsymbol{C}_\Pi$. We write $M\sim M'$ if $M_p\cong M'_p$ for every prime p. Further we write $M\approx M'$ if $M\sim M'$ and $\Omega_{Z\Pi}M\cong\Omega_{Z\Pi}M'$. For $M\in\boldsymbol{C}_\Pi$ we put $\gamma_M=\{X\in\boldsymbol{C}_\Pi\,|\,X\approx M\}$ and denote by $|\gamma_M|$ the number of all isomorphism types in γ_M.

PROPOSITION 2.1. *For any finite group* Π *the following statements are equivalent:*

(1) *Any* Π*-module* L *with* $L\approx T_\Pi$ *is a quasi-permutation module.*

(2) $\tilde{C}(Z\Pi)\subseteqq C^q(Z\Pi)$.

PROOF. (1)$\Rightarrow$(2): Let $[\mathfrak{A}]-[Z\Pi]\in\tilde{C}(Z\Pi)$. There is a Π-module L with $L\approx T_\Pi$ such that $\mathfrak{A}\oplus T_\Pi\cong Z\Pi\oplus L$. By hypothesis L is a quasi-permutation Π-module. Therefore [**5**], (1.4) shows that $\mathfrak{A}$ is a quasi-permutation Π-module. (2)$\Rightarrow$(1): Let L be a Π-module with $L\approx T_\Pi$. Now there is a projective ideal $\mathfrak{A}$ of $Z\Pi$ such that $T_\Pi\oplus Z\Pi\cong L\oplus\mathfrak{A}$. Hence $[\mathfrak{A}]-[Z\Pi]\in\tilde{C}(Z\Pi)$. Then by hypothesis $\mathfrak{A}$ is a quasi-permutation Π-module, and therefore L is so.

PROPOSITION 2.2. *For any finite group* Π *the following statements are equivalent:*

(1) $|\gamma_{T_\Pi}|=1$.

(2) *There exists a faithful quasi-permutation* Π*-module* N *satisfying* (ε) *such that* $|\gamma_N|=1$.

(3) $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. (1)$\Rightarrow$(2) is evident and (1)$\Leftrightarrow$(3) can be shown in the same way as in the proof of (2.1). Hence we only need to prove (2)$\Rightarrow$(1). To prove this let L be a Π-module with $L\approx T_\Pi$. Then $L\oplus N\cong T_\Pi\oplus N$ because $|\gamma_N|=1$. Since N is a quasi-permutation Π-module, there exists an exact sequence

$$0\longrightarrow N\longrightarrow S\longrightarrow S'\longrightarrow 0$$

where S and S' are permutation Π-modules. Taking the pushout of

$$\begin{array}{l} L\oplus N\cong T_{\Pi}\oplus N\rightarrow T_{\Pi}\oplus S, \\ \quad\downarrow \\ L\oplus S \end{array}$$
we get the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L\oplus N & \longrightarrow & T_{\Pi}\oplus S & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L\oplus S & \longrightarrow & X & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S' & = & S' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The second row and column of this diagram split and so

$$L\oplus S\oplus S'\cong X\cong T_{\Pi}\oplus S\oplus S'.$$

Using the cancelation theorem we get $L\cong T_{\Pi}$. This shows that $|\gamma_{T_{\Pi}}|=1$.

REMARK 2.3. Let Π be a finite abelian group. Let $\mathfrak{S}$ be the set of all subgroups, Π', of Π such that Π/Π' is cyclic and put $T=\bigoplus_{\Pi'\in\mathfrak{S}} Z\Pi/\Pi'$. In [5], (4.2) we have shown that if $\mathfrak{A}$ is a quasi-permutation projective ideal of $Z\Pi$ then $\mathfrak{A}\oplus T\cong Z\Pi\oplus T$. However the proof of it in [5] was fairly complicated. To prove this it suffices to show that $|\gamma_T|=1$ because $\tilde{C}(Z\Pi)=C^q(Z\Pi)$ by [5], (2.5). Using $\Omega_{Z\Pi}$ instead of N in the proof of (2.2), (2) $\Rightarrow$ (1) we can easily show that $|\gamma_T|=1$ along the same line as in the proof of (2.2).

LEMMA 2.4. *Let Π be a finite group and let Λ be a hereditary order in $Q\Pi$ containing $Z\Pi$. Then $|\gamma_{\Lambda^{(2)}}|=1$. Let Ω be a maximal order in $Q\Pi$ containing Λ. Then the natural map $i: C(\Lambda)\rightarrow C(\Omega)$ is an isomorphism.*

PROOF. First we will prove the second assertion. It is clear that i is an epimorphism. Hence we only need to show that i is a monomorphism. Let $\mathfrak{A}$ be a locally free ideal of Λ such that $Q\mathfrak{A}\cong Q\Pi$. Then we can show as in [5], (2.4) that $\mathfrak{A}\oplus\Omega\cong\Lambda\oplus\Omega\mathfrak{A}$. If $\Omega\mathfrak{A}\oplus\Omega\cong\Omega\oplus\Omega$, then $\mathfrak{A}\oplus\Omega^{(2)}\cong\Lambda\oplus\Omega^{(2)}$. Since Λ is hereditary, Ω is Λ-projective, and therefore $\mathfrak{A}\oplus\Lambda^{(l)}\cong\Lambda\oplus\Lambda^{(l)}$ for some $l\geqq 0$. This implies that i is a monomorphism. Let M be a Π-module such that $M\approx\Lambda^{(2)}$. Then M can be regarded as a Λ-module. Since $\Lambda^{(2)}$ satisfies (ε), we have $\Omega M\cong\Omega^{(2)}$. Hence the second assertion shows that $M=\Lambda M\cong\Lambda^{(2)}$. This proves that $|\gamma_{\Lambda^{(2)}}|=1$.

PROPOSITION 2.5. *Let Π be a finite group and suppose that there exists a hereditary order Λ in $Q\Pi$ containing $Z\Pi$ which is a quasi-permutation Π-*

module. Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. By (2.4) we have $|\gamma_{\Lambda^{(2)}}|=1$. Hence the Π-module $\Lambda^{(2)}$ satisfies the condition (2) in (2.2).

Let Π be a finite group. As usual we define the representation ring $A(Z\Pi)$ of $Z\Pi$ to be the abelian group with one generator for each $M\in \boldsymbol{C}_\Pi$ and relations $[M_1\oplus M_2]=[M_1]+[M_2]$ and $[M]=[M']$ when $M\cong M'$. There exists the natural ring homomorphism $\omega_\Pi: B(Z\Pi)\to A(Z\Pi)$. We denote the image of ω_Π by $B^A(Z\Pi)$.

The torsion part of an abelian group A is denoted by $t(A)$.

PROPOSITION 2.6. *For any finite group* Π *the following statements are equivalent:*

(1) *Any quasi-permutation* Π*-module* L *with* $L\sim T_\Pi$ *satisfies* $L\approx T_\Pi$.

(2) $t(B^A(Z\Pi))=0$.

(3) $C^q(Z\Pi)\subseteqq \tilde{C}(Z\Pi)$.

PROOF. (1)$\Rightarrow$(2): Let $[S]-[S']\in t(B^A(Z\Pi))$. Then we have $S\sim S'$. There is a quasi-permutation Π-module L such that $T_\Pi\oplus S\cong L\oplus S'$. Since $L\sim T_\Pi$, $L\approx T_\Pi$ by assumption. Hence $[S]-[S']=[L]-[T_\Pi]=0$ in $A(Z\Pi)$. Thus $t(B^A(Z\Pi))=0$. (2)$\Rightarrow$(3): Let $\mathfrak{A}$ be a quasi-permutation projective ideal of $Z\Pi$. Then there is a quasi-permutation Π-module L such that $\mathfrak{A}\oplus T_\Pi \cong Z\Pi\oplus L$. By definition there exist permutation Π-modules S, S' such that $L\oplus S'\cong S$. Hence $[\mathfrak{A}]-[Z\Pi]=[L]-[T_\Pi]=[S]-[T_\Pi\oplus S']=0$ in $A(Z\Pi)$. This shows that $[\mathfrak{A}]-[Z\Pi]\in\tilde{C}(Z\Pi)$. (3)$\Rightarrow$(1): Let L be a quasi-permutation Π-module with $L\sim T_\Pi$. Then we can find a quasi-permutation projective ideal $\mathfrak{A}$ of $Z\Pi$ such that $L\oplus Z\Pi\cong T_\Pi\oplus\mathfrak{A}$. Since $C^q(Z\Pi)\subseteqq\tilde{C}(Z\Pi)$, $\mathfrak{A}\oplus Z\Pi\cong Z\Pi\oplus Z\Pi$. Therefore $L\oplus Z\Pi^{(2)}\approx T_\Pi\oplus Z\Pi^{(2)}$ so that $L\approx T_\Pi$.

PROPOSITION 2.7. *For any finite group* Π *the following statements are equivalent:*

(1) *Any quasi-permutation* Π*-module* L *with* $L\sim T_\Pi$ *is isomorphic to* T_Π.

(2) $t(B(Z\Pi))=0$.

(3) $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. This can be proved in the same way as in (2.6).

REMARK 2.8. We can show that $\omega_\Pi: B(Z\Pi)\to A(Z\Pi)$ is a monomorphism (i.e., $B^A(Z\Pi)=B(Z\Pi)$) if and only if $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)\cap C^q(Z\Pi)$.

§ 3. Nilpotent groups and cyclic extensions of p-groups.

We begin with

PROPOSITION 3.1. *Let* Π *be a finite group which is a cyclic extension of a* p*-subgroup. Then* $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. By the Conlon's theorem ([**3**], (8.1)), we have $t(B(Z\Pi))=0$. Therefore this follows immediately from (2.7).

The following theorem is a generalization of [**5**], (3.4), (1) and (3.9), (1).

THEOREM 3.2. *Let Π be a finite nilpotent group. Then $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)$. Furthermore suppose that the 2-Sylow subgroup of Π is of split type over Q. Then $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$.*

PROOF. For each prime $p \mid |\Pi|$ we denote the p-Sylow subgroup of Π by $\Pi^{(p)}$. By (1.1) and (1.6) it suffices to prove the theorem in the case of $\Pi=\Pi^{(p)}\times\Pi'$ where Π' is a cyclic group with $p\nmid m=|\Pi'|$. In this case the first part of the theorem follows directly from (3.1). Further, if p is odd, we have shown in the proof of [**5**], (3.4), (1) that there is a faithful quasi-permutation Π-module N such that $|\gamma_N|=1$. Hence (2.2) implies that $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

Now it remains to prove that $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$ under the assumption that $\Pi=\Pi^{(2)}\times\Pi'$ where $\Pi^{(2)}$ is of split type over Q and Π' is a cyclic group of odd order m. Let $U_1, U_2, \cdots, U_t$ be the isomorphism types of irreducible $Q\Pi$-modules. We will construct a quasi-permutation Π-module N_i such that $Q\underset{Z}{\otimes}N_i\cong U_i$ and $|\gamma_{N_i}|=1$. Since $2\nmid m$ there exist an irreducible $Q\Pi^{(2)}$-module V_i and $m_i|m$ such that $U_i\cong V_i\underset{Q}{\otimes}Q[\zeta_{m_i}]$. Let ξ_i be the rational character of $\Pi^{(2)}$ afforded by V_i and let χ_i be an absolutely irreducible character of $\Pi^{(2)}$ such that $(\chi_i, \xi_i)\neq 0$. By the Feit's theorem ([**6**], (14.3)) there exist a subgroup Π_i' of $\Pi^{(2)}$ and an absolutely irreducible character χ_i' of Π_i' such that $\chi_i=\chi_i'^*$, $Q(\chi_i)=Q(\chi_i')$ and $\Pi_i''=\Pi_i'/\mathrm{Ker}\,\chi_i'$ has a cyclic normal subgroup of index 2. Then it is clear that $m_Q(\chi_i)=m_Q(\chi_i')$. Accordingly we can find a rational character ξ_i' such that $\xi_i=\xi_i'^*$ and $(\chi_i', \xi_i')\neq 0$. Since $\Pi^{(2)}$ is of split type over Q, $m_Q(\chi_i)=m_Q(\chi_i')=1$, and so each Π_i'' must be cyclic, dihedral or semidihedral. Let V_i' be the irreducible $Q\Pi_i'$-module with character ξ_i' and let A_i be the maximal order in $Q(\chi_i')$. Then there exists a quasi-permutation Π_i'-module N_i' such that $V_i'=Q\underset{Z}{\otimes}N_i'$ and $\mathrm{End}_{Z\Pi_i'}(N_i')=A_i$ (see the proof of [**5**], (3.4)). We put $N_i=(Z\Pi^{(2)}\underset{Z\Pi_i'}{\otimes}N_i')\underset{Z}{\otimes}Z[\zeta_{m_i}]$. Then N_i is clearly a quasi-permutation Π-module. Because $2\nmid m$, $A_i\underset{Z}{\otimes}Z[\zeta_{m_i}]$ is a Dedekind domain. It is easily seen that $\mathrm{End}_{Z\Pi}(N_i)=A_i\underset{Z}{\otimes}Z[\zeta_{m_i}]$ and therefore, by [**5**], §3, (E′), $|\gamma_{N_i}|=1$. Finally put $N=\bigoplus_{i=1}^{t}N_i$. Then N is a faithful quasi-permutation Π-module such that $|\gamma_N|=1$. Again by (2.2) this concludes that $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

REMARK 3.3. J. Ritter proved that, if Π is a finite nilpotent group whose 2-Sylow subgroup is of split type over Q, then $B(Q\Pi)=G(Q\Pi)$. However this result follows immediately from (1.4) and [**6**], (14.3).

Let Π be a finite group which is a semidirect product of a cyclic normal subgroup $C=\langle\sigma\rangle$ of order n and an abelian p-subgroup P such that $p\nmid n$.

Then we have $Q\Pi=\bigoplus_{m|n} Q\Pi/(\Phi_m(\sigma))$. For every $m|n$ the abelian p-group P acts naturally on C. Denote the kernel of this action by P_m and let $QP_m = \bigoplus_{i=1}^{s(m)} Q(\zeta_{pl_i^{(m)}})$ be the decomposition of QP_m into simple algebras. Then $Q\Pi/(\Phi_m(\sigma))$ can be expressed as the direct sum of the crossed products

$$\Sigma_{m,i}=\Delta(\varphi_i^{(m)},\, Q(\zeta_{mpl_i^{(m)}}),\, P/P_m)$$

where each $\varphi_i^{(m)}$ is a $\langle\zeta_{pl_i^{(m)}}\rangle$-valued 2-cocycle of P/P_m. Now it is easily seen that the image of $Z\Pi$ in $\Sigma_{m,i}$ coincides with the crossed product

$$\Lambda_{m,i}=\Delta(\varphi_i^{(m)},\, Z[\zeta_{mpl_i^{(m)}}],\, P/P_m)\,.$$

Put $\Lambda=\bigoplus_{m|n}\bigoplus_{i=1}^{s(m)}\Lambda_{m,i}$. Then Λ is an order of $Q\Pi$ containing $Z\Pi$.

LEMMA 3.4. *Let Π, $\Sigma_{m,i}$ and $\Lambda_{m,i}$ be as above. Then each $\Lambda_{m,i}$ is a hereditary order in $\Sigma_{m,i}$ which is a quasi-permutation Π-module.*

PROOF. Since $p\nmid n$ the extension $Q(\zeta_{mpl_i^{(m)}})/Q(\zeta_{mpl_i^{(m)}})^{P/P_m}$ is tamely ramified. Hence the crossed product $\Lambda_{m,i}$ is a hereditary order in $\Sigma_{m,i}$ (e.g. [18]). We denote the kernel of the natural projection $P_m\to Q(\zeta_{pl_i^{(m)}})$ by $P_{m,i}$. Then $P_m/P_{m,i}$ is cyclic and $\Lambda_{m,i}=\Delta(\varphi_i^{(m)},\, Z[\zeta_{mpl_i^{(m)}}],\, P/P_{m,i}\big/P_m/P_{m,i})$. Therefore we may assume that P_m is cyclic. Let $|P_m|=p^l$ and $P_m=\langle\tau\rangle$. Then $\Lambda_{m,i}\cong Z\Pi/(\Phi_{mp^l}(\sigma\tau))$. As in the proof of [5], (2.3) we can show that $Z\Pi/(\Phi_{mp^l}(\sigma\tau))$ is a quasi-permutation Π-module. Consequently $\Lambda_{m,i}$ is a quasi-permutation Π-module.

PROPOSITION 3.5. *Let Π be a finite group whose Sylow subgroups are abelian. Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.*

PROOF. To prove this we may assume by (1.1) and (1.6) that Π is hyperelementary. Then Π is expressible as the semidirect product of a cyclic normal subgroup C and an abelian p-subgroup P such that $p\nmid n=|C|$. Let Λ be the order of $Q\Pi$ containing $Z\Pi$ as given in the preceding lines of (3.4). Then (3.4) shows that Λ is a hereditary order in $Q\Pi$ which is a quasi-permutation Π-module. By (2.5) this concludes that $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

THEOREM 3.6. *Let Π be a finite group which is an extension of a p-group P by a cyclic group C with $p\nmid|C|$. In case $p=2$ suppose that all subgroups of P are of split type over Q. Then $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$.*

PROOF. By (3.1) we have $C^q(Z\Pi)=\tilde{C}^q(Z\Pi)$. Hence we only need to show that $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$. Let Π' be a hyperelementary subgroup of Π. We can write

$$1\longrightarrow C'\longrightarrow \Pi'\longrightarrow P'\longrightarrow 1$$

where C' is a cyclic group and P' is a p'-group such that $p'\nmid|C'|$. When $p'=p$, $P'\subseteqq P$ so that $\Pi'=P'\times C'$. Hence $\tilde{C}^q(Z\Pi')=\tilde{C}(Z\Pi')$ by (3.2). On the other hand, when $p'\neq p$, P' can be considered as a subgroup of C' and therefore P' is cyclic. So we can deduce the same conclusion from (3.5). Using (1.1) and (1.6) we get $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$.

§ 4. Metacyclic groups.

Let Π be a finite group which is a semidirect product of a cyclic normal subgroup $C=\langle\sigma\rangle$ of order n and an abelian p-subgroup P with $p\nmid n$. Then for each $m|n$ there exists the natural homomorphism $\mu_m: P\rightarrow \mathrm{Aut}\, C/\langle\sigma^{n/m}\rangle$. We denote the kernel of μ_m by P_m.

Now we suppose that $P_n=\{1\}$. Then both $Q\Pi/(\Phi_n(\sigma))$ and $Z\Pi/(\Phi_n(\sigma))$ can be identified with the trivial crossed products $\Delta(1, Q(\zeta_n), P)$ and $\Delta(1, Z[\zeta_n], P)$, respectively. We denote $\Delta(1, Q(\zeta_n), P)$ and $\Delta(1, Z[\zeta_n], P)$ by Σ_n and Λ_n, respectively. By (3.4) Λ_n is a hereditary order in Σ_n. Further let $A_n=Z[\zeta_n]$ and $R_n=Z[\zeta_n]^P$. Then A_n can be considered as a Λ_n-module. If M is a Λ_n-module, $\mathrm{Hom}_{\Lambda_n}(A_n, M)$ can be regarded as an R_n-module.

LEMMA 4.1. *Let Π, Λ_n, A_n, R_n be as above.*

(1) *Let Π' be a subgroup of Π. If $|\Pi'|$ is a power of p, then* $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\otimes_{Z\Pi} Z\Pi/\Pi')\cong A_n^{\Pi'}$ *as R_n-modules, while, if $|\Pi'|$ is not a power of p,* $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\otimes_{Z\Pi} Z\Pi/\Pi')$ *is a torsion R_n-module.*

(2) *Let $\mathfrak{A}$ be a projective ideal of $Z\Pi$. Then $\Lambda_n\mathfrak{A}\cong\Lambda_n$ as Λ_n-modules if and only if* $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\mathfrak{A})\cong A_n$ *as R_n-modules.*

PROOF. The first assertion can easily be proved, and the second assertion is only a special case of the Rosen's theorem ([**14**], p. 22, Theorem 8).

Here we return to the general situation. Let $n=q_1^{k_1}q_2^{k_2}\cdots q_s^{k_s}$ be the decomposition of $n=|C|$ into prime factors where $q_1, q_2, \cdots, q_s$ are distinct primes, and put $r_i=\prod_{j\neq i} q_j^{k_j}$ and $C_i=\langle\sigma^{r_i}\rangle$. Then P acts on each C_i. We denotes the number of the suffixes, i, such that P acts nontrivially on C_i by $m(\Pi)$.

THEOREM 4.2. *Let Π be a finite group which is a semidirect product of a cyclic normal subgroup C and a cyclic p-subgroup P with $p\nmid|C|$. Suppose that Π satisfies one of the following conditions:*

a) *p is odd;*

b) *$p=2$ and $m(\Pi)\leqq 1$;*

c) *$p=2$ and $P_n\neq\{1\}$;*

d) *$p=2$ and $|P|=2$.*

Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

PROOF. By (3.5) we have $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$. Hence we only need to prove that $C^q(Z\Pi)\subseteqq\tilde{C}(Z\Pi)$. Let $n=|C|$. Let $\Lambda=\bigoplus_{m|n}\bigoplus_{i=1}^{s(m)}\Lambda_{m,i}$ be the order of $Q\Pi$ containing $Z\Pi$ as given in the preceding line of (3.4). Let $\mathfrak{A}$ be a projective ideal of $Z\Pi$ such that $\mathfrak{A}\oplus S_1\cong Z\Pi\oplus S_2$ for some permutation Π-modules S_1

and S_2. Now to prove that $C^q(Z\Pi)\subseteqq\tilde{C}(Z\Pi)$ it suffices to show that $\Lambda_{m,i}\mathfrak{A}\oplus\Lambda_{m,i}\cong\Lambda_{m,i}\oplus\Lambda_{m,i}$ for each $m|n$ and each $1\leqq i\leqq s^{(m)}$. Using the induction on n it suffices to show this in case $m=n$. Let $\Pi'\neq\{1\}$ be a subgroup of Π. If $|\Pi'|$ is not a power of p, then $\Pi'\cap C\neq\{1\}$, hence $\Lambda_{n,i}\otimes_{Z\Pi}Z\Pi/\Pi'$ is a torsion module. If $|\Pi'|$ is a power of p, then Π' is conjugate to a subgroup P' of P. Suppose that $P_n\neq\{1\}$. Because P is cyclic, we have $P'\cap P_n\neq\{1\}$, and therefore $\Lambda_{n,i}\otimes_{Z\Pi}Z\Pi/\Pi'\cong\Lambda_{n,i}\otimes_{Z\Pi}Z\Pi/P'$ is also a torsion module. Tensoring $\mathfrak{A}\oplus S_1\cong Z\Pi\oplus S_2$ with $\Lambda_{n,i}$ over $Z\Pi$ and eliminating the torsion parts from both sides, we get $\Lambda_{n,i}\mathfrak{A}\oplus\Lambda_{n,i}\cong\Lambda_{n,i}\oplus\Lambda_{n,i}$.

Next suppose that $P_n=\{1\}$. Then $s^{(n)}=1$ and $\Lambda_n=\Lambda_{n,1}$ is the trivial crossed product $\Delta(1, Z[\zeta_n], P)$. We put $A_n=Z[\zeta_n]$ and $R_n=Z[\zeta_n]^P$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ be all the primes of A_n ramified over R_n, and, for each $1\leqq j\leqq r$, denote by T_j the inertia group of $\mathfrak{p}_j$. Since P is a cyclic p-group, the set of all subgroups of P is linearly ordered. Therefore there is the largest subgroup $T=T_{j_0}$ in $\{T_j\}_{1\leqq j\leqq r}$. Then the extension A_n^T/R_n is unramified and the prime ideal $\mathfrak{p}=\mathfrak{p}_{j_0}$ has the ramification index $|T|$ in A_n/A_n^T. Let $\Lambda'_n=\sum_{\tau\in T}A_nu_\tau\subseteqq\Lambda_n$. Then Λ'_n is the R_n-subalgebra of Λ_n. Let Π' be a p-subgroup of Π. Then Π' is conjugate to a subgroup P' of P. Regarding $\Lambda_n\otimes_{Z\Pi}Z\Pi/\Pi'$ as a Λ'_n-module, we easily see that

$$\Lambda_n\otimes_{Z\Pi}Z\Pi/\Pi'\cong\Lambda_n\otimes_{Z\Pi}Z\Pi/P'\cong\begin{cases}A_n^{[P:P']} & \text{when } T\subseteqq P'\\ [\Lambda_{n'}\otimes_{ZCT}ZCT/P']^{[P:P']} & \text{when } T\supseteqq P'.\end{cases}\quad\cdots\cdots(*)$$

Tensor $\mathfrak{A}\oplus S_1\cong Z\Pi\oplus S_2$ with Λ_n over $Z\Pi$ and eliminate the torsion parts from both sides. Then we have

$$\Lambda_n\mathfrak{A}\oplus\bigoplus_{P'\subseteqq P}[\Lambda_n\otimes_{Z\Pi}Z\Pi/P']^{(r_{P'})}\cong\Lambda_n\oplus\bigoplus_{P'\subseteqq P}[\Lambda_n\otimes_{Z\Pi}Z\Pi/P']^{(s_{P'})}$$

for some integers $r_{P'}$ and $s_{P'}$. Localize both sides at $\mathfrak{p}$ and regard them as $(\Lambda'_n)_\mathfrak{p}$-modules. Using the same argument as in [14], pp. 14~15, it follows from (*) that $r_{P'}=s_{P'}$ for any $P'\subset T$. Hence we may assume that $r_{P'}=s_{P'}=0$ when $P'\subset T$. Applying the functor $\mathrm{Hom}_{\Lambda_n}(A_n,\ )$ to them, we get, by (4.1), (1),

$$\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\mathfrak{A})\oplus\bigoplus_{T\subseteqq P'\subseteqq P}[A_n^{P'}]^{(r_{P'})}\cong A_n\oplus\bigoplus_{T\subseteqq P'\subseteqq P}[A_n^{P'}]^{(s_{P'})}.$$

Here every $A_n^{P'}/R_n$ is unramified because $T\subseteqq P'\subseteqq P$. Therefore, if p is odd, it follows from the Artin's theorem ([1], [7]) that every $A_n^{P'}$ is R_n-free. Hence we have $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\mathfrak{A})\cong A_n$ as R_n-modules. By (4.1), (2) this shows that $\Lambda_n\mathfrak{A}\cong\Lambda_n$. If $p=2$ and $|P|=2$, we have $P'=P$ or $P'=\{1\}$ and, if $p=2$ and $m(\Pi)=1$, we have $T=P$. In each of these cases we also have $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\mathfrak{A})\cong A_n$ as R_n-modules. Hence it follows from (4.1), (2) that $\Lambda_n\mathfrak{A}\cong\Lambda_n$. In the case where $p=2$ and $P_n\neq\{1\}$, the assertion has already been proved. Thus the proof of the theorem is completed.

If Π does not satisfy any of the conditions a)~d) in (4.2), it does not always hold that $\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$. In fact we have

EXAMPLE 4.3. Let $C=\langle\sigma\rangle$ be a cyclic group of order 15 and let $P=\langle\tau\rangle$ be a cyclic group of order 4. Define the homomorphism $\mu: P\to \mathrm{Aut}\, C$ by $\mu(\tau)(\sigma)=\sigma^2$ and let Π be the semidirect product of C and P defined by μ. Then we have $P_{15}=\{1\}$ and $m(\Pi)=2$. $R_{15}=A_{15}^P$ is the maximal order in $Q(\sqrt{-15})$. Further we have $T=\langle\sigma^2\rangle$. It is easily seen that A_{15}^T is the maximal order in $Q(\sqrt{-3}, \sqrt{5})$. Using the Artin's theorem we can show that A_{15}^T is not R_{15}-free. Then by (4.1) and (*) in the proof of (4.2) there exists a non-principal ideal $\mathfrak{b}$ of R_{15} such that $\Lambda_{15}\otimes_{Z\Pi} Z\Pi/T\cong A_{15}\oplus A_{15}\mathfrak{b}$ as Λ_{15}-modules. Since both $\Lambda_{15}\otimes_{Z\Pi} Z\Pi/T$ and A_{15} are quasi-permutation Π-modules, $A_{15}\mathfrak{b}$ is also a quasi-permutation Π-module. Now there exists a projective ideal $\mathfrak{A}$ of $Z\Pi$ such that $\mathfrak{A}\oplus A_{15}\cong Z\Pi\oplus A_{15}\mathfrak{b}$. Then $\mathfrak{A}$ is clearly a quasi-permutation Π-module. However we have $A_{15}\mathfrak{A}\not\cong A_{15}$. This implies that $C^q(Z\Pi)\subsetneqq\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

COROLLARY 4.4. *If Π is a finite group of squarefree order, then* $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

PROOF. This follows directly from (1.1), (1.6) and (4.2).

Next we consider another type of metacyclic groups.

PROPOSITION 4.5. *Let Π be a finite group which is a semidirect product of a cyclic normal subgroup C and a p-subgroup P of order p. Then* $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

PROOF. Let $n=|C|$ and put $C=\langle\sigma\rangle$ and $P=\langle\tau\rangle$. There exists an integer r such that $\tau^{-1}\sigma\tau=\sigma^r$. Then $(n, r)=1$ and $r^p\equiv 1 \bmod n$. Now we have $Q\Pi=\bigoplus_{m|n} Q\Pi/(\Phi_m(\sigma))$. For $m|n$ $Q\Pi/(\Phi_m(\sigma))$ is commutative if and only if $m|r-1$. If $m|r-1$ we denote the maximal order in $Q\Pi/(\Phi_m(\sigma))$ by A_m. Then A_m is clearly a quasi-permutation Π-module. On the other hand, if $m\nmid r-1$, $Q\Pi/(\Phi_m(\sigma))=M_p(Q(\zeta_m)^P)$. Put $\Lambda_m=Z\Pi/(\Phi_m(\sigma))$ and $A_m=Z[\zeta_m]$. Then Λ_m can be considered as an order in $M_p(Q(\zeta_m)^P)$. Regarding A_m as a Λ-module, we easily see that $A_m\cong\Lambda_m\otimes_{Z\Pi} Z\Pi/P$ as Λ_m-modules. Hence A_m is also a quasi-permutation Π-module. Further it is seen that $\mathrm{End}_{\Lambda_m}(A_m)=A_m^P$. Therefore, by [5], §3, (E′), we have $|\gamma_{A_m}|=1$. Let $N=\bigoplus_{m|n} A_m$. Then N is a faithful quasi-permutation Π-module such that $|\gamma_N|=1$. Therefore it follows from (2.2) that $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)$.

It should be noted that (4.5) is not a special case of (3.5).

THEOREM 4.6. *Let D_n be the dihedral group of order $2n$. Then* $\tilde{C}(ZD_n)=\tilde{C}^q(ZD_n)=C^q(ZD_n)$.

PROOF. By (4.5) we have $\tilde{C}(ZD_n)=\tilde{C}^q(ZD_n)$. If $2\nmid n$, we have $\tilde{C}(ZD_n)=C^q(ZD_n)$ by (4.2) and, if n is a power of 2, we also have $\tilde{C}(ZD_n)=C^q(ZD_n)$

by [5], (3.9). Hence we only need to prove that $C^q(ZD_n)\subseteqq\tilde{C}(ZD_n)$ under the assumption that $2|n$ and n is not a power of 2. Let $\{\sigma, \tau\}$ be the generators of D_n satisfying the relations $\sigma^n=\tau^2=1$ and $\tau^{-1}\sigma\tau=\sigma^{-1}$. The group D_n contains subgroups $\langle\sigma^i\tau\rangle$, $0\leqq i\leqq n-1$, of order 2 and every $\langle\sigma^i\tau\rangle$ is conjugate to $\langle\tau\rangle$ or $\langle\sigma\tau\rangle$. Now we have $QD_n=\bigoplus_{m|n} QD_n/(\Phi_m(\sigma))$. When $m=1$ or 2, $QD_n/(\Phi_m(\sigma))$ is commutative and we denote by Ω_m the maximal order in $QD_n/(\Phi_m(\sigma))$. On the other hand, when $m>2$, $QD_n/(\Phi_m(\sigma))=\Delta(1, Q(\zeta_m), \langle\tau\rangle)=M_2(Q(\zeta_m+\zeta_m^{-1}))$. Put $\Lambda_m=ZD_n/(\Phi_m(\sigma))=\Delta(1, Z[\zeta_m], \langle\tau\rangle)$. Then Λ_m is a $Z[\zeta_m+\zeta_m^{-1}]$-order in $QD_n/(\Phi_m(\sigma))$. Let Ω_m be a maximal order in $QD_n/(\Phi_m(\sigma))$ containing Λ_m. We put $\Omega=\bigoplus_{m|n}\Omega_m$. Then Ω is a maximal order in QD_n containing ZD_n. Let $\mathfrak{A}$ be a projective ideal of ZD_n such that $\mathfrak{A}\oplus S_1\cong ZD_n\oplus S_n$ for some permutation D_n-modules S_1 and S_2. To prove that $C^q(ZD_n)\subseteqq\tilde{C}(ZD_n)$ it suffices to show that $\Omega_m\mathfrak{A}\cong\Omega_m$ for each $m|n$. Using the induction on n we only need to show that $\Omega_n\mathfrak{A}\cong\Omega_n$.

Suppose that $\frac{n}{2}$ is odd. Then Λ_n is a hereditary order in $QD_n/(\Phi_n(\sigma))$. Regarding $A_n=Z[\zeta_n]$ as a Λ_n-module, we have $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\otimes_{ZD_n} ZD_n/\langle\tau\rangle)=A_n^{<\tau>}$ and $\mathrm{Hom}_{\Lambda_n}(A_n, \Lambda_n\otimes_{ZD_n} ZD_n/\langle\sigma\tau\rangle)=A_n^{<\sigma\tau>}$ by (4.1). However we have $A_n^{<\sigma>}=A_n^{<\sigma\tau>}=Z[\zeta_n+\zeta_n^{-1}]$. Therefore, as in the proof of (4.2), we get $\Lambda_n\mathfrak{A}\cong\Lambda_n$, hence $\Omega_n\mathfrak{A}\cong\Omega_n$.

Next suppose that $4|n$. Now put $v=\zeta_n-\zeta_n^{-1}$ and $w=1+\zeta_n$. Then $\sigma(v)=-v$ and $\sigma(w)=\zeta_n^{-1}w$. Since n is not a power of 2, both v and w are units of A_n. We can define Λ_n-homomorphisms $f: \Lambda_n(1+\tau)\to\Lambda_n(1-\tau)$ and $g: \Lambda_n(1+\sigma\tau)\to\Lambda_n(1+\tau)$ by $f(1+\tau)=v(1-\tau)$ and $g(1+\sigma\tau)=w(1+\tau)$, respectively. Then it is easily seen that both f and g are isomorphisms. Accordingly we have $\Lambda_n(1-\tau)\cong\Lambda_n(1+\tau)\cong\Lambda_n(1+\sigma\tau)$ as Λ_n-modules. Let $\Pi'\neq\{1\}$ be a subgroup of D_n. If $\Pi'\cap\langle\sigma\rangle\neq\{1\}$, $\Lambda_n\otimes_{ZD_n} ZD_n/\Pi'$ is a torsion module. On the other hand, if $\Pi'\cap\langle\sigma\rangle=\{1\}$, Π' is conjugate to $\langle\tau\rangle$ or $\langle\sigma\tau\rangle$ and so $\Lambda_n\otimes_{ZD_n} ZD_n/\Pi'\cong\Lambda_n(1+\tau)\cong\Lambda_n(1+\sigma\tau)$. Tensoring $\mathfrak{A}\oplus S_1\cong ZD_n\oplus S_2$ with Λ_n over ZD_n and eliminating the torsion parts from both sides, we get

$$\Lambda_n\mathfrak{A}\oplus\Lambda_n^{(l_1)}\oplus[\Lambda_n(1+\tau)]^{(l_2)}\cong\Lambda_n\oplus\Lambda_n^{(k_1)}\oplus[\Lambda_n(1+\tau)]^{(k_2)}.$$

Hence we have

$$\Omega_n\mathfrak{A}\oplus\Omega_n^{(l_1)}\oplus[\Omega_n(1+\tau)]^{(l_2)}\cong\Omega_n\oplus\Omega_n^{(k_1)}\oplus[\Omega_n(1+\tau)]^{(k_2)}.$$

There exists an exact sequence:

$$0\longrightarrow\Lambda_n(1-\tau)\longrightarrow\Lambda_n\longrightarrow\Lambda_n(1+\tau)\longrightarrow 0.$$

From this we get $\Omega_n\cong\Omega_n(1-\tau)\oplus\Omega_n(1+\tau)$. Since $\Lambda_n(1-\tau)\cong\Lambda_n(1+\tau)$, this shows that $\Omega_n\cong[\Omega_n(1+\tau)]^{(2)}$. Thus we get $\Omega_n\mathfrak{A}\cong\Omega_n$.

§ 5. The projective special linear group, the symmetric group, the alternating group, etc.

In this section we will apply the induction theorems to some types of finite groups.

LEMMA 5.1. *Let Π be a finite group and let P be an elementary abelian 2-group. If $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$, then $\tilde{C}(Z(\Pi\times P))=\tilde{C}^q(Z(\Pi\times P))=C^q(Z(\Pi\times P))$.*

PROOF. As this is easy, we omit it.

THEOREM 5.2. *Let Π be one of the following groups:*

(1) *the projective special linear group $PSL(2, p^f)$ where p is a prime and $f\geqq 0$;*

(2) *the Janko simple group J_1;*

(3) *the symmetric group S_n, $n\leqq 7$.*

Then $\tilde{C}(Z\Pi)=\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$.

PROOF. By the induction theorems (1.1) and (1.6) it suffices to show that $\tilde{C}(Z\Pi')=\tilde{C}^q(Z\Pi')=C^q(Z\Pi')$ for every (maximal) hyperelementary subgroup Π' of Π.

(1) Let $\Pi=PSL(2, p^f)$. Then all the subgroups of Π are completely determined by the Dickson's theorem (e. g. [8], (8.27)). It can easily be shown that any hyperelementary subgroup Π' of Π has one of the following forms:

a) an abelian group;

b) a dihedral group;

c) a semidirect product of a cyclic normal subgroup of order p and a cyclic q-subgroup where q is a prime such that $q\mid p-1$.

Therefore the result follows from (3.2), (4.6) and (4.2).

(2) Let $\Pi=J_1$ be the Janko simple group ([**10**]). The order of Π is $2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$. A 2-Sylow subgroup of Π is elementary abelian and all Sylow subgroups of Π of odd order are cyclic. All the maximal subgroups of Π are given in [**10**]. We easily see that each maximal hyperelementary subgroup Π' of Π has one of the following forms:

a) an abelian group;

b) a semidirect product of a cyclic normal subgroup of order m and a cyclic p-group of order p such that $p\nmid m$;

c) a direct product of a cyclic group of order 2 and a dihedral group;

d) a maximal hyperelementary subgroup of $PSL(2, 11)$;

e) a semidirect product of a cyclic normal subgroup C of order 15 and an elementary abelian 2-subgroup P of order 4 such that P acts faithfully on C.

In the cases a), b) and c) the assertion follows from (3.2), (4.2), (4.6) and (5.1), and in the case d) the assertion has been proved in (1). Suppose that Π' has the form e). Then, for every subgroup P' of P, $Z[\zeta_{15}]^{P'}$ is $Z[\zeta_{15}]^{P}$-free. Therefore we can prove the assertion in the same way as in the proof of (4.1).

(3) All the maximal hyperelementary subgroups of S_n, $n\leqq 7$, can easily be determined. If $n\leqq 6$, the assertion follows directly from (3.2), (4.2), (4.6) and (5.1). A maximal hyperelementary subgroup Π' of S_7 for which the assertion does not follow directly from the preceding results is conjugate to $\langle(1\ 2\ 3),(2\ 3)\rangle\times\langle(4\ 5\ 6\ 7),(4\ 6)\rangle$ $(\cong D_3\times D_4)$. However, in this case, it is clear that $C^q(Z\Pi')\subsetneqq C(Z\Pi')=\tilde{C}(Z\Pi')$. Further, using [5], § 3, (E′), we can construct a faithful quasi-permutation Π'-module N such that $|\gamma_N|=1$. Therefore we get $\tilde{C}^q(Z\Pi')=\tilde{C}(Z\Pi')$ by (2.2).

REMARK 5.3. It can be shown that $B(Q\Pi)=G(Q\Pi)$ for Π as in (5.2), (1) and (2), and it is well known that $B(QS_n)=G(QS_n)$ for any n. We will show in our forthcoming paper that $C(ZS_n)=\tilde{C}(ZS_n)=\tilde{C}^q(ZS_n)=C^q(ZS_n)$ for any n.

LEMMA 5.4. *Let Π be a finite group and let Π' be a subgroup of Π such that $N_\Pi(\Pi')=\Pi'$. Suppose that $C^q(Z\Pi')\not\subseteq\tilde{C}(Z\Pi')$ and that $C^q(Z\Pi'')\subseteqq\tilde{C}(Z\Pi'')$ for every proper subgroup Π'' of Π'. Then $C^q(Z\Pi)\not\subseteq\tilde{C}(Z\Pi)$.*

PROOF. Let $\mathfrak{A}'$ be a quasi-permutation projective ideal of $Z\Pi'$ such that $[\mathfrak{A}']-[Z\Pi']\notin\tilde{C}(Z\Pi')$. Then $Z\Pi\otimes_{Z\Pi'}\mathfrak{A}'$ is a quasi-permutation projective ideal of $Z\Pi$. Suppose that $[Z\Pi\otimes_{Z\Pi'}\mathfrak{A}']-[Z\Pi]\in\tilde{C}(Z\Pi)$. According to [5], (2.4), there exists a Π-module M such that $(Z\Pi\otimes_{Z\Pi'}\mathfrak{A}')\oplus M\cong Z\Pi\oplus M$. Regarding both sides as Π'-modules, the Mackey's subgroup theorem shows that

$$\bigoplus_{\Pi'\sigma\Pi'}[Z\Pi'\otimes_{Z(\Pi'\cap\sigma\Pi'\sigma^{-1})}(\sigma Z\Pi'\otimes_{Z\Pi'}\mathfrak{A}')]\oplus M\cong\bigoplus_{\Pi'\sigma\Pi'}[Z\Pi'\otimes_{Z(\Pi'\cap\sigma\Pi'\sigma^{-1})}(\sigma Z\Pi'\otimes_{Z\Pi'}Z\Pi')]\oplus M,$$

where the sum is taken over all (Π',Π')-double cosets of Π. Since $N_\Pi(\Pi')=\Pi'$, $\Pi'\cap\sigma\Pi'\sigma^{-1}\subsetneqq\Pi'$ for any $\Pi'\sigma\Pi'\neq\Pi'$. However each $\sigma Z\Pi'\otimes_{Z\Pi'}\mathfrak{A}'$ is a quasi-permutation projective $\Pi'\cap\sigma\Pi'\sigma^{-1}$-module. Hence by hypothesis $[\sigma Z\Pi'\otimes_{Z\Pi'}\mathfrak{A}']-[\sigma Z\Pi'\otimes_{Z\Pi'}Z\Pi']\in\tilde{C}(Z(\Pi'\cap\sigma\Pi'\sigma^{-1}))$ for any $\Pi'\sigma\Pi'\neq\Pi'$, so that $[Z\Pi'\otimes_{Z(\Pi'\cap\sigma\Pi'\sigma^{-1})}(\sigma Z\Pi'\otimes_{Z\Pi'}\mathfrak{A}')]-[Z\Pi'\otimes_{Z(\Pi'\cap\sigma\Pi'\sigma^{-1})}(\sigma Z\Pi'\otimes_{Z\Pi'}Z\Pi')]\in\tilde{C}(Z\Pi')$ for any $\Pi'\sigma\Pi'\neq\Pi'$. Therefore we can find a Π'-module M' such that $\mathfrak{A}'\oplus M'\cong Z\Pi'\oplus M'$. This implies that $[\mathfrak{A}']-[Z\Pi']\in\tilde{C}(Z\Pi')$ which is a contradiction. Thus we have $[Z\Pi\otimes\mathfrak{A}']-[Z\Pi]\notin\tilde{C}(Z\Pi)$.

PROPOSITION 5.5. *Let A_n be the alternating group of degree n. For $n\leqq 6$ $\tilde{C}(ZA_n)=\tilde{C}^q(ZA_n)=C^q(ZA_n)$. But, for $n=8$, 9, $C^q(ZA_n)\not\subseteq\tilde{C}(ZA_n)$.*

PROOF. It is well known that $A_6\cong PSL(2,9)$. Hence the first part of the proposition follows directly from (the proof of) (5.2), (1). Suppose that $\Pi=A_8$ or A_9 and put $\Pi'=\langle(1\ 2\ 3\ 4\ 5)(6\ 7\ 8),(2\ 3\ 5\ 4)(6\ 7)\rangle$. Then Π' is a subgroup of Π with $N_\Pi(\Pi')=\Pi'$ which is isomorphic to the group as in (4.3). Hence we have $C^q(Z\Pi') \subsetneqq \tilde{C}(Z\Pi')$. Furthermore by (4.2) $C^q(Z\Pi'')=\tilde{C}(Z\Pi'')$ for any proper subgroup Π'' of Π'. Thus (5.4) concludes that $C^q(Z\Pi) \subsetneqq \tilde{C}(Z\Pi)$.

References

[1] E. Artin, Questions de la base minimale dans la théorie des nombres algebriques, Centre Nat. Rech. Sci. (Colloq. Int., Paris), 24 (1949), 19-20.
[2] H. Bass, Algebraic K-theory, Benjamin, New York, 1968.
[3] S.B. Conlon, A basis for monomial algebras, J. Algebra, 20 (1972), 396-414.
[4] S. Endo and T. Miyata, Invariants of finite abelian groups, J. Math. Soc. Japan, 25 (1973), 7-26.
[5] S. Endo and T. Miyata, Quasi-permutation modules over finite groups, J. Math. Soc. Japan, 25 (1973), 397-421.
[6] W. Feit, Characters of finite groups, Benjamin, New York, 1967.
[7] A. Fröhlich, Discriminants of algebraic number fields, Math. Z., 74 (1960), 18-28.
[8] B. Huppert, Endliche Gruppen, I, Springer, Berlin, 1967.
[9] H. Jacobinski, Genera and decompositions of lattices over orders, Acta Math., 121 (1968), 1-29.
[10] Z. Janko, A new finite simple group with abelian Sylow 2-subgroups and its characterization, J. Algebra, 3 (1966), 147-186.
[11] T.Y. Lam, Induction theorems for Grothendieck groups and Whitehead groups of finite groups, Ann. École Norm. Sup., (4) 1 (1968), 91-148.
[12] I. Reiner and S. Ullom, Class groups of integral group rings, Trans. Amer. Math. Soc., 170 (1972), 1-30.
[13] J. Ritter, Ein Induktionssatz für rationale Charaktere von nilpotenten Gruppen, J. Reine Angew. Math., 254 (1972), 133-151.
[14] M. Rosen, Representations of twisted group rings, Thesis, Princeton Univ., 1963.
[15] J.-P. Serre, Représentations linéaires des groupes finis, Deuxième édition, Hermann, Paris, 1971.
[16] R.G. Swan, Induced representations and projective modules, Ann. of Math., 71 (1960), 552-578.
[17] S. Takahashi, On rational characters of a finite group, Notices Amer. Math. Soc., 12 (1965), 368.
[18] S. Williamson, Crossed products and hereditary orders, Nagoya Math. J., 23 (1963), 103-120.

Shizuo Endo
Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya-ku
Tokyo, Japan

Takehiko Miyata
Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka, Japan