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Symmetric spaces with invariant locally Hessian structures

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Let M be a differentiable manifold with a locally flat linear connection D. Then, for each point $p\in M$, there exists a local coordinate system $\{x^1, \cdots, x^n\}$ in a neighbourhood of p such that $D(dx^i)=0$, which we call an affine local coordinate system. A Riemannian metric g on M is said to be *locally Hessian* with respect to D, if there exists, for each point $p\in M$, a real-valued function ϕ of class C^∞ on a neighbourhood of p such that

$$g=\sum_{i,j}\frac{\partial^2\phi}{\partial x^i\partial x^j}dx^i dx^j,$$

where $\{x^1, \cdots, x^n\}$ is an affine local coordinate system around p. Such the pair (D, g) is called a *locally Hessian structure* on M. We know that for a locally flat Riemannian manifold the pair of the Riemannian connection and the Riemannian metric is a locally Hessian structure, and that a homogeneous self-dual convex cone has a canonical locally Hessian structure (cf. [**6**]).

Let G be a connected Lie group and B a closed subgroup of G. The pair (G, B) is called a symmetric pair if there exists an involutive automorphism σ of G such that $(B_\sigma)_0\subset B\subset B_\sigma$, where B_σ is the set of fixed points of σ and $(B_\sigma)_0$ is the identity component of B_σ. If, in addition, B contains no non-trivial normal subgroup of G, (G, B) is said to be an effective symmetric pair.

The aim of this paper is to prove the following

THEOREM. *Let (G, B) be an effective symmetric pair. If $M=G/B$ admits a locally Hessian structure (D, g) such that D and g are invariant under G, then M is affinely diffeomorphic and isometric (with respect to D and g respectively) to a direct product*

$$M_0\times M_1\times\cdots\times M_r,$$

where M_0 is a locally flat Riemannian manifold and the universal covering manifold of M_i $(1\leqq i\leqq r)$ is an irreducible homogeneous self-dual convex cone with a canonical locally Hessian structure.

1. In this section let (G, B) be a pair of a connected Lie group G and its closed subgroup B which needs not be symmetric. Assume that G/B admits an invariant locally Hessian structure (D, g). Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{b}$ be the Lie subalgebra of $\mathfrak{g}$ corresponding to B. For $X\in\mathfrak{g}$ we denote by X^* the vector field on M induced by the 1-parameter group of transformations $\exp(-tX)$. We put $A_{X^*}Y^*=-D_{Y^*}X^*$ for $X, Y\in\mathfrak{g}$. Let V be the tangent space of G/B at $o=\{B\}$ and let $f(X)$ and $q(X)$ denote the values of A_{X^*} and X^* at o respectively. Then we have (cf. [**6**])

(1) *f is a linear representation of $\mathfrak{g}$ in V,*

(2) *q is a linear map of $\mathfrak{g}$ onto V such that*

$$q([X, Y])=f(X)q(Y)-f(Y)q(X),$$

and the kernel of q coincides with $\mathfrak{b}$.

Let ω be an invariant volume element on G/B. If ω has the expression

$$\omega=K\,dx^1\wedge\cdots\wedge dx^n$$

in an affine local coordinate system $\{x^1, \cdots, x^n\}$, then the forms

$$\alpha=\sum_i \frac{\partial \log K}{\partial x^i}dx^i,$$

$$D\alpha=\sum_{i,j}\frac{\partial^2 \log K}{\partial x^i\partial x^j}dx^i dx^j,$$

are called the *Koszul form* and the *canonical bilinear form* respectively. Let α_o, $D\alpha_o$ denote the values of α, $D\alpha$ at o. Then we have

$$\alpha_o(q(X))=\mathrm{Tr}\,f(X), \tag{3}$$

$$D\alpha_o(q(X), q(Y))=\alpha_o(f(X)q(Y)), \tag{4}$$

for $X, Y\in\mathfrak{g}$ (cf. [**4**], [**6**]).

Let $\langle\,,\rangle$ denote the inner product on V given by the Riemannian metric g. Then $\langle\,,\rangle$ satisfies the following condition (cf. [**6**])

$$\text{(C)}\qquad \langle f(X)q(Y), q(Z)\rangle+\langle q(Y), f(X)q(Z)\rangle =\langle f(Y)q(X), q(Z)\rangle+\langle q(X), f(Y)q(Z)\rangle.$$

Let V^* be the dual space of V and let f^* be the representation of $\mathfrak{g}$ contragredient to f. We define a linear map $\gamma:\mathfrak{g}\to V^*$ by $(\gamma(X))(v)=\langle q(X), v\rangle$ for $X\in\mathfrak{g}$, $v\in V$. Let d_{f^*} denote the coboundary operator for the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in (V^*, f^*). Then the condition (C) is equivalent to

$$\text{(C')}\qquad d_{f^*}\gamma=0.$$

In fact, for $X, Y, Z\in\mathfrak{g}$ we have

$$((d_{f^*}\gamma)(X, Y))(q(Z))$$
$$=(f^*(X)\gamma(Y))(q(Z))-(f^*(Y)\gamma(X))(q(Z))-\gamma([X, Y])q(Z)$$
$$=-\langle q(Y), f(X)q(Z)\rangle+\langle q(X), f(Y)q(Z)\rangle-\langle q([X, Y]), q(Z)\rangle$$
$$=-\langle q(Y), f(X)q(Z)\rangle+\langle q(X), f(Y)q(Z)\rangle$$
$$-\langle f(X)q(Y), q(Z)\rangle+\langle f(Y)q(X), q(Z)\rangle .$$

PROPOSITION 1. *If G/B admits an invariant locally Hessian structure, then G is not semi-simple.*

PROOF. Let d_f denote the coboundary operator for the cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in (V, f). Regarding q as a 1-dimensional (V, f)-cochain, we have $(d_f q)(X, Y)=f(X)q(Y)-f(Y)q(X)-q([X, Y])=0$ for all $X, Y \in \mathfrak{g}$. Now assume that $\mathfrak{g}$ is semi-simple. Since the cohomology group $H^1(\mathfrak{g}, (V, f))$ of the Lie algebra $\mathfrak{g}$ with coefficients in (V, f) is zero, there exists an element $e \in V$ such that $q=d_f e$. Choosing an element $E \in \mathfrak{g}$ such that $q(E)=e$, we have $q(X)=f(X)q(E)$ for all $X \in \mathfrak{g}$. Since the cohomology group $H^1(\mathfrak{g}, (V^*, f^*))$ of the Lie algebra $\mathfrak{g}$ with coefficients in (V^*, f^*) is zero and since $d_{f^*}\gamma=0$, there exists an element $c^* \in V^*$ such that $\gamma=d_{f^*}c^*$. Therefore

$$\langle q(X), q(Y)\rangle=(\gamma(X))(q(Y))=((d_{f^*}c^*)(X))(q(Y))$$
$$=-c^*(f(X)q(Y))$$

for all $X, Y \in \mathfrak{g}$. In particular

$$\langle q(E), q(X)\rangle=\langle q(X), q(E)\rangle=-c^*(f(X)q(E))=-c^*(q(X))$$

for $X \in \mathfrak{g}$. Combining these with (C), we have

$$\langle f(E)q(X), q(Y)\rangle+\langle q(X), f(E)q(Y)\rangle$$
$$=\langle f(X)q(E), q(Y)\rangle+\langle q(E), f(X)q(Y)\rangle$$
$$=\langle q(X), q(Y)\rangle-c^*(f(X)q(Y))$$
$$=2\langle q(X), q(Y)\rangle .$$

This implies that $f(E)+{}^t f(E)=2$, where ${}^t f(E)$ is the transpose of $f(E)$ with respect to $\langle\, ,\rangle$. Taking the trace of the both sides of this formula we get $\mathrm{Tr}\, f(E)=\dim V$. On the other hand, since $\mathfrak{g}=[\mathfrak{g}, \mathfrak{g}]$, we have $\mathrm{Tr}\, f(E)=0$, which is a contradiction. Thus Proposition is completely proved.

2. In the following we always assume that (G, B) is an effective symmetric pair. Then there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g}=\mathfrak{b}+\mathfrak{m} \text{ (a vector space direct sum)},$$
(5)
$$[\mathfrak{b}, \mathfrak{b}]\subset\mathfrak{b}, \quad [\mathfrak{b}, \mathfrak{m}]\subset\mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{b} .$$

Since q is a linear isomorphism from $\mathfrak{m}$ onto V, for each $u\in V$ there exists a unique element $X_u\in\mathfrak{m}$ such that

$$q(X_u)=u\,.$$

We put

$$L_u=f(X_u)\,,$$

and define a multiplication low in V by

$$u\cdot v=L_u v\,. \tag{6}$$

Then, by (5) the algebra V is commutative.

LEMMA 1. *Let R_o be the value of the curvature tensor for the Riemannian metric g at o. Then, for $u, v\in V$ we have*

$$R_o(u, v)=-[L_u, L_v]\,.$$

PROOF. Identifying $\mathfrak{m}$ with V by q, it is known that

$$R_o(X, Y)Z=-[[X, Y], Z]$$

for $X, Y, Z\in\mathfrak{m}$ (cf. [**2**]). Therefore $R_o(u, v)w=q(R_o(X_u, X_v)X_w)=-q([[X_u, X_v], X_w])=-f([X_u, X_v])q(X_w)+f(X_w)q([X_u, X_v])=-[L_u, L_v]w$, for $u, v, w\in V$.

QED.

LEMMA 2. *For $W\in\mathfrak{h}$, $f(W)$ is a derivation of the algebra V.*

PROOF. Let $u\in V$. Since $q([W, X_u])=f(W)q(X_u)-f(X_u)q(W)=f(W)u$ and since $[W, X_u]\in\mathfrak{m}$, we obtain $[W, X_u]=X_{f(W)u}$. Therefore we have $(f(W)u)\cdot v=f(X_{f(W)u})v=f([W, X_u])v=f(W)f(X_u)v-f(X_u)f(W)v=f(W)(u\cdot v)-u\cdot(f(W)v)$.

QED.

For simplicity, we put $\alpha=\alpha_o$, $\tau=D\alpha_o$. Then the following formulas follow from (3) and (4).

$$\alpha(u)=\mathrm{Tr}\, L_u\,, \tag{3'}$$

$$\tau(u, v)=\alpha(u\cdot v)\,. \tag{4'}$$

LEMMA 3. *For $u, v, w\in V$, we have*

(i) $$[[L_u, L_v], L_w]=L_{[u\cdot w\cdot v]}\,,$$

(ii) $$\tau(u\cdot v, w)=\tau(v, u\cdot w)\,,$$

where $[u\cdot w\cdot v]=u\cdot(w\cdot v)-(u\cdot w)\cdot v$.

PROOF. Since $q([[X_u, X_v], X_w])=f([X_u, X_v])q(X_w)-f(X_w)q([X_u, X_v])=[L_u, L_v]w=[u\cdot w\cdot v]$ and since $[[X_u, X_v], X_w]\in\mathfrak{m}$, we have $[[X_u, X_v], X_w]=X_{[u\cdot w\cdot v]}$. Therefore we obtain $[[L_u, L_v], L_w]=f([[X_u, X_v], X_w])=f(X_{[u\cdot w\cdot v]})=L_{[u\cdot w\cdot v]}$. Applying this we have $\tau(u\cdot v, w)-\tau(v, u\cdot w)=\mathrm{Tr}\, L_{(u\cdot v)\cdot w-v\cdot(u\cdot w)}=-\mathrm{Tr}\, L_{[v\cdot u\cdot w]}=-\mathrm{Tr}\,[[L_v, L_w], L_u]=0$. QED.

LEMMA 4. *Let* ${}^t f(X)$ *denote the transpose of* $f(X)$ *with respect to* $\langle\,,\rangle$. *Then we have*

(i) $${}^t f(X)=f(X) \quad \text{for all } X\in\mathfrak{m},$$

(ii) $${}^t f(W)=-f(W) \quad \text{for all } W\in\mathfrak{b}.$$

In particular $f(\mathfrak{g})$ *is self-adjoint with respect to* $\langle\,,\rangle$.

PROOF. We recall the condition (C);

$$\begin{aligned}&\langle f(Y)q(Z), q(X)\rangle+\langle q(Z), f(Y)q(X)\rangle\\ &=\langle f(Z)q(Y), q(X)\rangle+\langle q(Y), f(Z)q(X)\rangle.\end{aligned}$$

Let $X, Y, Z\in\mathfrak{m}$. Since V is commutative, we have $\langle q(Z), f(X)q(Y)\rangle=\langle f(X)q(Z), q(Y)\rangle$, which implies (i). Let $X, Y\in\mathfrak{m}$ and $Z=W\in\mathfrak{b}$. Since $q(W)=0$, it follows that $\langle f(W)q(Y), q(X)\rangle+\langle q(Y), f(W)q(X)\rangle=0$. Thus (ii) is proved.

QED.

LEMMA 5. *Let* $\mathrm{Ker}\, f$ *denote the kernel of* f. *Then we have* $\mathrm{Ker}\, f\subset\mathfrak{m}$.

PROOF. Let $Z\in\mathrm{Ker}\, f\cap\mathfrak{b}$. For $X\in\mathfrak{m}$ we have $[Z, X]\in\mathfrak{m}$ and $q([Z, X])=f(Z)q(X)-f(X)q(Z)=0$. Hence $[Z, X]=0$. For $W\in\mathfrak{b}$ we have $[Z, W]\in[\mathrm{Ker}\, f, \mathfrak{b}]\cap[\mathfrak{b}, \mathfrak{b}]\subset\mathrm{Ker}\, f\cap\mathfrak{b}$. These imply that $\mathrm{Ker}\, f\cap\mathfrak{b}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{b}$. Since (G, B) is effective, it follows $\mathrm{Ker}\, f\cap\mathfrak{b}=\{0\}$. Now suppose that $X\in\mathfrak{m}$ and $W\in\mathfrak{b}$. Then $f(X+W)+{}^t f(X+W)=2f(X)$ and $f(X+W)-{}^t f(X+W)=2f(W)$ by Lemma 4. It follows that $\mathrm{Ker}\, f=\mathrm{Ker}\, f\cap\mathfrak{b}+\mathrm{Ker}\, f\cap\mathfrak{m}$. Thus we get $\mathrm{Ker}\, f\subset\mathfrak{m}$. QED.

We set

(6) $$\mathfrak{g}(\mathfrak{m})=[\mathfrak{m}, \mathfrak{m}]+\mathfrak{m},$$

Then $\mathfrak{g}(\mathfrak{m})$ is a subalgebra of $\mathfrak{g}$. Let $G(\mathfrak{m})$ be the connected Lie subgroup of G generated by $\mathfrak{g}(\mathfrak{m})$ and put $B(\mathfrak{m})=G(\mathfrak{m})\cap B$. Then we have

(7) $$G/B=G(\mathfrak{m})/B(\mathfrak{m}).$$

By the definition of $\mathfrak{g}(\mathfrak{m})$ and by Lemma 4 (i), $f(\mathfrak{g}(\mathfrak{m}))$ is self-adjoint with respect to $\langle\,,\rangle$. Let $V_0=\{v_0\in V\,;\, f(X)v_0=0 \text{ for all } X\in\mathfrak{g}(\mathfrak{m})\}$ and let V' be the orthogonal complement of V_0 with respect to $\langle\,,\rangle$. Then, under the action of $f(\mathfrak{g}(\mathfrak{m}))$, V' is invariant and is decomposed into a direct sum $V'=\sum_{i=1}^{r} V_i$ of mutually orthogonal, invariant and irreducible subspaces. We have then

(8) $$\begin{aligned}&V=V_0+V_1+\cdots+V_r,\\ &V_i\cdot V_i\subset V_i, \qquad V_i\cdot V_j=\{0\} \qquad (i\neq j).\end{aligned}$$

LEMMA 6. *Put* $\mathfrak{m}_i=\{X\in\mathfrak{m}\,;\, q(X)\in V_i\}$ $(0\leqq i\leqq r)$. *Then we have*

(i) $$\mathrm{Ker}\, f=\mathfrak{m}_0,$$

(ii) $$[\mathfrak{m}_i, \mathfrak{m}_j]=\{0\} \quad (0\leqq i<j\leqq r).$$

PROOF. Let $X\in\mathfrak{m}_0$. Then $f(X)q(Y)=f(Y)q(X)=0$ for all $Y\in\mathfrak{m}$, and therefore $X\in\operatorname{Ker} f$. Conversely, let $X\in\operatorname{Ker} f$. Since $X\in\mathfrak{m}$ by Lemma 5, we have $f(Y)q(X)=f(X)q(Y)=0$ for all $Y\in\mathfrak{m}$. Since $\mathfrak{g}(\mathfrak{m})$ is generated by $\mathfrak{m}$, we obtain $f(Y)q(X)=0$ for all $Y\in\mathfrak{g}(\mathfrak{m})$ and therefore $X\in\mathfrak{m}_0$. Thus (i) is proved. Let $X_i\in\mathfrak{m}_i$ and $X_j\in\mathfrak{m}_j$. By (8) we have $f([X_i, X_j])v=f(X_i)f(X_j)v-f(X_j)f(X_i)v=0$ for all $v\in V$. Therefore it follows $[X_i, X_j]\in\operatorname{Ker} f\cap\mathfrak{b}$ and so $[X_i, X_j]=0$ by Lemma 5. QED.

LEMMA 7. *If we set*

$$\mathfrak{g}_i=[\mathfrak{m}_i, \mathfrak{m}_i]+\mathfrak{m}_i$$

for $0\leqq i\leqq r$, *then we have*

(i) $$\mathfrak{g}(\mathfrak{m})=\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_r,$$

(ii) $\mathfrak{g}_i$ *is an ideal of* $\mathfrak{g}(\mathfrak{m})$.

PROOF. Since $[\mathfrak{m}_i, \mathfrak{m}_j]=\{0\}$ $(i\neq j)$, (i) is trivial, and $[\mathfrak{g}_i, \mathfrak{g}_j]=\{0\}$ $(i\neq j)$. Therefore, to prove (ii), it is sufficient to see $[[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i]\subset\mathfrak{m}_i$. This follows from the fact that $[[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i]\subset\mathfrak{m}$ and $q([[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i])=f([\mathfrak{m}_i, \mathfrak{m}_i])q(\mathfrak{m}_i)\subset f(\mathfrak{g})V_i\subset V_i$. QED.

LEMMA 8. *For* $X\in\mathfrak{g}_i$ *we denote by* $f_i(X)$ *the restriction of* $f(X)$ *to* V_i. *Then* f_i $(1\leqq i\leqq r)$ *is a faithful irreducible representation of* $\mathfrak{g}_i$ *in* V_i.

PROOF. We fix i between 1 and r and suppose $f_i(X)=0$. Since $f(\mathfrak{g}_i)$ is generated by $f(\mathfrak{m}_i)$ and since $f(\mathfrak{m}_i)V_j=\{0\}$, we have $f(X)V_j=\{0\}$ for $j\neq i$. Therefore $f(X)=0$. Since $\operatorname{Ker} f\cap\mathfrak{g}_i=\mathfrak{m}_0\cap\mathfrak{g}_i=\{0\}$ by Lemma 6, $X=0$ and so f_i is faithful. Let U_i be a subspace of V_i invariant under $f_i(\mathfrak{g}_i)$. Since $f(\mathfrak{g}_j)$ is generated by $f(\mathfrak{m}_j)$ and since $f(\mathfrak{m}_j)U_i=\{0\}$, we have $f(\mathfrak{g}_j)U_i=\{0\}$ for $j\neq i$. Therefore U_i is a subspace of V_i invariant under $f(\mathfrak{g}(\mathfrak{m}))$, and so $U_i=\{0\}$ or V_i, which proves that f_i is an irreducible representation. QED.

3. In this section we prove the following

PROPOSITION 2. *Under the same assumptions as in Theorem, assume further that* f *is a faithful irreducible representation of* $\mathfrak{g}$ *in* V. *Then the universal covering manifold of* G/B *is an irreducible homogeneous self-dual convex cone.*

Before proving this proposition, we prepare some results.

We put $U_0=\{u_0\in V;\ \tau(u_0, v)=0 \text{ for all } v\in V\}$. By Lemma 3 (ii) we have $\tau(u\cdot u_0, v)=\tau(u_0, u\cdot v)=0$ for $u_0\in U_0$, $u, v\in V$. Hence $f(\mathfrak{m})U_0\subset U_0$. Let $u_0\in U_0$, $v\in V$ and $W\in\mathfrak{b}$. Then, from Lemma 2 it follows that $\tau(f(W)u_0, v)=\operatorname{Tr} L_{(f(W)u_0)\cdot v}$ $=\operatorname{Tr} L_{f(W)(u_0\cdot v)}-\operatorname{Tr} L_{u_0\cdot(f(W)v)}=\operatorname{Tr} f(X_{f(W)(u_0\cdot v)})-\tau(u_0, f(W)v)=\operatorname{Tr} f([W, X_{u_0\cdot v}])=0$. This implies that $f(\mathfrak{b})U_0\subset U_0$. Therefore $f(\mathfrak{g})U_0\subset U_0$. Since f is an irreducible representation, we have $U_0=\{0\}$ or V, which means that τ is non-degenerate on V or $\tau=0$ on V.

Assume that $\tau=0$ on V. Let e be an element of V such that $\langle e, u\rangle=\alpha(u)$ for all $u\in V$. By (4′) and Lemma 4 (i) it follows that $\langle e\cdot u, v\rangle=\langle u\cdot e, v\rangle=\langle e, u\cdot v\rangle=\alpha(u\cdot v)=\tau(u, v)=0$ for all $u, v\in V$. Hence $L_e=0$. Therefore we have $\langle e, e\rangle=\alpha(e)=\mathrm{Tr}\, L_e=0$, and so $e=0$. Thus we obtain

(a) $$\alpha(u)=0 \quad \text{for all } u\in V.$$

Since $\mathfrak{g}$ admits a faithful irreducible representation, $\mathfrak{g}$ is a reductive Lie algebra (cf. [**1**]). Let $\mathfrak{g}=\mathfrak{c}+\mathfrak{s}$, where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{s}$ is a semi-simple part of $\mathfrak{g}$. Then it is clear $\mathfrak{c}=\mathfrak{c}\cap\mathfrak{b}+\mathfrak{c}\cap\mathfrak{m}$. Let $C\in\mathfrak{c}\cap\mathfrak{b}$. Then $f(C)q(X)=f(C)q(X)-f(X)q(C)=q([C, X])=0$ for all $X\in\mathfrak{m}$. Hence $f(C)=0$. Since f is faithful, it follows that $C=0$. Thus we have

(b) $$\mathfrak{c}\subset\mathfrak{m}.$$

Let $C\in\mathfrak{c}$ and let $P(x)$ be the minimal polynomial of $f(C)$. We shall prove that $P(x)$ is an irreducible polynomial over the real number field $\boldsymbol{R}$. In fact, assume $P(x)=Q(x)R(x)$, where $Q(x)$ and $R(x)$ are polynomials over $\boldsymbol{R}$ whose degrees are less than that of $P(x)$. If we put $U=Q(f(C))V$, then the subspace U of V is invariant under $f(\mathfrak{g})$. Since f is an irreducible representation, it follows that $U=\{0\}$ or V. In the case $U=\{0\}$, we have $Q(f(C))=0$, which contradicts the fact that $P(x)$ is the minimal polynomial of $f(C)$. In the case $U=V$, $Q(f(C))$ is non-singular. From $Q(f(C))R(f(C))=P(f(C))=0$, it follows $R(f(C))=0$, which is also a contradiction. Therefore $P(x)$ is an irreducible polynomial over $\boldsymbol{R}$.

The polynomial $P(x)$ is thus one of the forms $x-\lambda$ $(\lambda\in\boldsymbol{R})$ or $(x-\lambda)(x-\bar{\lambda})$ $(\lambda\in\boldsymbol{C},\ \bar{\lambda}\neq\lambda)$. Since ${}^t f(C)=f(C)$ by Lemma 4 (i) and (b), the eigenvalues of $f(C)$ are real. Hence it follows $P(x)=x-\lambda$ $(\lambda\in\boldsymbol{R})$ and $f(C)=\lambda$. By (a) we have $0=\alpha(q(C))=\mathrm{Tr}\, f(C)=\lambda \dim V$. Therefore $f(C)=0$. Since f is a faithful representation, we obtain $C=0$. Therefore we have $\mathfrak{c}=\{0\}$ and $\mathfrak{g}$ is a semi-simple Lie algebra, which contradicts Proposition 1. Thus we have so far proved the following

LEMMA 9. *τ is non-degenerate.*

We recall now the following known results (A), (B).

(A) *Let V be a commutative algebra with a multiplication $u\cdot v=L_u v$. Suppose*

(i) $[[L_u, L_v], L_w]=L_{[u\cdot w\cdot v]}$ *for* $u, v, w\in V$,

(ii) *the bilinear form* $\tau(u, v)=\mathrm{Tr}\, L_{u\cdot v}$ *is non-degenerate, where* $[u\cdot w\cdot v]=u\cdot(w\cdot v)-(u\cdot w)\cdot v$. *Then V is a semi-simple Jordan algebra* (cf. [7]).

(B) *Let V be a real Jordan algebra. Then the following conditions are equivalent.*

(i) *V is a formal real Jordan algebra.*

(ii) *The bilinear form $\tau(u, v)=\mathrm{Tr}\, L_{u\cdot v}$ is positive definite.*

(iii) *There exists an inner product* $\langle\,,\rangle$ *on* V *such that* $\langle u\cdot v, w\rangle=\langle v, u\cdot w\rangle$ *for all* $u, v, w\in V$ (cf. [3]).

In view of Lemmas 3, 4, 9, (A) and (B), we obtain

LEMMA 10. *V is a simple formal real Jordan algebra.*

We are now in a position to prove Proposition 2. Let $\tilde{G}$ be the universal covering group of G and let π be the covering projection of $\tilde{G}$ onto G. Denoting by $\tilde{B}_0$ the identity component of $\pi^{-1}(B)$, $\tilde{G}/\tilde{B}_0$ is the universal covering manifold of G/B. Since $\tilde{G}$ is simply connected, there exists a linear representation $\tilde{f}$ of $\tilde{G}$ in V such that $d\tilde{f}=f$, where $d\tilde{f}$ is the differential of $\tilde{f}$. Let e be an element in V such that $\tau(e, u)=\alpha(u)$ for all $u\in V$. Then e is the unit element in V and we have

$$f(X)e=q(X) \qquad \text{for all } X\in\mathfrak{g}. \tag{9}$$

In fact,

$$\begin{aligned}\tau(e\cdot u, v)&=\tau(u\cdot e, v)=\tau(e, u\cdot v)\\&=\alpha(u\cdot v)=\tau(u, v)\end{aligned}$$

for $u, v\in V$. Since τ is positive definite, it follows that $e\cdot u=u\cdot e=u$ for all $u\in V$. Hence $f(X)e=q(X)$ for all $X\in\mathfrak{m}$. By Lemma 2, $f(W)$ is a derivation of V for $W\in\mathfrak{b}$. Hence $f(W)e=0=q(W)$ for $W\in\mathfrak{b}$. We have thus proved (9).

Let $\Omega=\tilde{f}(\tilde{G})e$ be the orbit of $\tilde{f}(\tilde{G})$ through e. Then Ω is an irreducible homogeneous self-dual convex cone in V (cf. [3], [7]). Let $\tilde{H}=\{\tilde{h}\in\tilde{G}\,;\ \tilde{f}(\tilde{h})e=e\}$ and let $\tilde{\mathfrak{h}}$ be the subalgebra of $\mathfrak{g}$ corresponding to $\tilde{H}$. Then $\tilde{\mathfrak{h}}=\mathfrak{b}$. Indeed, by (9) X is contained in $\tilde{\mathfrak{h}}$ if and only if $f(X)e=q(X)=0$. Hence $\tilde{B}_0\subset\tilde{H}$. Therefore we have the natural projection $p: \tilde{G}/\tilde{B}_0\to\Omega=\tilde{G}/\tilde{H}$ and this map is a covering projection. Since Ω is convex, it is simply connected. Hence p gives an isomorphism from $\tilde{G}/\tilde{B}_0$ onto Ω. This completes the proof of Proposition 2.

4. Proof of Theorem. Let G_i be the connected Lie subgroup of G generated by $\mathfrak{g}_i$ and let $B_j=B(\mathfrak{m})\cap G_j$. Then it follows from (7), (8) and Lemma 7 that M is affinely diffeomorphic and isometric (with respect to D and g respectively) to the direct product

$$G_0/B_0\times G_1/B_1\times\cdots\times G_r/B_r.$$

According to Lemmas 1 and 6, $M_0=G_0/B_0$ is a locally flat Riemannian manifold. By Lemma 8 and by Proposition 2, the universal covering manifold of $M_i=G_i/B_i$ $(1\leqq i\leqq r)$ is an irreducible homogeneous self-dual convex cone. Thus our Theorem is completely proved.

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