Analytic approximation of continuous ovals of constant width

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In his paper [6] S. Tanno proved that for any positive number δ the δ-neighbourhood of a continuous oval of constant width contains a C^∞-oval of the same constant width. The purpose of this note is to show that this approximation can be assumed to be real analytic. Furthermore Tanno's proof can be simplified essentially by using results on ovals of constant width in the plane which have been presented by P. C. Hammer and A. Sobczyk in [4] and [5].

In these papers Hammer and Sobczyk gave the following description of plane convex curves of constant width: Let $p: R \to R$ be a continuous periodic function with period 2π such that $p(t+\pi)=-p(t)$ for all $t \in R$. Suppose that p satisfies the following Lipschitz condition

$$(1) \qquad \begin{aligned} &|p(t_1)-p(t_2)| < k_1|t_1-t_2| \\ &|p(t_1)+p(t_2)| < k_2|\pi-(t_2-t_1)| \end{aligned}$$

for suitable constants $k_1, k_2 > 0$ and for all pairs t_1, t_2 such that $0 < t_2-t_1 < \pi$. Then the line family

$$(2) \qquad x \sin t - y \cos t - p(t) = 0,$$

$0 \leq t \leq \pi$, is outwardly simple, i.e. they intersect each other within a bounded domain ([5]). Under this condition the curve

$$(3) \qquad \mathfrak{x}(t) = -p(t)\mathfrak{z}(t) + (c - \int_0^t p(t')dt')\mathfrak{z}(t),$$

$t \in R$, $\mathfrak{e}(t)=(\cos t, \sin t)$, is an oval of constant width (here the point denotes differentiation with respect to t), if the real number c satisfies the inequality

$$(4) \qquad c > \sup \{Dp(t) + \int_0^t p(t')dt' \mid t \in R\}$$

where $Dp(t)$ represents any of the possible limits of sequences of difference quotients of p at t.

In this case (3) describes a C^1-curve by a continuous parametrization. The

curve (3) is orthogonal to the line family (2). If c approaches the right-hand side of (4) from above, then (3) is also an oval of constant width or a single point [4]. Hammer and Sobczyk show that any curve of constant width can be constructed in this way.

THEOREM. *Let* $\mathfrak{x}: S^1 \to E^2$ *be a continuous convex curve of constant width* d. *Then, for any positive number* δ, *there exists a real analytic convex curve* $\mathfrak{x}: S^1 \to E^2$ *of the same constant width in the* δ-*neighbourhood of* $\mathfrak{x}$ *in* E^2.

PROOF. Let ε be a positive real number.

I: By Blaschke's approximation theorem (see [1] and [2] or [3]) there is a Reuleaux-polygon $\mathfrak{x}_1$ (i.e. a convex curve of constant width which can be described by a finite number of rotations) within the ε-neighbourhood of $\mathfrak{x}$.

II: Taking an outer parallel curve $\mathfrak{x}_2$ to $\mathfrak{x}_1$ ([3]) in a distance smaller then ε, we get a C^1-Reuleaux-polygon $\mathfrak{x}_2$ within the ε-neighbourhood of $\mathfrak{x}_1$.

III: Using the result of Hammer and Sobczyk described above we have the existence of a real constant c and a continuous periodic function $p_2: R \to R$ with period 2π satisfying the Lipschitz condition (1) such that

$$p_2(t+\pi) = -p_2(t) ,$$

$$\mathfrak{x}_2(t) = -p_2(t)\dot{\mathfrak{e}}(t) + (c - \int_0^t p_2(t')dt')\mathfrak{e}(t)$$

for all $t \in R$ and the inequality (4) is valid for p_2. Since p_2 is the support function of the normal lines to $\mathfrak{x}_2$, it is piecewise of the class C^∞ in this special case. Therefore p_2 can be approximated by a periodic C^∞-function $p_3: R \to R$ with period 2π such that

$$p_3(t+\pi) = -p_3(t) ,$$

$$|p_3(t) - p_2(t)| < \frac{\eta}{4\pi} ,$$

$$\dot{p}_3(t) \leq \frac{d}{dt}p_2|_{t+0} + \frac{\eta}{2} ,$$

$$\dot{p}_3(t) \leq \frac{d}{dt}p_2|_{t-0} + \frac{\eta}{2} ,$$

for all $t \in R$ where η is a fixed positive real number such that

$$\eta < \min \{\varepsilon, \ c - \sup \{Dp_2(t) + \int_0^t p_2(t')dt' \,|\, t \in R\}\} .$$

In the case under consideration $Dp_2(t)$ is the closed interval between

$$\frac{d}{dt}p_2|_{t+0} \quad \text{and} \quad \frac{d}{dt}p_2|_{t-0} .$$

The function p_3 satisfies the Lipschitz condition (1) for suitable real constants k_1, k_2 and the inequality (4) for the constant c defined above:

$$\sup\left\{ Dp_3(t) + \int_0^t p_3(t')dt' \,|\, t \in R \right\}$$

$$\leq \sup\left\{ Dp_2(t) + \frac{\eta}{2} + \int_0^t p_2(t')dt' + \int_0^t |p_2(t') - p_3(t')| dt' \,|\, t \in R \right\}$$

$$\leq \sup\left\{ Dp_2(t) + \int_0^t p_2(t')dt' \,|\, t \in R \right\} + \frac{\eta}{2}$$

$$+ t \sup \{ |p_2(t') - p_3(t')| \,|\, t' \in R \}$$

$$\leq \sup\left\{ Dp_2(t) + \int_0^t p_2(t')dt' \,|\, t \in R \right\} + \frac{\eta}{2} + 2\pi \frac{\eta}{4\pi} < c$$

if $t \in [0, 2\pi]$. Thus

$$\mathfrak{x}_3(t) = -p_3(t)\dot{\mathfrak{e}}(t) + \left(c - \int_0^t p_3(t')dt' \right)\mathfrak{e}(t)$$

describes a C^∞-oval of constant width which is contained in the ε-neighbourhood of $\mathfrak{x}_2$ since

$$|| \mathfrak{x}_2(t) - \mathfrak{x}_3(t) || \leq |p_2(t) - p_3(t)| + \left| \int_0^t (p_2(t)) - p_3(t')dt' \right|$$

$$< \eta < \varepsilon \quad \text{if} \quad t \in [0, 2\pi].$$

IV: The C^∞-function $\dot{p}_3$ satisfies the identities $\dot{p}_3(t) = \dot{p}_3(t + 2\pi)$ and $\dot{p}_3(t) = -\dot{p}_3(t + \pi)$ for all $t \in R$. It is easy to see that for an arbitrary positive number η there exists a real analytic function $q: R \to R$ which is periodic with period 2π such that $q(t + \pi) = -q(t)$ and $|\dot{p}_3(t) - q(t)| < \frac{\eta}{8\pi^2}$ for all $t \in R$. Then there is a value $t_0 \in [0, \pi]$ such that

$$(5) \qquad \int_{t_0}^{t_0 + \pi} q(t')dt' = 0.$$

The real analytic function $p_4: R \to R$ which is defined by $p_4(t) = \int_{t_0}^t q(t')dt'$ is periodic with period 2π and satisfies

$$p_4(t) + p_4(t + \pi) = \int_{t_0}^t q(t')dt' + \int_{t_0 + \pi}^{t + \pi} q(t')dt' = 0$$

for all $t \in R$. Furthermore we have

$$|p_4(t) - p_3(t)| = \left| \int_{t_0}^t (q(t') - \dot{p}_3(t'))dt' - p_3(t_0) \right|$$

$$\leq \frac{\eta}{8\pi^2} |t - t_0| + \frac{1}{2} |p_3(t_0 + \pi) - p_3(t_0)|$$

$$\leq \frac{\eta}{8\pi} + \left| \int_{t_0}^{t_0 + \pi} (\dot{p}_3(t') - q(t'))dt' \right| < \frac{\eta}{4\pi} \quad \text{using (5)},$$

if $t \in [0, \pi]$. Hence this inequality is valid for any real t. Since

$$\dot{p}_4(t) = q(t) < \dot{p}_3(t) + \frac{\eta}{2}$$

we can show as in the third section that for a sufficiently small $\eta > 0$

$$\underline{x}_4(t) = -p_4(t)\dot{e}(t) + \left(c - \int_0^t p_4(t')dt'\right)\underline{e}(t)$$

describes a real analytic oval of constant width d_4 which is contained in the ε-neighbourhood of $\underline{x}_3$.

V: From the previous sections we conclude that $\underline{x}_4$ stays within the 4ε-neighbourhood of $\underline{x}$ and we have $|d - d_4| < 8\varepsilon$. By a suitable homothetic transformation h of E^2 we get a real analytic oval $h \circ \underline{x}_4$ of constant width d such that

$$|\underline{x}_4(t) - h(\underline{x}_4(t))| \leq |d - d_4| < 8\varepsilon,$$

i. e., $h \circ \underline{x}_4$ is contained in the 16ε-neighbourhood of $\underline{x}$. Hence our theorem is proved, if we choose $\varepsilon < \frac{\delta}{16}$.

REMARK: If $\underline{x}$ is symmetric with respect to a straight line, then the approximating real analytic oval of constant width can be chosen symmetric too. This implies the validity of an analogous approximation theorem for convex surfaces of revolution which have constant width.

References

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