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On cohomology mod 2 of the classifying spaces of non-simply connected classical Lie groups

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§1. Introduction.

Let G be a compact connected Lie group. Consider the following two statements:

(1.1) $H^*(G; Z_2)$ has a simple system of primitive generators.

(1.2) $H^*(BG; Z_2)$ is a polynomial algebra.

As is well known, (1.2) implies (1.1). On the other hand for any simple G except $Spin(2^k+1)$, $k \geqq 4$, and $PSp(2n+1)$, (1.1) implies (1.2).

According to Borel [4] $Spin(2^k+1)$, $k \geqq 4$, does not satisfy (1.2).

In this paper, we show (1.2) for $PSp(2n+1)$ namely $H^*(BPSp(2n+1); Z_2)$ is a polynomial algebra (see Theorem 4.4).

In the section 2 we consider some 2-groups in $Sp(n)$ and $PSp(n)$. In the section 3 we consider a kind of stability in the cohomology $H^*(BPSp(n); Z_2)$ and $H^*(PSp(n); Z_2)$. The section 4 is devoted to the determination of $H^*(BPSp(2n+1); Z_2)$. In the section 5 we can classify the compact connected simple Lie groups whose mod 2 cohomology rings have simple systems of universally transgressive generators (Theorem 5.2).

Throughout this paper the map $BG \rightarrow BH$ induced by a homomorphism $f: G \rightarrow H$ of Lie groups is also denoted by the symbol f (Milnor [8]).

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§2. Some 2-groups in $Sp(n)$ and $PSp(n)$.

In this section various finite subgroups of $Sp(n)$ and $PSp(n)$ are considered. We use the following notations

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} = (a_1, \cdots, a_n) \in Sp(n) \qquad \text{for} \quad a_i \in Sp(1).$$

DEFINITION 2.1.

$$\boldsymbol{L}=\{\pm 1,\ \pm i,\ \pm j,\ \pm k\}\subset \boldsymbol{Sp}(1),$$

$$\tilde{\boldsymbol{L}}(n)=\{\alpha\cdot E_n\,;\ \alpha\in L\}\qquad \text{for}\quad E_n=(1,\cdots,1),$$

$$\tilde{\boldsymbol{V}}_0(n)=\{(\varepsilon_1,\cdots,\varepsilon_n)\in \boldsymbol{Sp}(n)\,;\ \varepsilon_i=\pm 1\},$$

$$\tilde{\boldsymbol{V}}(n)=\{A\cdot B\,;\ A\in\tilde{\boldsymbol{L}}(n),\ B\in\tilde{\boldsymbol{V}}_0(n)\},$$

$$\boldsymbol{L}(n)=\tilde{\boldsymbol{L}}(n)/\Delta(n)\qquad \text{for}\quad \Delta(n)=\{\pm E_n\},$$

$$\boldsymbol{V}_0(n)=\tilde{\boldsymbol{V}}_0(n)/\Delta(n)$$

and

$$\boldsymbol{V}(n)=\tilde{\boldsymbol{V}}(n)/\Delta(n).$$

Note that $\Delta(n)$ is the center of $\boldsymbol{Sp}(n)$, and the following diagram is commutative.

$$\begin{array}{ccccc} \tilde{\boldsymbol{V}}_0(n) & \longrightarrow & \tilde{\boldsymbol{V}}(n) & \longrightarrow & \boldsymbol{Sp}(n) \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ \boldsymbol{V}_0(n) & \longrightarrow & \boldsymbol{V}(n) & \xrightarrow{\ i\ } & \boldsymbol{PSp}(n) \end{array} \tag{2.1}$$

where π are natural projections and the horizontal lines are natural inclusions.

PROPOSITION 2.2.

i) $\tilde{\boldsymbol{V}}_0(n)\cong(\boldsymbol{Z}_2)^n$,

ii) $\boldsymbol{V}_0(n)\cong(\boldsymbol{Z}_2)^{n-1}$,

iii) $\boldsymbol{V}(n)\cong(\boldsymbol{Z}_2)^{n+1}$,

iv) $\boldsymbol{L}(n)\cong(\boldsymbol{Z}_2)^2$,

and

v) $\boldsymbol{V}(1)=\boldsymbol{L}(1)\cong(\boldsymbol{Z}_2)^2$.

The proposition follows directly from the definitions.

The following facts are well known (see for example Borel-Hirzebruch [**5**]).

PROPOSITION 2.3.

i) *$\tilde{\boldsymbol{V}}_0(n)$ is the maximal elementary 2-group of a maximal torus $\boldsymbol{T}^n$ of $\boldsymbol{Sp}(n)$. The inclusions $\tilde{\boldsymbol{V}}_0(n)\xrightarrow{j'}\boldsymbol{T}^n\xrightarrow{j}\boldsymbol{Sp}(n)$ induce injections $H^*(B\boldsymbol{Sp}(n)\,;\boldsymbol{Z}_2)\to H^*(B\boldsymbol{T}^n\,;\boldsymbol{Z}_2)\to H^*(B\tilde{\boldsymbol{V}}_0(n)\,;\boldsymbol{Z}_2)$ where $H^*(B\boldsymbol{Sp}(n)\,;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[q_1,\cdots,q_n]$, $\deg q_i=4i$.*

ii) *$\boldsymbol{Sp}(1)=\boldsymbol{Spin}(3)$, so $\boldsymbol{PSp}(1)=\boldsymbol{SO}(3)$ and $H^*(B\boldsymbol{PSp}(1)\,;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[y_2,y_3]$, $\deg y_i=i$.*

iii) *$\boldsymbol{L}(1)$ is a maximal elementary 2-group of $\boldsymbol{SO}(3)$ and so $\iota^*: H^*(B\boldsymbol{PSp}(1)\,;\boldsymbol{Z}_2)\to H^*(B\boldsymbol{L}(1)\,;\boldsymbol{Z}_2)$ is injective for the inclusion $\iota:\boldsymbol{L}(1)\to\boldsymbol{PSp}(1)$.*

iv) *$H^*(B\boldsymbol{V}(n)\,;\boldsymbol{Z}_2)=\boldsymbol{Z}_2[t_1,\cdots,t_{n+1}]$, $\deg t_i=1$.*

§ 3. Stability of $H^*(\boldsymbol{PSp}(n)\,;\boldsymbol{Z}_2)$.

The following result is well known (Borel [3]).

(3.1) $$H^*(\boldsymbol{Sp}(n)\,;\boldsymbol{Z})=\wedge(e_3, e_7, \cdots, e_{4n-1})$$

where each e_{4i-1} is universally transgressive and $\tau(e_{4i-1})\equiv q_i$: the i-th univeral symplectic Pontrjagin class.

Let

$$\Delta_k=\Delta_{k,n}: \boldsymbol{Sp}(n) \longrightarrow (\boldsymbol{Sp}(n))^k \subset \boldsymbol{Sp}(kn)$$

be the k-fold diagonal map and we put

$$\boldsymbol{Q}(n, k)=\boldsymbol{Sp}(kn)/\Delta_k\boldsymbol{Sp}(n)\,.$$

Then we have a fibering

(3.2) $$\boldsymbol{Sp}(n) \xrightarrow{\Delta k} \boldsymbol{Sp}(kn) \xrightarrow{\rho} \boldsymbol{Q}(n, k)\,.$$

THEOREM 3.2. *If* $(p, k)=1$ *then*

$$H^*(\boldsymbol{Q}(n, k)\,;\boldsymbol{Z}_p)=\wedge(\bar{x}_{4n+3}, \cdots, \bar{x}_{4kn-1})$$

where $\rho^*(\bar{x}_{4i-1})=e_{4i-1}$ *for* $i=n+1, \cdots, kn$.

PROOF. Since the map $\Delta_k: (B)\boldsymbol{Sp}(n)\to(B)\boldsymbol{Sp}(kn)$ corresponds to the k-fold Whitney sum of quaternionic vector bundles,

(3.3) $$\Delta_k^*(q_i)=\sum_{i_1+\cdots+i_k=i} q_{i_1}\cdots q_{i_k}=kq_i+\text{decomposable}\,.$$

Consider the Serre spectral sequence for the fibering $\boldsymbol{Sp}(kn)\to\boldsymbol{Q}(n, k)\to B\boldsymbol{Sp}(n)$:

$$E_2^{*,*}=\boldsymbol{Z}_p[q_1, \cdots, q_n]\otimes\wedge(e_3, e_7, \cdots, e_{4kn-1})$$
$$\Longrightarrow H^*(\boldsymbol{Q}(n, k)\,;\boldsymbol{Z}_p)\,.$$

By (3.1) and (3.3) we have $d_{4i}(1\otimes e_{4i-1})=kq_i\otimes 1$ for $i=1, 2, \cdots, n$, and

$$Gr(H^*(\boldsymbol{Q}(n, k)\,;\boldsymbol{Z}_p))=E_\infty^{*,*}=E_{4n+1}^{*,*}=\wedge(e_{4n+3}, \cdots, e_{4kn-1})\,.$$

Next, it follows from (3.3) $\Delta_k^*(e_{4i-1})=k\cdot e_{4i-1}$, $i=1, \cdots, n$, for $\Delta_k^*: H^*(\boldsymbol{Sp}(nk)\,;\boldsymbol{Z}_p)\to H^*(\boldsymbol{Sp}(n)\,;\boldsymbol{Z}_p)$. Thus $\boldsymbol{Sp}(n)$ is totally non-homologous to zero in the fibering (3.2), and $H^*(\boldsymbol{Q}(n, k)\,;\boldsymbol{Z}_p)$ is mapped injectively into $H^*(\boldsymbol{Sp}(kn)\,;\boldsymbol{Z}_p)$ under ρ^*. Consequently the theorem follows. Q. E. D.

Since $\Delta_k(-E_n)=-E_{kn}$ the map Δ_k induces a map, denoted by the same symbol Δ_k, such that the following diagram is commutative.

$$\begin{array}{ccc} \boldsymbol{Sp}(n) & \xrightarrow{\Delta_k} & \boldsymbol{Sp}(kn) \\ \downarrow & & \downarrow \\ \boldsymbol{PSp}(n) & \xrightarrow{\Delta_k} & \boldsymbol{PSp}(kn)\,. \end{array}$$

Note that $\boldsymbol{PSp}(kn)/\Delta_k\boldsymbol{PSp}(n)$ is homeomorphic to $\boldsymbol{Q}(n, k)$.

THEOREM 3.3. *Let k be an odd integer >0, then the homomorphism*

$$\Delta_k^*: H^i(B\boldsymbol{PSp}(kn); \boldsymbol{Z}_2) \longrightarrow H^i(B\boldsymbol{PSp}(n); \boldsymbol{Z}_2)$$

is isomorphic if $i \leqq 4n+2$ and monomorphic if $i \leqq 4n+3$.

PROOF. Apply Theorem 3.2 to the Serre exact sequence for the fibering $\boldsymbol{Q}(n, k) \to B\boldsymbol{PSp}(n) \to B\boldsymbol{PSp}(kn)$. Q. E. D.

For each positive integer n, define $\nu_p(n)$ by

$$n = \prod_{p:\,\text{prime}} p^{\nu_p(n)}.$$

COROLLARY 3.4. *If $\nu_2(n) = \nu_2(m)$, then as algebras $H^*(B\boldsymbol{PSp}(n); \boldsymbol{Z}_2)$ is isomorphic to $H^*(B\boldsymbol{PSp}(m); \boldsymbol{Z}_2)$ for $* \leqq 4\,\mathrm{Min}\,(m, n)+2$.*

PROPOSITION 3.5. *For odd integer $k>0$, the Serre spectral sequence for the fibering $\boldsymbol{PSp}(n) \to \boldsymbol{PSp}(kn) \to \boldsymbol{Q}(n, k)$ collapses.*

PROOF. By Theorem 11.3 of [4] and Theorem 3.3, $\boldsymbol{P.S.}(\boldsymbol{PSp}(kn)) = \boldsymbol{P.S.}(\boldsymbol{PSp}(n)) \cdot \boldsymbol{P.S.}(\boldsymbol{Q}(n, k))$, where $\boldsymbol{P.S.}(X) = \sum \{\mathrm{rank}\, H^i(X; \boldsymbol{Z}_2)\} t^i \in \boldsymbol{Z}[[t]]$. So we get the proposition. Q. E. D.

Similar discussions hold for $\boldsymbol{SO}(2n)/\{\pm E_{2n}\} = \boldsymbol{PO}(2n)$ and $\boldsymbol{SU}(n)/\Gamma_l$, where Γ_l is a central subgroup of order $l \mid n$.

THEOREM 3.6. i) *Let k be an odd integer >0, then*

$$\Delta_k^*: H^i(B\boldsymbol{PO}(2kn); \boldsymbol{Z}_2) \longrightarrow H^i(B\boldsymbol{PO}(2n); \boldsymbol{Z}_2)$$

is isomorphic if $i \leqq n-1$ and monomorphic if $i \leqq n$.

ii) *If $(k, p) = 1$ and $l \mid n$, then*

$$\Delta_k^*: H^i(B(\boldsymbol{SU}(kn)/\Gamma_l); \boldsymbol{Z}_p) \longrightarrow H^i(B(\boldsymbol{SU}(n)/\Gamma_l); \boldsymbol{Z}_p)$$

is isomorphic if $i \leqq 2n$ and monomorphic if $i \leqq 2n+1$.

§ 4. Mod 2 cohomology of $B\boldsymbol{PSp}(2n+1)$.

Let $\boldsymbol{G}$ be a compact connected Lie group and $\boldsymbol{H}$ its closed subgroup. Consider the following commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{H} & \longrightarrow & \boldsymbol{EH} & \longrightarrow & B\boldsymbol{H} \\
\downarrow{\scriptstyle j} & & \downarrow & & \downarrow \\
\boldsymbol{G} & \longrightarrow & \boldsymbol{EG} & \longrightarrow & B\boldsymbol{G} \\
\downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle =} \\
\boldsymbol{G}/\boldsymbol{H} & \xrightarrow{i} & B\boldsymbol{H} & \xrightarrow{j} & B\boldsymbol{G}
\end{array}$$

where the horizontal lines are fiberings and $\boldsymbol{EG}$ (resp. $\boldsymbol{EH}$) is the total space of universal $\boldsymbol{G}$- (resp. $\boldsymbol{H}$-) bundle. Let k be a field.

LEMMA 4.1. i) *If* $x \in H^*(\boldsymbol{G}/\boldsymbol{H}; k)$ *is transgressive with respect to the bottom fibering, then the element* $p^*(x) \in H^*(\boldsymbol{G}; k)$ *is universally transgressive.*

ii) *If* $x \in H^*(\boldsymbol{G}; k)$ *is universally transgressive then so is* $j^*(x) \in H^*(\boldsymbol{H}; k)$.

iii) *If* $H^i(\boldsymbol{G}/\boldsymbol{H}; k) = 0$ *for* $i < n$, $\deg x < n-1$ *for* $x \in H^*(\boldsymbol{G}; k)$ *and if* $j^*(x)$ *is universally transgressive, then* x *is also universally transgressive.*

These follow from the naturality of the transgression.

The following result is due to Borel [4] (see also Baum-Browder [2]).

LEMMA 4.2. *We can choose generators* $a, x_7, x_{11}, \cdots, x_{8n+3} \in H^*(\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2)$ *such that* $H^*(\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[a]/(a^4) \otimes \wedge(x_7, x_{11}, \cdots, x_{8n+3})$ *where* $\deg a = 1$ *and* $\pi^*(x_{4i-1}) = e_{4i-1}$, $i = 2, 3, \cdots, 2n+1$, *for the projection* $\pi: \boldsymbol{Sp}(2n+1) \to \boldsymbol{PSp}(2n+1)$.

LEMMA 4.3. *Let* $i^*: H^*(B\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2) \to H^*(B\boldsymbol{V}(2n+1); \boldsymbol{Z}_2)$ *be induced by the inclusion* $i: \boldsymbol{V}(2n+1) \to \boldsymbol{PSp}(2n+1)$.

i) i^* *is injective for degree* $\leqq 7$.

ii) *If* $x_7, x_{11}, \cdots, x_{4s-1}$, $s \leqq 2n+1$, *are universally transgressive, then* $i^*\tau(x_{4i-1}) \neq 0$ *for* $i = 2, 3, \cdots, s$.

PROOF. i) Consider the following commutative diagram

$$\begin{array}{ccccc} B\boldsymbol{L}(1) & & \xrightarrow{\quad\iota\quad} & & B\boldsymbol{PSp}(1) \\ \downarrow \Delta_{2n+1} & & & & \downarrow \Delta_{2n+1} \\ B\boldsymbol{L}(2n+1) & \xrightarrow{k} & B\boldsymbol{V}(2n+1) & \xrightarrow{i} & B\boldsymbol{PSp}(2n+1). \end{array}$$

By Theorem 3.3, $\Delta_{2n+1}^*: H^\varepsilon(B\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2) \to H^\varepsilon(B\boldsymbol{PSp}(1); \boldsymbol{Z}_2)$ is injective for $\varepsilon \leqq 7$ and, by iii) of Proposition 2.3, ι^* is injective. Thus $\Delta_{2n+1}^* k^* i^* = \iota^* \Delta_{2n+1}^*$ is injective and the result follows.

ii) Consider the following commutative diagram

$$\begin{array}{ccccc} B\tilde{\boldsymbol{V}}_0(2n+1) & \xrightarrow{j'} & B\boldsymbol{T}^{2n+1} & \xrightarrow{j} & B\boldsymbol{Sp}(2n+1) \\ \downarrow \pi & & & & \downarrow \pi \\ B\boldsymbol{V}_0(2n+1) & \xrightarrow{i'} & B\boldsymbol{V}(2n+1) & \xrightarrow{i} & B\boldsymbol{PSp}(2n+1). \end{array}$$

Then $\pi^*(\tau(x_{4i-1})) = \tau(e_{4i-1}) = q_{4i} \neq 0$, by i) of Proposition 2.3, j^* and j'^* are injective. Thus the result follows. Q. E. D.

THEOREM 4.4. *The elements* $a, x_7, x_{11}, \cdots, x_{8n+1} \in H^*(\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2)$ *can be chosen to be universally transgressive. Thus*

$$H^*(B\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[y_2, y_3, y_8, y_{12}, \cdots, y_{8n+4}]$$

for transgression images $\tau(a) = y_2$, $\tau(a^2) = y_3$, $\tau(x_7) = y_8, \cdots, \tau(x_{8n+3}) = y_{8n+4}$.

PROOF. We prove the theorem by induction on n. The case $n = 0$ is seen in ii) of Proposition 2.3. Let $n \geqq 1$ and assume that $H^*(B\boldsymbol{PSp}(2n-1); \boldsymbol{Z}_2) = \Delta(a, a^2, x_7, x_{11}, \cdots, x_{8n-5})$ for universally transgressive elements $a, x_7, x_{11}, \cdots,$

x_{8n-5}. By Proposition 3.5 and Theorem 3.2

$$\Delta_{2n+\varepsilon}^{*}: H^{i}(\boldsymbol{PSp}((2n-1)(2n+1)); \boldsymbol{Z}_2) \longrightarrow H^{i}(\boldsymbol{PSp}(2n-\varepsilon); \boldsymbol{Z}_2),$$

$\varepsilon=\pm 1$, is bijective for $i<8n-4\varepsilon+3$. Then we may choose generators of $H^*(\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2)$ in Lemma 4.2 as follows:

$$a=\Delta_{2n-1}^{*}\Delta_{2n+1}^{*-1}(a),$$

$x_7=\Delta_{2n-1}^{*}\Delta_{2n+1}^{*-1}(x_7), \cdots, x_{8n-5}=\Delta_{2n-1}^{*}\Delta_{2n+1}^{*-1}(x_{8n-5})$, $x_{8n-1}=\Delta_{2n-1}^{*}p^*(\bar{x}_{8n-1})$ and $x_{8n+3}=\Delta_{2n-1}^{*}p^*(\bar{x}_{8n+3})$ for the projection $p: \boldsymbol{PSp}((2n-1)(2n+1)) \rightarrow \boldsymbol{Q}(2n-1, 2n+1)$ and for the elements $\bar{x}_{8n-1}, \bar{x}_{8n+3}$ of Theorem 3.2. Applying Lemma 4.1 to the case $\boldsymbol{G}=\boldsymbol{PSp}((2n-1)(2n+1))$ and $\boldsymbol{H}=\Delta_{2n-1}(\boldsymbol{PSp}(2n+1))$, we have that the elements $a, x_7, x_{11}, \cdots, x_{8n-1}$ are universally transgressive.

Now assume that $\bar{x}_{8n+3}$ is not transgressive in the spectral sequence $(E_r^{p,q})$ associated with the fibering

$$\boldsymbol{Q}(2n-1, 2n+1) \longrightarrow B\boldsymbol{PSp}(2n-1) \xrightarrow{\Delta_{2n+1}} B\boldsymbol{PSp}((2n-1)(2n+1)).$$

Then the only possibility is

$$d_5(1\otimes \bar{x}_{8n+3})=y\otimes \bar{x}_{8n-1}\neq 0$$

where y belongs to $H^5(B\boldsymbol{PSp}((2n-1)(2n+1)); \boldsymbol{Z}_2)$ $(\cong H^5(B\boldsymbol{PSp}(1); \boldsymbol{Z}_2)$, thus $y=y_2y_3)$. It follows $0=d_{8n}(y\otimes \bar{x}_{8n-1})=\tau(\bar{x}_{8n-1})\cdot y\otimes 1$ in $E_{8n}^{8n+5,0}\cong E_2^{8n+5,0}\cong H^{8n+5}(B\boldsymbol{PSp}((2n-1)(2n+1)); \boldsymbol{Z}_2)$. Apply Δ_{2n-1}^{*}, then the same relation $\tau(\bar{x}_{8n-1})\cdot y=0$ holds in $H^{8n+5}(B\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2)$. By the naturality of τ, $\tau(\bar{x}_{8n-1})=\tau(x_{8n-1})$. Then applying Lemma 4.3 for $s=2n$ we have that $i^*y\neq 0$ and $i^*\tau(\bar{x}_{8n-1})\neq 0$, which contradicts to $i^*\tau(\bar{x}_{8n-1})i^*y=i^*(\tau(\bar{x}_{8n-1})\cdot y)=0$ since $H^*(B\boldsymbol{V}(2n+1); \boldsymbol{Z}_2)$ is a polynomial algebra by iv) of Proposition 2.3. Consequently we have proved that $\bar{x}_{8n+3}$ is transgressive. Therefore x_{8n+3} is transgressive by i) of Lemma 4.1.

Finally, by use of Proposition 16.1 of Borel [3], we have the second assertion of the theorem. Q. E. D.

Corollary 4.5 (Baum-Browder [2]). *$H^*(\boldsymbol{PSp}(2n+1); \boldsymbol{Z}_2)$ is primitively generated.*

For it is transgressively generated.

Corollary 4.6. *$\boldsymbol{V}(2n+1)$ is a maximal elementary 2-group in $\boldsymbol{PSp}(2n+1)$.*

This follows easily from Corollary of Proposition 7 of Borel-Serre [6] and Theorem 4.4.

§5. Mod 2 cohomology rings of compact connected simple Lie groups.

Each compact connected simple Lie group is the quotient, by a central subgroup, of a compact connected simply connected simple Lie group which is classified as follows:

type	A_n	B_n	C_n	D_{2n}	D_{2n+1}	G_2	F_4	E_6	E_7	E_8
center	Z_{n+1}	Z_2	Z_2	$Z_2 \oplus Z_2$	Z_4	0	0	Z_3	Z_2	0

The Hopf algebra structure of the mod 2 cohomology rings of the compact connected simple Lie groups are known by Araki [1], Baum-Browder [2], Kojima [7] and Toda [11], in particular we have the following.

THEOREM 5.1. *Let $\boldsymbol{G}$ be a compact connected simple Lie group such that $H^*(\boldsymbol{G}; \boldsymbol{Z}_2)$ is primitively generated. Then $\boldsymbol{G}$ is isomorphic to one of the followings:*

$$\boldsymbol{SU}(n), \quad \boldsymbol{Sp}(n), \quad n \geqq 2;$$

$$\boldsymbol{Spin}(7), \quad \boldsymbol{Spin}(8), \quad \boldsymbol{Spin}(9), \quad \boldsymbol{Spin}(2^k+1), \quad k \geqq 4;$$

$$\boldsymbol{G}_2, \quad \boldsymbol{F}_4;$$

$$\boldsymbol{SU}(n)/\Gamma_l, \quad n \geqq 3; \quad \boldsymbol{PSp}(2m-1), \quad m \geqq 1; \quad \boldsymbol{SO}(n), \quad n \geqq 5;$$

where Γ_l is a central subgroup of $\boldsymbol{SU}(n)$ with an odd order $l > 1$.

According to Borel [4], Quillen [9], $H^*(\boldsymbol{Spin}(2^k+1); \boldsymbol{Z}_2)$ is not transgressively generated for $k \geqq 4$, and using known results in Borel [3], [4] and Theorem 4.4 we get the following

THEOREM 5.2. *Let $\boldsymbol{G}$ be a compact connected simple Lie group. Then the following three conditions are equivalent.*

1) *$H^*(\boldsymbol{G}; \boldsymbol{Z}_2)$ has a simple system $\{x_i\}$ of universally transgressive generators $x_i: H^*(\boldsymbol{G}; \boldsymbol{Z}_2) = \Delta(x_1, \cdots, x_n)$.*

2) *$H^*(B\boldsymbol{G}; \boldsymbol{Z}_2)$ is a polynomial algebra* (generated by $\tau(x_1), \cdots, \tau(x_n)$).

3) *$\boldsymbol{G}$ is one of the groups listed in Theorem* 5.1 *excluding* $\boldsymbol{Spin}(2^k+1)$, $k \geqq 4$.

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