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On the prolongation of local holomorphic solutions of partial differential equations, II, prolongation across the pluri-harmonic hypersurface

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§1. Introduction.

Recently many results about the holomorphic continuation of solutions of a partial differential equation in a complex domain are obtained. Let Ω be a domain in C^n with a smooth boundary, denoted by $\partial\Omega$ and $P(z, D)$ be a linear partial differential operator whose coefficients are holomorphic in some neighborhood of a given point $p\in\partial\Omega$. The main results already obtained are the following:

(i) if the boundary $\partial\Omega$ is *non-characteristic* with respect to $P(z, D)$ at p, then every holomorphic solution $u(z)$ of $P(z, D)u(z)=0$ in Ω is holomorphic at p. (Zerner [5].)

(ii) if the boundary $\partial\Omega$ is *simply characteristic* at p and the normal curvature in some direction in the complex bi-characteristic curve is negative, then every solution $u(z)$ of $P(z, D)u(z)=0$ in Ω is holomorphic at p. (Tsuno [3].)

(iii) if the boundary $\partial\Omega$ is *strictly pseudo-convex* and *simply characteristic* at p and the normal curvature in every direction in the complex bicharacteristic curve is positive, then under some additional conditions we can construct a solution $u(z)$ of $P(z, D)u(z)=0$ in Ω which is not holomorphic at p. (Tsuno [3].)

(iv) if the boundary $\partial\Omega$ is *real-analytic*, P. Pallu de La Barrière ([1] and his thesis) studied the existence and prolongation of holomorphic solutions in the framework of the hyperfunction theory and obtained the following result as an application (Théorème 4.1 in his thesis): under the same situation of our theorem in the next section, if the holomorphic function $P_m(z, \mathrm{grad}_z\Phi(z))$ has at most *simple zero* on the hypersurface $\{\Phi(z)=0\}$, then every solution $u(z)$ of $P(z, D)u(z)=0$ in Ω is holomorphic at 0.

In this paper we are concerned with the case where the boundary $\partial\Omega$ is a level surface of a *pluri-harmonic function* and $p\in\partial\Omega$ is a *general characteristic point* with respect to $P(z, D)$.

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§ 2. Theorem and its proof.

Let U be an open ball containing 0 in $\boldsymbol{C}^n$ with the coordinates $(z_1, \cdots, z_n)$ and $\phi(z)$ be a non-degenerate real valued C^2 function in U which satisfies the equation

$$\partial\bar{\partial}\phi=0.$$

Such a function ϕ is called *pluri-harmonic* and it is well known that the pluri-harmonic function $\phi(z)$ is locally the real part of a holomorphic function $\Phi(z)$. (R. C. Gunning and H. Rossi [2], p. 271.) Thus we may assume that $\phi(z)=\operatorname{Re}\Phi(z)$ in U and $\phi(0)=0$, and set

$$\Omega=\{z\in U \mid \phi(z)<0\}.$$

Let $P(z, D)=\sum_{|\alpha|\leqq m} a_\alpha(z)(\partial/\partial z)^\alpha$ be a linear differential operator whose coefficients are holomorphic in U and denote by $P_m(z, \xi)$ the principal part $P_m(z, \xi)=\sum_{|\alpha|=m} a_\alpha(z)\xi^\alpha$ of $P(z, \xi)$. From Zerner's theorem [5] if $P_m(0, \operatorname{grad}_z\Phi(0))\neq 0$ where $\operatorname{grad}_z\Phi=(\partial\Phi/\partial z_1, \cdots, \partial\Phi/\partial z_n)$, every holomorphic solution $u(z)$ of $P(z, D)u(z)=0$ in Ω is holomorphic at 0. Therefore we study in this paper the case where the origin is characteristic with respect to $P(z, D)$. That is, we assume that

$$P_m(0, \operatorname{grad}_z\Phi(0))=0.$$

Then we have the following theorem.

THEOREM. *If the holomorphic function $P_m(z, \operatorname{grad}_z\Phi(z))$ is not identically zero on the complex hypersurface $\{\Phi(z)=0\}$, then every solution $u(z)$ of $P(z, D)u(z)=0$ in Ω is holomorphic at the origin.*

PROOF. Since $\Phi(z)$ is a non-degenerate holomorphic function, we may assume that $\Phi(z)=z_1$ after a suitable change of variables. Then by the assumption $P_m(0, z'; N)$, where $z'=(z_2, \cdots, z_n)$ and $N=(1, 0, \cdots, 0)$, is not identically zero in z'. Therefore after one more change of variables in z'-space, if necessary, we may suppose that $P_m(0, z_2, 0, \cdots, 0; N)$ is not identically zero. Since the set of zeros of a holomorphic function $P_m(0, z_2, 0, \cdots, 0; N)$ must be discrete, we can choose a constant $r>0$ so small that $\partial\Omega$ is non-characteristic at any point of the compact set $\Gamma=\{(0, z_2, 0, \cdots, 0) \mid |z_2|=r\}$ in $\partial\Omega\cap U$.

Let $u(z)$ be any solution of $P(z, D)\, u(z)=0$ in $\Omega=\{z\in U \mid \operatorname{Re} z_1<0\}$. Then by Zerner's theorem [5] $u(z)$ is holomorphic in a neighborhood of Γ. Since Γ is compact, there exists a constant ρ $(0<\rho<r)$ such that $u(z)$ is holomor-

phic in $\Omega\cup\{z\mid r-\rho<|z_2|<r+\rho,\ |z_j|<\rho\ (j\neq 2)\}\subset U$. Now we define the function $\tilde{u}(z)$ for $z\in\Delta=\{z\mid |z_j|<\rho,\ j=1,\cdots,n\}$ by

$$\tilde{u}(z_1, z_2, \cdots, z_n)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{u(z_1, \zeta, z_3, \cdots, z_n)}{\zeta-z_2}d\zeta .$$

Then $\tilde{u}(z)$ is holomorphic in Δ and $\tilde{u}(z)=u(z)$ for $z\in\Delta\cap\Omega$ by Cauchy's integral formula. Therefore $u(z)$ is holomorphically continued across the origin, which completes the proof.

REMARK. A non-degenerate holomorphic function $\Phi(z)$ which does not satisfy the assumption in the above theorem is not always characteristic with respect to the operator $P(z, D)$; this means that $P_m(z, \operatorname{grad}\Phi(z))$ does not always vanish identically in a neighborhood of 0. We will study in the forthcoming paper [4] the case where the function $\Phi(z)$ is not characteristic but satisfies $P_m(z, \operatorname{grad}\Phi(z))=0$ on the complex surface $\Phi(z)=0$. For example when $\Phi(z)=z_1$ and $P(z, D)=z_1\frac{\partial^2}{\partial z_1^2}+\frac{\partial}{\partial z_1}$, the equation $P(z, D)\ u(z)=0$ has the solution $u(z)=\log z_1$. In [4] we will analyse these operators, which is said to be of the Fuchsian type.

References

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