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On the intransitive Lie algebras whose transitive parts are infinite and primitive

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Introduction.

At the beginning of this century E. Cartan developed the theory of infinite Lie groups in his series of papers [1], [2], [3], and one of his main achievements was the classification of the simple infinite Lie groups [3].

With the increasing interest in this field, modern formulations and treatments were made around 1960 (especially in [9], [5], [13]), and then the rigorous and systematic proof of the classification of the simple infinite Lie groups was given in the transitive case ([8], [6], [10]).

On the other hand in the intransitive case, only a few attempts were made by several authors. N. Tanaka [14] and K. Ueno [15] studied the generalized G-structures towards the equivalence problem of the intransitive Lie groups. V. Guillemin [7] studied a Jordan-Hölder decomposition of the transitive Lie algebras and introduced simple intransitive Lie algebras occuring in the decomposition. But in his treatment the intransitive Lie algebras considered are limited to those which are ideals of some Lie algebras.

It seems that there are no satisfactory formulations and treatments of the intransitive infinite Lie groups. In particular, in spite of the early work of Cartan, the classification of the simple intransitive infinite Lie groups has not yet been rigorously settled.

The purpose of this paper is to contribute to the classification of the simple intransitive infinite Lie groups.

According to Cartan [3], the classification problem is divided into the following three problems:

(I) To reduce the problem to (II) and (III).

(II) To determine all intransitive Lie groups whose restrictions to the orbits are infinite and primitive.

(III) To determine all intransitive infinite Lie groups whose restrictions to the orbits are finite, simple, and simply transitive on the orbits.

In this paper we take up the problem (II). We shall formulate it in the category of the formal Lie algebras, and carry it through.

Now we explain our main results, describing the construction of this paper.

Let V be a finite dimensional vector space, and $D(V)$ be the Lie algebra of all formal vector fields on V (see § 1). The closed Lie subalgebras A of $D(V)$ are the formal objects corresponding to the Lie algebra sheaves on a manifold, and are what we are concerned with.

In § 2 we establish a formal version of the Frobenius' theorem with singularities which was obtained by T. Nagano [**12**] in the analytic category. By the theorem we can consider the (formal) orbit of A through the origin of V, and the restriction of A to the orbit, which we call the transitive part of A.

Then our problem is stated as follows: Determine all closed intransitive Lie subalgebras A of $D(V)$ whose transitive parts are infinite and primitive.

To carry out the problem, we assume that the ground field is the complex number field $\boldsymbol{C}$, and we impose on A the following reasonable regularity conditions:

(A_1) The orbit of A through the origin is a regular orbit.

(A_2) The associated graded Lie algebra $\mathfrak{gr}^k(A)$ of A satisfies

$$(R) \qquad [V, \mathfrak{gr}^k(A)] \subset \mathfrak{gr}^{k-1}(A) \quad \text{for all} \quad k\,.$$

The condition (R) is suggested by the work of Tanaka, loc. cit., where he proved that (R) is always satisfied if A comes from the Lie algebra sheaf $\mathcal{A}$ which satisfies the condition that $\dim \mathcal{A}_x/\mathcal{A}_x^k$ is independent of the point x, where $\mathcal{A}_x^k$ is the subalgebra of the stalk $\mathcal{A}_x$ consisting of those germs vanishing to order k.

The formal version of this fact is explained in § 4, after we introduce the prolongation of $D(V)$ in § 3.

Our main theorem is then stated as follows (see § 6):

THEOREM. *Let A be a closed Lie subalgebra of $D(V)$ satisfying (A_1) and (A_2). Assume that the transitive part L of A is infinite and primitive. Then A is isomorphic to a subalgebra of $L[W^*]$ containing $L'[W^*]$ for a certain vector space W.*

Here $L[W^*]$ denotes the intransitive extension of L by W (see § 5 for the definition). It is well-known that there are six classes of infinite primitive Lie algebras. Four of them are simple and the other two are not; each has the ideal of codimension 1. We have denoted by L' either L itself or the ideal of L, according as L is simple or not.

Thus the algebra A is completely determined up to W if the transitive part L is simple. Though A is not uniquely determined if L is not simple, it is not so essential in the sense that $L[W^*]/L'[W^*]$ is abelian and isomorphic to the ring $F(W)$ of formal function on W (see § 11).

In § 5 we introduce the notion of essential invariants due to E. Cartan [**4**]. If we reduce A by excluding the inessential invariants out of A, then W is determined by $V=U\oplus W$, and the isomorphism of A to a subalgebra of $L[W^*]$ is given by a formal (power series) isomorphism of V.

In § 7 we review briefly some known results for the primitive infinite Lie algebras, especially for irreducible ones.

In § 8 we study the contact Lie algebra and determine its structure thoroughly.

In § 9, 10 and 11, the proof of the theorem is carried out. First we determine the associated bi-graded Lie algebra $\sum \mathfrak{a}^{p,q}$ of A in § 9. Then starting from the associated bi-graded Lie algebra, we determine A in § 10 and 11. Since the most difficult is the case where L is contact, we devote ourselves mainly to this case and the proof for the case where L is irreducible is only outlined in § 11.

In this way the problem (II) is solved. For the other problems, especially for (I), it seems that further studies of intransitive Lie groups will be needed.

The author would like to express his deep gratitude to Professor N. Tanaka who first introduced him to infinite Lie groups and has encouraged him with kind advice.

§ 1. Formal functions, vector fields and mappings.

Let V be a finite dimensional vector space over a field k of characteristic 0. Let V^* be the dual space of V and $S^p(V^*)$ be the p-times symmetric tensor of V^*. The complete direct sum $\prod_{p=0}^{\infty} S^p(V^*)$ is denoted by $F(V)$ and elements of $F(V)$ are called *formal functions* on V. If we fix a basis $x^1, \cdots, x^n$ of V^*, formal functions are regarded as formal power series in n indeterminates $x^1, \cdots, x^n$. We endow a topology on $F(V)$ by assigning a filtration $\{F^p(V)\}$ as a fundamental system of neighbourhoods of the origin, where we set $F^p(V) = \prod_{k\geqq p} S^k(V^*)$.

By a *formal vector field* on V we mean a continuous derivation of $F(V)$. We denote by $D(F(V))$ or $D(V)$ the Lie algebra of all formal vector fields on V. Since any continuous derivation is uniquely determined by its value on V^*, $D(V)$ is identified with $V\otimes F(V)$. Let $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ be the basis of V dual to $x^1, \cdots, x^n$. Then each $X\in D(V)$ is written uniquely as

$$X=\sum f^i \frac{\partial}{\partial x^i},$$

where f^i is given by $f^i=X\cdot x^i$. Let $D^p(V)$ be the subspace of $D(V)$ consisting of formal vector fields X such that $X\cdot F(V)\subset F^{p+1}(V)$. Then $D^p(V)$ is identified with $V\otimes F^{p+1}(V)$ and it is easy to verify that

$$[D^p(V), D^q(V)]\subset D^{p+q}(V).$$

We also endow a topology on $D(V)$ by the filtration $\{D^p(V)\}$

Let A be a Lie subalgebra of $D(V)$. The filtration $\{D^p(V)\}$ of $D(V)$ induces a filtration $\{A^p\}$ of A, where we set $A^p=A\cap D^p(V)$. Putting $\mathfrak{gr}^p(A)=A^p/A^{p+1}$, we get the associated graded Lie algebra $\mathfrak{gr}(A)=\sum \mathfrak{gr}^p(A)$ of A. Since $\mathfrak{gr}^p(D(V))$ is canonically isomorphic to $V\otimes S^{p+1}(V^*)$, $\mathfrak{gr}^p(A)$ is identified with a subspace of $V\otimes S^{p+1}(V^*)$. In particular $\mathfrak{gr}^{-1}(A)$ is a subspace of V. A is called *transitive* if $\mathfrak{gr}^{-1}(A)=V$ and *intransitive* otherwise.

Let W be another vector space. By a *formal mapping* from V to W we mean a ring homomorphism $\varphi^*: F(W)\to F(V)$ with $\varphi^*(1)=1$. If φ^* is an isomorphism, φ is called a *formal isomorphism.*

Let φ_k be the $S^{k+1}(V^*)$-component of $\varphi^*|W^*$. φ_k is considered as an element of $W\otimes S^{k+1}(V^*)$. Since φ^* is uniquely determined by its value on W^*, the correspondence

$$\varphi \longrightarrow (\varphi_k)_{k\geq -1}, \qquad \varphi_k\in W\otimes S^{k+1}(V^*), \tag{1.1}$$

is bijective. Note that φ is an isomorphism if and only if φ_0 is.

Let $f^1, \cdots, f^s$ be a set of formal functions. It is called *independent* if the projection of $1, f^1, \cdots, f^s$ on $F(V)/F^2(V)$ is independent. In other words, if we denote by $f(0)$ the constant term of f and by d_0f the projection of $f-f(0)$ on V^*, $f^1, \cdots, f^s$ is independent if and only if $d_0f^1, \cdots, d_0f^s$ is linearly independent. If $n=\dim V$, an independent set of n formal functions $f^1, \cdots, f^n$ on V forms a *formal coordinate system* of V.

Let $f^1, \cdots, f^s$ and $g^1, \cdots, g^s$ be independent, then there exists a formal isomorphism φ of V such that $\varphi^*g^i=f^i$ for $i=1, \cdots, s$.

Let φ be a formal isomorphism of V to W. Then it induces the Lie algebra isomorphism

$$\varphi_*: \ D(V) \longrightarrow D(W)$$

defined by $\varphi_*X=(\varphi^*)^{-1}\circ X\circ\varphi^*$ for $X\in D(V)$.

Let φ be a formal isomorphism of V such that $\varphi^*F^1(V)\subset F^1(V)$. Then clearly we have $\varphi^*F^p(V)\subset F^p(V)$. From this we see that

$$\varphi_*D^p(V)\subset D^p(V) \qquad \text{for all } p.$$

Therefore φ_* induces the map

$$\varphi_*^\sharp: \ V\otimes S^{p+1}(V^*) \longrightarrow V\otimes S^{p+1}(V^*). \tag{1.2}$$

Let $\varphi=(\varphi_k)$, where $\varphi_k\in V\otimes S^{k+1}(V^*)$, with respect to the identification of (1.1), then $\varphi_{-1}=0$ and φ_0 is non-singular in this case. It is easily verified that the map (1.2) coincides with the following map:

$$\varphi_0\otimes\otimes^{p+1}\varphi_0^{-1}: \ V\otimes S^{p+1}(V^*) \longrightarrow V\otimes S^{p+1}(V^*).$$

Moreover we have

PROPOSITION 1.1. *Let φ be a formal isomorphism of V, and let $\varphi=(\varphi_k)_{k\geqq -1}$ with $\varphi_k\in V\otimes S^{k+1}(V^*)$. Assume that $\varphi_{-1}=0$, $\varphi_0=id$, and $\varphi_i=0$ for $0<i<p$. Then*

$$\varphi_* X_q = X_q - [\varphi_p, X_q] \qquad (\text{mod } D^{p+q+1}(V))$$

for any $X_q\in V\otimes S^{q+1}(V^*)$.

The proof is omitted.

Throughout this paper we confine ourselves to the formal category. So we shall hereafter often omit the adjective "formal".

§ 2. Formal version of Frobenius' theorem with singularities.

Let A be a Lie subalgebra of $D(V)$. A function f is called an *invariant* of A if $X\cdot f=0$ for all $X\in A$, and an ideal $\mathcal{J}$ of $F(V)$ is called *invariant* by A if $X\cdot\mathcal{J}\subset\mathcal{J}$ for all $X\in A$.

The following theorem is a formal version of the Frobenius' theorem with singularities (for the analytic case refer to [**12**]):

THEOREM 2.1. *Let A be a Lie subalgebra of $D(V)$, and s be the codimension of $\mathfrak{gr}^{-1}(A)$. Then there exist s independent functions $f^1,\cdots,f^s$ such that the closed ideal $(f^1,\cdots,f^s)$ generated by $f^1,\cdots,f^s$ is invariant by A.*

PROOF. Let A^F be the $F(V)$-module generated by A. Then A^F is a Lie subalgebra of $D(V)$ and satisfies $\mathfrak{gr}^{-1}(A^F)=\mathfrak{gr}^{-1}(A)$. Note that an ideal $\mathcal{J}$ is invariant by A^F if and only if it is invariant by A. Therefore we assume, without loss of generality, that $A^F=A$.

For the proof we need the following lemma:

LEMMA 2.1. *Let A be a Lie subalgebra of $D(V)$ satisfying $A^F=A$, with $\dim\mathfrak{gr}^{-1}(A)=r$. Then there exist r vectors $X_1,\cdots,X_r$ in A such that by a suitable coordinate system $x^1,\cdots,x^r,y^1,\cdots,y^s$ of V with $x^i,y^\alpha\in F^1(V)$ each X_i is expressed in the following form:*

$$X_i=\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{s}\xi_i^\alpha\frac{\partial}{\partial y^\alpha} \qquad \textit{for}\quad 1\leqq i\leqq r,$$

where each ξ_i^α belongs to the ideal $(y^1,\cdots,y^s)$.

Taking this lemma for granted, we can prove the theorem immediately: Let $X_1,\cdots,X_r$ and $x^1,\cdots,x^r,y^1,\cdots,y^s$, be as in the lemma. Then we assert that the ideal $(y^1,\cdots,y^s)$ is invariant by A.

Let U be the vector subspace of A spanned by $X_1,\cdots,X_r$ and B be the Lie subalgebra of A consisting of those $Y\in A$ of the form

$$Y=\sum_{\alpha=1}^{s}\eta^\alpha\frac{\partial}{\partial y^\alpha}$$

in the coordinate system $x^1,\cdots,x^r,y^1,\cdots,y^s$. Then it is obvious that $A=U^F+B$. Therefore it suffices to prove that $(y^1,\cdots,y^s)$ is invariant by B.

If $\sum_{\alpha=1}^{s}\eta^{\alpha}\frac{\partial}{\partial y^{\alpha}}\in B$, then we see easily that $\eta^{\alpha}(0)=0$ for $\alpha=1,\cdots,s$. It is also easy to see that

$$(2.1)\qquad \begin{cases}[U, U]\subset B\\ [B, U]\subset B.\end{cases}$$

Therefore, for $Y=\sum\eta^{\alpha}\frac{\partial}{\partial y^{\alpha}}\in B$, if we set

$$[X_k,\cdots[X_j,[X_i, Y]],\cdots]=\sum_{\alpha=1}^{s}\eta_{ij\cdots k}^{\alpha}\frac{\partial}{\partial y^{\alpha}}.$$

We have $\eta_{ij\cdots k}^{\alpha}(0)=0$, from which by a simple calculation we get

$$\frac{\partial\cdots\partial\eta^{\alpha}}{\partial x^{k}\cdots\partial x^{j}\partial x^{i}}(0)=0\qquad\left(\begin{matrix}1\leqq\alpha\leqq s\\ 1\leqq i, j,\cdots k\cdots\leqq r\end{matrix}\right).$$

This implies that $Y\cdot y$ belongs to the ideal $(y^1,\cdots,y^s)$. Thus the proof of the theorem is completed.

The proof of the lemma proceeds as follows: Taking account of the assumption that $A^F=A$, we can find $X_1,\cdots,X_r\in A$ and a coordinate system $u^1,\cdots,u^r, v^1,\cdots,v^s$, with $u^1(0)=\cdots=v^s(0)=0$ such that

$$(2.2)\qquad X_i=\frac{\partial}{\partial u^i}+\sum_{\alpha=1}^{s}\xi_i^{\alpha}\frac{\partial}{\partial v^{\alpha}},\qquad \xi_i^{\alpha}(0)=0.$$

We want to find functions φ^{α}, $\lambda_{i\beta}^{\alpha}$ ($1\leqq\alpha$, $\beta\leqq s$, $1\leqq i\leqq r$) satisfying the equations:

$$(2.3)\qquad \begin{cases}X_i f^{\alpha}=\sum_{\beta=1}^{s}\lambda_{i\beta}^{\alpha}f^{\beta}\\ f^{\alpha}=v^{\alpha}-\varphi^{\alpha}\end{cases}$$

plus the condition:

(2.4) $\qquad\varphi^{\alpha}$ is a power series in $u^1,\cdots,u^r$ and $\varphi^{\alpha}(0)=0$.

In order to obtain φ^{α}, $\lambda_{i\beta}^{\alpha}$, we expand them in power series of $u^1,\cdots,u^r$:

$$(2.5)\qquad \begin{cases}\varphi^{\alpha}=\varphi^{\alpha(1)}+\varphi^{\alpha(2)}+\cdots+\varphi^{\alpha(p)}+\cdots\\ \lambda_{i\beta}^{\alpha}=\lambda_{i\beta}^{\alpha(0)}+\cdots+\lambda_{i\beta}^{\alpha(p)}+\cdots,\end{cases}$$

and we are to determine inductively $\varphi^{\alpha(p)}$ and $\lambda_{i\beta}^{\alpha(p)}$ so as to satisfy (2.3) and (2.4). In each inductive step, we need certain compatibility conditions in order to find $\varphi^{\alpha(p)}$. It is in effect satisfied by the following fact: If we put

$$(2.6)\qquad [X_k,\cdots[X_j, X_i]\cdots]=\sum_{\alpha=1}^{s}\eta_{ij\cdots k}^{\alpha}\frac{\partial}{\partial v^{\alpha}},$$

then we have

$$(2.7)\qquad \eta_{ij\cdots k}^{\alpha}(0)=0.$$

In this way, though we omit the details, we can determine $\varphi^{\alpha(p)}$ and $\lambda_{i\beta}^{\alpha(p)}$ uniquely. Then each X_i is expressed in the desired form by the coordinate system $u^1, \cdots, u^r, f^1, \cdots, f^s$, which completes the proof of Lemma 2.1.

The ordinary Frobenius' theorem has the following formal version:

THEOREM 2.2. *Let $X_1, \cdots, X_r$ be independent vectors in $D(V)$ and satisfy*

$$[X_i, X_j] = \sum_{k=1}^{r} f_{ij}^k X_k \quad \textit{for some} \quad f_{ij}^k \in F(V).$$

Then there exist s functions $y^1, \cdots, y^s$ ($s = \dim V - r$) which are independent and are invariants of $X_1, \cdots, X_r$, i.e.,

$$X_i \cdot y^\alpha = 0 \quad \textit{for} \quad 1 \leqq i \leqq r, \quad 1 \leqq \alpha \leqq s.$$

The proof is also carried out quite formally.

§ 3. Prolongation of $D(V)$.

For a vector space V, we put $V^{(k)} = V \otimes \sum_{i=0}^{k} S^k(V^*)$ and consider the ring $F(V^{(k)})$ of the formal functions on $V^{(k)}$. Since $S^p(V)^*$ is identified with $S^p(V^*)$ by the natural pairing $\langle\ ,\ \rangle$, we identify $V^{(k)*}$ with $V^* \otimes \sum_{i=0}^{k} S^i(V)$. By this identification, there is a natural inclusion:

$$\iota_{lk}: \ F(V^{(k)}) \longrightarrow F(V^{(l)}) \qquad \text{for} \quad l \geqq k.$$

We set $F(V^{(\infty)}) = \varinjlim_k F(V^{(k)})$.

Let $\{x^1, \cdots, x^n\}$ be a basis of V^*, and $\{p_\alpha^i; i=1, \cdots, n, \alpha=(\alpha_1, \cdots, \alpha_n), |\alpha| \leqq k\}$ be a basis of $V^{(k)*}$ such that $\iota_{k0} x^i = p_0^i$, then each element of $F(V^{(k)})$ is regarded as a (formal) function of $\{p_\alpha^i\}$. For $f \in F(V)$, by the substitution:

$$\bar{x}^i = \sum_{|\alpha| \leqq k} \frac{1}{\alpha!} p_\alpha^i x^\alpha.$$

We get a function $j^k f$

$$j^k f = f(\bar{x}^1, \cdots, \bar{x}^n).$$

Then $j^k f$ is regarded as an element of $F(V \oplus V^{(k)})$. Thus we get a map $j^k: F(V) \rightarrow F(V \oplus V^{(k)})$. We note that $F(V \oplus V^{(k)})$ is identified with $F(V) \hat{\otimes} F(V^{(k)})$ ($F(V) \otimes F(V^{(k)})$ is endowed with the topology defined by the filtration $\{\Phi^p\}$, where $\Phi^p = \sum_{r+s=p} F^r(V) \otimes F^s(V^{(k)})$ and $\hat{\otimes}$ denotes the completion with respect to this topology). Since $F(V)$ is the dual space of $S(V)$ by the pairing $\langle\ ,\ \rangle$, there is defined the pairing $S(V) \times F(V + V^{(k)}) \rightarrow F(V^{(k)})$, which we also denote by $\langle\ ,\ \rangle$.

Now for $f \in F(V)$, we define the derivative Δf of f as an element of $\mathrm{Hom}(S(V), F(V^{(\infty)}))$ in the following way: For $\xi \in \sum_{i=0}^{k} S^i(V)$ and $l \geqq k$, we put

$$\Delta_\xi^l f = \langle \xi, j^l f \rangle,$$

then we see that

$$\iota_{l'l}\Delta_\xi^l f=\Delta_\xi^{l'} f \qquad \text{for} \quad l'\geqq l.$$

We define Δf to be the injective limit of $\Delta^l f$.

It being prepared, for $X\in D(V)$ the *k-th prolongation* $\bar{p}^k X$ is defined to be the derivation of $F(V^{(k)})$ i.e. the element of $D(V^{(k)})$ by the following condition:

$$\bar{p}^k X(\alpha\otimes\xi)=\Delta_\xi(X\cdot\alpha) \qquad \text{for} \quad \alpha\otimes\xi\in V^*\otimes\sum_{i=0}^{k}S^i(V). \tag{3.1}$$

Since any derivation of $F(V^{(k)})$ is uniquely determined by its value on $V^{(k)*}$, $\bar{p}^k X$ is uniquely determined by (3.1).

PROPOSITION 3.1. *The map*

$$\bar{p}^k:\ D(V)\longrightarrow D(V^{(k)})$$

is an injective Lie algebra homomorphism.

PROOF. This proposition can be verified by a direct calculation.

It is convenient for our purpose to modify $\bar{p}^k$ by a translation. Let θ be the formal isomorphism of $V^{(k)}$, defined by $\theta^*(p_\alpha^i)=p_\alpha^i-\delta_\alpha^i$, where

$$\delta_\alpha^i=\begin{cases}1 & \text{if} \quad \alpha=(0,\cdots \overset{i}{\overbrace{1}}\cdots 0)\\ 0 & \text{otherwise}.\end{cases}$$

We set $p^k=\theta_*\cdot\bar{p}^k$, then we have

PROPOSITION 3.2. *The map*

$$p^k:\ D(V)\longrightarrow D(V^{(k)})$$

is an injective Lie algebra homomorphism, and for $l\geqq 0$, $X\in D^{k+l}(V)$ *if and only if* $p^kX\in D^l(V^{(k)})$.

The proof is omitted, since it is obtained by a simple calculation.

§ 4. Regularity conditions.

Let A be an intransitive Lie subalgebra of $D(V)$. By the Frobenius' theorem, Theorem 3.1, we see that there exists an A-invariant ideal $(f^1,\cdots,f^s)$ of $F(V)$, such that $s=\operatorname{codim}\mathfrak{gr}^{-1}(A)$ and $f^1,\cdots,f^s$ is independent. Then the ideal $(f^1,\cdots,f^s)$ can be considered to define the *orbit* of A through the origin. Thus we may speak of the orbit of A through the origin, and we can consider the restriction of A to the orbit. More precisely, since $(f^1,\cdots,f^s)$ is invariant by A and $X\in A$ induces the derivation $\varphi(X)$ of $F(V)/(f^1,\cdots,f^s)$. Since $f^1,\cdots,f^s$ is independent, $F(V)/(f^1,\cdots,f^s)$ is identified with $F(U)$, where U is the subspace of V annihilated by $d_0f^1,\cdots,d_0f^s$. Let L be the image of $\varphi: A\rightarrow D(U)$. Then L is a transitive Lie subalgebra of $D(U)$, which we call the *transitive part* of A.

In general the orbit of A through the orign may be singular. We say that

A is *regular* if it is a regular orbit, that is, if A satisfies the following condition: There exist independent s-functions $f^1, \cdots, f^s \in F(U)$ such that $s=\operatorname{codim} \mathfrak{gr}^{-1}(A)$ and that $X \cdot f^i=0$ for all $X \in A$ and $i=1, \cdots, s$.

We say that A is *k-regular* if the orbit of the k-th prolongation $p^k A$ is regular.

We have the following proposition which was originally proved by N. Tanaka [**14**] in the category of Lie algebra sheaves.

PROPOSITION 4.1. *If a subalgebra A of $D(V)$ is k-regular, then*

$$[\mathfrak{gr}^k(A), V] \subset \mathfrak{gr}^{k-1}(A).$$

PROOF. First of all note that the statement is equivalent to the following statement

$$[A^k, D(V)] \subset A^{k-1}+D^k(V). \tag{4.1}$$

Let ρ be the codimension of $\mathfrak{gr}^{-1}(p^k A)$ in $V^{(k)}$. Since the orbit of $p^k A$ is regular, we can find independent ρ functions $h_1, \cdots, h_\rho$ in $F(V^{(k)})$ such that

$$Z \cdot h_i=0 \quad \text{for} \quad Z \in p^k A, \quad i=1, \cdots, \rho.$$

Let $X \in A^k$ and $Y \in D(V)$. Then we have

$$[p^k X, p^k Y] h_i=(p^k X)(p^k Y) h_i.$$

On the other hand since we see $p^k X \in D^0(V^{(k)})$ by Proposition 3.2, we have

$$[p^k X, p^k Y] h_i(0)=0.$$

Then there exists a $Z \in A$ such that

$$[p^k X, p^k Y]-p^k Z \in D^0(V^{(k)}).$$

Again by Proposition 3.2, we have

$$[X, Y]-Z \in D^k(V).$$

While it is obvious that $[X, Y] \in D^{k-1}(V)$. Hence we have $Z \in A^{k-1}$, and the proposition is proved.

COROLLARY. *If $A \subset D(V)$ is k-regular for all $k \geqq 0$, then the graded Lie algebra $\mathfrak{gr}(A)$ satisfies the following condition (R):*

$$[\mathfrak{gr}^p(A), V] \subset \mathfrak{gr}^{p-1}(A) \quad \textit{for all} \quad p \geqq 0. \tag{R}$$

REMARK. If A is transitive, the condition (R) is always satisfied.

Now assuming (R), we can introduce the associated bi-graded Lie algebra $\sum \mathfrak{a}^{p,q}$ of A.

Let $\sum \mathfrak{gr}^p(A)$ be the associated graded Lie algebra of A. Put $\mathfrak{gr}^{-1}(A)=U$. Then by condition (R), we have $\mathfrak{gr}^p(A) \subset U \otimes S^{p+1}(V^*)$ for all p. For any integer r we define a subspace $\Phi^r \mathfrak{gr}^p(A)$ by

$$\Phi^r \mathfrak{gr}^p(A) = \left\{ X \in \mathfrak{gr}^p(A) \,\middle|\, \begin{array}{l} [\cdots[X, u_1]\cdots, u_s] = 0, \\ \text{for all } u_1, \cdots, u_s \in U, \text{ with } s = p+2-r \end{array} \right\} \tag{4.2}$$

where we define that $\Phi^r \mathfrak{gr}^p(A) = 0$ for $r \geqq p+2$. We set $\Phi^r \mathfrak{gr}(A) = \sum_p \Phi^r \mathfrak{gr}^p(A)$, then we have

$$\begin{cases} \cdots \supset \Phi^r \mathfrak{gr}(A) \supset \Phi^{r+1} \mathfrak{gr}(A) \supset \cdots \\ [\Phi^r \mathfrak{gr}(A), \Phi^s \mathfrak{gr}(A)] \subset \Phi^{r+s} \mathfrak{gr}(A). \end{cases} \tag{4.3}$$

Thus we get a filtration $\{\Phi^r \mathfrak{gr}(A)\}$ of $\mathfrak{gr}(A)$. We put $\mathfrak{a}^{p,r} = \Phi^r \mathfrak{gr}^p(A) / \Phi^{r+1} \mathfrak{gr}^p(A)$, then we have

$$[\mathfrak{a}^{p,r}, \mathfrak{a}^{q,s}] \subset \mathfrak{a}^{p+q,r+s} \tag{4.4}$$

with respect to the induced Lie algebra structure, and we get the associated bi-graded Lie algebra $\sum \mathfrak{a}^{p,r}$.

If we choose a complementary subspace W to U, then $\mathfrak{a}^{p,r}$ is considered as $\mathfrak{a}^{p,r} \subset U \otimes S^{p-r+1}(U^*) \otimes S^r(W^*)$.

Let L be the transitive part of A. Then we see obviously that

$$\mathfrak{a}^{p,0} = \mathfrak{gr}^p(L), \tag{4.5}$$

where $\sum \mathfrak{gr}^p(L)$ is the associated graded Lie algebra of L.

§5. Essential invariants.

Let A be a Lie subalgebra of $D(V)$ which is regular and satisfies (R), then the orbit of A is defined by independent invariant functions, say $y^1, \cdots, y^s$. Let η be the subspace of V^* generated by $d_0 y^1, \cdots, d_0 y^s$ and put $U = \eta^\perp$. Then we have $\mathfrak{gr}^{-1}(A) = U$, moreover, in view of the condition (R), we see that

$$\mathfrak{gr}^p(A) \subset U \otimes S^{p+1}(V^*) \qquad \text{for all } p.$$

Let L be the transitive part of A, which is identified with a subalgebra of $D(U)$. We define $\mathfrak{n}^p \subset U \otimes S^{p+1}(V^*)$ by the following exact sequence:

$$0 \longrightarrow \mathfrak{n}^p \longrightarrow \mathfrak{gr}^p(A) \longrightarrow \mathfrak{gr}^p(L) \longrightarrow 0. \tag{5.1}$$

We put

$$V_0 = \{v \in V;\ [v, \mathfrak{n}^0] = 0\}, \tag{5.2}$$

then clearly we have $V_0 \supset U$. Let $\eta_0 = V_0^\perp$. Then we have $\mathfrak{n}^0 \subset U \otimes \eta_0$. From this, making use of (R), we have

$$\mathfrak{n}^p \subset U \otimes S^{p+1}(\eta_0).$$

DEFINITION. If $V_0 = U$, we say that A is *effective* in $D(V)$ and that the invariants $y^1, \cdots, y^s$ are *essential*.

We can always exclude from A inessential invariants in the following way:

PROPOSITION 5.1. *Let A be a regular Lie subalgebra of $D(V)$ satisfying (R). Then there exist independent invariant functions $f^1, \cdots, f^r$ $(r\geqq 0)$ such that the map*

$$\varphi: \quad A \longrightarrow D(F'), \qquad \textit{where} \quad F'=F/(f^1, \cdots, f^r)$$

is injective and the image A' is regular and satisfies (R) and effective in $D(F')$.

PROOF. Let $y^1, \cdots, y^s$ be independent invariants of A with $s=\dim \mathfrak{gr}^{-1}(A)$. Assume that $U\subsetneqq V_0$. Let η_1 be a complementary subspace to η_0 in η. By a suitable linear change, we may assume that η_1 is generated by $d_0y^1, \cdots, d_0y^r$. Let $F'=F(V)/(y^1, \cdots, y^r)$, then it is easy to see that $\varphi: A\rightarrow D(F')$ satisfies the statements of the proposition. q. e. d.

Here we give a typical example of intransitive Lie algebras: Let L be a closed transitive Lie subalgebra of $D(U)$ and W be a finite dimensional vector space. Then the topological completion of the Lie algebras $L\otimes F(W)$ is also endowed with the Lie algebra structure, which we denote by $L[W^*]$ and call the *intransitive extension* of L by W^*.

Let $V=U\oplus W$, then there is a natural imbedding of $L[W^*]$ into $D(V)$. It is uniquely determined by the following condition: $(X\otimes f)(\alpha)=fX(\alpha)$ and $(X\otimes f)(\beta)=0$ for $X\in L$, $f\in F(W)$, $\alpha\in U^*$, $\beta\in W^*$, where U^*, W^* and $F(W)$ are regarded as the subspaces of $F(V)$ in the natural manner. By the above imbedding we always regard $L[W^*]$ as the closed subalgebra of $D(V)$. In particular, $D(U)[W^*]$ is identified with the closed subalgebra $D(V;U)$ of $D(V)$ consisting of those vector fields X such that $X\beta=0$ for all $\beta\in W^*$.

Evidently, the transitive part of $L[W^*]$ is L, and $L[W^*]$ is effective in $D(V)$.

§ 6. Statement of the main theorem.

Now our main problem can be stated as follows.

Problem: Determine all closed Lie subalgebras A of $D(V)$ whose transitive parts are infinite and primitive.

As explained in introduction, this problem is deeply related to the classification of simple intransitive infinite Lie algebras.

It is reasonable to impose on A some regularity conditions. In view of § 4, we assume the following conditions (A_1), (A_2).

(A_1) A is regular, i.e., the orbit of A through the origin is regular,

(A_2) the associated graded Lie algebra $\mathfrak{gr}(A)$ of A satisfies the condition (R).

In this paper we assume:

(A_0) the ground field is the complex number field $\boldsymbol{C}$.

Under these assumptions we shall solve the problem in the following way.

MAIN THEOREM. *Let A be a closed Lie subalgebra of* $D(V)$ *satisfying* (A_0), (A_1) *and* (A_2). *Assume that its transitive part L is infinite and primitive. Then A is isomorphic to a subalgebra of* $L[W^*]$ *containing* $L'[W^*]$ *for a certain vector space W.*

In the statement L' denotes either L itself or the ideal of codimension 1 according as L is simple or not (cf. § 7).

Thus the theorem completely determines A up to the dimension of W^* in the case where L is simple.

If L is not simple, then there is some ambiguity. However, it is not essential in the sense that $L[W^*]/L'[W^*]$ is abelian and isomorphic to $F(W)$.

In view of Proposition 5.1, we may further assume the following condition:

(A_3) A is effective in $D(V)$.

Under this assumption $\dim W^*$ is uniquely determined, and the isomorphism is given by a formal isomorphism of V (Theorem 10.2 and 11.2).

The proof of the main theorem will be carried out in the subsequent sections.

§ 7. Primitive infinite Lie algebras.

It is a well known fact that there are only six classes of infinite primitive Lie algebras over $\boldsymbol{C}$, and any primitive infinite subalgebra L of $D(U)$ is isomorphic to one of the following Lie algebras:

(I) $L_{\mathfrak{gl}}(U)=D(U)$,

(II) $L_{\mathfrak{sl}}(U)$: the Lie subalgebra of $D(U)$ consisting of all vector fields of divergence zero,

(II′) $L_{\mathfrak{csl}}(U)$: the Lie subalgebra of $D(U)$ consisting of all vector fields of constant divergence,

(III) $L_{\mathfrak{sp}}(U)$: the Lie subalgebra of $D(U)$ consisting of all vector fields which leave invariant a hamiltonian form,

(III′) $L_{\mathfrak{csp}}(U)$: the Lie subalgebra of $D(U)$ consisting of all vector fields which leave invariant a hamiltonian form up to constant factors,

(IV) $L_{\mathfrak{ct}}$: the Lie subalgebra of $D(U)$ consisting of vector fields which leave invariant a contact structure.

In the list above, (I), (II), (III) and (IV) are simple, (II′) and (III′) are not simple: $L_{\mathfrak{sl}}$ and $L_{\mathfrak{sp}}$ are ideals of $L_{\mathfrak{csl}}$ and $L_{\mathfrak{csp}}$ respectively of codimension 1. All except (IV) are irreducible.

The structures of irreducible infinite Lie algebras are well known. We shall explain some facts which will be necessary for our purpose.

Let L be one of irreducible infinite Lie algebras, (I), (II), (II′), (III) and (III′), and $\mathfrak{g}(L)=\sum\mathfrak{g}^p(L)$ be its associated graded Lie algebra. Then L is isomorphic to the complete graded Lie algebra $\prod\mathfrak{g}^p(L)$. The structures of the graded Lie algebras $\mathfrak{g}(L)$ are as follows:

(I) $\mathfrak{g}(L_{\mathfrak{l}}(U))=U+\mathfrak{gl}(U)+\mathfrak{gl}(U)^{(1)}+\cdots+\mathfrak{gl}(U)^{(p)}+\cdots,$

(II) $\mathfrak{g}(L_{\mathfrak{sl}}(U))=U+\mathfrak{sl}(U)+\mathfrak{sl}(U)^{(1)}+\cdots+\mathfrak{sl}(U)^{(p)}+\cdots,$

(II′) $\mathfrak{g}(L_{\mathfrak{csl}}(U))=U+\mathfrak{gl}(U)+\mathfrak{sl}(U)^{(1)}+\cdots+\mathfrak{sl}(U)^{(p)}+\cdots,$

(III) $\mathfrak{g}(L_{\mathfrak{s}}(U))=U+\mathfrak{sp}(U)+\mathfrak{sp}(U)^{(1)}+\cdots+\mathfrak{sp}(U)^{(p)}+\cdots,$

(III′) $\mathfrak{g}(L_{\mathfrak{csp}}(U))=U+\mathfrak{csp}(U)+\mathfrak{sp}(U)^{(1)}+\cdots+\mathfrak{csp}(U)^{(p)}+\cdots,$

where $\mathfrak{sl}(U)$, $\mathfrak{sp}(U)$ and $\mathfrak{csp}(U)$ are the subalgebras of $\mathfrak{gl}(U)$ in the usual notation, and $\mathfrak{sl}(U)^{(p)}$ (resp. $\mathfrak{sp}(U)^{(p)}$) denotes the p-th prolongation of $\mathfrak{sl}(U)$ (resp. $\mathfrak{sp}(U)$).

Thus the linear isotropy algebra $\mathfrak{g}^0(L)$ is either simple, or a direct sum of the simple ideal and the 1-dimensional center.

By the representation theory it is known that the representation of $\mathfrak{sl}(U)$ on $\mathfrak{sl}(U)^{(p)}$ is irreducible, from which it is easily seen that

$$[U, \mathfrak{sl}(U)^{(p)}]=\mathfrak{sl}(U)^{(p-1)} \qquad \text{for } p\geqq 1.$$

The same statement holds also for $\mathfrak{sp}(U)$. Hence we see that $\sum\mathfrak{g}^p(L)$ is generated by $\mathfrak{g}^{-1}(L)$, $\mathfrak{g}^0(L)$ and $\mathfrak{g}^1(L)$.

Finally we mention some facts about the Spencer cohomology group $\sum H^{p,q}(\mathfrak{g}(L))$ of $\mathfrak{g}(L)$. For the definition refer e.g., to [**5**].

1) $H^{p,q}(\mathfrak{g}(L))=0$ for $p\geqq 1$ and $q\geqq 0$, except that $H^{1,1}(\mathfrak{g}(L_{\mathfrak{csl}}(U))\cong \boldsymbol{C}$.

2) Let $c\in H^{0,2}(\mathfrak{g}(L))$. If $X\cdot c=0$ for any elements X of the simple part of $\mathfrak{g}^0(L)$, then $c=0$. (Refer to [8].)

§ 8. Contact Lie algebras.

Let V be a $(2n+1)$-dimensional vector space and $z, x^1, \cdots, x^n, y^1, \cdots, y^n$ be a basis of the dual space V^*, which will be fixed throughout this section.

The contact Lie algebra $\mathfrak{c}$ (or more precisely $\mathfrak{c}(V)$) on V is by definition the Lie algebra consisting of those formal vector fields X on V which preserve the contact form, $\omega=dz+\sum_{i=1}^{n}x^i dy^i-y^i dx^i$, up to functional factors, i.e., $L_X\omega=f\omega$ for some $f\in F(V)$.

As a subalgebra of $D(V)$, $\mathfrak{c}$ has a usual filtration $\Phi^p\mathfrak{c}$, where $\Phi^p\mathfrak{c}=\mathfrak{c}\cap D^p(V)$. Putting $\mathfrak{g}^p(\mathfrak{c})=\Phi^p\mathfrak{c}/\Phi^{p+1}\mathfrak{c}$, we have the usual associated graded Lie algebra, $\mathfrak{g}(\mathfrak{c})=\sum\mathfrak{g}^p(\mathfrak{c})$, of $\mathfrak{c}$. Note that $\mathfrak{c}$ is not flat, that is, $\mathfrak{c}$ is not isomorphic to the complete graded Lie algebra $\prod\mathfrak{g}^p(\mathfrak{c})$, since the 1st structure function of $\mathfrak{c}$ does not vanish. (For the definition of structure function refer to [**12**].)

To clarify the structure of $\mathfrak{c}$, first of all we observe that there is a bijective correspondence between $\mathfrak{c}(V)$ and $F(V)$. It is given by the map assigning $X\in\mathfrak{c}$, to $X\lrcorner\omega\in F(V)$. Then the inverse map; $\xi: F(V)\to\mathfrak{c}$ is given by

$$(8.1)\qquad \begin{cases} \xi(f)\lrcorner\omega=f \\ \xi(f)\lrcorner d\omega=\dfrac{\partial f}{\partial z}\omega-df\,, \end{cases}$$

or explicitly by

$$(8.2)\qquad \xi(f)=\sum_i -\frac{1}{2}\Big(\frac{\partial f}{\partial y^i}-x^i\frac{\partial f}{\partial z}\Big)\frac{\partial}{\partial x^i}+\sum_i\frac{1}{2}\Big(\frac{\partial f}{\partial x^i}+y^i\frac{\partial f}{\partial z}\Big)\frac{\partial}{\partial y^i}$$
$$+\Big(f-\frac{1}{2}\sum_j x^j\frac{\partial f}{\partial x^j}+y^j\frac{\partial f}{\partial y^j}\Big)\frac{\partial}{\partial z}\,.$$

The generalized Poisson bracket is defined by $[f, g]=\xi^{-1}[\xi(f), \xi(g)]$ for $f, g\in F(V)$. From (8.2) we have the following classical formula:

$$(8.3)\qquad [f, g]=\frac{1}{2}\{f, g\}+\Big(f-\frac{1}{2}If\Big)\frac{\partial g}{\partial z}-\Big(g-\frac{1}{2}Ig\Big)\frac{\partial f}{\partial z}\,,$$

where

$$\{f, g\}=\sum_i\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i}-\frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}\,,$$

and

$$I=\sum_i x^i\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial y^i}\,.$$

Using the letters $u^1, \cdots, u^{2n}$ instead of $x^1, \cdots, x^n, y^1, \cdots, y^n$, we denote by u^α the monomial $(u^1)^{\alpha_1}\cdots(u^{2n})^{\alpha_{2n}}$ of degree $|\alpha|$, where $\alpha=(\alpha_1, \cdots, \alpha_{2n})$ and $|\alpha|=\sum\alpha_i$. The following formula is a direct consequence of (8.3), but is important for examining the structure of the contact Lie algebra:

$$(8.4)\qquad [z^r u^\alpha, z^s u^\beta]=\frac{1}{2}z^{r+s}\{u^\alpha, u^\beta\}$$
$$+\Big(-\frac{1}{2}|\alpha|s+\frac{1}{2}|\beta|r+s-r\Big)z^{r+s-1}u^{\alpha+\beta}\,.$$

In particular we have

$$(8.5)\qquad [1, z^s u^\beta]=s\cdot z^{s-1}u^\beta\,.$$

Let us introduce the order of f with respect to the contact structure. For a monomial $z^r u^\alpha$ it is defined as $ord\,(z^r u^\alpha)=2r+|\alpha|-2$, and for any function f, $ord\,(f)$ is defined to be the minimum of orders of non-zero monomials of f.

We define the subspace $\mathfrak{c}^p$ of $\mathfrak{c}$ by

$$\mathfrak{c}^p=\{\xi(f)\mid f \text{ is homogeneous of order } p\}\,.$$

Then we have

PROPOSITION 8.1. *The complete direct sum decomposition* $\mathfrak{c}=\prod_p\mathfrak{c}^p$ *satisfies:*

i) $\dim \mathfrak{c}^p < \infty$ *for all* p, *and* $\mathfrak{c}^p = (0)$ *for* $p < -2$.

ii) $[\mathfrak{c}^p, \mathfrak{c}^q] \subset \mathfrak{c}^{p+q}$ *for all* p, q.

iii) *For* $p \geqq -1$ *and* $X_p \in \mathfrak{c}^p$, *the condition that* $[X_p, X_{-1}] = 0$ *for all* $X_{-1} \in \mathfrak{c}^{-1}$ *implies* $X_p = 0$.

PROOF. i) is clear. ii) and iii) follow from (8.4).

Now we make a more refined decomposition of $\mathfrak{c}^p$. We define the subspace $\mathfrak{k}_p^r$ of $\mathfrak{c}^{p+2r}$ to be the linear hull of $\{\xi(z^r u^\alpha) \mid |\alpha| = p+2\}$. Then we have $\mathfrak{c}^p = \sum_{q+2r=p} \mathfrak{k}_q^r$ and $\mathfrak{k}_q^r = (0)$ for $q < -2$ or $r < 0$, and in particular $\mathfrak{c}^{-2} = \mathfrak{k}_{-2}^0$, $\mathfrak{c}^{-1} = \mathfrak{k}_{-1}^0$. Note that $\frac{\partial}{\partial z}$ $(=\xi(1))$ is a basis of $\mathfrak{c}^{-2}$. By Proposition 8.1, ii) and iii), the skew-symmetric 2-form θ on $\mathfrak{c}^{-1}$ defined by

$$[u_{-1}, v_{-1}] = \theta(u_{-1}, v_{-1}) \frac{\partial}{\partial z}, \qquad u_{-1}, v_{-1} \in \mathfrak{c}^{-1},$$

is non-degenerate. From (8.5) we see that

$$\left[X_p, \frac{\partial}{\partial z}\right] = 0 \qquad \text{for} \quad X_p \in \mathfrak{k}_p^0.$$

Hence we have

$$[[X_p, u_{-1}], v_{-1}] = [u_{-1}, [X_p, v_{-1}]] = 0, \tag{8.6}$$

for $X_p \in \mathfrak{k}_p^0$ and $u_{-1}, v_{-1} \in \mathfrak{k}_{-1}^0$. Noting that $[\mathfrak{k}_p^0, \mathfrak{c}^{-1}] \subset \mathfrak{k}_{p-1}^0$, we define the map $\rho : \mathfrak{k}_p^0 \to \mathrm{Hom}(\mathfrak{c}^{-1}, \mathfrak{k}_{p-1}^0)$ by $\rho(X_p) \cdot u = [X_p, u]$ for $X_p \in \mathfrak{k}_p^0$, $u \in \mathfrak{c}^{-1}$. Then ρ is injective by Proposition 8.1, iii). (8.6) implies that $\rho(\mathfrak{k}_p^0) = \mathfrak{sp}(\mathfrak{c}^{-1})^{(p)}$ for $p \geqq 0$, where $\mathfrak{sp}(\mathfrak{c}^{-1})$ is the Lie algebra consisting of all endomorphisms of $\mathfrak{c}^{-1}$ which leave invariant θ, and $\mathfrak{sp}(\mathfrak{c}^{-1})^{(p)}$ is the p-th prolongation of $\mathfrak{sp}(\mathfrak{c}^{-1})$. Moreover we see that

$$\rho([X_p, X_q]) = [\rho(X_p), \rho(X_q)] \tag{8.7}$$

for $X_p \in \mathfrak{k}_p^0$, $X_q \in \mathfrak{k}_q^0$, $p, q \geqq -1$ except $p = q = -1$, where the bracket operation on the right hand side of (8.7) is that of the graded Lie algebra $\sum \mathfrak{sp}(\mathfrak{c}^{-1})^{(p)}$.

From (8.4) we see also that $[\mathfrak{k}_0^0, \mathfrak{k}_p^r] \subset \mathfrak{k}_p^r$ i.e., $\mathfrak{k}_p^r$ is $\mathfrak{k}_0^0$-invariant and that the map $ad\left(\frac{\partial}{\partial z}\right) : \mathfrak{k}_p^{r+1} \to \mathfrak{k}_p^r$ $(r \geqq 0)$ is a $\mathfrak{k}_0^0$-equivariant bijection. Thus we have

PROPOSITION 8.2. *The contact Lie algebra* $\mathfrak{c}$ *is canonically decomposed as* $\mathfrak{c} = \prod \mathfrak{c}^p$ *and* $\mathfrak{c}^p = \sum_{q+2r=p} \mathfrak{k}_q^r$ *and satisfies*

i) $\mathfrak{k}_q^r = 0$ *for* $r < 0$ *or* $q < -2$.

ii) $\dim \mathfrak{k}_{-2}^0 = 1$, $\dim \mathfrak{k}_{-1}^0 = 2n$, *and* $\mathfrak{k}_p^0 \cong \mathfrak{sp}(\mathfrak{c}^{-1})^{(p)}$ *for* $p \geqq 0$.

iii) $\mathfrak{k}_p^r \cong \mathfrak{k}_0^0$ *for* $r \geqq 0$.

iv) $\mathfrak{k}_p^r$ *is* $\mathfrak{k}_0^0$*-invariant and irreducible.*

PROOF. iv) follows from the fact that representation of $\mathfrak{sp}(\mathfrak{c}^{-1})$ on $\mathfrak{sp}(\mathfrak{c}^{-1})^{(p)}$ is irreducible (cf. [8] or [**15**]).

PROPOSITION 8.3. *The bracket operation of the contact Lie algebra satisfies*

i) $[\mathfrak{k}_p^r, \mathfrak{k}_q^s] \subset \mathfrak{k}_{p+q}^{r+s} + \mathfrak{k}_{p+q+2}^{r+s-1}$

and moreover,

ii) $[\mathfrak{k}_p^r, \mathfrak{k}_q^s] = \mathfrak{k}_{p+q}^{r+s} + (rq-ps)\mathfrak{k}_{p+q+2}^{r+s-1}$ *for* $r, s \geqq 0$ *and* $p, q \geqq -1$,

iii) $[\mathfrak{k}_{-2}^r, \mathfrak{k}_q^s] = (rq+2s)\mathfrak{k}_q^{r+s-1}$ *for* $r, s \geqq 0$ *and* $q \geqq -2$.

PROOF. i) follows from (8.4). ii) and iii) follows from (8.4) and the irreducibility of $\mathfrak{k}_p^r$.

PROPOSITION 8.4.

i) $\sum \mathfrak{c}^p$ *is generated by* $\mathfrak{k}_{-1}^0, \mathfrak{k}_{-1}^1$ *and* $\mathfrak{k}_1$ *as Lie algebra.*

ii) $\sum_{p \geqq 0} \mathfrak{c}^p$ *is generated by* $\mathfrak{k}_0^0, \mathfrak{k}_{-1}^1, \mathfrak{k}_{-2}^1$ *and* $\mathfrak{k}_1$ *as Lie algebra.*

Proposition 8.4 follows easily from Proposition 8.3.

For simplicity, hereafter $\mathfrak{k}_p^0$ is denoted by $\mathfrak{k}_p$.

Let us return to the usual associated graded Lie algebra of $\mathfrak{c}$. Put

$$G^p(\mathfrak{c}) = \mathfrak{k}_p + \mathfrak{k}_{p-1}^1 + \cdots + \mathfrak{k}_{-1}^{p+1} + \mathfrak{k}_{-2}^{p+1},$$

then we see from (8.2) that $\Phi^l \mathfrak{c} = \prod_{p \geqq l} G^p(\mathfrak{c})$ and that $G^p(\mathfrak{c}) \cong \mathfrak{g}^p(\mathfrak{c})$. If we denote by $\bar{\mathfrak{k}}_p^r$ the image of $\mathfrak{k}_p^r$ by the projection $\Phi^p \mathfrak{c} \to \mathfrak{g}^p(\mathfrak{c})$, then we have

PROPOSITION 8.5.

$$\mathfrak{g}^p(\mathfrak{c}) = \bar{\mathfrak{k}}_p + \bar{\mathfrak{k}}_{p-1}^1 + \cdots + \bar{\mathfrak{k}}_{-1}^{p+1} + \bar{\mathfrak{k}}_{-2}^{p+1}.$$

Combining Proposition 8.3 and 8.5, we have

PROPOSITION 8.6.

i) $[\mathfrak{g}^p(\mathfrak{c}), \mathfrak{g}^{-1}(\mathfrak{c})] = \mathfrak{g}^{p-1}(\mathfrak{c})$ *for* $p \geqq 0$.

ii) $[\mathfrak{g}^p(\mathfrak{c}), \bar{\mathfrak{k}}_{-1}] = \check{\mathfrak{g}}^{p-1}(\mathfrak{c})$ *for* $p \geqq 0$.

where $\check{\mathfrak{g}}^p(\mathfrak{c}) = \bar{\mathfrak{k}}_p + \cdots + \bar{\mathfrak{k}}_{-1}^{p+1}$.

PROPOSITION 8.7. *The Spencer cohomology group* $H^{p,q}(\mathfrak{g}(\mathfrak{c}))$ *of the associated graded Lie algebra* $\mathfrak{g}(\mathfrak{c})$ *of* $\mathfrak{c}$ *vanishes for* $p \geqq 1$ *and* $q \geqq 0$.

PROOF. By a calculation we see that the p-th prolongation $\mathfrak{g}^0(\mathfrak{c})^{(p)}$ coincides with $\mathfrak{g}^p(\mathfrak{c})$ and that $\mathfrak{g}^0(\mathfrak{c})$ is involutive as a linear subspace of $\mathrm{Hom}(V, V)$. From this the proposition follows (see [5]).

§ 9. Determination of the associated bi-graded Lie algebra.

Let A be a closed Lie subalgebra of $D(V)$ satisfying (A_1), (A_2) and (A_3). In this section we shall determine the associated bi-graded Lie algebra of A under the assumption that its transitive part L is infinite and primitive.

In view of the assumption (A_1), if necessary, by transforming A by a formal isomorphism, we may assume that A is a closed Lie subalgebra of $D(V; U)$ for a subspace U of V, and that L is a transitive infinite primitive subalgebra of $D(U)$.

Let $\sum \mathfrak{a}^p$ be the associated graded Lie algebra of A. Then $\sum \mathfrak{a}^p$ is considered as a subspace of $U \otimes S^{p+1}(V^*)$. Let $\sum \mathfrak{a}^{p,q}$ be the associated bi-graded

Lie algebra of A. Choosing a subspace W of V complementary to U, we identify $\mathfrak{a}^{p,q}$ with a subspace of $U\otimes S^{p+1-q}(U^*)\otimes S^q(W^*)$. Let $\sum\mathfrak{g}^p(L)$ be the associated graded Lie algebra of L, where $\mathfrak{g}^p(L)$ is considered as a subspace of $U\otimes S^{p+1}(U^*)$. Then by our assumption we have

$$\mathfrak{a}^{p,0}=\mathfrak{g}^p(L) \qquad \text{for all } p. \tag{9.1}$$

The first non-trivial term of $\sum\mathfrak{a}^{p,q}$ is $\mathfrak{a}^{0,1}$. By definition we have the following exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U\otimes W^* & \longrightarrow & U\otimes U^*+U\otimes W^* & \longrightarrow & U\otimes U^* & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathfrak{a}^{0,1} & \longrightarrow & \mathfrak{a}^0 & \longrightarrow & \mathfrak{g}^0 & \longrightarrow & 0\,. \end{array}$$

Since $\mathfrak{a}^{0,1}$ is an abelian ideal of $\mathfrak{a}^0$, the representation of $\mathfrak{g}^0$ on $\mathfrak{a}^{0,1}$ is induced by the adjoint representation of $\mathfrak{a}^0$. It is nothing but the restriction to $\mathfrak{a}^{0,1}$ of the natural representation of $\mathfrak{g}^0$ on $U\otimes W^*$, where the representation of $\mathfrak{g}^0$ on W^* is considered to be trivial.

The space $\mathfrak{a}^{0,1}$ is characterized by the following properties:

$$\text{(9.2)}\qquad \begin{array}{ll} \text{i)} & \mathfrak{a}^{0,1} \text{ is a } \mathfrak{g}^0\text{-invariant subspace of } U\otimes W^*, \\ \text{ii)} & \delta(w)\mathfrak{a}^{0,1}\neq(0) \text{ for any non zero } w\in W, \end{array}$$

where $\delta(w)$ denotes the contraction by w.

The property ii) follows from (A_3).

To determine $\mathfrak{a}^{0,1}$, we prepare the following lemma:

LEMMA 9.1. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\boldsymbol{C}$, and $\rho:\mathfrak{g}\rightarrow\mathfrak{gl}(E)$ be a representation of $\mathfrak{g}$ on a finite dimensional vector space E, and assume that none of the $\mathfrak{g}$-irreducible components of E are isomorphic to another. Given a subspace P of $E\otimes W^*$, where W is a finite dimensional vector space. If P is $\mathfrak{g}$-invariant and satisfies $\delta(w)P=E$, for all non-zero $w\in W$, where $\delta(w)$ denotes the contraction by w. Then $P=E\otimes W^*$.*

PROOF. First we shall prove this lemma under the assumption that ρ is irreducible. Since $\mathfrak{g}$ is simple, we can decompose P to $\mathfrak{g}$-irreducible components: $P=P_1+\cdots+P_r$. For a fixed i $(1\leqq i\leqq r)$ there exists a $w\in W$ such that $\delta(w)P_i\neq 0$. Then $\delta(w)$ gives an isomorphism of P_i onto E, for $\delta(w)$ is a $\mathfrak{g}$-homomorphism and P_i and E are $\mathfrak{g}$-simple. Suppose that $\delta(w')P_i\neq 0$ for another $w'\in W$. Then $\delta(w)\circ\delta(w')^{-1}$ is a $\mathfrak{g}$-automorphism of E. Since E is $\mathfrak{g}$-simple, in view of Schur's Lemma we see that $\delta(w)\circ\delta(w')^{-1}=\lambda id_E$ for some complex number λ. Hence we have $\delta(w-\lambda w')P_i=0$. Therefore, if we put $N(P_i)=\{w\in W\,|\,\delta(w)P_i=0\}$, we see that $N(P_i)$ is a subspace of W of condimension 1. We can thus find non-zero $\alpha_i\in W^*$ such that $\alpha_i(N(P_i))=0$. Now it is easy to see that $P_i=E\otimes\{\alpha_i\}$. By the assumption that $\delta(w)P\neq 0$ for any non-zero $w\in W$, we see that $\alpha_1,\cdots,\alpha_r$ is a basis of W^* and we have $P=E\otimes W^*$.

Next we shall prove without the assumption that ρ is irreducible. Let $E=E_1+\cdots+E_s$ be a $\mathfrak{g}$-irreducible decomposition of E. By our assumption E_i is not isomorphic to E_j as a $\mathfrak{g}$-module if $i\neq j$. Let $P=P_1+\cdots+P_r$ be a $\mathfrak{g}$-irreducible decomposition of P. Since there exists a $w_i\in W$ such that $\delta(w_i)P_i\neq 0$, P_i is isomorphic to only one of $\{E_j\}$. Let $P_1^{(j)},\cdots,P_{r_j}^{(j)}$ be the set of those P_i that are isomorphic to E_j and put $P^{(j)}=P_1^{(j)}+\cdots+P_{r_j}^{(j)}$. Then we see that $P^{(j)}\subset E_j\otimes W^*$ and that $\delta(w)P^{(j)}=E_j$ for all non-zero $w\in W$. Hence we have $P^{(j)}=E_j\otimes W^*$ by our previous argument. Therefore we have $P=E\otimes W^*$.

q. e. d.

In the case where L is irreducible, the simple part of $\mathfrak{g}^0(L)$ being transitive on U, we can apply Lemma 9.1 to this case and obtain the following

PROPOSITION 9.1. *If the transitive part L of A is irreducible, then*

$$\mathfrak{a}^{0,1}=U\otimes W^*. \tag{9.3}$$

From this proposition we can determine the bi-graded Lie algebra $\sum\mathfrak{a}^{p,q}$ in the case L is irreducible.

PROPOSITION 9.2. *Assume that L is irreducible. Then the associated bi-graded Lie algebra $\sum\mathfrak{a}^{p,q}$ of A is determined by*

i) *For $p\neq q$, $\mathfrak{a}^{p,q}=\mathfrak{g}^{p-q}(L)\otimes S^q(W^*)$.*

ii) *For $p=q$, $\mathfrak{g}^0(L')\otimes S^p(W^*)\subset\mathfrak{a}^{p,p}\subset\mathfrak{g}^0(L)\otimes S^p(W^*)$.*

PROOF. By virtue of our assumption (A_2). We see easily that

$$\mathfrak{a}^{p,q}\subset\mathfrak{g}^{p-q}(L)\otimes S^q(W^*).$$

Generally it holds that $[\mathfrak{a}^{p,r},\mathfrak{a}^{q,s}]\subset\mathfrak{a}^{p+q,r+s}$. Since $\mathfrak{a}^{0,1}=\mathfrak{a}^{-1}(L)\otimes W^*$ by Proposition 9.1, in particular we have

$$[\mathfrak{a}^{p,r},\ \mathfrak{g}^{-1}(L)\otimes W^*]\subset\mathfrak{a}^{p-1,r+1}. \tag{9.4}$$

As we explained in § 7, the graded Lie algebra $\sum\mathfrak{g}^p(L)$ satisfies

$$[\mathfrak{g}^p(L),\ \mathfrak{g}^{-1}(L)]=\mathfrak{g}^{p-1}(L)\qquad\text{for}\quad p\neq 1$$

and

$$[\mathfrak{g}^1(L),\ \mathfrak{g}^{-1}(L)]=\mathfrak{g}^0(L').$$

Applying (9.4) repeatedly to (9.1) we see that

$$\mathfrak{a}^{p,r}=\mathfrak{g}^{p-r}(L)\otimes S^r(W^*)\qquad\text{for}\quad p\neq r$$

and

$$\mathfrak{g}^0(L')\otimes S^p(W^*)\subset\mathfrak{a}^{p,p}\subset\mathfrak{g}^0(L)\otimes S^p(W^*).$$

q. e. d.

REMARK. Since $\mathfrak{g}^0(L)=\mathfrak{g}^0(L')+\{I\}$, $\mathfrak{a}^{p,p}$ is represented in the following form:

$$\mathfrak{a}^{p,p}=\mathfrak{g}^0(L')\otimes S^p(W^*)+\{I\}\otimes\mathfrak{b}^p$$

where $\mathfrak{b}^p$ is a subspace of $S^p(W^*)$ and satisfies $\delta(w)\mathfrak{b}^p\subset\mathfrak{b}^{p-1}$ for any $w\in W$.

Now we turn to the case where L is a contact Lie algebra $\mathfrak{c}$. Let U_0 be the subspace of U of codimension 1 which is left invariant by $\mathfrak{g}^0(\mathfrak{c})$.

Put $W_0=\{w\in W\,|\,\delta(w)\mathfrak{a}^{0,1}\subset U_0\}$.

Then we have:

PROPOSITION 9.3. *There exists a subspace W_1 of W complementary to W_0 such that*

$$\mathfrak{a}^{0,1}=U_0\otimes W_1^{\perp}\oplus U\otimes W_0^{\perp}$$

where $W_i^{\perp}$ denotes the subspace of W^* consisting of the annihilators of W_i.

PROOF. In the proof, for simplicity we write $\mathfrak{a}^{0,1}$ as $\mathfrak{a}$. Let W_1 be a subspace of W complementary to W_0. We make the following identification:

$$W_0^{\perp}=W_1^*\,,\quad W_1^{\perp}=W_0^* \qquad \text{and} \qquad U\otimes W^*=U\otimes W_0^*+U\otimes W_1^*\,.$$

Let p_i $(i=0,1)$ be the projection of $U\otimes W^*$ to $U\otimes W_i^*$. Then we have $p_0(\mathfrak{a})=U_0\otimes W_0^*$ and $p_1(\mathfrak{a})=U\otimes W_1^*$. In fact, $p_1(\mathfrak{a})$ is invariant by the action of the simple part $\mathfrak{k}_0$ of $\mathfrak{g}_0(\mathfrak{c})$. From the definition of W_0 and (9.2) ii) we see that $\delta(w)p_1(\mathfrak{a})=U$ for any non-zero $w\in W_1$. Moreover U decomposes to $\mathfrak{k}_0$-irreducible subspaces; $U=U_0+U_1$ and $\mathfrak{k}_0$ acts trivially on U_1. Hence by Lemma 9.1 we have $p_1(\mathfrak{a})=U\otimes W_1^*$. Analogously we have $p_0(\mathfrak{a})=U_0\otimes W_0^*$. Thus we get $\mathfrak{a}\subset U_0\otimes W_0^*+U\otimes W_1^*$.

In the following exact sequence of $\mathfrak{k}_0$-morphisms,

$$0\longrightarrow \mathfrak{a}\cap U\otimes W_1^*\longrightarrow \mathfrak{a}\underset{\sigma_0}{\overset{p_0}{\rightleftarrows}} U_0\otimes W_0^*\longrightarrow 0\,,$$

since $\mathfrak{k}_0$ is simple, there exists a $\mathfrak{k}_0$-splitting σ_0 of p_0. If we put $\tau=\sigma_0-p_0\circ\sigma_0$, then τ is a $\mathfrak{k}_0$-morphism: $U_0\otimes W_0^*\to U\otimes W_1^*$. Since $\bar{\mathfrak{k}}_0$ acts trivially on U_1, we see that τ maps $U_0\otimes W_0^*$ to $U_0\otimes W_1^*$. Moreover, using the $\mathfrak{k}_0$-simplicity of U_0, we can find a linear map, $f\colon W_0^*\to W_1^*$ such that $\tau=id_{U_0}\otimes f$. Put

$$W_1'=\{w\in W\,|\,\langle w,\alpha+f(\alpha)\rangle=0 \text{ for all } \alpha\in W_0^*\}\,.$$

Then we have $W=W_0+W_1'$ (direct sum), and now it is easy to see that $\mathfrak{a}^{0,1}=U_0\otimes W_1'^{\perp}+U\otimes W_0^{\perp}$. Replacing W_1 by W_1', we have obtained the proposition.

q. e. d.

In the special case when W_0 is trivial, making use of Proposition 9.3, we can determine the bi-graded Lie algebra $\sum\mathfrak{a}^{p,q}$.

PROPOSITION 9.4. *Notation and assumption be as above.*

i) *If $W_0=(0)$, then $\mathfrak{a}^{p,r}=\mathfrak{g}^{p-r}(\mathfrak{c})\otimes S^r(W^*)$.*

ii) *If $W_0=W$, then $\mathfrak{a}^{p,r}=\check{\mathfrak{g}}^{p-r}(\mathfrak{c})\otimes S^r(W^*)$,*

where $\check{\mathfrak{g}}^q(\mathfrak{c})=\bar{\mathfrak{k}}_q+\bar{\mathfrak{k}}_{q-1}^1+\cdots+\bar{\mathfrak{k}}_{-1}^{q+1}$.

PROOF. By the assumption (A_2) we see that

$$\mathfrak{a}^{p,q}\subset\mathfrak{g}^{p-q}(\mathfrak{c})\otimes S^q(W^*)\,.$$

On the other hand we have

$$[\mathfrak{a}^{p,q}, \mathfrak{a}^{0,1}] \subset \mathfrak{a}^{p,q+1}.$$

While we know that

$$[\mathfrak{g}^p(\mathfrak{c}), \mathfrak{g}^{-1}(\mathfrak{c})] = \mathfrak{g}^{p-1}(\mathfrak{c}) \qquad \text{for } p \geqq 0$$

and

$$[\check{\mathfrak{z}}^p(\mathfrak{c}), \mathfrak{g}^{-1}(\mathfrak{c})] = \check{\mathfrak{z}}^{p-1}(\mathfrak{c}) \qquad \text{for } p \geqq 0.$$

Hence it follows easily the statement of i) and that

$$\check{\mathfrak{z}}^{p-q}(\mathfrak{c}) \otimes S^q(W^*) \subset \mathfrak{a}^{p,q} \subset \mathfrak{g}^{p-q}(\mathfrak{c}) \otimes S^q(W^*) \qquad \text{if } W_0 = W.$$

But in this case taking account of the facts that $\mathfrak{a}^{0,1} = \bar{\mathfrak{k}}_{-1} \otimes W^*$ and $[\bar{\mathfrak{k}}^p_{-2}, \bar{\mathfrak{k}}_{-2}] = \bar{\mathfrak{k}}^{p-1}_{-2}$ for $p \geqq 0$, we have $\mathfrak{a}^{p,q} = \check{\mathfrak{z}}^{p-q}(\mathfrak{c}) \otimes S^q(W^*)$. q. e. d.

Though we have heretofore reserved the possibility that W_0 might not be trivial, we shall prove the following proposition in the next section:

PROPOSITION 9.5. $W_0 = (0)$.

§ 10. Proof of the main theorem (contact case).

In the preceding sections we have studied the associated bi-graded Lie algebra $\mathfrak{a} = \sum \mathfrak{a}^{p,q}$ of A. In this and the next section we shall determine the Lie algebra A starting from its associated bi-graded Lie algebra $\mathfrak{a}$, and obtain our main theorem.

In this section we shall be concerned with the case where the transitive part is a contact Lie algebra. Our present aim is the following

THEOREM 10.1. *Let A be a closed Lie subalgebra of $D(V; U)$. Assume that the bi-graded Lie algebra $\mathfrak{a} = \sum \mathfrak{a}^{p,q}$ of A satisfies*

$$\mathfrak{a}^{p,q} = \mathfrak{g}^{p-q}(\mathfrak{c}) \otimes S^q(W^*) \qquad \textit{for all } p, q$$

and that the first structure function C_A of A is not zero. Here $\mathfrak{g}(\mathfrak{c}) = \sum \mathfrak{g}^p(\mathfrak{c})$ is the associated graded Lie algebra of the contact Lie algebra $\mathfrak{c}$ with $\mathfrak{g}^{-1}(\mathfrak{c}) = U$, and W is a complementary subspace to U in V. Then A is isomorphic to the intransitive extension $\mathfrak{c}[W^*]$ of $\mathfrak{c}$ by W^*, and the isomorphism is induced by a formal isomorphism of V.

PROOF. We follow the notation in § 8. Since $\mathfrak{c} = \prod G^p$ and $\mathfrak{c}[W^*] = \prod_{p,r} G^p \otimes S^r(W^*)$, the theorem follows immediately if we prove the following lemma:

LEMMA 10.1. *Under the same assumption as in Theorem 10.1, there exists a sequence $\{\varphi^{(l)}\}_{l \geqq 0}$ of formal isomorphisms of V satisfying the following conditions:*

i) $\varphi^{(l)} = I_0 + \xi^{(l)}$, *where I_0 is the identity in $V \otimes V^*$ and $\xi^{(l)} \in D^l(V; U) = D^l(V) \cap D(V; U)$,*

ii) $\mathfrak{k}^0_0 \subset \varphi^{(l)}_* \circ \cdots \circ \varphi^{(0)}_* A + D^{l+1}(V; U)$,

iii) $G^q\otimes S^r(W^*)\subset\varphi_*^{(l)}\circ\cdots\circ\varphi_*^{(0)}A+D^l(V;U)$ *for all* q, r.

PROOF. We shall construct $\varphi^{(l)}$ by induction on l. For $l=0$ we put $\varphi^{(0)}=I_0$. Then it is obvious that i), ii) and iii) are satisfied.

Now supposing that we are given $\varphi^{(i)}$ for $0\leqq i\leqq l$, we construct $\varphi^{(l+1)}$ so as to satisfy i), ii), iii) for $l+1$ instead of l. The procedure is rather long, and we carry it out in several steps. For simplicity, the modified $\varphi_*^{(l)}\circ\cdots\circ\varphi_*^{(0)}A$ is also denoted by A.

1) *We can modify* A *by a formal isomorphism* ϕ_1 *so as to satisfy*

$$\mathfrak{k}_0\subset A+D^{l+2}(V;U). \tag{10.1}$$

Moreover ϕ_1 *can be taken to satisfy*

$$\phi_1=I_0+\xi,\qquad \xi\in U\otimes S^{l+2}(V^*).$$

First of all note that $\mathfrak{k}_0$ is a simple Lie algebra. By induction assumption we can find a subspace $\underline{\mathfrak{k}}_0$ of A such that $\underline{\mathfrak{k}}_0$ is isomorphic to $\mathfrak{k}_0$ and that

$$\underline{\mathfrak{k}}_0\equiv\mathfrak{k}_0\quad(\text{mod. } D^{l+1}(V;U)),$$

that is, for $X\in\mathfrak{k}_0$ the corresponding $\underline{X}\in\underline{\mathfrak{k}}_0$ is written as

$$\underline{X}=X+\sum_{p+q\geqq l+1}X_{p,q},\qquad X_{p,q}\in U\otimes S^{p+1}(U^*)\otimes S^q(W^*).$$

Let $E^{p,q}$ be a $\mathfrak{k}_0$-invariant subspace of $U\otimes S^{p+1}(U^*)\otimes S^q(W^*)$ complementary to $\mathfrak{g}^p(\mathfrak{c})\otimes S^q(W^*)$. Since $\mathfrak{a}^{p,q}=\mathfrak{g}^{p-q}(\mathfrak{c})\otimes S^q(W^*)$ by our assumption, we may assume that $X_{p,q}\in E^{p,q}$ by a suitable choice of $\underline{\mathfrak{k}}_0$. We define the map $f:\mathfrak{k}_0\to\sum_{p+q=l+1}E^{p,q}$, by $f(X)=\sum_{p+q=l+1}X_{p,q}$, $X\in\mathfrak{k}_0$. Then, expanding $[\underline{X},\underline{Y}]$, we see easily that

$$[f(X),Y]+[X,f(Y)]=f[X,Y]\qquad\text{for}\quad X,Y\in\mathfrak{k}_0.$$

Since the 1st cohomology group of the representation of $\mathfrak{k}_0$ vanishes, we can find $\xi\in\sum_{p+q=l+1}E_{p,q}\subset U\otimes S^{l+2}(V^*)$, such that

$$f(X)=[\xi,X]\qquad\text{for all}\quad X\in\mathfrak{k}_0.$$

Let ϕ_1 be a formal isomorphism defined by $\phi_1=I_0+\xi$. In view of Proposition 1.1, we have $\phi_{1*}X=X$ (mod. $D^{l+2}(V;U)$) for $X\in\mathfrak{k}_0$. Thus, if we change A by $\phi_{1*}A$, we get (10.1).

Note that the induction assumption is not violated by this transformation.

2) *We can modify* A *by a formal isomorphism* ϕ_2 *with* $\phi_2=I_0+\xi_2$, $\xi_2\in U\otimes S^{l+1}(V^*)$, *so as to satisfy*

$$G^{-1}\subset A+U\otimes U^*\otimes S^l(W^*)+D^{l+1}(V;D), \tag{10.2}$$

without violating (10.1).

If $l=0$, (10.2) is always satisfied, since $\mathfrak{a}^{0,1}=U\otimes W^*$. Therefore we assume $l\geqq1$. By the induction assumption we can find a subspace $\underline{G}^{-1}$ of A which is isomorphic to G^{-1} and satisfies

$$\underline{G}^{-1}\equiv G^{-1} \quad (\text{mod.}\, D^{l}(V\,;\,U))\,.$$

Therefore, each $\underline{X}\in\underline{G}^{-1}$ corresponding to $X\in G^{-1}$, is written as

$$\underline{X}\equiv X+\sum_{p+q=l}X_{p,q} \quad (\text{mod.}\, D^{l+1}(V\,;\,U))\,,$$

where we may assume that $X_{p,q}\in E^{p,q}$.

Note that any $X\in G^{-1}$ is written uniquely as $X=X_{-1}+X_0$ for $X_{-1}\in\mathfrak{g}^{-1}(\mathfrak{c})$ $(=U)$ and $X_0\in\mathfrak{g}^{0}(\mathfrak{c})$, and that the projection $X\to X_{-1}$ is bijective.

For $X, Y\in G^{-1}$, we have

$$[\underline{X}, \underline{Y}]=[X, Y]+\sum_{p+q=l}[X_{-1}, Y_{p,q}]+[X_{p,q}, Y_{-1}] \quad (\text{mod.}\, D^{l}(V\,;\,U))$$

and

$$[X, Y]=\underline{[X, Y]} \quad (\text{mod.}\, D^{l}(V\,;\,U))\,.$$

Therefore we have

$$\sum_{p+q=l}[X_{-1}, Y_{p,q}]+[X_{p,q}, Y_{-1}]\in A+D^{l}(V\,;\,U)\,.$$

Suppose that $X_{p,q}=0$ for $p>r\geqq 2$ and for all $X\in G^{-1}$, then we see that

$$\underline{C}(X_{-1}, Y_{-1})=[X_{-1}, Y_{r,l-r}]+[X_{r,l-r}, Y_{-1}]\in g^{r-1}(\mathfrak{c})\otimes S^{l-r}(W^{*})$$

and we get an element $\underline{C}$ of $\text{Hom}\,(\wedge^{2}\mathfrak{g}^{-1}(\mathfrak{c}), \mathfrak{g}^{r-1}(\mathfrak{c}))\otimes S^{l-r}(W^{*})$. It is not difficult to see that

$$\partial\underline{C}=0\,, \tag{10.3}$$

$$a\cdot\underline{C}=0\,, \qquad a\in\mathfrak{k}_0\,, \tag{10.4}$$

where ∂ is the coboundary operator of the Spencer complex associated to the graded Lie algebra $\sum_{p}\mathfrak{g}^{p}(\mathfrak{c})$. Since the Spencer cohomology group $H^{p,q}(\sum\mathfrak{g}^{p}(\mathfrak{c}))$ vanishes for $p\geqq 1$ (Proposition 8.7), we can find $\alpha\in\text{Hom}\,(\mathfrak{g}^{-1}(\mathfrak{c}), \mathfrak{g}^{r}(\mathfrak{c}))\otimes S^{l-r}(W^{*})$ so as to satisfy

$$\partial\alpha+\underline{C}=0\,, \tag{10.5}$$

$$a\cdot\alpha=0 \qquad \text{for all} \quad a\in\mathfrak{k}_0\,. \tag{10.6}$$

Since $\alpha\in\text{Hom}\,(\mathfrak{g}^{-1}(\mathfrak{c}), \mathfrak{g}^{r}(\mathfrak{c})\otimes S^{l-r}(W^{*}))$, there exists a subspace $\underline{\underline{G}}^{-1}$ of A such that $\underline{\underline{G}}^{-1}$ is isomorphic to G^{-1} and that

$$\underline{\underline{X}}=X_{-1}+X_0+\sum_{\substack{p+q=l\\ p\leqq r}}X_{p,q}+\alpha(X_{-1})\in\underline{\underline{G}}^{-1}+D^{l+1}(V\,;\,U) \tag{10.7}$$

$$\text{for} \quad X=X_{-1}+X_0\in G^{-1} \text{ and } X_{p,q}\in E^{p,q}\,.$$

If we replace $\underline{G}^{-1}$ by $\underline{\underline{G}}^{-1}$, then we see from (10.5) and (10.7) that $\underline{C}=0$. Hence there exists a $\xi_{r+1,l-r}\in U\otimes S^{r+2}(U^{*})\otimes S^{l-r}(W^{*})$ such that

(10.8) $$X_{-1}+X_0+[\xi_{r+1,l-r}, X_{-1}] \in A+\sum_{p<r} U\otimes S^p(U^*)\otimes S^{l-p}(W^*)+D^{l+1}(V; U) \quad \text{for all} \quad X_{-1}+X_0\in G^{-1}.$$

Moreover (10.6) and (10.7) imply that

(10.9) $$[a, \xi_{r+1,l-r}]=0, \qquad a\in\mathfrak{k}_0.$$

Therefore if we modify A by the formal isomorphism $I_0+\xi_{r+1,l-r}$, we get

$$G^{-1}\subset A+\sum_{p<r} U\otimes S^p(U^*)\otimes S^{l-p}(W^*)+D^{l+1}(V; U),$$

and finally by induction on r we have (10.2).

Observe that by such modification of A, (10.1) remains valid by virtue of (10.9).

3) By the argument above and the fact that $\mathfrak{a}^{l,l+1}=U\otimes S^{l+1}(W^*)$, we may assume that there exist subspaces $\underline{\mathfrak{k}}_0$, $\underline{\mathfrak{m}}$ of A isomorphic to $\mathfrak{k}_0$ and $\mathfrak{m}$ respectively and each $\underline{a}\in\underline{\mathfrak{k}}_0$ and $\underline{X}\in\underline{\mathfrak{m}}$ is expressed in the following form:

$$\begin{cases} \underline{a}\equiv a & (\text{mod. } D^{l+2}(V; U)), \\ \underline{X}\equiv X+X_{0,l} & (\text{mod. } D^{l+1}(V; U)), \end{cases}$$

where $a\in\mathfrak{k}_0$, $X\in\mathfrak{m}$ and $X_{0,l}\in E^{0,l}$. Since $[\underline{a}, \underline{X}]-\underline{[a, X]}\in A$ and $E^{0,l}$ is $\mathfrak{k}_0$-invariant, we see that

$$f[a, X]=[a, f(x)] \qquad a\in\mathfrak{k}_0, \quad X\in\mathfrak{m},$$

where f is the map from $\mathfrak{m}$ to $E^{0,l}$ defined by $f(X)=X_{0,l}$ for $X\in\mathfrak{m}$.

So we examine the representation of $\mathfrak{k}_0$ on $E^{0,l}$. It is convenient to write down explicitly in a coordinate system. Let $z, x^1, \cdots, x^n, y^1, \cdots, y^n, t^1, \cdots, t^m$ be a basis of V^* such that $z\in\bar{\mathfrak{k}}^*_{-2}$, $x^i, y^j\in\bar{\mathfrak{k}}^*_{-1}$ and $t^\alpha\in W^*$ and $\frac{\partial}{\partial z}, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^m}$ be the dual basis of V. Then $\mathfrak{k}_{-2}$, $\mathfrak{k}_{-1}$ are generated by the following elements:

$$\begin{cases} \mathfrak{k}_{-2}: & \frac{\partial}{\partial z} \\ \mathfrak{k}_{-1}: & \frac{\partial}{\partial x^i}-y^i\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y^i}+x^i\frac{\partial}{\partial z} \qquad (1\leqq i\leqq n). \end{cases}$$

The decomposition of $U\otimes U^*$;

$$U\otimes U^*=U_0\otimes U_0^*+U_0\otimes Z^*+Z^*\otimes U_0^*+Z\otimes Z^*$$

is $\mathfrak{k}_0$-invariant, where we put $Z=\left\{\frac{\partial}{\partial z}\right\}$, $U_0=\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$. Since $\mathfrak{k}_0$ is the Lie algebra consisting of those linear maps of U_0 which leave invariant the non-degenerate two form $\theta=\sum_{i=1}^{n} x^i \wedge y^i$, the isomorphism $\mu: U_0 \to U_0^*$, given by $\langle \mu(u), u' \rangle = \theta(u, u')$ for $u, u' \in U_0$, is a $\mathfrak{k}_0$-isomorphism. The representations of $\mathfrak{k}_0$ on Z and Z^* are trivial. Hence $U_0 \otimes Z^*$ and $Z \otimes U_0^*$ are $\mathfrak{k}_0$-isomorphic to U_0 and irreducible. On the other hand, $U_0 \otimes U_0$ decomposes to three $\mathfrak{k}_0$-irreducible components:

$$U_0 \otimes U_0 = S^2(U_0) + \mathfrak{q} + \{\theta\} ,$$

where $\mathfrak{q}$ is the $\mathfrak{k}_0$-invariant subspace of $\wedge^2(U_0)$ complementary to the 1-dimensional subspace $\{\theta\}$ generated by θ. By the isomorphism $id_{U_0} \otimes \mu$, $U_0 \otimes U_0^*$ decomposes to three $\mathfrak{k}_0$-irreducible subspaces:

$$U_0 \otimes U_0^* = \mathfrak{k}_0 + \mathfrak{p} + \{I\}$$

where

$$\mathfrak{p} = (\mathrm{id}_{U_0} \otimes \mu)(\mathfrak{q}) ,$$

$$I = (\mathrm{id}_{U_0} \otimes \mu)\theta = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} .$$

Note that $\mathfrak{k}_0$ and $\mathfrak{p}$ are not isomorphic to each other.

On the other hand, we see that $\mathfrak{g}^0(\mathfrak{c}) = \mathfrak{k}_0 + \bar{\mathfrak{k}}^1_{-2} + \bar{\mathfrak{k}}^1_{-1}$, and that

$$\bar{\mathfrak{k}}^1_{-2} = \left\{ J = 2z \frac{\partial}{\partial z} + I \right\} ,$$

$$\bar{\mathfrak{k}}^1_{-1} = \left\{ z \frac{\partial}{\partial x^i},\ z \frac{\partial}{\partial y^i} \right\} = U_0 \otimes Z^* .$$

Therefore we may put

$$E^{0,0} = \mathfrak{p} + Z \otimes U_0^* + Z \otimes Z^*$$

and

$$E^{0,l} = \mathfrak{p} \otimes S^l(W^*) + (Z \otimes U_0^*) \otimes S^l(W^*) + (Z \otimes Z^*) \otimes S^l(W^*) .$$

Since $f: U \to E^{0,l}$ is a $\mathfrak{k}_0$-homomorphism we can find $\lambda(t)$, $\tau(t) \in S^l(W^*)$, such that

$$f\left(\frac{\partial}{\partial x^i}\right) = -\lambda(t) y^i \frac{\partial}{\partial z} , \quad f\left(\frac{\partial}{\partial y^i}\right) = \lambda(t) x^i \frac{\partial}{\partial z} , \quad f\left(\frac{\partial}{\partial z}\right) = \tau(t) z \frac{\partial}{\partial z} .$$

Hence $\underline{\mathfrak{m}}$ is generated by

$$\begin{cases} \dfrac{\partial}{\partial x^i} - \lambda(t) y^i \dfrac{\partial}{\partial z} + \cdots \\[2ex] \dfrac{\partial}{\partial y^i} + \lambda(t) x^i \dfrac{\partial}{\partial z} + \cdots \\[2ex] \dfrac{\partial}{\partial z} + \tau(t) z \dfrac{\partial}{\partial z} + \cdots \end{cases}$$

whe e the higher order terms written by dots are in $D^{l+1}(V; U)$.

Here we note that $\lambda(t)$ is a scalar for $l=0$, and that it is not zero becausc of the assumption that the 1st structure function C_A does not vanish. Let ϕ_3 be the formal isomorphism defined by

$$\begin{cases} \phi_3^* x^i = \sqrt{1+\lambda(t)}\, x^i \\ \phi_3^* y^i = \sqrt{1+\lambda(t)}\, y^i \\ \phi_3^* z = \int_0^z \dfrac{d\zeta}{1+\tau(t)\zeta} \\ \phi_3^* t = t\,. \end{cases}$$

By a simple calculation we see that

$$\begin{aligned} &\phi_{3*}\Big(\frac{\partial}{\partial x^i} - \lambda(t) y^i \frac{\partial}{\partial z} + \cdots\Big) \\ &\qquad = (1+\nu(t))\Big(\frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial z}\Big) + \cdots \\ &\phi_{3*}\Big(\frac{\partial}{\partial y^i} + \lambda(t) x^i \frac{\partial}{\partial z} + \cdots\Big) \\ &\qquad = (1+\nu(t))\Big(\frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z}\Big) + \cdots \\ &\phi_{3*}\Big(\frac{\partial}{\partial z} + \tau(t) z \frac{\partial}{\partial z} + \cdots\Big) \\ &\qquad = \frac{\partial}{\partial z} + \cdots, \end{aligned}$$

where dots are in $D^{l+1}(V; U)$. It is easy to check that ϕ_3 is expressed as $\phi_3 = I_0 + \xi_3$, $\xi_3 \in D^l(V; U)$ and that (10.1) is preserved by this modification. Applying the formal isomorphisms ϕ_1, ϕ_2 and ϕ_3 to A, we have the following:

We can modify A by a formal isomorphism $\varphi^{(l+1)}$ *with* $\varphi^{(l+1)} = \mathrm{id} + \xi$, $\xi \in D^l(V; U)$, *so as to satisfy:*

$$(10.10)\qquad \begin{cases} \text{i)}\ \ \mathfrak{k}_0 \subset A + D^{l+2}(V; U) \\ \text{ii)}\ \ \dfrac{\partial}{\partial z} \in A + D^{l+1}(V; U) \\ \text{iii)}\ \ \textit{there exists } \nu(t) \in S^l(W^*) \textit{ such that} \\ \qquad \left.\begin{array}{l} (1+\nu(t))\Big(\dfrac{\partial}{\partial x^i} - y^i \dfrac{\partial}{\partial z}\Big) \\ (1+\nu(t))\Big(\dfrac{\partial}{\partial y^i} + x^i \dfrac{\partial}{\partial z}\Big) \end{array}\right\} \in A + D^{l+1}(V; U)\,. \end{cases}$$

4) To complete our induction, we shall show that (10.10) implies

$$(10.11)\qquad G^q \otimes S^r(W^*) \subset A + D^{l+1}(V; U) \qquad \text{for all } q, r\,.$$

In view of Proposition 8.4 and the fact that $[\mathfrak{c}, \mathfrak{m}]=\mathfrak{c}$, we see that $\sum_{q+r\geqq 0} G^q\otimes S^r(W^*)$ is generated by G^0, $\mathfrak{k}_1$, $G^{-1}\otimes W^*$ as a Lie algebra. Therefore to prove (10.11) it suffices to verify (10.11) only for G^{-1}, $\mathfrak{m}^1$, $\mathfrak{k}_1$ and $G^{-1}\otimes W^*$, where $\mathfrak{m}^1=\mathfrak{k}^1_{-2}+\mathfrak{k}^1_{-1}$.

5) First we show that

$$\mathfrak{m}^1\subset A+U\otimes U^*\otimes S^l(W^*)+D^{l+1}(V\,;U)\,. \tag{10.12}$$

For $l\leqq 0$ it is obvious, so we assume that $l\geqq 1$. Let $\underline{\mathfrak{m}}^1$ be a subspace of A which is isomorphic to $\mathfrak{m}^1$ and $\underline{\mathfrak{m}}^1\equiv\mathfrak{m}^1$ (mod. $D^l(V\,;U)$). Then each $\underline{X}\in\underline{\mathfrak{m}}^1$ is written as

$$\underline{X}=X+\sum_{p+q\geqq l} X_{p,q}\,,\qquad X\in\mathfrak{m}^1$$

and we may also assume that $X_{p,q}\in E^{p,q}$. Then we have

$$X_{p,q}=0\qquad \text{for}\quad p+q=l\,,\ p\geqq 1\,. \tag{10.13}$$

In fact, let $\underline{u}$ be any element of A such that

$$\underline{u}=u_{-1,0}+u_{0,0}+u_{0,1}+\sum_{p+q>l} u_{p,q}$$

where $u=u_{-1,0}+u_{0,0}\in G^{-1}$ and $u_{p,q}\in U\otimes S^{p+1}(U^*)\otimes S^q(W^*)$, then we have

$$[\underline{X}, \underline{u}]=[X, u]+\sum_{p+q=l}[X_{p,q}, u_{-1,0}]\quad (\text{mod.}\, D^l(V\,;U))\,.$$

Since $[X, u]\in G^{-1}+G^0$ and $G^{-1}+G^0\subset A+D^l(V\,;U)$, we have

$$\sum_{p+q=l}[X_{p,q}, u_{-1,0}]\in A+D^l(V\,;U)\,.$$

If $X_{p,q}=0$ for $p>r\geqq 1$ and $p+q=l$, we have

$$[X_{r,l-r}, u_{-1,0}]\in\mathfrak{g}^{r-1}(\mathfrak{c})\otimes S^{l-r}(W^*)\qquad \text{for all}\quad u_{-1,0}\in U\,.$$

Hence $X_{r,l-r}\in\mathfrak{g}^{r-1}(\mathfrak{c})^{(1)}\otimes S^{l-r}(W^*)$, where $\mathfrak{g}^{r-1}(\mathfrak{c})^{(1)}$ is the prolongation of $\mathfrak{g}^{r-1}(\mathfrak{c})$. By Proposition 8.7, we see that $\mathfrak{g}^{r-1}(\mathfrak{c})^{(1)}=\mathfrak{g}^r(\mathfrak{c})$. Therefore $X_{r,l-r}\in\mathfrak{g}^r(\mathfrak{c})\otimes S^{l-r}(W^*)$, which implies $X_{r,l-r}=0$. Finally we see that $X_{p,q}=0$ for $p+q=l$, $p\geqq 1$, proving (10.13).

REMARK. If $G^{-1}\subset A+D^{l+1}(V\,;U)$, by the same argument as above we see that $X_{p,q}=0$ for $p+q=l$ or $l+1$ and $p\geqq 1$.

Next we examine the $(0, l)$-component of $\underline{X}\in\underline{\mathfrak{m}}^1$. Note that $\mathfrak{k}^1_{-2}$ and $\mathfrak{k}^1_{-1}$ are generated by

$$\mathfrak{k}^1_{-2}=\left\{J=2z\frac{\partial}{\partial z}+\sum x^i\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial y^i}\right\},$$

$$\mathfrak{k}^1_{-1}=\left\{\begin{array}{l} z\dfrac{\partial}{\partial y^i}+x^ie \\[2ex] z\dfrac{\partial}{\partial x^i}-y^ie\end{array}\right\}$$

where $e=z\frac{\partial}{\partial z}+\sum x^i\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial y^i}$.

By examining the representation of $\mathfrak{k}_0$ on $U\otimes U^*\otimes S^l(W^*)$, we see that the following elements:

$$\begin{cases} \underline{Y}_i=z\frac{\partial}{\partial y^i}+x^ie+g(t)x^i\frac{\partial}{\partial z}\cdots \\ \underline{X}_i=z\frac{\partial}{\partial x^i}-y^ie-g(t)y^i\frac{\partial}{\partial z}\cdots \\ \underline{J}=J+f(t)z\frac{\partial}{\partial z}\cdots, \end{cases} \qquad (i=1,\cdots,n)$$

are contained in A for some $g(t)$, $f(t)\in S^l(W^*)$, where dots are in $D^{l+1}(V;U)$. Then we have

$$\underline{Y}_i-[\underline{J},\underline{Y}_i]\equiv 2gx^i\frac{\partial}{\partial z}-fz\frac{\partial}{\partial y^i} \quad (\text{mod. } D^{l+1}(V;U))$$

and

$$2gx^i\frac{\partial}{\partial z}-fz\frac{\partial}{\partial y^i}\in\mathfrak{g}^0\otimes S^l(W^*).$$

From this we see that $g=0$. Thus we have $\mathfrak{k}^1_{-1}\subset A+D^{l+1}(V;U)$. The fact that $\mathfrak{k}^1_{-2}\subset A+D^{l+1}(V;U)$ can be proved immediately as soon as we get $G^{-1}\subset A+D^{l+1}(V;U)$. In fact it follows easily from the remark after (10.13) and the fact that

$$\left[z\frac{\partial}{\partial x^i}-y^ie,\ \frac{\partial}{\partial y^i}+x^i\frac{\partial}{\partial z}\right]=J-\sum x^i\frac{\partial}{\partial x^i}+y^i\frac{\partial}{\partial y^i}.$$

6) Now we show that

$$G^{-1}\otimes W^*\subset A+D^{l+1}(V;U).$$

In the same way as in 5) we see that

$$\underline{t^\alpha\frac{\partial}{\partial z}}=t^\alpha\frac{\partial}{\partial z}+\tau_\alpha(t)z\frac{\partial}{\partial z}\in A+D^{l+1}(V;U)$$

for some $\tau_\alpha(t)\in S^l(W^*)$. On the other hand we have seen that

$$\underline{J}=J+f(t)z\frac{\partial}{\partial z}\in A+D^{l+1}(V;U)$$

for some $f(t)\in S^l(W^*)$. From this we see

$$\left[\underline{J},\underline{t^\alpha\frac{\partial}{\partial z}}\right]=-2t^\alpha\frac{\partial}{\partial z}-f(t)\cdot t^\alpha\frac{\partial}{\partial z}\in A+D^{l+1}(V;U).$$

Since $f(t)\cdot t^\alpha\frac{\partial}{\partial z}\in A+D^{l+1}$, we have

$$t^\alpha\frac{\partial}{\partial z}\in A+D^{l+1}(V;U).$$

From the fact that

$$\left[t^\alpha \frac{\partial}{\partial z}, z\frac{\partial}{\partial x^i} - y^i e\right] = t^\alpha\left(\frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial z}\right),$$

we see that

$$t^\alpha\left(\frac{\partial}{\partial x^i} - y^i \frac{\partial}{\partial z}\right) \in A + D^{l+1}.$$

Thus we have

$$G^{-1} \otimes W^* \subset A + D^{l+1}(V; U).$$

7) The verification of the fact that

$$\mathfrak{k}_1 \subset A + D^{l+1}(V; U)$$

can be carried out similarly, and we omit the proof.

8) Observe that $[\mathfrak{k}_1, \mathfrak{k}_{-1} \otimes W^*] = \mathfrak{k}_0 \otimes W^*$ and that $[\mathfrak{k}_0 \otimes W^*, \mathfrak{k}_{-1} \otimes S^q(W^*)] = \mathfrak{k}_{-1} \otimes S^{q+1}(W^*)$. From this and the results of 6) and 7), we see that $\mathfrak{k}_{-1} \otimes S^{l-1}(W^*) \subset A + D^{l+1}(V; U)$. Hence, recalling (10.10) iii), we have

$$\mathfrak{k}_{-1} \subset A + D^{l+1}(V; U).$$

Therefore we have

$$G^{-1} \subset A + D^{l+1}(V; U).$$

Thus we have proved (10.11) and completed our induction.

This completes the proof of Lemma 10.1 and Theorem 10.1.

Here we are in a position to give a proof of Proposition 9.5.

PROOF OF PROPOSITION 9.5. Suppose that $W_0 = (0)$. Let us regard $F(U + W_0)$ as the quotient ring of $F(V)$ factored by the ideal generated by $(U + W_0)^\perp$. Since $(U + W_0)^\perp$ is invariant by A, there is a homomorphism of A into $D(U + W_0)$. Let A' be its image, then A' satisfies the same conditions as those of A except that $W_1 = (0)$ for A'. So we may assume that A satisfies that

$$W_0 = W \neq (0).$$

Let $t^\alpha \frac{\partial}{\partial x^i}$, $t^\alpha \frac{\partial}{\partial y^i} \in \bar{\mathfrak{k}}_{-1} \otimes W^*$ and $\frac{\partial}{\partial y^i} \in \bar{\mathfrak{k}}_{-1}$. In the same way as in the proof of Lemma 10.1, we can find representatives $\underline{X}_i^\alpha$, $\underline{Y}_i^\alpha$, $\underline{Y}_i \in A$ of those elements such that

$$\text{(10.14)} \quad \begin{cases} \underline{X}_i^\alpha \equiv t^\alpha \dfrac{\partial}{\partial x^i} - \tau^\alpha y^i \dfrac{\partial}{\partial z} \quad (\text{mod. } D^2(V; U)) \\ \underline{Y}_i^\alpha \equiv t^\alpha \dfrac{\partial}{\partial y^i} + \tau^\alpha x^i \dfrac{\partial}{\partial z} \quad (\text{mod. } D^2(V; U)) \end{cases} \quad \text{where } \tau^\alpha \in W^*$$

and

$$\text{(10.15)} \quad \underline{Y}_i = \frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z} \quad (\text{mod. } D^2(V; U) + W^* \cdot D^2(V; U)).$$

From (10.14) we see that

$$[\underline{X}_i^\alpha, \underline{Y}_i^\alpha] = 2t^\alpha\tau^\alpha\frac{\partial}{\partial z} \quad (\mathrm{mod.}\ D^2(V;U)).$$

Hence we have $\tau^\alpha=0$, because we know that $\mathfrak{a}^{1,2}=\bar{\mathfrak{k}}_{-1}\otimes S^2(W^*)$ by Proposition 9.4. Therefore, in view of (10.15) we see that

$$[\underline{X}_i^\alpha, \underline{Y}_i] = t^\alpha\frac{\partial}{\partial z} \quad (\mathrm{mod.}\ D^1(V;U)).$$

Hence we have $t^\alpha\frac{\partial}{\partial z}\in\mathfrak{a}^{0,1}$, which contradicts the fact that $\mathfrak{a}^{0,1}=\bar{\mathfrak{k}}_{-1}\otimes W^*$. Thus we have proved that $W_0=0$. q. e. d.

Combining Theorem 10.1 with Proposition 9.4 and Proposition 9.5, we have:

THEOREM 10.2. *Let A be a closed Lie subalgebra of $D(V)$ satisfying (A_1), (A_2) and (A_3). Assume that the transitive part L of A is a contact Lie algebra. Then A is isomorphic to $L[W^*]$ by a formal isomorphism of V, where W^* is the subspace of V^* annihilating $\mathfrak{gr}^{-1}(A)$.*

§11. Proof of the main theorem (irreducible case).

In this section we consider the case where the transitive part L of A is infinite and irreducible. For an irreducible infinite Lie algebra L, L' denotes the ideal of L of codimension 1 if it exists and denotes L itself otherwise.

THEOREM 11.1. *Let A be a closed Lie subalgebra of $D(V;U)$. Assume that the associated bi-graded Lie algebra $\sum\mathfrak{a}^{p,q}$ satisfies*

$$\mathfrak{g}^{p-q}(L')\otimes S^q(W^*)\subset\mathfrak{a}^{p,q}\subset\mathfrak{g}^{p-q}(L)\otimes S^q(W^*) \qquad \textit{for all}\ \ p, q$$

where L is an irreducible infinite transitive Lie algebra in $D(V)$ and W is a complementary subspace of V to U. Then A is isomorphic, by a formal isomorphism of V, to a Lie subalgebra of $L[W^*]$ containing $L'[W^*]$.

This theorem is obtained by the following lemmas.

LEMMA 11.1. *Under the same assumption as in Theorem 11.1, there exists a sequence $\{\varphi^{(l)}\}_{l\geq 0}$ of formal isomorphisms of V satisfying the following conditions:*

i) $\varphi^{(l)}=I_0+\xi^{(l)}$ *where* $\xi^{(l)}\in D^l(V;U)$.

ii) $\mathfrak{g}^0(L')\subset\varphi_*^{(l)}\circ\cdots\circ\varphi_*^{(0)}A+D^{l+1}(V;U)$.

iii) $\mathfrak{g}^q(L')\otimes S^r(W^*)\subset\varphi_*^{(l)}\circ\cdots\circ\varphi_*^{(0)}A+D^l(V;U)$.

We can prove this lemma on the same lines as the Lemma 10.1. But the proof is much simpler than that of Lemma 10.1, mainly because L is flat. So we omit the proof.

LEMMA 11.2. *Under the same assumption as in Theorem 11.1, further assume that $A\supset L'[W^*]$. Then $A\subset L[W^*]$.*

PROOF. Since $\mathfrak{g}^0(L)=\mathfrak{g}^0(L')+\{I\}$, we have only to verify that for any element of $\mathfrak{a}^{q,q}$ of the form $I\otimes\tau$, $\tau\in S^q(W^*)$, we can find a representative

$\underline{I\otimes\tau}\in A^q$ satisfying $\underline{I\otimes\tau}\in L[W^*]$.

Let $E^{p,q}$ be a $\mathfrak{g}^0(L')$-invariant complementary subspace of $U\otimes S^{p+1}(U^*)\otimes S^q(W^*)$ to $\mathfrak{g}^p(L')\otimes S^q(W^*)$. Then we can find a representative $\underline{I\otimes\tau}\in A^q$ of $I\otimes\tau$ such that it is written as

$$\underline{I\otimes\tau}=I\otimes\tau+\sum_{p+r>q}X_{p,r}, \qquad X_{p,r}\in E^{p,r}.$$

By the same way as in Lemma 10.1, we see that $X_{p,r}=0$ for $p>0$. As to $X_{0,r}$, taking into consideration the representation of $\mathfrak{g}^0(L')$, we see that

$$X_{0,r}=I\otimes\sigma^r$$

for some $\sigma^r\in S^r(W^*)$. The components $X_{-1,r}$ are trivial of course. Hence we have proved Lemma 11.2 and Theorem 11.1.

Combining Theorem 11.1 and Proposition 9.2, we have the following

THEOREM 11.2. *Let A be a closed Lie subalgebra of $D(V)$ satisfying (A_1), (A_2) and (A_3). Assume that the transitive part L of A is irreducible and infinite. Then A is isomorphic, by a formal isomorphism of V, to a subalgebra of $L[W^*]$ containing $L'[W^*]$, where W^* is the subspace of V^* annihilating* $\mathfrak{gr}^{-1}(A)$.

REMARK. In the case L is not simple, i.e., (II′) or (III′), $L[W^*]$ is identified with $L'[W^*]+F(W)$ and $F(W)$ is the center of $L[W^*]$. There is a bijective correspondence between the subalgebra A of $L[W^*]$ containing $L'[W^*]$ and the subspace P of $F(W)$. Put $P^k=P\cap F^k(W)$ and $\mathfrak{p}^k=P^k/P^{k+1}\subset S^k(W^*)$. Then $L'[W^*]+P$ satisfies (A_2) if and only if the graded vector space $\sum\mathfrak{p}^k$ satisfies

$$\delta(w)\mathfrak{p}^k\subset\mathfrak{p}^{k-1} \qquad \text{for all } k \text{ and } w\in W.$$

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