J. Math. Soc. Japan
Vol. 27, No. 1, 1975

On the characterization of complex projective space by differential equations

By David E. BLAIR

(Received April 18, 1973)
(Revised Feb. 4, 1974)

§ 1. Introduction.

The existence of a non-trivial solution of certain differential equations on a Riemannian manifold often determines some geometric and topological properties of the manifold. For example in [5] M. Obata announced the following results.

THEOREM A (see also [4]). *Let M^n be a complete connected Riemannian manifold of dimension $n \geqq 2$. Then M^n admits a non-trivial solution f of*

$$\nabla\nabla f + kfg = 0, \qquad k = \text{const.} > 0$$

if and only if M^n is globally isometric to a Euclidean sphere S^n of radius $1/\sqrt{k}$.

THEOREM B. *Let M^n be a complete connected, simply connected Riemannian manifold. Then M^n admits a non-trivial solution f of*

$$(\nabla\nabla\omega)(Z, X, Y) + k(2\omega(Z)g(X, Y) + \omega(X)g(Y, Z) + \omega(Y)g(X, Z)) = 0$$

where $\omega = df$ if and only if M^n is isometric to a Euclidean sphere of radius $1/\sqrt{k}$.

THEOREM C. *Let M^{2n} be a complete connected, simply connected Kähler manifold. Then M^{2n} admits a non-trivial solution f of*

$$4(\nabla\nabla\theta)(Z, X, Y) + c(2\theta(Z)G(X, Y) + \theta(X)G(Y, Z) + \theta(Y)G(X, Z)$$
$$-\theta(JX)\Omega(Y, Z) - \theta(JY)\Omega(X, Z)) = 0, \qquad c > 0$$

where $\theta = df$ if and only if M^{2n} is isometric to complex projective space PC^n with the Fubini-Study metric of constant holomorphic sectional curvature c.

In [5] Obata gives a proof of Theorem A and an indication of the proofs of Theorems B and C. Our purpose here is to show the relation between Theorems B and C by deducing Theorem C from Theorem B in the case of Hodge manifolds.

In Theorem B, grad f is an infinitesimal projective transformation and we show that on an odd-dimensional sphere S^{2n+1} we can find such a vector field orthogonal to the distinguished direction of the contact structure on the sphere and invariant with respect to that direction.

Thus the idea of our proof will be to project the equation of Theorem B via the Hopf fibration $\pi: S^{2n+1} \to PC^n$ giving the desired equation on PC^n. Conversely the principal circle bundles over a simply connected manifold M form a group isomorphic to $H^2(M, Z)$, (see e. g. Kobayashi [2]). Now selecting the bundle corresponding to the fundamental 2-form Ω of the Hodge manifold M^{2n} (Hatakeyama [1]) we lift the equation of Theorem C using some properties of grad f on M^{2n} (an infinitesimal H-projective transformation, see e. g. [8]) to show that the bundle space is S^{2n+1}.

We remark that grad f in Theorem A is an infinitesimal conformal transformation on the sphere. However as there is no known Kähler analogue of such a vector field, i. e. no "H-conformal" transformation, one would not expect an equation similar to that of Theorem A for PC^n.

§ 2. Preliminaries.

First let J denote the usual almost complex structure on C^{n+1} ($J^2=-I$) and let $\iota: S^{2n+1} \to R^{2n+2} \approx C^{n+1}$ be the standard imbedding of the sphere of radius 2, S^{2n+1}, into R^{2n+2}. Then S^{2n+1} inherits an almost contact structure (φ, ξ, η) defined by

$$J\iota_* X = \iota_* \varphi X + \eta(X) N, \qquad JN = -\iota_* \xi$$

where N is the unit outer normal, that is we have

$$\varphi^2 = -I + \eta \otimes \xi$$

$$\eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0.$$

With respect to the metric g induced from the usual inner product on R^{2n+2} we have

$$g(X, \xi) = \eta(X)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

i. e. S^{2n+1} has an almost contact metric structure.

Letting D denote the Riemannian connexion on $R^{2n+2} \approx C^{n+1}$ and $\bar{\nabla}$ the Riemannian connexion of g, the Gauss-Weingarten equations are

$$D_{\iota_* X} \iota_* Y = \iota_* \bar{\nabla}_X Y - \frac{1}{2} g(X, Y) N,$$

$$D_{\iota_* X} N = \frac{1}{2} \iota_* X$$

since the second fundamental form of $\iota: S^{2n+1} \to R^{2n+2}$ with respect to the outer normal N is $-\frac{1}{2} g$. Then since J is parallel with respect to D we have

$$0=(D_{\iota_* X}J)\iota_* Y=\iota_*(\bar{\nabla}_X\varphi)Y-\frac{1}{2}g(X,\varphi Y)N+(\bar{\nabla}_X\eta)(Y)N$$
$$+\frac{1}{2}\eta(Y)\iota_* X-\frac{1}{2}g(X,Y)\iota_*\xi$$

from which we obtain

$$\begin{cases}(\bar{\nabla}_X\varphi)Y=\dfrac{1}{2}(g(X,Y)\xi-\eta(Y)X)\\ \bar{\nabla}_X\xi=-\dfrac{1}{2}\varphi X\end{cases}\tag{2.1}$$

that is, S^{2n+1} carries a *Sasakian* or *normal contact metric* structure. On a Sasakian manifold of dimension $2n+1$ we also have that $\eta\wedge(d\eta)^n\neq 0$, i.e. η is a contact structure and that $\mathcal{L}_\xi g=0$ where $\mathcal{L}$ denotes Lie differentiation. Defining the fundamental 2-form Φ of a Sasakian structure by $\Phi(X, Y)=g(X,\varphi Y)$ we have

$$\begin{aligned}d\eta(X,Y)&=(\bar{\nabla}_X\eta)(Y)-(\bar{\nabla}_Y\eta)(X)\\&=2(\bar{\nabla}_X\eta)(Y)\\&=2g(\bar{\nabla}_X\xi,Y)\\&=g(-\varphi X,Y)\\&=\Phi(X,Y),\end{aligned}$$

using the fact that η is Killing.

We now consider the well-known Hopf fibration $\pi: S^{2n+1}\to PC^n$. As $JN=-\iota_*\xi$ above we see that the vector field ξ is vertical. Thus since ξ is Killing, π is a Riemannian submersion. Moreover

$$\begin{aligned}(\mathcal{L}_\xi\varphi)X=[\xi,\varphi X]-\varphi[\xi,X]&=\bar{\nabla}_\xi\varphi X-\bar{\nabla}_{\varphi X}\xi-\varphi\bar{\nabla}_\xi X+\varphi\bar{\nabla}_X\xi\\&=(\bar{\nabla}_\xi\varphi)X=0.\end{aligned}$$

Thus both φ and g are projectable and we shall show that the usual Kähler structure on PC^n is obtained by this projection.

We define J and G on PC^n by

$$JX=\pi_*\varphi\tilde{\pi}X,\qquad G(X,Y)\circ\pi=g(\tilde{\pi}X,\tilde{\pi}Y)$$

where $\tilde{\pi}$ denotes the horizontal lift with respect to the Riemannian connexion $\bar{\nabla}$ of g. For future reference we give all the differential equations of the submersion (see e.g. [**6**]). Let ∇ denote the Riemannian connexion of G; then we have

$$\left\{\begin{aligned} &\bar{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y=\tilde{\pi}\nabla_X Y-\frac{1}{2}\Phi(\tilde{\pi}X,\tilde{\pi}Y)\xi \\ &\bar{\nabla}_{\tilde{\pi}X}\xi=-\frac{1}{2}\varphi\tilde{\pi}X \\ &\bar{\nabla}_{\xi}\tilde{\pi}X=-\frac{1}{2}\varphi\tilde{\pi}X \\ &\bar{\nabla}_{\xi}\xi=0\,. \end{aligned}\right. \tag{2.2}$$

All but the third equation are immediate and the third one follows from the second and the following lemma.

LEMMA 2.1. $[\xi, \tilde{\pi}X]=0$.

PROOF. First of all $\pi_*[\xi, \tilde{\pi}X]=[\pi_*\xi, X]=0$ and hence $[\xi, \tilde{\pi}X]$ is vertical. Now

$$g([\xi,\tilde{\pi}X],\xi)=g(\bar{\nabla}_{\xi}\tilde{\pi}X,\xi)-g(\bar{\nabla}_{\tilde{\pi}X}\xi,\xi)=-g(\tilde{\pi}X,\bar{\nabla}_{\xi}\xi)=0\,.$$

It is easy to check that the J and G defined above on PC^n satisfy

$$J^2=-I\,,\qquad G(JX,JY)=G(X,Y)\,,\qquad \nabla_X J=0\,.$$

Thus the induced structure on PC^n is Kählerian. We define its fundamental 2-form Ω by $\Omega(X,Y)=G(X,JY)$. Note that $\Omega(X,Y)\circ\pi=g(\tilde{\pi}X,\varphi\tilde{\pi}Y)=\Phi(\tilde{\pi}X,\tilde{\pi}Y)$, that is $\Phi=\pi^*\Omega$.

Finally we show that the Kähler structure just defined on PC^n has constant holomorphic curvature equal to 1, in fact we give the curvature tensor completely.

$$\begin{aligned} G(R_{XY}Z,W)\circ\pi &= g(\tilde{\pi}\nabla_X\nabla_Y Z-\tilde{\pi}\nabla_Y\nabla_X Z-\tilde{\pi}\nabla_{[X,Y]}Z,\tilde{\pi}W) \\ &= g(\bar{\nabla}_{\tilde{\pi}X}\tilde{\pi}\nabla_Y Z-\bar{\nabla}_{\tilde{\pi}Y}\tilde{\pi}\nabla_X Z-\bar{\nabla}_{\tilde{\pi}[X,Y]}\tilde{\pi}Z,\tilde{\pi}W) \\ &= g(\bar{\nabla}_{\tilde{\pi}X}\bar{\nabla}_{\tilde{\pi}Y}\tilde{\pi}Z-\frac{1}{4}\Phi(\tilde{\pi}Y,\tilde{\pi}Z)\varphi\tilde{\pi}X-\bar{\nabla}_{\tilde{\pi}X}\bar{\nabla}_{\tilde{\pi}Y}\tilde{\pi}Z \\ &\qquad\qquad +\frac{1}{2}\Phi(\tilde{\pi}X,\tilde{\pi}Z)\varphi\tilde{\pi}Y-\bar{\nabla}_{[\tilde{\pi}X,\tilde{\pi}Y]}\tilde{\pi}Z \\ &\qquad\qquad +\eta([\tilde{\pi}X,\tilde{\pi}Y])\bar{\nabla}_{\xi}\tilde{\pi}Z,\tilde{\pi}W) \\ &= \frac{1}{4}(g(\tilde{\pi}X,\tilde{\pi}W)g(\tilde{\pi}Y,\tilde{\pi}Z)-g(\tilde{\pi}Y,\tilde{\pi}W)g(\tilde{\pi}X,\tilde{\pi}Z) \\ &\quad +\Phi(\tilde{\pi}X,\tilde{\pi}W)\Phi(\tilde{\pi}Y,\tilde{\pi}Z)-\Phi(\tilde{\pi}Y,\tilde{\pi}W)\Phi(\tilde{\pi}X,\tilde{\pi}Z) \\ &\quad -2\Phi(\tilde{\pi}X,\tilde{\pi}Y)\Phi(\tilde{\pi}Z,\tilde{\pi}W)) \\ &= \frac{1}{4}(G(X,W)G(Y,Z)-G(Y,W)G(X,Z) \\ &\quad +\Omega(X,W)\Omega(Y,Z)-\Omega(Y,W)\Omega(X,Z) \\ &\quad -2\Omega(X,Y)\Omega(Z,W))\circ\pi \end{aligned}$$

where we have used $\eta(\tilde{\pi}X)=0$ and

$$\eta([\tilde{\pi}X, \tilde{\pi}Y])=-d\eta(\tilde{\pi}X, \tilde{\pi}Y)=-\Phi(\tilde{\pi}X, \tilde{\pi}Y).$$

We close this section with some matters of notation. For a 1-form ω and Riemannian connexion ∇ we write

$$(\nabla\omega)(X, Y)=(\nabla_X\omega)(Y)$$

$$(\nabla\nabla\omega)(X, Y, Z)=(\nabla_X\nabla_Y\omega)(Z)-(\nabla\omega)(\nabla_X Y, Z).$$

For a vector field U we write

$$(\nabla U)X=\nabla_X U$$

$$(\nabla\nabla U)(X, Y)=\nabla_X\nabla_Y U-(\nabla U)\nabla_X Y.$$

§ 3. Infinitesimal projective transformations.

In this section we give some properties of infinitesimal projective transformations and of the differential equation of Theorem B. Recall that a vector field U is an infinitesimal projective transformation on a Riemannian manifold M^n with Riemannian connexion ∇ if it satisfies

$$(\nabla\nabla U)(X, Y)+R_{UX}Y=(Xf)Y+(Yf)X$$

where $f=\frac{1}{n+1}\operatorname{tr}(\nabla U)$. It is known [7] that if M^n is an Einstein space then this f satisfies the differential equation of Theorem B.

We now show that if a function f satisfies the differential equation of Theorem B, then $\operatorname{grad} f$ is an infinitesimal projective transformation. First of all from the equation we have

$$\begin{aligned}(\nabla\nabla\omega)(X, Y, Z)-(\nabla\nabla\omega)(Y, X, Z)&=(R_{XY}\omega)(Z)\\ &=k(\omega(Y)g(X, Z)-\omega(X)g(Y, Z))\end{aligned}$$

from which

$$(\nabla\nabla\omega)(Z, X, Y)+(R_{XY}\omega)(Z)=-2k(\omega(Z)g(X, Y)+\omega(X)g(Z, Y)).$$

But if U is the contravariant form of $\omega=df$,

$$(\nabla\nabla\omega)(Z, X, Y)=g((\nabla\nabla U)(Z, X), Y)$$

and

$$(R_{XY}\omega)(Z)=g(R_{XY}U, Z)=g(R_{UZ}X, Y)$$

giving

$$(\nabla\nabla U)(Z, X)+R_{UZ}X=-2k((Zf)X+(Xf)Z).$$

We now give some lemmas that will be needed later.

LEMMA 3.1. *On S^{2n+1} there exists a non-trivial solution $\omega=df$ of the equation of Theorem B such that* $\operatorname{grad} f$ *is orthogonal to the vector field ξ of the* *usual Sasakian structure on* S^{2n+1}.

PROOF. First suppose $\iota: S^{2n+1}\to R^{2n+2}$ is given by $\sum_{\alpha=1}^{2n+2}(X^\alpha)^2=4$; then a function f satisfying the equation of Theorem B is the restriction of

$$\tilde{f}=\sum_{\alpha,\beta} a_{\alpha\beta}X^\alpha X^\beta\,, \qquad a_{\alpha\beta}=a_{\beta\alpha}\in R\,.$$

Now $N=\frac{1}{2}(X^1,\cdots,X^{2n+2})$ is the outer normal to S^{2n+1} in R^{2n+2}. Thus grad f is orthogonal to ξ if and only if grad $\tilde{f}$ is orthogonal to JN in R^{2n+2}, i.e.

$$\sum_\alpha (\sum_\beta a_{\alpha\beta}X^\beta)(\sum_\gamma J^\alpha_\gamma X^\gamma)=0$$

which means we must have

$$\sum_\alpha (J^\alpha_\gamma a_{\alpha\beta}+J^\alpha_\beta a_{\alpha\gamma})=0$$

but such $a_{\alpha\beta}$'s can easily be chosen.

LEMMA 3.2. *The vector fields* $v=\operatorname{grad} f$ *and* ξ *in Lemma* 3.1 *statisfy* $\mathcal{L}_\xi v=0$.

PROOF. By Lemma 3.1, $\xi f=g(v,\xi)=0$; moreover ξ is Killing so that

$$g(\mathcal{L}_\xi v, X)=\xi(Xf)-[\xi, X]f=X(\xi f)=0\,.$$

§4. Proof of Theorem C.

We first note that it suffices to prove the theorem for $c=1$. For if $c\neq 1$, the homothetic change of metric $\bar{G}=cG$, transforms the differential equation into the corresponding one for $c=1$ and transforms the curvature tensor of the Fubini-Study metric of constant holomorphic sectional curvature c into the corresponding curvature tensor for $c=1$. We now give the proof of Theorem C in two parts.

I. Sufficiency. Let S^{2n+1} be the sphere of radius 2 in R^{2n+2} with its induced normal contact metric structure (φ,ξ,η,g) and let $\omega=d\tilde{f}$ be a non-trivial solution of the equation of Theorem B on S^{2n+1} which by virtue of Lemma 3.1 we assume is orthogonal to ξ. We now consider the Hopf fibration $\pi: S^{2n+1}\to PC^n$ as discussed in section 2. By Lemma 3.2 and the fact that ξ is Killing we have $\mathcal{L}_\xi\omega=0$ and hence that ω is projectable. Thus we can define θ on PC^n by

$$\theta(X)\circ\pi=\omega(\tilde{\pi}X)$$

and since $\xi\tilde{f}=0$, we define f on PC^n by $f\circ\pi=\tilde{f}$ from which we have $\theta=df$. Now using the differential equations of the submersion we have

$$\begin{aligned}(\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y)&=\tilde{\pi}X\omega(\tilde{\pi}Y)-\omega(\bar{\nabla}_{\tilde{\pi}X}\tilde{\pi}Y)=(X\theta(Y))\circ\pi-\theta(\nabla_XY)\circ\pi\\&=(\nabla_X\theta)(Y)\circ\pi\end{aligned}$$

and

$$(\bar{\nabla}_{\tilde{\pi}X}\omega)(\xi)=-\omega(\bar{\nabla}_{\tilde{\pi}X}\xi)=\frac{1}{2}\omega(\varphi\tilde{\pi}X)=\frac{1}{2}\omega(\tilde{\pi}JX)=\frac{1}{2}\theta(JX)\circ\pi\,.$$

By Lemma 2.1 we also have

$$(\bar{\nabla}_{\xi}\omega)(\tilde{\pi}X)=\xi\omega(\tilde{\pi}X)-\omega(\bar{\nabla}_{\xi}\tilde{\pi}X)=-\omega(\bar{\nabla}_{\tilde{\pi}X}\xi)=\frac{1}{2}\theta(JX)\circ\pi\,.$$

We now differentiate $(\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y)=(\nabla_X\theta)(Y)\circ\pi$ with respect to $\tilde{\pi}Z$.

$$\begin{aligned}&(\bar{\nabla}_{\tilde{\pi}Z}\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y)+(\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}\nabla_Z Y-\frac{1}{2}\Phi(\tilde{\pi}Z,\tilde{\pi}Y)\xi)\\ &\qquad=(\nabla_Z\nabla_X\theta)(Y)\circ\pi+(\nabla_X\theta)(\nabla_Z Y)\circ\pi\,.\end{aligned}$$

Therefore

$$\begin{aligned}&(\bar{\nabla}_{\tilde{\pi}Z}\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y)-(\bar{\nabla}_{\bar{\nabla}_{\tilde{\pi}Z}\tilde{\pi}X}\omega)(\tilde{\pi}Y)-\frac{1}{4}\Phi(\tilde{\pi}Z,\tilde{\pi}Y)\omega(\varphi\tilde{\pi}X)\\ &\qquad=(\nabla_Z\nabla_X\theta)(Y)\circ\pi-(\nabla_{\nabla_Z X}\theta)(Y)\circ\pi\\ &\qquad\quad+\frac{1}{2}\Phi(\tilde{\pi}Z,\tilde{\pi}X)(\bar{\nabla}_{\xi}\omega)(\tilde{\pi}Y)\end{aligned}$$

from which

$$\begin{aligned}(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z,\tilde{\pi}X,\tilde{\pi}Y)&=(\nabla\nabla\theta)(Z,X,Y)\circ\pi+\frac{1}{4}\Omega(Z,Y)\theta(JX)\circ\pi\\ &\quad+\frac{1}{4}\Omega(Z,X)\theta(JY)\circ\pi\,.\end{aligned}$$

On the other hand

$$\begin{aligned}(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z,\tilde{\pi}X,\tilde{\pi}Y)&=-\frac{1}{4}(2\omega(\tilde{\pi}Z)g(\tilde{\pi}X,\tilde{\pi}Y)+\omega(\tilde{\pi}X)g(\tilde{\pi}Y,\tilde{\pi}Z)\\ &\quad+\omega(\tilde{\pi}Y)g(\tilde{\pi}X,\tilde{\pi}Z)=-\frac{1}{4}(2\theta(Z)G(X,Y)\\ &\quad+\theta(X)G(Y,Z)+\theta(Y)G(X,Z))\circ\pi\,.\end{aligned}$$

Thus $\theta=df$ satisfies

$$\begin{aligned}&4(\nabla\nabla\theta)(Z,X,Y)+2\theta(Z)G(X,Y)+\theta(X)G(Y,Z)+\theta(Y)G(X,Z)\\ &\qquad-\theta(JX)\Omega(Y,Z)-\theta(JY)\Omega(X,Z)=0\end{aligned}$$

as desired.

II. Necessity. The necessity part of the proof is, of course, more difficult and we begin some lemmas.

LEMMA 4.1. *The vector field $U=\operatorname{grad} f$ is an infinitesimal H-projective transformation, i.e. we have*

$$\begin{aligned}(\nabla\nabla\theta)(Z,X,Y)+G(R_{UZ}X,Y)&=-\frac{c}{2}(\theta(Z)G(X,Y)+\theta(X)G(Y,Z)\\ &\quad-\theta(JX)\Omega(Y,Z)-\theta(JZ)\Omega(Y,X))\,.\end{aligned}$$

The proof of this lemma is similar to the corresponding computation in the projective case given in section 3.

LEMMA 4.2. *The vector field* $U=\operatorname{grad} f$ *is analytic* (*i.e.* $\mathcal{L}_U J=0$) *on* M^{2n}.

PROOF. We first show that M^{2n} is irreducible. Suppose M^{2n} is a product of Kähler manifolds. From the equations of Theorem C and Lemma 4.1 we have

$$4G(R_{UZ}X, Y)=c(\theta(Y)G(Z, X)-\theta(X)G(Y, Z)+\theta(JX)\Omega(Y, Z)$$
$$-\theta(JY)\Omega(X, Z)+2\theta(JZ)\Omega(Y, X)).$$

If now X and Y are tangent to one factor and Z is tangent to another we have $G(R_{UZ}X, Y)=-G(R_{XY}Z, U)=0$ and hence

$$0=c(2\theta(JZ)G(Y, JX)).$$

Taking $Y=JX$ we then have $\theta(JZ)=0$. But J acts non-singularly on each factor, therefore choosing a basis compatible with the product structure we find that $\theta\equiv 0$ on M^{2n} contradicting the fact that $\theta=df$ is a non-trivial solution of the equation of Theorem C.

Next we show that the Ricci tensor does not vanish on M^{2n}. From the above expression for $G(R_{UZ}X, Y)$ we have

$$4G(R_{UZ}U, Z)=c(\theta(Z)^2-\theta(U)G(Z, Z)+3\theta(JZ)\Omega(Z, U))$$

so that taking Z to be a unit vector field orthogonal to U we have for the sectional curvature $K(U, Z)$

$$K(U, Z)=\frac{c}{4}\left(1+3\frac{\theta(JZ)^2}{\theta(U)}\right).$$

Thus for $Z=\frac{JU}{\theta(U)^{1/2}}$, $K(U, JU)=c$ and for Z orthogonal to both U and JU, $K(U, Z)=\frac{c}{4}$. Now as the Ricci curvature in the direction of U is an average of sectional curvatures containing U, we see that the Ricci curvature in the direction of U is a positive constant on $M^*=\{m\in M \mid U(m)\neq 0\}$. But the zeros of U (i.e. the critical points of f) lie on a set of measure zero. Thus by the continuity of the Ricci tensor, we see that the Ricci tensor of M^{2n} does not vanish.

Lemma 4.2 now follows from the fact (see e.g. [8]) that on an irreducible Kähler manifold with non-vanishing Ricci tensor, every H-projective vector field is analytic.

LEMMA 4.3. *On* M^{2n} *we have* $(\nabla_Y\theta)(JX)+(\nabla_X\theta)(JY)=0$.

PROOF. Since $U=\operatorname{grad} f$ is analytic we have

$$0=-G((\mathcal{L}_U J)X, Y)=-G(\nabla_U JX-\nabla_{JX}U-J\nabla_U X+J\nabla_X U, Y)$$
$$=G(\nabla_{JX}U, Y)+G(\nabla_X U, JY)=(\nabla_{JX}\theta)(Y)+(\nabla_X\theta)(JY).$$

The result now follows from the fact that $d\theta=0$.

We now turn directly to the proof of the theorem. It is well known that the set of all principal circle bundles over a simply connected manifold M is a group isomorphic to $H^2(M, Z)$ (see e.g. Kobayashi [2]). Thus let $\pi: M^{2n+1} \to M^{2n}$ be the principal circle bundle over the Hodge manifold M^{2n} corresponding to the fundamental 2-form Ω. Let η' be a connexion form on M^{2n+1}. Then there exists a 2-form Ψ on M^{2n} such that $d\eta' = \pi^*\Psi$; but the characteristic class $[\Psi] \in H^2(M^{2n}, Z)$ is independent of the choice of connexions (see e.g. Kobayashi [2], Hatakeyama [1]), so that $[\Psi] = [\Omega]$. Thus there exists a 1-form α on M^{2n} such that $\Omega - \Psi = d\alpha$ and hence $\pi^*\Omega = d(\eta' + \pi^*\alpha)$. Setting $\eta = \eta' + \pi^*\alpha$ we can easily check that η is a connexion with $d\eta = \pi^*\Omega$ and moreover $\eta \wedge d\eta^n \neq 0$. Let ξ be a vertical vector field such that $\eta(\xi) = 1$ and define φ on M^{2n+1} by $\varphi = \tilde{\pi} J \pi_*$ where $\tilde{\pi}$ denotes the horizontal lift with respect to η. It is easy to check that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and $\varphi^2 = -I + \eta \otimes \xi$. Defining a metric g on M^{2n+1} by $g = \pi^*G + \eta \otimes \eta$, one can easily verify that M^{2n+1} has a contact metric structure. Finally, Morimoto [3] and Hatakeyama [1] showed that since M^{2n} is complex and Ω, the curvature form of η, is of bidegree (1, 1), the almost contact structure (φ, ξ, η) is normal. Thus M^{2n+1} is Sasakian and so equations (2.1) hold from which one can show that the differential equations of the submersion are again given by (2.2).

Now let $\theta = df$ be a non-trivial solution of the differential equation of Theorem C (with $c=1$) and define ω on M^{2n+1} by $\omega = \pi^*\theta$. Then by direct computation

$$\begin{cases} (\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y) = (\nabla_X\theta)(Y)\circ\pi\,, \\ (\bar{\nabla}_{\tilde{\pi}X}\omega)(\xi) = \dfrac{1}{2}\theta(JX)\circ\pi\,, \\ (\bar{\nabla}_{\xi}\omega)(\tilde{\pi}X) = \dfrac{1}{2}\theta(JX)\circ\pi\,, \\ (\bar{\nabla}_{\xi}\omega)(\xi) = 0\,. \end{cases} \tag{4.1}$$

Now as $(\bar{\nabla}\bar{\nabla}\omega)$ is a tensor it suffices to compute it on vector fields of the form $\tilde{\pi}X$ and ξ. Differentiating the first of equations (4.1) with respect to $\tilde{\pi}Z$ we have

$$\begin{aligned} &(\bar{\nabla}_{\tilde{\pi}Z}\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}Y) + (\bar{\nabla}_{\tilde{\pi}X}\omega)(\tilde{\pi}\nabla_Z Y - \frac{1}{2}\Phi(\tilde{\pi}Z, \tilde{\pi}Y)\xi) - (\bar{\nabla}_{\bar{\nabla}_{\tilde{\pi}Z}\tilde{\pi}X}\omega)(\tilde{\pi}Y) \\ &\quad = ((\nabla_Z\nabla_X\theta)(Y) + (\nabla_X\theta)(\nabla_Z Y))\circ\pi - (\nabla_{\nabla_Z X}\theta)(Y)\circ\pi \\ &\qquad + \frac{1}{2}\Phi(\tilde{\pi}Z, \tilde{\pi}X)(\bar{\nabla}_{\xi}\omega)(\tilde{\pi}Y) \end{aligned}$$

and therefore

$$(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z, \tilde{\pi}X, \tilde{\pi}Y)=(\nabla\nabla\theta)(Z, X, Y)\circ\pi+\frac{1}{4}(\Omega(Z, Y)\theta(JX)$$
$$+\Omega(Z, X)\theta(JY))\circ\pi$$
$$=-\frac{1}{4}(2\theta(Z)G(X, Y)+\theta(X)G(Y, Z)+\theta(Y)G(X, Z))\circ\pi$$
$$=-\frac{1}{4}(2\omega(\tilde{\pi}Z)g(\tilde{\pi}X, \tilde{\pi}Y)+\omega(\tilde{\pi}X)g(\tilde{\pi}Y, \tilde{\pi}Z)$$
$$+\omega(\tilde{\pi}Y)g(\tilde{\pi}X, \tilde{\pi}Z)).$$

Differentiating the second of equations (4.1) with respect to $\tilde{\pi}Z$ we have

$$(\bar{\nabla}_{\tilde{\pi}Z}\bar{\nabla}_{\tilde{\pi}X}\omega)(\xi)+(\bar{\nabla}_{\tilde{\pi}X}\omega)\Big(-\frac{1}{2}\varphi\tilde{\pi}Z\Big)-(\bar{\nabla}_{\bar{\nabla}_{\tilde{\pi}Z}\tilde{\pi}X}\omega)(\xi)$$
$$=\frac{1}{2}((\nabla_Z\theta)(JX)+\frac{1}{2}\theta(J\nabla_Z X))\circ\pi-\frac{1}{2}\theta(J\nabla_Z X)\circ\pi$$
$$+\frac{1}{2}\Phi(\tilde{\pi}Z, \tilde{\pi}X)(\bar{\nabla}_\xi\omega)(\xi)$$

giving

$$(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z, \tilde{\pi}X, \xi)=\frac{1}{2}((\nabla_Z\theta)(JX)+(\nabla_X\theta)(JZ))\circ\pi=0$$

by Lemma 4.3.

Differentiating the fourth of equations (4.1) with respect to $\tilde{\pi}Z$ we have

$$(\bar{\nabla}_{\tilde{\pi}Z}\bar{\nabla}_\xi\omega)(\xi)+(\bar{\nabla}_\xi\omega)\Big(-\frac{1}{2}\varphi\tilde{\pi}Z\Big)-(\bar{\nabla}_{\bar{\nabla}_{\tilde{\pi}Z}\xi}\omega)(\xi)=\frac{1}{2}(\bar{\nabla}_{\varphi\tilde{\pi}Z}\omega)(\xi)$$

and therefore

$$(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z, \xi, \xi)=-\frac{1}{2}\theta(Z)\circ\pi=-\frac{1}{2}\omega(\tilde{\pi}Z).$$

In like manner we obtain

$$(\bar{\nabla}\bar{\nabla}\omega)(\tilde{\pi}Z, \xi, \tilde{\pi}Y)=\frac{1}{2}((\nabla_Z\theta)(JY)+(\nabla_{JZ}\theta)(Y))\circ\pi=0,$$
$$(\bar{\nabla}\bar{\nabla}\omega)(\xi, \tilde{\pi}X, \tilde{\pi}Y)=\frac{1}{2}((\nabla_X\theta)(JY)+(\nabla_{JX}\theta)(Y))\circ\pi=0,$$
$$(\bar{\nabla}\bar{\nabla}\omega)(\xi, \tilde{\pi}X, \xi)=-\frac{1}{4}\omega(\tilde{\pi}X),$$
$$(\bar{\nabla}\bar{\nabla}\omega)(\xi, \xi, \tilde{\pi}Y)=-\frac{1}{4}\omega(\tilde{\pi}Y),$$
$$(\bar{\nabla}\bar{\nabla}\omega)(\xi, \xi, \xi)=0.$$

Thus we see that ω satisfies the differential equation of Theorem B with $k=\frac{1}{4}$ and since $\theta=df$, $\omega=\pi^*\theta=d(f\circ\pi)$. M^{2n+1} or its simply connected covering space is therefore globally isometric to a Euclidean sphere S^{2n+1} of radius 2. Hence the fibration $\pi: M^{2n+1} \rightarrow M^{2n}$ is the Hopf fibration and M^{2n} is isometric to PC^n with $c=1$ as desired.

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David E. BLAIR

Department of Mathematics
Michigan State University
East Lansing
Michigan 48823
U.S.A.