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Nonlinear oscillation of second order functional differential equations with advanced argument

By Takaŝi KUSANO and Hiroshi ONOSE

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§ 1. Introduction.

In this paper we consider the nonlinear second order functional differential equation with advanced argument

$$[r(t)y'(t)]'+f(y(g(t)), t)=0\,. \tag{1}$$

The conditions we always assume for r, g, f are as follows:

(a) $r(t)$ is continuous and positive for $t\geqq\alpha$;
(b) $g(t)$ is continuous for $t\geqq\alpha$, and $g(t)\geqq t$;
(c) $f(y, t)$ is continuous for $|y|<\infty$, $t\geqq\alpha$, and $yf(y, t)>0$ for $y\neq 0$, $t\geqq\alpha$.

It is convenient to classify equations of the form (1) according to the nonlinearity of $f(y, t)$ with respect to y. Equation (1) is called *superlinear* if, for each fixed $t, f(y, t)/y$ is nondecreasing in y for $y>0$ and nonincreasing in y for $y<0$. It is called *strongly superlinear* if there exists a number $\sigma>1$ such that, for each fixed t, $f(y, t)/|y|^{\sigma}\operatorname{sgn} y$ is nondecreasing in y for $y>0$ and nonincreasing in y for $y<0$. Equation (1) is called *sublinear* if, for each fixed $t, f(y, t)/y$ is nonincreasing in y for $y>0$ and nondecreasing in y for $y<0$. It is called *strongly sublinear* if there exists a number $\tau<1$ such that, for each t, $f(y, t)/|y|^{\tau}\operatorname{sgn} y$ is nonincreasing in y for $y>0$ and nondecreasing in y for $y<0$. This classification includes the corresponding classification of the equations of the form

$$[r(t)y'(t)]'+y(g(t))F([y(g(t))]^{2}, t)=0 \tag{2}$$

as given in [**6**]. (See also [**5**] and [**9**].)

In what follows we restrict our discussion to those solutions $y(t)$ of (1) which exist on some ray $[T_{y}, \infty)$ and satisfy $\sup\{|y(t)| : t\geqq T\}>0$ for every $T\geqq T_{y}$. Such a solution is said to be *oscillatory* if the set of its zeros is not bounded; otherwise, it is said to be *nonoscillatory*. Equation (1) itself is called oscillatory if all of its solutions are oscillatory.

The problem of oscillation of solutions of functional differential equations with deviating arguments has received a wide attention during the last several years. Among numerous papers dealing with this problem we refer in particular to [**1-4**, **6**, **7**, **10**] for standard results regarding Equation (1) and/or related second order equations. Most of the literature, however, has been devoted to the investigation of differential equations with *retarded* arguments, and very little is known about the oscillatory behavior of differential equations with *advanced* arguments.

The main purpose of this paper is to undertake an effort in the direction of establishing oscillation and nonoscillation results for second order advanced differential equations. Use is made of some of the results and techniques developed in [**6**] for retarded differential equations of the form (2). We distinguish the two cases

(3) $$\int_{\alpha}^{\infty} r(t)^{-1}dt<\infty\,,$$

(4) $$\int_{\alpha}^{\infty} r(t)^{-1}dt=\infty\,,$$

and examine them separately in Sections 2 and 3. In each section to follow, we present necessary and sufficient conditions for Equation (1) which is either superlinear or sublinear to have nonoscillatory solutions with specific asymptotic properties, and then we provide oscillation criteria for Equation (1) which is either strongly superlinear or strongly sublinear. In particular, we are able to give a characterization for the oscillation situation of the strongly superlinear advanced equation (1). Thus it turns out that there is a remarkable difference between the oscillatory character of advanced differential equations and that of retarded differential equations.

§ 2. The case where $\int_{\alpha}^{\infty} r(t)^{-1}dt<\infty$.

In this section we examine the advanced differential equation (1) for which (3) is satisfied. We use the function $\rho(t)$ defined by

$$\rho(t)=\int_{t}^{\infty} r(s)^{-1}ds\,.$$

A) *Behavior of nonoscillatory solutions.* We begin with two lemmas which give useful information on the behavior of nonoscillatory solutions of (1).

LEMMA 1. *Let* (3) *hold. If* $y(t)$ *is an eventually positive solution of* (1), *then there are positive numbers* t_1, a_1, a_2 *such that*

(5) $$y(t)\geqq -r(t)y'(t)\rho(t) \qquad \textit{for} \quad t\geqq t_1\,,$$

and

$$a_1\rho(t)\leqq y(t)\leqq a_2 \qquad \text{for} \quad t\geqq t_1. \tag{6}$$

This lemma can be proved exactly as in the proof of the lemma of Kusano and Naito [**6**] concerning retarded differential equations of the form (2). The proof is therefore omitted. Evidently, similar inequalities hold for an eventually negative solutions of (1).

LEMMA 2. *In addition to* (3) *assume that*

$$\int^{\infty}\rho(t)|f(c, t)|dt=\infty \qquad \text{for all } c\neq 0. \tag{7}$$

Then all nonoscillatory solutions of (1) *tend to zero as* $t\to\infty$.

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). Without loss of generality we may suppose that $y(t)>0$ for $t\geqq t_1$. By Lemma 1 $y(t)$ is bounded above. Multiplying (1) by $\rho(t)$ and integrating from t_1 to t, we get

$$\begin{aligned}\rho(t)r(t)y'(t)+y(t)-\rho(t_1)r(t_1)y'(t_1)-y(t_1)&\\ +\int_{t_1}^{t}\rho(s)f(y(g(s)), s)ds=0&.\end{aligned} \tag{8}$$

We examine the following two cases:

$$\int_{t_1}^{\infty}\rho(t)f(y(g(t)), t)dt=\infty, \tag{9}$$

$$\int_{t_1}^{\infty}\rho(t)f(y(g(t)), t)dt<\infty. \tag{10}$$

Suppose (9) holds. Then, from (8) we see that $\rho(t)r(t)y'(t)\to-\infty$ as $t\to\infty$. Hence, there are numbers $t_2\geqq t_1$ and $M>0$ such that

$$\rho(t)r(t)y'(t)\leqq -M \qquad \text{for} \quad t\geqq t_2.$$

Dividing the above inequality by $\rho(t)r(t)$ and integrating give

$$y(t)-y(t_2)\leqq -M\int_{t_2}^{t}(\rho(s)r(s))^{-1}ds=M\log[\rho(t)/\rho(t_2)].$$

Since $\rho(t)\to 0$ as $t\to\infty$, it follows that $y(t)\to-\infty$ as $t\to\infty$. This contradicts the positivity of $y(t)$. Consequently, (9) cannot hold. Now, since (10) holds, letting $t\to\infty$ in (8), we see that

$$\lim_{t\to\infty}[\rho(t)r(t)y'(t)+y(t)]$$

exists and is finite. We claim that $\lim_{t\to\infty} y(t)$ exists as a finite number. In fact, if this is false, then there are numbers ξ and η such that

$$\liminf_{t\to\infty} y(t) < \xi < \eta < \limsup_{t\to\infty} y(t).$$

We are able to choose an increasing sequence $\{\tau_\nu\}_{\nu=1}^{\infty}$ with the following properties:

(11) $$\lim_{\nu\to\infty} \tau_\nu = \infty, \qquad y'(\tau_\nu) = 0, \qquad \nu = 1, 2, \cdots,$$

(12) $$y(\tau_{2\nu-1}) < \xi, \qquad y(\tau_{2\nu}) > \eta, \qquad \nu = 1, 2, \cdots.$$

In view of (11) the limit

$$\lim_{\nu\to\infty} [\rho(\tau_\nu) r(\tau_\nu) y'(\tau_\nu) + y(\tau_\nu)] = \lim_{\nu\to\infty} y(\tau_\nu)$$

exists. This, however, is a contradiction to (12). Therefore, there exists a finite limit $y(\infty) = \lim_{t\to\infty} y(t)$. If $y(\infty) > 0$, then there are positive constants c_1, c_2 such that

(13) $$c_1 \leqq y(g(t)) \leqq c_2 \qquad \text{for} \quad t \geqq t_1.$$

Using (10) and (13) we readily conclude that

$$\int_{t_1}^{\infty} \rho(t) f(c_1, t) dt < \infty \qquad \text{if (1) is superlinear,}$$

$$\int_{t_1}^{\infty} \rho(t) f(c_2, t) dt < \infty \qquad \text{if (1) is sublinear.}$$

This contradiction shows that $y(\infty) = 0$. Thus the proof is complete.

B) *Existence of nonoscillatory solutions.* According to inequality (6) of Lemma 1, among possible nonoscillatory solutions of Equation (1), those which are asymptotic to nonzero constants as $t \to \infty$ may be regarded as the "maximal" solutions, and those which are asymptotic to functions of the form $a\rho(t)$, $a \neq 0$, as $t \to \infty$ may be regarded as the "minimal" solutions. In case Equation (1) is either superlinear or sublinear necessary and sufficient conditions for the existence of nonoscillatory solutions of these two special kinds can be established without difficulty.

THEOREM 1. *Let* (1) *be either superlinear or sublinear. Assume that* (3) *holds. Then, a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/\rho(t) = \text{const.} \neq 0$ *is that*

(14) $$\int^{\infty} |f(c\rho(g(t)), t)| dt < \infty \qquad \textit{for some} \quad c \neq 0.$$

THEOREM 2. *Let* (1) *be either superlinear or sublinear. Assume that* (3) *holds. Then, a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t) = \text{const.} \neq 0$ *is that*

$$\int^{\infty}\rho(t)|f(c,t)|dt<\infty \qquad \textit{for some} \quad c\neq 0. \tag{15}$$

In [6, Theorems 1 and 2] it is shown that the same conclusions of the above theorems hold for the retarded differential equation (2). It is easily seen that the arguments used in [6] apply equally well to the case of the advanced equation (1). Since the theorem corresponding to Theorem 2 is proved in full detail in [6], we give here only the proof of Theorem 1.

PROOF OF THEOREM 1. (*Necessity*) Let $y(t)$ be a nonoscillatory solution with the property $\lim_{t\to\infty} y(t)/\rho(t)=k\neq 0$. We may suppose that $k>0$. There are positive numbers t_1, c_1, c_2 such that

$$c_1\rho(g(t))\leqq y(g(t))\leqq c_2\rho(g(t)) \qquad \text{for} \quad t\geqq t_1. \tag{16}$$

Since $y(t)\to 0$ as $t\to\infty$, $y'(t)\leqq 0$ eventually. Take t_1 so large that $y'(t)\leqq 0$ and (5) hold for $t\geqq t_1$. From (5) and (16) we get

$$-r(t)y'(t)\leqq c_2 \qquad \text{for} \quad t\geqq t_1. \tag{17}$$

Integrating (1) from t_1 to t and using (17), we find

$$\int_{t_1}^{\infty} f(y(g(t)),t)dt<\infty. \tag{18}$$

From (16) and (18) it follows that

$$\int_{t_1}^{\infty} f(c_1\rho(g(t)),t)dt<\infty \qquad \text{if (1) is superlinear},$$

$$\int_{t_1}^{\infty} f(c_2\rho(g(t)),t)dt<\infty \qquad \text{if (1) is sublinear}.$$

(*Sufficiency*) Suppose (14) holds with $c>0$. A similar argument can be applied if $c<0$. Put $a=c/2$ if (1) is superlinear and $a=c$ if (1) is sublinear. Choose $T>0$ so large that

$$\int_{T}^{\infty} f(c\rho(g(t)),t)dt<\frac{a}{2}. \tag{19}$$

We wish to obtain the required solution as a solution of the following integral equation

$$y(t)=a\rho(t)+\rho(t)\int_{T}^{t} f(y(g(s)),s)ds+\int_{t}^{\infty}\rho(s)f(y(g(s)),s)ds. \tag{20}$$

To solve (20) via Schauder's fixed point theorem we introduce the linear space $C_\rho[T,\infty)$ of all continuous functions $y:[T,\infty)\to R$ such that

$$\|y\|_\rho=\sup\{\rho(t)^{-1}|y(t)|:\ t\geqq T\}<\infty.$$

It is clear that $C_\rho[T,\infty)$ is a Banach space with norm $\|\cdot\|_\rho$. We define the operator Φ by

$$(\Phi y)(t)=a\rho(t)+\rho(t)\int_T^t f(y(g(s)), s)ds+\int_t^\infty \rho(s)f(y(g(s)), s)ds,$$

and seek a fixed point of Φ in the set $Y=\{y\in C_\rho[T,\infty): a\rho(t)\leqq y(t)\leqq 2a\rho(t)$ for $t\geqq T\}$, which is a bounded, convex and closed subset of $C_\rho[T,\infty)$. For this purpose we shall show that Φ is continuous and maps Y into a compact subset of Y.

i) *Φ maps Y into Y.* If $y\in Y$, then clearly, $(\Phi y)(t)\geqq a\rho(t)$, $t\geqq T$, and in view of (19)

$$\begin{aligned}(\Phi y)(t)&\leqq a\rho(t)+\rho(t)\int_T^\infty f(y(g(s)), s)ds\\ &\leqq a\rho(t)+2\rho(t)\int_T^\infty f(c\rho(g(s)), s)ds\\ &\leqq 2a\rho(t), \qquad t\geqq T.\end{aligned}$$

ii) *Φ is continuous.* Let $\{y_n\}$ be a sequence of elements of Y such that $\lim_{n\to\infty}\|y_n-y\|_\rho=0$. Since Y is closed, $y\in Y$ and

$$|(\Phi y_n)(t)-(\Phi y)(t)|\leqq\rho(t)\int_T^\infty F_n(s)ds,$$

where

$$F_n(s)=|f(y_n(g(s)), s)-f(y(g(s)), s)|.$$

It follows that

$$\|\Phi y_n-\Phi y\|_\rho\leqq\int_T^\infty F_n(s)ds. \tag{21}$$

Since

$$\lim_{n\to\infty}F_n(s)=0 \quad\text{and}\quad F_n(s)\leqq 4f(c\rho(g(s)), s) \qquad \text{for } s\geqq T,$$

we apply the Lebesgue dominated convergence theorem to conclude from (21) that $\lim_{n\to\infty}\|\Phi y_n-\Phi y\|_\rho=0$. This proves the continuity of Φ.

iii) *ΦY is precompact.* It suffices to show that the family of functions $\{\rho^{-1}\Phi y: y\in Y\}$ is uniformly bounded and equicontinuous on $[T,\infty)$. Since the uniform boundedness is obvious, we need only to demonstrate the equicontinuity. This will be accomplished if we show that, for any given $\varepsilon>0$, the interval $[T,\infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ε; see Levitan [8, § 3].

If $y\in Y$, then we have for $t_2>t_1\geqq T$

$$\begin{aligned}(\rho^{-1}\Phi y)(t_2)-(\rho^{-1}\Phi y)(t_1)&=\int_{t_1}^{t_2}f(y(g(s)), s)ds\\ &+\rho(t_2)^{-1}\int_{t_2}^\infty\rho(s)f(y(g(s)), s)ds-\rho(t_1)^{-1}\int_{t_1}^\infty\rho(s)f(y(g(s)), s)ds\end{aligned} \tag{22}$$

$$
(23)\qquad \begin{aligned}
&=\int_{t_1}^{t_2} f(y(g(s)), s)ds-\rho(t_2)^{-1}\int_{t_1}^{t_2}\rho(s)f(y(g(s)), s)ds \\
&\quad+[\rho(t_2)^{-1}-\rho(t_1)^{-1}]\int_{t_1}^{\infty}\rho(s)f(y(g(s)), s)ds .
\end{aligned}
$$

From (22) we obtain

$$
|(\rho^{-1}\Phi y)(t_2)-(\rho^{-1}\Phi y)(t_1)|\leqq 3\int_{t_1}^{\infty} f(y(g(s)), s)ds\leqq 6\int_{t_1}^{\infty} f(c\rho(g(s)), s)ds .
$$

By (19) the last integral tends to zero as $t_1\to\infty$, so, given an $\varepsilon>0$, there exists $T^*>T$ such that

$$
|(\rho^{-1}\Phi y)(t_2)-(\rho^{-1}\Phi y)(t_1)|<\varepsilon \qquad \text{if} \quad t_2>t_1\geqq T^* .
$$

This shows that the oscillations of all $\rho^{-1}\Phi y$, $y\in Y$, on $[T^*, \infty)$ are less than ε. Now, let $T\leqq t_1<t_2\leqq T^*$. Then, in view of (23) and (19) we see that

$$
\begin{aligned}
|(\rho^{-1}\Phi y)(t_2)-(\rho^{-1}\Phi y)(t_1)|&\leqq 2\int_{t_1}^{t_2} f(c\rho(g(s)), s)ds \\
&\quad+2\rho(t_1)\rho(t_2)^{-1}\int_{t_1}^{t_2} f(c\rho(g(s)), s)ds \\
&\quad+2\rho(t_2)^{-1}|\rho(t_2)-\rho(t_1)|\int_{t_1}^{\infty} f(c\rho(g(s)), s)ds \\
&\leqq a\rho(T^*)^{-1}|\rho(t_2)-\rho(t_1)|+2[\rho(T)\rho(T^*)^{-1}+1]\int_{t_1}^{t_2} f(c\rho(g(s)), s)ds .
\end{aligned}
$$

This ensures the existence of $\delta>0$ such that for all $y\in Y$

$$
|(\rho^{-1}\Phi y)(t_2)-(\rho^{-1}\Phi y)(t_1)|<\varepsilon \qquad \text{if} \quad t_2-t_1<\delta .
$$

Consequently, we can divide the interval $[T, T^*]$, and hence the whole interval $[T, \infty)$, into a finite number of subintervals on each of which every $\rho^{-1}\Phi y$, $y\in Y$, has oscillation less than ε. It follows that $\overline{\Phi Y}$ is a compact subset of Y.

From the preceding observations we see that Schauder's fixed point theorem can be applied to the operator Φ. Let $y\in Y$ be a fixed point of Φ. Then, by the definition of Φ, $y(t)$ is a solution of the integral equation (20) for $t\geqq T$. Since, by l'Hospital's rule,

$$
\begin{aligned}
\lim_{t\to\infty}\frac{y(t)}{\rho(t)}=\lim_{t\to\infty}\frac{y'(t)}{\rho'(t)}&=a+\int_{T}^{\infty} f(y(g(s)), s)ds \\
&\leqq a+2\int_{T}^{\infty} f(c\rho(g(s)), s)ds\leqq 2a ,
\end{aligned}
$$

we conclude that $y(t)$ is a solution of Equation (1) with the required asymptotic property. This completes the proof of Theorem 1.

An important special case of (1) is the following generalized Emden-Fowler equation

$$[r(t)y'(t)]'+p(t)|y(g(t))|^{\gamma}\operatorname{sgn} y(g(t))=0 \tag{24}$$

where γ is a positive constant and $p(t)$ is a continuous and nonnegative function on $[\alpha, \infty)$.

COROLLARY 1. (i) *Equation* (24) *has a nonoscillatory solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/\rho(t)=\text{const.} \neq 0$ *if and only if*

$$\int^{\infty}[\rho(g(t))]^{\gamma}p(t)dt<\infty .$$

(ii) *Equation* (24) *has a nonocsillatory solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)=\text{const.}$ $\neq 0$ *if and only if*

$$\int^{\infty}\rho(t)p(t)dt<\infty .$$

C) *Oscillation criteria.* The object of this subsection is to present oscillation criteria for Equation (1) which is either strongly superlinear or strongly sublinear. We start with the strongly superlinear case for which a characterization of the oscillation situation can be established.

THEOREM 3. *Let* (1) *be strongly superlinear. Assume that* (3) *holds. Then, a necessary and sufficient condition for* (1) *to be oscillatory is that*

$$\int^{\infty}|f(c\rho(g(t)), t)|dt=\infty \qquad \text{for all} \quad c\neq 0 . \tag{25}$$

PROOF. That (25) is necessary follows from Theorem 1. To prove the sufficiency part suppose there exists a nonoscillatory solution $y(t)$ of (1). Without loss of generality we may suppose that $y(t)>0$ for $t\geqq t_0$. Obviously, $y(g(t))>0$ for $t\geqq t_0$. From (1) $[r(t)y'(t)]'\leqq 0$ for $t\geqq t_0$, so that $r(t)y'(t)$ is nonincreasing. Hence, $y'(t)$ is eventually of constant sign. We show that $y'(t)<0$ eventually. Let $c>0$ be given arbitrarily but fixed. Take $t_1\geqq t_0$ so large that $\rho(g(t))\leqq c$ for $t\geqq t_1$. By the superlinearity and the monotonicity of $\rho(t)$ we have

$$cf(c\rho(g(t)), t)\leqq \rho(t)f(c, t) \qquad \text{for} \quad t\geqq t_1 . \tag{26}$$

Since (25) and (26) imply (7), it follows from Lemma 2 that $y(t)\to 0$ as $t\to\infty$. This rules out the possibility that $y'(t)>0$ for $t\geqq t_0$, so there exists $t_2>t_1$ such that $y'(t)<0$ for $t\geqq t_2$.

By Lemma 1 the following inequalities hold:

$$y(t)\geqq -r(t)y'(t)\rho(t)\geqq k\rho(t) , \qquad t\geqq t_2 , \tag{27}$$

where $k=-r(t_2)y'(t_2)>0$. Using (27) and the strong superlinearity, we get

$$[y(g(t))]^{-\sigma}f(y(g(t)), t)\geqq [k\rho(g(t))]^{-\sigma}f(k\rho(g(t)), t) , \tag{28}$$

where $\sigma>1$ is the strong superlinearity constant. With the help of (27), (28) and the decreasing property of $r(t)y'(t)$ we obtain

$$\begin{aligned}\{-[-r(t)y'(t)]^{1-\sigma}\}' &= (\sigma-1)[-r(t)y'(t)]^{-\sigma}f(y(g(t)), t)\\ &= (\sigma-1)[-r(t)y'(t)]^{-\sigma}\cdot[y(g(t))]^{\sigma}\cdot[y(g(t))]^{-\sigma}f(y(g(t)), t)\\ &\geqq (\sigma-1)[-r(g(t))y'(g(t))]^{-\sigma}\cdot[-r(g(t))y'(g(t))\rho(g(t))]^{\sigma}\cdot\\ &\qquad\cdot[k\rho(g(t))]^{-\sigma}f(k\rho(g(t)), t)\\ &= (\sigma-1)k^{-\sigma}f(k\rho(g(t)), t).\end{aligned} \tag{29}$$

An integration of (29) yields

$$(\sigma-1)k^{-\sigma}\int_{t_2}^{t}f(k\rho(g(s)), s)ds \leqq [-r(t_2)y'(t_2)]^{1-\sigma}-[-r(t)y'(t)]^{1-\sigma},$$

which implies $\int_{t_2}^{\infty}f(k\rho(g(t)), t)dt<\infty$, a contradiction to (25). It follows that Equation (1) cannot possess nonoscillatory solutions, and the proof is complete.

COROLLARY 2. *Consider Equation* (24) *with* $\gamma>1$. *Assume that* (3) *holds. Then, a necessary and sufficient condition for* (24) *to be oscillatory is that*

$$\int^{\infty}[\rho(g(t))]^{\gamma}p(t)dt=\infty.$$

Next, we provide an oscillation criterion for the strongly sublinear equation (1).

THEOREM 4. *Let* (1) *be strongly sublinear. In addition to* (3) *assume that* $g'(t)\geqq 0$ *for* $t\geqq\alpha$. *Then, a sufficient condition for* (1) *to be oscillatory is that*

$$\int^{\infty}\rho(g(t))|f(c, t)|dt=\infty \qquad \textit{for all} \quad c\neq 0. \tag{30}$$

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). Suppose that $y(t)>0$ for $t\geqq t_0$. Since (30) implies (7), by Lemma 2 it follows that $y(t)\to 0$ as $t\to\infty$. Hence there is $t_1>t_0$ such that $y'(t)<0$ for $t\geqq t_1$. Integrating (1) from t_1 to t and using the fact that $r(t)y'(t)$ is nonincreasing, we obtain

$$-r(g(t))y'(g(t))\geqq\int_{t_1}^{t}f(y(g(s)), s)ds, \qquad t\geqq t_1,$$

which yields

$$-y'(g(t))g'(t)\geqq\frac{g'(t)}{r(g(t))}\int_{t_1}^{t}f(y(g(s)), s)ds, \qquad t\geqq t_1. \tag{31}$$

Now it holds that

$$\begin{aligned}\{-[y(g(t))]^{1-\tau}\}' &= (1-\tau)[y(g(t))]^{-\tau}\frac{g'(t)}{r(g(t))}\int_{t_1}^{t}f(y(g(s)), s)ds\\ &\geqq (1-\tau)\frac{g'(t)}{r(g(t))}\int_{t_1}^{t}[y(g(s))]^{-\tau}f(y(g(s)), s)ds,\end{aligned} \tag{32}$$

where $\tau<1$ is the strong sublinearity constant. Here we have used (31) and the decreasing character of $y(t)$. There is a constant $k>0$ such that $y(g(t))\leqq k$ for $t\geqq t_1$, so by the strong sublinearity of (1)

$$[y(g(t))]^{-\tau}f(y(g(t)),t)\geqq k^{-\tau}f(k,t)\,,\qquad t\geqq t_1\,. \tag{33}$$

From (32) and (33) it follows that

$$\{-[y(g(t))]^{1-\tau}\}'\geqq(1-\tau)k^{-\tau}\frac{g'(t)}{r(g(t))}\int_{t_1}^{t}f(k,s)ds\,.$$

An integration of the above inequality yields

$$(1-\tau)k^{-\tau}\int_{t_1}^{t}\Big(\int_{g(s)}^{g(t)}\frac{d\sigma}{r(\sigma)}\Big)f(k,s)ds\leqq[y(g(t_1))]^{1-\tau}-[y(g(t))]^{1-\tau}\,. \tag{34}$$

Letting $t\to\infty$ in (34) we obtain

$$\int_{t_1}^{\infty}\rho(g(s))f(k,s)ds<\infty\,,$$

which contradicts (30). A parallel argument holds if we assume that $y(t)<0$ for $t\geqq t_0$. This finishes the proof.

COROLLARY 3. *Consider Equation* (24) *with* $0<\gamma<1$. *In addition to* (3) *assume tkat* $g'(t)\geqq 0$ *for* $t\geqq\alpha$. *A sufficient condition for* (24) *to be oscillatory is that*

$$\int^{\infty}\rho(g(t))p(t)dt=\infty\,.$$

From Theorem 2 it follows that if Equation (1), whether superlinear or sublinear, is oscillatory, then

$$\int^{\infty}\rho(t)|f(c,t)|dt=\infty\qquad\text{for all}\quad c\neq 0\,. \tag{35}$$

For the oscillation of the strongly sublinear equation (1) there is a gap between the necessary condition (35) and the sufficient condition (30). Clearly, this gap is caused by the presence of the advanced argument $g(t)$. The following example shows that (35) is not sufficient for the strongly sublinear equation (1) to be oscillatory.

EXAMPLE 1. Consider the advanced equation

$$[t^3y'(t)]'+t[y(t^3)]^{1/3}=0\,,$$

which has a nonoscillatory solution $y(t)=t^{-1}$. Obviously, (35) is satisfied but (30) is violated.

It may happen, however, that (35) is identical with (30). This is the case, for instance, if $g(t)\equiv t$, or if $r(t)$ and $g(t)$ are subject to one of the following restrictions:

(I) $r(t)$ is asymptotic to $ct(\log t)^{\alpha}$ as $t\to\infty$, and $\limsup_{t\to\infty} g(t)/t^{\beta}<\infty$, where $c>0$, $\alpha>1$ and $\beta\geqq 1$;

(II) $r(t)$ is asymptotic to ct^{p} as $t\to\infty$, and $\limsup_{t\to\infty} g(t)/t<\infty$, where $c>0$ and $p>1$;

(III) $r(t)$ is asymptotic to ce^{qt} as $t\to\infty$, and $g(t)=t+\tau(t)$, $0\leqq\tau(t)\leqq M$, where $c>0$, $q>0$ and $M>0$.

In each of these cases (35) becomes a necessary and sufficient condition for the strongly sublinear equation (1) to be oscillatory. It might be a question of interest to characterize completely the oscillation of the general advanced equation (1) which is strongly sublinear.

§ 3. The case where $\int_{\alpha}^{\infty} r(t)^{-1}dt=\infty$.

We now turn to the advanced differential equation (1) for which (4) is satisfied. In what follows we use the function $R(t)$ defined by

$$R(t)=\int_{\alpha}^{t} r(s)^{-1}ds\,.$$

A) *Behavior of nonoscillatory solutions.* We need the following lemma which describes the possible behavior of nonoscillatory solutions of (1).

LEMMA 3. *Let* (4) *hold. If $y(t)$ is an eventually positive solution of* (1), *then there are positive numbers t_1, a_1, a_2 such that*

$$y'(t)>0 \qquad for \quad t\geqq t_1\,, \tag{36}$$

and

$$a_1\leqq y(t)\leqq a_2R(t) \qquad for \quad t\geqq t_1\,. \tag{37}$$

We omit the proof of this lemma, since it is essentially the same as that of the lemma of Bykov and Merzlyakova [3]. Clearly, similar inequalities hold for an eventually negative solution of (1).

B) *Existence of nonoscillatory solutions.* Lemma 3 tells us that, among all nonoscillatory solutions of (1), those which are bounded can be regarded as the "minimal" solutions, and those which are asymptotic to functions of the form $aR(t)$, $a\neq 0$, as $t\to\infty$ can be regarded as the "maximal" solutions. Necessary and sufficient conditions for the existence of these two special types of nonoscillatory solutions are given in the following theorems.

THEOREM 5. *Let* (1) *be either superlinear or sublinear. Assume that* (4) *holds. Then, a necessary and sufficient condition for* (1) *to have a nonoscillatory solution $y(t)$ such that $\lim_{t\to\infty} y(t)/R(t)=$const. $\neq 0$ is that*

(38) $$\int^{\infty} |f(cR(g(t)), t)|\,dt<\infty \qquad \textit{for some } c\neq 0.$$

THEOREM 6. *Let* (1) *be either superlinear or sublinear. Assume that* (4) *holds. Then, a necessary and sufficient condition for* (1) *to have a bounded nonoscillatory solution is that*

(39) $$\int^{\infty} R(t)|f(c, t)|\,dt<\infty \qquad \textit{for some} \quad c\neq 0.$$

We shall only prove Theorem 6, since the proof of Theorem 5 is very similar.

PROOF OF THEOREM 6. (*Necessity*) Let $y(t)$ be a bounded nonoscillatory solution of (1). We may suppose that $y(t)>0$ for $t\geqq t_1$, since the case $y(t)<0$ can be handled similarly. By (36) $y'(t)>0$ for $t\geqq t_1$, so there are positive constants c_1, c_2 such that

(40) $$c_1\leqq y(g(t))\leqq c_2 \qquad \text{for} \quad t\geqq t_1.$$

Multiplying (1) by $R(t)$ and integrating from t_1 to t, we have

$$R(t)r(t)y'(t)-y(t)-R(t_1)r(t_1)y'(t_1)+y(t_1)$$
$$+\int_{t_1}^{t} R(s)f(y(g(s)), s)\,ds=0,$$

which implies in view of the boundedness of $y(t)$ and the positivity of $y'(t)$ that

(41) $$\int_{t_1}^{\infty} R(t)f(y(g(t)), t)\,dt<\infty.$$

From (40) and (41) we readily have

$$\int_{t_1}^{\infty} R(t)f(c_1, t)\,dt<\infty \qquad \text{if (1) is superlinear,}$$

$$\int_{t_1}^{\infty} R(t)f(c_2, t)\,dt<\infty \qquad \text{if (1) is sublinear.}$$

(*Sufficiency*) Suppose (39) holds with $c>0$. Put $a=c/2$ if (1) is superlinear and $a=c$ if (1) is sublinear. Choose $T>0$ so large that

(42) $$\int_{T}^{\infty} R(t)f(c, t)\,dt<\frac{a}{2}$$

and consider the integral equation

(43) $$y(t)=a+\int_{T}^{t} R(s)f(y(g(s)), s)\,ds+R(t)\int_{t}^{\infty} f(y(g(s)), s)\,ds.$$

A solution of (43) is clearly a solution of Equation (1). We introduce the

Banach space $CB[T, \infty)$ of all bounded continuous functions $y: [T, \infty) \to R$ with norm

$$\|y\| = \sup\{|y(t)| : t \geqq T\}.$$

We define the operator Ψ by

$$(\Psi y)(t) = a + \int_T^t R(s) f(y(g(s)), s) ds + R(t) \int_t^\infty f(y(g(s)), s) ds.$$

A fixed point of Ψ is sought for in the set $Y = \{y \in CB[T, \infty) : a \leqq y(t) \leqq 2a$ for $t \geqq T\}$, which is a bounded, closed and convex subset of $CB[T, \infty)$.

i) *Ψ maps Y into Y.* If $y \in Y$, then $(\Psi y)(t) \geqq a$, $t \geqq T$, and by (42)

$$(\Psi y)(t) \leqq a + \int_T^\infty R(s) f(y(g(s)), s) ds$$

$$\leqq a + 2\int_T^\infty R(s) f(c, s) ds \leqq 2a, \qquad t \geqq T.$$

ii) *Ψ is continuous.* Let $\{y_n\}$ be a sequence of elements of Y such that $\lim_{n\to\infty} \|y_n - y\| = 0$. Since Y is closed, $y \in Y$, and

$$|(\Psi y_n)(t) - (\Psi y)(t)| \leqq \int_T^\infty G_n(s) ds, \qquad t \geqq T,$$

where

$$G_n(s) = R(s) |f(y_n(g(s)), s) - f(y(g(s)), s)|.$$

It follows that

$$\|\Psi y_n - \Psi y\| \leqq \int_T^\infty G_n(s) ds. \tag{44}$$

Noting that $\lim_{n\to\infty} G_n(s) = 0$ and $G_n(s) \leqq 4R(s) f(c, s)$ for $s \geqq T$ and applying the Lebesgue dominated convergence theorem, we obtain from (44) that $\lim_{n\to\infty} \|\Psi y_n - \Psi y\| = 0$, proving the continuity of Ψ.

iii) *ΨY is precompact.* It suffices to prove that ΨY is equicontinuous on $[T, \infty)$. Let $y \in Y$ and $t_2 > t_1 \geqq T$. Then, we have

$$\begin{aligned}(\Psi y)(t_2) - (\Psi y)(t_1) = &\int_{t_1}^{t_2} R(s) f(y(g(s)), s) ds \\ &+ R(t_2) \int_{t_2}^\infty f(y(g(s)), s) ds - R(t_1) \int_{t_1}^\infty f(y(g(s)), s) ds\end{aligned} \tag{45}$$

$$\begin{aligned}= &\int_{t_1}^{t_2} R(s) f(y(g(s)), s) ds - R(t_2) \int_{t_1}^{t_2} f(y(g(s)), s) ds \\ &+ [R(t_2) - R(t_1)] \int_{t_1}^\infty f(y(g(s)), s) ds.\end{aligned} \tag{46}$$

Using (45) we have

$$|(\Psi y)(t_2)-(\Psi y)(t_1)| \leqq 3\int_{t_1}^{\infty} R(s)f(y(g(s)), s)ds$$
$$\leqq 6\int_{t_1}^{\infty} R(s)f(c, s)ds\,.$$

Since by (39) the last integral tends to zero as $t_1 \to \infty$, given an $\varepsilon>0$, there exists $T^*>T$ such that for all $y\in Y$,

$$(47) \qquad |(\Psi y)(t_2)-(\Psi y)(t_1)|<\varepsilon \quad \text{if} \quad t_2>t_1\geqq T\,.$$

Using (46) we see that if $T\leqq t_1<t_2\leqq T^*$, then

$$|(\Psi y)(t_2)-(\Psi y)(t_1)| \leqq \int_{t_1}^{t_2} R(s)f(y(g(s)), s)ds$$
$$+R(t_2)\int_{t_1}^{t_2} f(y(g(s)), s)ds+|R(t_2)-R(t_1)|\int_{t_1}^{\infty} f(y(g(s)), s)ds$$
$$\leqq aR(T)^{-1}|R(t_2)-R(t_1)|+4aR(T^*)\int_{t_1}^{t_2} f(c, s)ds\,,$$

which shows that there exists $\delta>0$ such that for all $y\in Y$,

$$(48) \qquad |(\Psi y)(t_2)-(\Psi y)(t_1)|<\varepsilon \quad \text{if} \quad t_2-t_1<\delta\,.$$

In view of (47) and (48) we are able to decompose the interval $[T, \infty)$ into a finite number of subintervals on each of which all functions Ψy, $y\in Y$, have oscillations less than ε. Thus, ΨY is precompact.

By Schauder's fixed point theorem there exists $y\in Y$ such that $y=\Psi y$. Then, $y(t)$ is a solution to the integral equation (43), and since

$$y'(t)=r(t)^{-1}\int_{t}^{\infty} f(y(g(s)), s)ds>0\,,$$

$y(t)$ goes monotonically to a positive limit in $[a, 2a]$ as $t\to\infty$. This completes the proof.

COROLLARY 4. (i) *Equation* (24) *which satisfies* (4) *has a nonoscillatory solution* $y(t)$ *such that* $\lim_{t\to\infty} y(t)/R(t)=\text{const.}\neq 0$ *if and only if*

$$\int^{\infty} [R(g(t))]^{\gamma} p(t)dt<\infty\,.$$

(ii) *Equation* (24) *which satisfies* (4) *has a bounded nonoscillatory solution if and only if*

$$\int^{\infty} R(t)p(t)dt<\infty\,.$$

C) *Oscillation criteria.* We provide oscillation criteria for Equation (1) which is either strongly superlinear or strongly sublinear. As in the preceding section a necessary and sufficient condition is obtained for the oscillation of the strongly superlinear equation (1).

THEOREM 7. *Let* (1) *be strongly superlinear. Suppose* (4) *holds. Then, a necessary and sufficient condition for* (1) *to be oscillatory is that*

$$\int^{\infty} R(t)|f(c, t)|dt=\infty \qquad \text{for all} \quad c\neq 0\,. \tag{49}$$

PROOF. The necessity of (49) follows immediately from Theorem 6. To prove that (49) is sufficient let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(g(t))>0$ for $t\geqq t_1$. By Lemma 3 $y'(t)>0$ for $t\geqq t_1$. Integrating (1) from t to infinity, we have

$$r(t)y'(t)\geqq\int_t^{\infty} f(y(g(s)), s)ds\,, \qquad t\geqq t_1\,.$$

Dividing the above by $r(t)$ and integrating from t_1 to t, we find

$$y(t)\geqq\int_{t_1}^{t} R(s\,;\,t_1)f(y(g(s)), s)ds\,, \qquad t\geqq t_1\,,$$

where

$$R(t\,;\,t_1)=\int_{t_1}^{t} r(s)^{-1}ds\,.$$

Since $y(t)$ is increasing and $g(t)\geqq t$, it follows that

$$y(g(t))\geqq\int_{t_1}^{t} R(s\,;\,t_1)f(y(g(s)), s)ds\,, \qquad t\geqq t_1\,. \tag{50}$$

On the other hand, there exists a constant $k>0$ such that $y(g(t))\geqq k$ for $t\geqq t_1$, so that by the strong superlinearity

$$\begin{aligned} f(y(g(s)), s)&=[y(g(s))]^{\sigma}\cdot[y(g(s))]^{-\sigma}f(y(g(s)), s) \\ &\geqq k^{-\sigma}[y(g(s))]^{\sigma}f(k, s)\,, \end{aligned} \tag{51}$$

where $\sigma>1$ is the strong superlinearity constant. Combining (50) with (51) gives

$$[y(g(t))]^{-\sigma}\leqq k^{\sigma^2}\Big(\int_{t_1}^{t} R(s\,;\,t_1)[y(g(s))]^{\sigma}f(k, s)ds\Big)^{-\sigma}. \tag{52}$$

Multiplying (52) by $R(t, t_1)[y(g(t))]^{\sigma}f(k, t)$ and integrating over $[t_2, t]$, $t_2>t_1$, we obtain

$$\int_{t_2}^{t} R(s\,;\,t_1)f(k, s)ds\leqq\frac{k^{\sigma^2}}{\sigma-1}\Big(\int_{t_1}^{u} R(s\,;\,t_1)[y(g(s))]^{\sigma}f(k, s)ds\Big)^{1-\sigma}\Big|_{u=t}^{u=t_2}$$

which yields $\int_{t_2}^{\infty} R(t\,;\,t_1)f(k, t)dt<\infty$, a contradiction to (49). The proof is therefore complete.

COROLLARY 5. *Consider Equation* (24) *with* $\gamma>1$. *Suppose* (4) *holds. Then,* (24) *is oscillatory if and only if*

$$\int^{\infty} R(t)p(t)dt=\infty .$$

We now pass to the strongly sublinear equation (1).

THEOREM 8. *Let* (1) *be strongly sublinear. Suppose* (4) *holds. If*

$$\int^{\infty}\frac{R(t)}{R(g(t))}|f(cR(g(t)), t)|dt=\infty \qquad \text{for all} \quad c\neq 0, \tag{53}$$

then (1) *is oscillatory.*

PROOF. Let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(g(t))>0$ for $t\geqq t_1$. From (1), $r(t)y'(t)$ is nonincreasing, and by Lemma 3 $y'(t)>0$ for $t\geqq t_1$. We observe that

$$y(t)>\int_{t_1}^{t} r(s)^{-1}r(s)y'(s)ds\geqq r(t)y'(t)\int_{t_1}^{t} r(s)^{-1}ds ,$$

that is,

$$y(t)>R(t\,;t_1)r(t)y'(t) , \qquad t\geqq t_1 . \tag{54}$$

On the other hand, from Lemma 3 it follows that there are positive numbers $t_2>t_1$ and k such that

$$y(t)\leqq kR(t\,;t_1) \quad \text{or} \quad y(g(t))\leqq kR(g(t)\,;t_1) \qquad \text{for} \quad t\geqq t_2 . \tag{55}$$

Noting that $y(g(t))\geqq y(t)$ and using (54) and (55) together with the strong sublinearity with constant $\tau<1$, we see that

$$\begin{aligned}
\{-[r(t)y'(t)]^{1-\tau}\}' &=(1-\tau)[r(t)y'(t)]^{-\tau}f(y(g(t)), t)\\
&=(1-\tau)[r(t)y'(t)]^{-\tau}\cdot[y(g(t))]^{\tau}\cdot[y(g(t))]^{-\tau}f(y(g(t)), t)\\
&\geqq(1-\tau)[r(t)y'(t)]^{-\tau}\cdot[y(t)]^{\tau}\cdot[y(g(t))]^{-\tau}f(y(g(t)), t)\\
&\geqq(1-\tau)k^{-\tau}[R(t\,;t_1)]^{\tau}[R(g(t)\,;t_1)]^{-\tau}f(kR(g(t)\,;t_1), t)\\
&\geqq(1-\tau)k^{-\tau}[R(t\,;t_1)/R(g(t)\,;t_1)]^{\tau}f(kR(g(t)\,;t_1), t) .
\end{aligned}$$

An integration of the above shows that

$$\int^{\infty}\frac{R(t\,;t_1)}{R(g(t)\,;t_1)}f(kR(g(t)\,;t_1), t)dt<\infty ,$$

which contradicts (53). This completes the proof.

REMARK. If we assume, in addition to the hypotheses of Theorem 8, that

$f(y, t)$ is nondecreasing in y for each t, then we can replace (53) by the following weaker condition:

$$\int^{\infty} |f(cR(t), t)| dt = \infty \qquad \text{for all} \quad c \neq 0 .$$

On the basis of the above remark we have the following

COROLLARY 6. *Consider Equation* (24) *with* $0<\gamma<1$. *Suppose* (4) *holds. Then,* (24) *is oscillatory if*

$$\int^{\infty} [R(t)]^{\gamma} p(t) dt = \infty .$$

From Theorem 5 it follows that

$$\int^{\infty} |f(cR(g(t)), t)| dt = \infty \qquad \text{for all} \quad c \neq 0 . \tag{56}$$

is a necessary condition for (1) which is either superlinear or sublinear to be oscillatory. For the strongly sublinear equation (1) it may happen that (53) is equivalent to (56). This is the case, for example, if $g(t) \equiv t$, or if $r(t)$ and $g(t)$ satisfy one of the following conditions:

(I) $r(t)$ is asymptotic to ct as $t \to \infty$, and $\limsup_{t\to\infty} g(t)/t^{\beta} < \infty$, where $c>0$ and $\beta \geqq 1$;

(II) $r(t)$ is asymptotic to ct^{p} as $t \to \infty$, and $\limsup_{t\to\infty} g(t)/t < \infty$, where $c>0$ and $p<1$;

(III) $r(t)$ is asymptotic to ce^{-qt} as $t \to \infty$, and $g(t) = t + \tau(t)$, $0 \leqq \tau(t) \leqq M$, where $c>0$, $q>0$ and $M>0$.

In each of the above cases (56) is a necessary and sufficient condition in order that the strongly sublinear equation (1) be oscillatory.

However, when $g(t)$ is a general advanced argument, as the following example shows, there is a gap between (53) and (56) which prevents (56) from being a sufficient condition for the oscillation of (1).

EXAMPLE 2. Consider the advanced equation

$$[t^{-2} y'(t)]' + 2t^{-4} [y(t^{3})]^{1/3} = 0 ,$$

which has a nonoscillatory solution $y(t) = t$. As is easily seen, (56) is satisfied but (53) is violated. We note that the associated ordinary differential equation

$$[t^{2} y'(t)]' + 2t^{-4} [y(t)]^{1/3} = 0$$

is oscillatory.

We have been unable to bridge the gap between (53) and (56) to establish a necessary and sufficient condition for the oscillation of the strongly sublinear equation (1).

CONCLUDING REMARKS. We remark that in the strongly superlinear case the oscillatory behavior of Equation (1) satisfying (3) is considerably different from that of Equation (1) satisfying (4). In fact, according to Theorem 7, when (4) holds, Equation (1) and the corresponding ordinary differential equation

$$[r(t)y'(t)]'+f(y(t), t)=0 \tag{57}$$

exhibit the same oscillatory behavior. On the contrary, by Theorem 3, if (3) holds, the oscillation of Equation (1) for some $g(t)$ implies that of Equation (57), but the converse is not always true as the following example shows.

EXAMPLE 3. Consider the differential equations

$$[t^3y'(t)]'+t^9[y(t)]^3=0,$$

$$[t^3y'(t)]'+t^9[y(t^3)]^3=0.$$

The former is oscillatory by Theorem 3, whereas the latter possesses a nonoscillatory solution $y(t)=t^{-1}$.

We conclude with the remark that there is a remarkable difference between the oscillatory character of *advanced* differential equations of the form (1) and that of *retarded* differential equations of the form (1). The difference will be apparent if we compare our results (in particular, Theorems 3, 4, 7, 8) with those obtained in [**6**] (in particular, Theorems 4, 5 of [**6**]).

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Takaŝi KUSANO
Department of Mathematics
Faculty of Science
Hiroshima University
Higashisenda-machi
Hiroshima, Japan

Hiroshi ONOSE
Department of Methematics
College of General Education
Ibaraki University
Bunkyo, Mito
Japan