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Groups of algebras over $A\otimes\bar{A}$

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Introduction.

Let A be an R-algebra, where R is a fixed commutative ring. An *algebra over* A is a pair (U, i) where U is an R-algebra and $i: A\rightarrow U$ an R-algebra map. They form a category. The definition of morphisms is obvious.

Sweedler [1] starts to try to classify algebras over A by their *underlying A-bimodules.* In almost all the chapters he assumes the algebra A is *commutative.* His method is useful for such algebras (U, i) over A as i sends A isomorphically onto the *centralizer* of A in U.

When A is commutative, he defines a product "$\times_{A}$" on the category of algebras over A. This product is neither in general associative nor unitary.

A $\times_{A}$*-bialgebra* is a triple $(B, \Delta, \mathcal{J})$ where B is an algebra over A and $\Delta: B\rightarrow B\times_{A}B, \mathcal{J}: B\rightarrow \mathrm{End}_{R}A$ are maps of algebras over A making some diagrams commute.

When Δ is an isomorphism and $\mathcal{J}$ is injective, he defines $\mathcal{E}_{B}$ to be the set of isomorphism classes of algebras (U, i) over A such that $U\cong B$ as *A-bimodules.* He shows that i then maps A isomorphically onto the centralizer in U of A. The product "$\times_{A}$" makes $\mathcal{E}_{B}$ into an abelian monoid with unit $\langle B\rangle$ the class of B.

Let $\mathcal{G}\langle B\rangle$ denote the group of invertible elements in $\mathcal{E}_{B}$.

Among other things he proves that if $\langle U\rangle$ the class of of U belongs to $\mathcal{G}\langle B\rangle$ then there is a canonical isomorphism of algebras over A

$$\zeta: (U^{0}\times_{A}U)^{0}\longrightarrow B$$

with the assumption of the existence of some isomorphism $\mathcal{S}: B\rightarrow(B^{0}\times_{A}B)^{0}$ of algebras over A, called an "Ess" map. Here we denote by U^{0} the *opposite* algebra to U considered as an algebra over A.

Based on this fact, he shows that if A is a simple B-module (via $\mathcal{J}: B\rightarrow \mathrm{End}_{R}A$), then all algebras (U, i) over A with $\langle U\rangle\in\mathcal{G}\langle B\rangle$ are *simple.* (Exactly, some additional hypothesis on B is needed).

Further, for a $\times_{A}$-bialgebra $(B, \Delta, \mathcal{J})$ where Δ is an isomorphism and $\mathcal{J}$ is injective he constructs some *semi-co-simplicial complex* consisting of *commutative* algebras and algebra homomorphisms. Taking the groups of invertible elements, he obtains some complex of abelian groups. He computes the cohomology groups $H^n(B)$ for $n=0, 1, 2$ and shows that $H^2(B)\cong\mathcal{G}\langle B\rangle$. The commutativity of this $\times_A$-bialgebra cohomology follows from the *cocommutativity* of B.

The purpose of this article is to re-obtain the above theory of Sweedler for algebra A which is not necessarily *commutative*.

Let $\bar{A}$ denote the opposite algebra to A with the *anti-isomorphism* $A\rightarrow\bar{A}$, $a\mapsto\bar{a}$.

If U is an algebra over A, then the opposite algebra U^0 is an algebra over $\bar{A}$. We consider *algebras over* $A\otimes\bar{A}$. Here and below we write $\otimes$ to denote $\otimes_R$. For example $\mathrm{End}_R A$ is an algebra over $A\otimes\bar{A}$ with structure map

$$A\otimes\bar{A}\longrightarrow \mathrm{End}_R A\,, \qquad a\otimes\bar{b}\longmapsto a^l b^r$$

where a^l (resp. b^r) denotes the left (resp. right) translation by a (resp. b).

If U is an algebra over $A\otimes\bar{A}$, then the opposite algebra U^0 is also an algebra over $A\otimes\bar{A}$, since $A\otimes\bar{A}$ is canonically anti-isomorphic with itself. Our analysis is useful for such algebras (U, i) over $A\otimes\bar{A}$ that i sends isomorphically $\bar{A}$ onto the centralizer of $i(A)$ in U. $\mathrm{End}_R A$ is such an algebra.

Our task begins with making a slight but important change of the definition of "$\times_A$".

Let M be an $\bar{A}$-bimodule and N an A-bimodule. Let

$$\int_a {}_{\bar{a}}M\otimes {}_aN$$

denote the quotient module of $M\otimes N$ by the submodule generated by the elements $\bar{a}m\otimes n-m\otimes an$ with $a\in A$, $m\in M$ and $n\in N$. Let

$$\int^b\int_a {}_{\bar{a}}M_{\bar{b}}\otimes {}_aN_b$$

denote the submodule of $\int_a {}_{\bar{a}}M\otimes_a N$ consisting of

$$\{\sum_i m_i\otimes n_i \,|\, \sum_i m_i\bar{b}\otimes n_i=\sum_i m_i\otimes n_i b,\ \forall b\in A\}\,.$$

We define $M\times_A N$ to be *this R-module*.

Let (U, i) be an algebra over $\bar{A}$ and (V, j) an algebra over A. Then U is an $\bar{A}$-bimodule and V an A-bimodule. The R-module $U\times_A V$ is an R-algebra, where $1\otimes 1$ is the unit and the multiplication is defined by

$$(\sum_i u_i\otimes v_i)(\sum_j u'_j\otimes v'_j)=\sum_{i,j} u_i u'_j\otimes v_i v'_j\,.$$

Suppose M and N are $A\otimes\bar{A}$-*bimodules*. Since the product "$\times_A$" is functorial, the A-bimodule operation on M induces an A-bimodule structure on

$M\times_A N$ and the $\bar{A}$-bimodule operation on N an $\bar{A}$-biomodule structure on $M\times_A N$. Then $M\times_A N$ is an $A\otimes\bar{A}$-bimodule. We use the following symbol to explain this structure

$$_{l,\bar{u}}(M\times_A N)_{r,\bar{v}}=\int^b\int_a {}_{l,\bar{a}}M_{r,\bar{b}}\otimes {}_{a,\bar{u}}N_{b,\bar{v}}$$

where l, u, r and v are general elements of A.

Thus "$\times_A$" defines a product on the category of $A\otimes\bar{A}$-bimodules. This product is not necessarily associative. But for three $A\otimes\bar{A}$-bimodules M, N and P, there is an $A\otimes\bar{A}$-bimodule $M\times_A N\times_A P$ and we have canonical $A\otimes\bar{A}$-bilinear maps $\alpha:(M\times_A N)\times_A P\to M\times_A N\times_A P$ and $\alpha': M\times_A(N\times_A P)\to M\times_A N\times_A P$. If both α and α' are injective having the same image, the triple (M, N, P) is said to *associate*.

Let (U, i) and (V, j) be *algebras over* $A\otimes\bar{A}$. Then $U\times_A V$ is also an algebra over $A\otimes\bar{A}$ with respect to the algebra map

$$h: A\otimes\bar{A}\longrightarrow U\times_A V,\ h(a\otimes\bar{b})=i(a)\otimes j(\bar{b})\,.$$

Thus "$\times_A$" induces a product on the category of algebras over $A\otimes\bar{A}$.

A "$\times_A$-bialgebra" can be defined to be a triple $(B, \Delta, \mathcal{J})$ where B is an algebra over $A\otimes\bar{A}$ and $\Delta: B\to B\times_A B$ and $\mathcal{J}: B\to \mathrm{End}_R A$ are maps of algebras over $A\otimes\bar{A}$ making some diagrams commute.

If A is a *finite projective* R-algebra, then $\mathrm{End}_R A$ has a unique $\times_A$-bialgebra structure where $\mathcal{J}$ is the identity.

If A is a *division R-algebra*, there is a unique maximal subalgebra B of $\mathrm{End}_R A$ which has a $\times_A$-bialgebra structure with $\mathcal{J}$ the inclusion.

The above are examples of $\times_A$-bialgebras where Δ is an isomorphism and $\mathcal{J}$ is injective.

When Δ is an isomorphism and $\mathcal{J}$ is injective, the monoid $\mathcal{E}_B$ and the group $\mathcal{G}\langle B\rangle$ are defined similarly as [**1**]. But they are *not abelian*.

To ensure the existence of an isomorphism

$$\zeta:(U^0\times_A U)^0\longrightarrow B$$

for an algebra U over $A\otimes\bar{A}$ with $\langle U\rangle\in\mathcal{G}\langle B\rangle$, we also need an "Ess" map for B. Some difficulty lies in the definition.

To define the Ess map, Sweedler [**1**] compares the bimodules

$$((U\times_A V)^0\times_A(W\times_A X)^0)^0 \quad\text{and}\quad (U^0\times_A W)^0\times_A(V^0\times_A X)^0\,.$$

When U is an $A\otimes\bar{A}$-bimodule, let U^0 denote the $A\otimes\bar{A}$-bimodule where $U\to U^0$, $u\mapsto u^0$ is an R-module isomorphism and $(a\otimes\bar{b})u^0(c\otimes\bar{d})=((d\otimes\bar{c})u(b\otimes\bar{a}))^0$, $a, b, c, d\in A$, $u\in U$.

If U, V, W and X are $A\otimes\bar{A}$-bimodules, then the left $l, \bar{u}$ $A\otimes\bar{A}$- and the right $r, \bar{v}$ $A\otimes\bar{A}$-bimodule structures of the above bimodules come from $({}_{\bar{u}}V_{\bar{v}}, {}_{\bar{r}}X_{\bar{l}})$ and $({}_{\bar{r}}W_{\bar{l}}, {}_{\bar{u}}V_{\bar{v}})$ respectively. Hence they are *not comparable.*

Instead of the latter we use the bimodule

$$(((U^0\times_A W)^0\times_A V)^0\times_A X)^0$$

where the $A\otimes\bar{A}$-bimodule structure comes from $({}_{\bar{u}}V_{\bar{v}}, {}_{\bar{r}}X_{\bar{l}})$. Making use of some natural $A\otimes\bar{A}$-bilinear maps from these bimodules into some bimodule, we define the Ess map $\mathcal{S}: B\to(B^0\times_A B)^0$.

We can prove that if $\mathcal{S}$ is an isomorphism, then for all algebra U over $A\otimes\bar{A}$ with $\langle U\rangle\in\mathcal{G}\langle B\rangle$ there is a natural isomorphism of algebras over $A\otimes\bar{A}$ $\zeta: (U^0\times_A U)^0\to B$. Hence [**1**, Theorem (3.7)] can be applied to re-obtain a similar result to [**1**, Theorem (10.3)].

In the same way as [**1**, Chapter 15] we can form a semi-co-simplicial complex consisting of R-algebras and their homomorphisms from the $\times_A$-bialgebra $(B, \Delta, \mathcal{J})$. Although the algebras appearing in the complex are *not commutative* except at 0 and 1, we can define and compute the cohomology groups $H^n(B)$ for $n=0$, 1 and 2 by taking the groups of units. $H^0(B)$ and $H^1(B)$ are abelian, but $H^2(B)$ not. It is shown that we also have $H^2(B)\cong\mathcal{G}\langle B\rangle$.

The interest of this article is concentrated on the above theory of $\mathcal{G}\langle B\rangle$. We are not dealing with the analogy of the "$\times_A$-bialgebra determined by some class of ideals $\{L_\alpha\}$ of $A\otimes A$" or the "$\times_A$-bialgebra D_A of differential operators". Sweedler gives some sufficient conditions in order for A to be a simple D_A-module. He also computes the center of D_A. Extending these accounts to the case when A is not commutative is left to the reader. We consider it is not too difficult.

§ 0. Conventions.

Throughout we fix a commutative ring R with unit.

We write $\otimes$, Hom and End to denote $\otimes_R$, Hom_R and End_R. All modules and algebras are R-modules and R-algebras. They are unitary. Subalgebras of an algebra have the same unit.

For an algebra A, let $\bar{A}$ denote the opposite algebra where

$$A\longrightarrow\bar{A}, \qquad a\longmapsto\bar{a}$$

is an algebra anti-isomorphism.

We shall treat such a module M as is given many representations and anti-representations of algebras $\rho_i: A\to\mathrm{End}\,M$. We always assume that they *commute* in the sense

$$\rho_i(a)\rho_j(b)=\rho_j(b)\rho_i(a)$$

for all $a, b\in A$ and $i\neq j$.

In many cases, each representation is indicated by "position". For example, let M be a left $A\otimes\bar{A}$- and right A-bimodule, N a left $\bar{A}$- and right $A\otimes\bar{A}$-bimodule and P a left A- and right $\bar{A}$-bimodule. Then $L=M\otimes N\otimes P$ has eight representations and anti-representations

$$\rho_i: A \longrightarrow \mathrm{End}\, L\,, \qquad i=1, 2, \cdots, 8$$

each of which corresponds to the letter a_i in

$$L={}_{a_1,\bar{a}_2}M_{a_3}\otimes{}_{\bar{a}_4}N_{a_5,\bar{a}_6}\otimes{}_{a_7}P_{\bar{a}_8}\,.$$

($\rho_1, \rho_6, \rho_7, \rho_8$ are representations and $\rho_2, \rho_3, \rho_4, \rho_5$ anti-representations). They commute with each other.

In such a case we can use the symbols $\int_x$ and $\int^x$ of Sweedler [**1**]. For example,

$$Q_1=\int_x {}_xM\otimes{}_{\bar{x}}N\otimes{}_xP\,,$$

$$Q_2=\int_y M_y\otimes N_{\bar{y}}\otimes P$$

denote L/X_1 and L/X_2 respectively, where X_1 and X_2 are the submodules of L generated by

$$\{\rho_i(a)(l)-\rho_j(a)(l) \mid i, j=1, 4, 7, a\in A, l\in L\}$$

$$\{\rho_i(a)(l)-\rho_j(a)(l) \mid i, j=3, 6, a\in A, l\in L\}$$

respectively. Dually

$$Q_3=\int^u {}_{\bar{u}}M\otimes N\otimes P_{\bar{u}}\,,$$

$$Q_4=\int^v M_v\otimes N_v\otimes{}_vP$$

denote the submodules of L

$$\{l\in L \mid \rho_i(a)(l)=\rho_j(a)(l),\ i, j=2, 8, a\in A\}$$

$$\{l\in L \mid \rho_i(a)(l)=\rho_j(a)(l),\ i, j=3, 5, 7, a\in A\}$$

respectively.

Since each representation commutes with one another, the rest of the representations used to define $\int_x$ or $\int^x$ induce representations on the resulting coequalizer or the equalizer.

For example there remain on Q_i the following representations:

$$\rho_2, \rho_3, \rho_5, \rho_6, \rho_8 \qquad \text{on } Q_1\,,$$

$$\rho_1, \rho_2, \rho_4, \rho_5, \rho_7, \rho_8 \quad \text{on } Q_2,$$

$$\rho_1, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7 \quad \text{on } Q_3,$$

$$\rho_1, \rho_2, \rho_4, \rho_6, \rho_8 \quad \text{on } Q_4.$$

Therefore we can form the following modules for instance:

$$Q_5 = \int_y \int_x {}_xM_y \otimes {}_{\bar{x}}N_{\bar{y}} \otimes {}_xP$$

$$Q_6 = \int_x \int_y {}_xM_y \otimes {}_{\bar{x}}N_{\bar{y}} \otimes {}_xP$$

$$Q_7 = \int^u \int^v {}_{\bar{u}}\, M_v \otimes N_v \otimes {}_vP_{\bar{u}}$$

$$Q_8 = \int^v \int^u {}_{\bar{u}}\, M_v \otimes N_v \otimes {}_vP_{\bar{u}}$$

$$Q_9 = \int^u \int_y {}_{\bar{u}}\, M_y \otimes N_{\bar{y}} \otimes P_{\bar{u}}$$

$$Q_{10} = \int_y \int^u {}_{\bar{u}}\, M_y \otimes N_{\bar{y}} \otimes P_{\bar{u}}.$$

In the above, we have $Q_5 \cong Q_6$ and $Q_7 \cong Q_8$, since colimits commute with each other and so do limits. We shall denote them by

$$Q_5 = Q_6 = \int_{x,y} {}_xM_y \otimes {}_{\bar{x}}N_{\bar{y}} \otimes {}_xP,$$

$$Q_7 = Q_8 = \int^{u,v} {}_{\bar{u}}M_v \otimes N_v \otimes {}_vP_{\bar{u}}.$$

Of course they inherit the representations other than used to define $\int_{x,y}$ or $\int^{u,v}$.

On the other hand, Q_9 and Q_{10} are not in general isomorphic, but the inclusion $\int^u {}_{\bar{u}}M \otimes N \otimes P_{\bar{u}} \subsetneq L$ induces a homomorphism

$$\int_y \int^u {}_{\bar{u}}M_y \otimes N_{\bar{y}} \otimes P_{\bar{u}} \longrightarrow \int_y M_y \otimes N_{\bar{y}} \otimes P.$$

Since its image lies clearly in Q_9, we have a natural homomorphism

$$\int_y \int^u {}_{\bar{u}}M_y \otimes N_{\bar{y}} \otimes P_{\bar{u}} \longrightarrow \int^u \int_y {}_{\bar{u}}M_y \otimes N_{\bar{y}} \otimes P_{\bar{u}}.$$

We call this last homomorphism "the exchange map from $\int_y \int^u$ to $\int^u \int_y$".

For example, the following chain of natural homomorphisms is induced from the exchange maps:

$$\int_y\int^v\int_x\int^u \longrightarrow \int^v\int_{x,y}\int^u \longrightarrow \int^v\int_y\int^u\int_x \longrightarrow \int^{u,v}\int_{x,y}.$$

Any composites may also be called the exchange maps.

In this paper we mainly treat $A\otimes\bar{A}$-bimodules. If M is an $A\otimes\bar{A}$-bimodule, we define M^0 to be the $A\otimes\bar{A}$-bimodule, R-isomorphic with M via $m\mapsto m^0$, $M\to M^0$, with structure determined by

$$(a\otimes\bar{b})m^0(c\otimes\bar{d})=((d\otimes\bar{c})m(b\otimes\bar{a}))^0$$

$a, b, c, d\in A$, $m\in M$. The isomorphism

$${}_{l,\bar{u}}M^0{}_{r,\bar{v}} \xrightarrow{\sim} {}_{v,\bar{r}}M_{u,\bar{l}}, \quad m^0\mapsto m$$

is compatible with each representation indicated by position l, u, r and v.

An *algebra over* A is a pair (U, i) where U is an algebra and $i: A\to U$ a map of algebras. A map of algebras over A from (U, i) to (V, j) is such an algebra map $f: U\to V$ that $j=f\circ i$. Then algebras over A form a category.

Each algebra (U, i) over A is an A-bimodule with structure $aub=i(a)ui(b)$, $a, b\in A$, $u\in U$. This is the *underlying* A-bimodule of (U, i).

If (U, i) is an algebra over $A\otimes\bar{A}$, let U^0 denote the *opposite* algebra to U with the anti-isomorphism $U\to U^0$, $u\mapsto u^0$. Then (U^0, i^0) is an algebra over $A\otimes\bar{A}$, where $i^0(a\otimes\bar{b})=i(b\otimes\bar{a})^0$, $a, b\in A$. If M denotes the underlying $A\otimes\bar{A}$-bimodule of (U, i), then the underlying $A\otimes\bar{A}$-bimodule of (U^0, i^0) is M^0.

A is a *left* $A\otimes\bar{A}$-module, where $(a\otimes\bar{b})c=acb$, $a, b, c\in A$.

End A is an algebra over $A\otimes\bar{A}$ with respect to the algebra map $A\otimes\bar{A}\to$ End A, $a\otimes\bar{b}\mapsto a^l b^r$, where $a^l b^r(c)=acb$, $a, b, c\in A$. The underlying $A\otimes\bar{A}$-bimodule structure of End A is explained by position:

$${}_{l,\bar{u}}(\mathrm{End}\,A)_{r,\bar{v}}=\mathrm{Hom}\,({}_{r,\bar{v}}A, {}_{l,\bar{u}}A)\,.$$

A family of submodules $\{M_\alpha\}$ of a module M is *directed* if for each indices α, β there is an index γ such that $M_\alpha+M_\beta\subset M_\gamma$. The union $\bigcup_\alpha M_\alpha$ is then called directed.

If we write $M\otimes_A N$ this denotes the tensor product of the right module M_A with the left module ${}_AN$.

If we write $\mathrm{Hom}_A(M, N)$ this is the "hom" from the left module ${}_AM$ to the left module ${}_AN$.

§ 1. $M\times_A N$ and $M\times_A P\times_A N$ as modules.

Until (1.10) let M be an $\bar{A}$-bimodule, N an A-bimodule and P an $A\otimes\bar{A}$-bimodule.

1.1. Definition. $M\times_A N=\int^y\int_x {}_{\bar{x}}M_{\bar{y}}\otimes_x N_y$, which is simply a module.

If $f: M\to M'$ is a map of $\bar{A}$-bimodules and $g: N\to N'$ a map of A-bimodules, then the map $f\otimes g\colon \int_x {}_{\bar{x}}M\otimes_x N\to\int_x {}_{\bar{x}}M'\otimes_x N'$ induces the following homomorphism:

1.2. $$f\times g\colon M\times_A N\longrightarrow M'\times_A N'.$$

"$\times_A$" gives a biadditive functor from (the category of $\bar{A}$-bimodules)$\times$(the category of A-bimodules) to (the category of modules).

1.3. REMARK. $P\times_A N$ has an A-bimodule structure determined by

$$_x(P\times_A N)_y={}_xP_y\times_A N.$$

$M\times_A P$ has an $\bar{A}$-bimodule structure determined by

$$_{\bar{x}}(M\times_A P)_{\bar{y}}=M\times_A {}_{\bar{x}}P_{\bar{y}}.$$

If P' is another $A\otimes\bar{A}$-bimodule, then the above structures make $P\times_A P'$ into an $A\otimes\bar{A}$-bimodule.

1.4. DEFINITION.

$$M\times_A P\times_A N=\int^{y,b}\int_{x,a} {}_{\bar{x}}M_{\bar{y}}\otimes_{x,\bar{a}}P_{y,\bar{b}}\otimes_a N_b.$$

If $f\colon M\to M'$ is a map of $\bar{A}$-bimodules, $g\colon P\to P'$ a map of $A\otimes\bar{A}$-bimodules and $h\colon N\to N'$ a map of A-bimodules, then $f\otimes g\otimes h\colon \int_{x,a} {}_{\bar{x}}M\otimes_{x,\bar{a}}P\otimes_a N\to \int_{x,a} {}_{\bar{x}}M'\otimes_{x,\bar{a}}P'\otimes_a N'$ induces the map

1.5. $$f\times g\times h\colon M\times_A P\times_A N\longrightarrow M'\times_A P'\times_A N'.$$

The functor "$-\times_A-\times_A-$" is additive in each variable.

1.6. REMARK. If M (resp. N) is an $A\otimes\bar{A}$-bimodule, then $M\times_A P\times_A N$ is an A-bimodule (resp. $\bar{A}$-bimodule), where the structure is indicated by

$$_l(M\times_A P\times_A N)_r={}_lM_r\times_A P\times_A N$$

$$(\text{resp. } {}_{\bar{u}}(M\times_A P\times_A N)_{\bar{v}}=M\times_A P\times_A {}_{\bar{u}}N_{\bar{v}}).$$

Hence if M, N and P are $A\otimes\bar{A}$-bimodules, then $M\times_A P\times_A N$ has the canonical $A\otimes\bar{A}$-bimodule structure.

1.7. PROPOSITION. *The image of the composite* $(M\times_A P)\times_A N\overset{\iota}{\hookrightarrow}\int_a {}_{\bar{a}}(M\times_A P)\otimes_a N\xrightarrow{\iota\otimes 1}\int_{x,a} {}_{\bar{x}}M\otimes_{x,\bar{a}}P\otimes_a N$ *is contained in* $M\times_A P\times_A N$. *Let* $\alpha\colon (M\times_A P)\times_A N\to M\times_A P\times_A N$ *denote the induced map.*

i) *If* $_AN$ *is flat, then* α *is injective. If in addition* $_{A\otimes\bar{A}}A$ *is finitely presented, then* α *is an isomorphism.*

ii) *If* $_AN$ *is projective, then* α *is an isomorphism.*

iii) *If* $_AN$ *is a directed union of projective submodules and* $\int_x {}_{\bar{x}}M\otimes_{x,\bar{a}}P$ *is*

${}_{\bar{a}}\bar{A}$*-flat, then* α *is an isomorphism.*

iv) *If* M *(resp.* N*) is an* $A\otimes\bar{A}$*-bimodule, then* α *is* A*-bilinear (resp.* $\bar{A}$*-bilinear.*

PROOF. iv) is easily checked. The proof of i), ii) and iii) is similar to [**1**, (2.5)]. Q.E.D.

There is a similar map

$$\alpha' : M\times_A(P\times_A N)\longrightarrow M\times_A P\times_A N$$

for which analogous results hold.

1.8. DEFINITION. The triple (M, P, N) *associates* if the maps α and α' are injective having the same image.

In this case there is the *association isomorphism* of modules

$$t:(M\times_A P)\times_A N\cong M\times_A(P\times_A N)$$

such that $\alpha'\circ t=\alpha$. This is A-bilinear (resp. $\bar{A}$-bilinear) when M (resp. N) is an $A\otimes\bar{A}$-bimodules, hence $A\otimes\bar{A}$-bilinear if both M and N are $A\otimes\bar{A}$-bimodules.

1.9. DEFINITION. The $A\otimes\bar{A}$-bimodule P is *associative* if the triple (P, P, P) associates.

1.10. PROPOSITION. *If* ${}_AP$ *and* ${}_{\bar{A}}P$ *are flat and* ${}_{\bar{A}}M$ *and* ${}_AN$ *are directed unions of projective submodules, then the* α *and* α' *maps are isomorphisms. Hence* (M, P, N) *associates.*

PROOF. See [**1**, (2.11)] or use (1.7), iii).

1.11. PROPOSITION. *Let* M *and* M' *be* $A\otimes\bar{A}$*-bimodules and* N *and* N' *be left* $A\otimes\bar{A}$*-modules. View* $N\otimes N'$ *as a left* $A\otimes\bar{A}$*-module by* $(a\otimes\bar{b})(n\otimes n')=an\otimes\bar{b}n'$, $a, b\in A$, $n\in N$, $n'\in N'$. *Consider the composite* $(M\times_A M')\otimes_{A\otimes\bar{A}}(N\otimes N')\xrightarrow{\iota\otimes 1}\int_{x,a,b}{}_{\bar{x}}M_a\otimes_x M'_{\bar{b}}\otimes_a N\otimes_{\bar{b}}N'\xrightarrow{\text{tw}}\int_x{}_{\bar{x}}M\otimes_A N\otimes_x M'\otimes_{\bar{A}}N'\xrightarrow{\text{cano}}\int_x{}_{\bar{x}}M\otimes_{A\otimes\bar{A}}N\otimes_x M'\otimes_{A\otimes\bar{A}}N'$, *where* $\text{tw}(m\otimes m'\otimes n\otimes n')=m\otimes n\otimes m'\otimes n'$ *and* cano *denotes the canonical projection. This induces a homomorphism*

$$\phi:(M\times_A M')\otimes_{A\otimes\bar{A}}\Big(\int_c{}_{\bar{c}}N\otimes_c N'\Big)\longrightarrow\int_x{}_{\bar{x}}M\otimes_{A\otimes\bar{A}}N\otimes_x M'\otimes_{A\otimes\bar{A}}N'$$

where note that $\int_c{}_{\bar{c}}N\otimes_c N'$ *is a quotient left* $A\otimes\bar{A}$*-module of* $N\otimes N'$.

PROOF. The left hand side is isomorphic to $\int_c(M\times_A M')\otimes_{A\otimes\bar{A}}({}_{\bar{c}}N\otimes_c N')$. Let $\sum_i m_i\otimes m'_i\in M\times_A M'$, $n\in N$ and $n'\in N'$. Then $\sum_i m_i\bar{c}\otimes n\otimes m'_i\otimes n'=\sum_i m_i\otimes n\otimes m'_ic\otimes n'$ in $\int_x{}_{\bar{x}}M\otimes_A N\otimes_x M'\otimes_{\bar{A}}N'$ for all $c\in A$. Hence $\sum_i m_i\otimes\bar{c}n\otimes m'_i\otimes n'=\sum_i m_i\otimes n\otimes m'_i\otimes cn'$ in $\int_x{}_{\bar{x}}M\otimes_{A\otimes\bar{A}}N\otimes_x M'\otimes_{A\otimes\bar{A}}N'$. Therefore the map ϕ is induced. Q.E.D.

In general if P and Q are B-bimodules, where B is an algebra, $P\otimes_B Q$ is a B-bimodule with structure determined by $b(p\otimes q)b'=bp\otimes qb'$, $b, b'\in B$, $p\in P$, $q\in Q$.

1.12. PROPOSITION. *Let M, M', N and N' be $A\otimes\bar{A}$-bimodules. The composite* $(M\times_A M')\otimes_{A\otimes\bar{A}}(N\times_A N') \xrightarrow{1\otimes\iota} (M\times_A M')\otimes_{A\otimes\bar{A}}\left(\int_c {}_{\bar{c}}N\otimes_c N'\right) \xrightarrow{\phi} \int_x {}_{\bar{x}}M\otimes_{A\otimes\bar{A}}N\otimes {}_x M'\otimes_{A\otimes\bar{A}}N'$ *induces the $A\otimes\bar{A}$-bilinear map*

$$\xi : (M\times_A M')\otimes_{A\otimes\bar{A}}(N\times_A N') \longrightarrow (M\otimes_{A\otimes\bar{A}}N)\times_A(M'\otimes_{A\otimes\bar{A}}N') .$$

PROOF. Left to the reader.

§ 2. The maps θ, θ' and θ''.

2.1. DEFINITION. If M is a left $\bar{A}$-module and N a left A-module, there are the maps

$$\Lambda : \int_x {}_{\bar{x}}M\otimes_x \mathrm{End}\, A \longrightarrow \mathrm{Hom}\,(A, M)$$

$$\Lambda(m\otimes c)(a)=\overline{c(a)}m ,$$

$$\Lambda' : \int_x {}_{\bar{x}}\mathrm{End}\, A\otimes_x N \longrightarrow \mathrm{Hom}\,(A, N)$$

$$\Lambda'(c\otimes n)(a)=c(a)n ,$$

$$\Lambda'' : \int_{x,y} {}_{\bar{x}}M\otimes_{x,\bar{y}}\mathrm{End}\, A\otimes_y N \longrightarrow \mathrm{Hom}\left(A, \int_z {}_{\bar{z}}M\otimes_z N\right)$$

$$\Lambda''(m\otimes c\otimes n)(a)=\overline{c(a)}m\otimes n=m\otimes c(a)n ,$$

$c\in \mathrm{End}\, A$, $m\in M$, $n\in N$, $a\in A$.

Sufficient conditions for Λ, Λ' or Λ'' to be injective are given in [**1**, (1.5)].

2.2. PROPOSITION. *If M is an $\bar{A}$-bimodule and N an A-bimodule, the maps Λ, Λ' and Λ'' "induce" the maps respectively:*

$$\theta : M\times_A \mathrm{End}\, A \longrightarrow M$$

$$\theta(\sum_i m_i\otimes c_i)=\sum_i \overline{c_i(1)}m_i ,$$

$$\theta' : \mathrm{End}\, A\times_A N \longrightarrow N$$

$$\theta'(\sum_j d_j\otimes n_j)=\sum_j d_j(1)n_j ,$$

$$\theta'' : M\times_A \mathrm{End}\, A\times_A N \longrightarrow M\times_A N$$

$$\theta''(\sum_i m_i\otimes c_i\otimes n_i)=\sum_i \overline{c_i(1)}m_i\otimes n_i=\sum_i m_i\otimes c_i(1)n_i .$$

The sense of "inducing" is explained in the proof.

i) *Tne map* θ *is* $\bar{A}$*-bilinear and the map* θ' A*-bilinear.*

ii) *If* M *is an* $A\otimes\bar{A}$*-bimodule, then* θ *is* $A\otimes\bar{A}$*-bilinear and* θ'' A*-bilinear.*

iii) *If* N *is an* $A\otimes\bar{A}$*-bimodule, then* θ' *is* $A\otimes\bar{A}$*-bilinear and* θ'' $\bar{A}$*-bilinear.*

PROOF. The map $\Lambda: \int_x {}_{\bar{x}}M_{\bar{y}}\otimes_x(\mathrm{End}\ A)_z \to \mathrm{Hom}\,({}_zA, M_{\bar{y}})$ is y, z A-bilinear. Taking the equalizer $y=z$, we obtain the θ map as the composite:

$$M\times_A \mathrm{End}\ A \xrightarrow{\ \Lambda\ } \int^y \mathrm{Hom}\,({}_yA, M_{\bar{y}}) \cong M$$

$$f \longmapsto f(1)\,.$$

The map Λ' induces θ' in the same way.

The map $\Lambda'': \int_{x,a} {}_{\bar{x}}M_{\bar{y}}\otimes \mathrm{Hom}\,({}_{z,\bar{c}}A, {}_{x,\bar{a}}A)\otimes_a N_b \to \mathrm{Hom}\left({}_{z,\bar{c}}A, \int_x {}_{\bar{x}}M_{\bar{y}}\otimes_x N_b\right)$ is y, z, b, c A-multilinear. Take the equalizer $y=z$ and $b=c$.

If in general P is a right $A\otimes\bar{A}$-module, we have a canonical isomorphism

$$\int^{y,b} \mathrm{Hom}\,({}_{y,\bar{b}}A, P_{y,\bar{b}}) \cong \int^y P_{y,\bar{y}}\,, \qquad f \longmapsto f(1)\,.$$

Hence we have the induced map

$$\int^{y,b}\int_{x,a} {}_{\bar{x}}M_{\bar{y}}\otimes \mathrm{Hom}\,({}_{y,\bar{b}}A, {}_{x,\bar{a}}A)\otimes_a N_b \xrightarrow{\ \Lambda''\ }$$

$$\int^{y,b} \mathrm{Hom}\,({}_{y,\bar{b}}A, \int_x {}_{\bar{x}}M_{\bar{y}}\otimes_x N_b) \cong \int^y\int_x {}_{\bar{x}}M_{\bar{y}}\otimes_x N_y\,.$$

This is the map $\theta'': M\times_A \mathrm{End}\ A\times_A N \to M\times_A N$.

i), ii) and iii) are straightforward. Q. E. D.

The maps θ, θ' and θ'' are functorial in each variable.

2.3. PROPOSITION. *Let* M *be an* $\bar{A}$*-bimodule and* N *an* A*-bimodule. The following diagram commutes.*

$$\begin{array}{ccc}
(M\times_A \mathrm{End}\ A)\times_A N & \xrightarrow{\ \alpha\ } & M\times_A \mathrm{End}\ A\times_A N \\
\Big\downarrow{\scriptstyle \theta\times 1} & \overset{\theta''}{\swarrow} & \Big\uparrow{\scriptstyle \alpha'} \\
M\times_A N & \xleftarrow[\ 1\times\theta'\ ]{} & M\times_A(\mathrm{End}\ A\times_A N)\,.
\end{array}$$

PROOF. This follows from a direct calculation.

2.4. We show that the maps θ, θ': End $A\times_A$ End $A \rightrightarrows$ End A coincide. Put ∂: End $A\to A$, $\partial(c)=c(1)$, $c\in$ End A.

2.5. LEMMA. *Let* X *be a right* A*-module and* Y *a right* $\bar{A}$*-module. The map* ∂ *induces isomorphisms*

$$\int^{x} \mathrm{Hom}\,(X_x, (\mathrm{End}\,A)_x) \cong \mathrm{Hom}\,(X, A)\,, \qquad F \mid \longrightarrow \partial\circ F\,,$$

$$\int^{y} \mathrm{Hom}\,(Y_{\bar{y}}, (\mathrm{End}\,A)_{\bar{y}}) \cong \mathrm{Hom}\,(Y, A)\,, \qquad G \mid \longrightarrow \partial\circ G\,.$$

The inverses are given respectively by

$$f \mid \longrightarrow \hat{f}\,, \quad \textit{where} \quad \hat{f}(s)(a)=f(sa)$$

$$g \mid \longrightarrow \check{g}\,, \quad \textit{where} \quad \check{g}(t)(a)=g(t\bar{a})$$

$a\in A$, $s\in X$, $t\in Y$, $f\in \mathrm{Hom}\,(X, A)$, $g\in \mathrm{Hom}\,(Y, A)$.

PROOF. Exercise (cf. [**1**, (5.2)]).

2.6. COROLLARY. *If Z is a right $A\otimes\bar{A}$-module, the map ∂ induces an isomorphism*

$$\int^{x,y} \mathrm{Hom}\,(Z_{x,\bar{y}}, (\mathrm{End}\,A)_{x,\bar{y}}) \cong \mathrm{Hom}\left(\int_z Z_{z,\bar{z}}, A\right), \qquad H \mid \rightarrow \partial\circ H\,.$$

2.7. PROPOSITION. *The maps θ, θ': $\mathrm{End}\,A\times_A \mathrm{End}\,A \rightrightarrows \mathrm{End}\,A$ coincide.*

PROOF. Since they are $A\otimes\bar{A}$-bilinear, we have only to show $\partial\circ\theta=\partial\circ\theta'$. If $\sum_i c_i\otimes d_i\in \mathrm{End}\,A\times_A \mathrm{End}\,A$, then $\partial(\theta(\sum_i c_i\otimes d_i))=\partial(\sum_i \overline{d_i(1)}c_i)=\sum_i \overline{d_i(1)}c_i(1)=\sum_i c_i(1)d_i(1)=\partial(\sum_i c_i(1)d_i)=\partial(\theta'(\sum_i c_i\otimes d_i))$. Hence $\partial\circ\theta=\partial\circ\theta'$.

§ 3. $U\times_A V$ and $U\times_A W\times_A V$ as algebras.

Proposition [**1**, (3.1)] can be read as follows:

3.1. PROPOSITION. *Let U be an algebra over $\bar{A}$ and V an algebra over A. $\int_x U_{\bar{x}}\otimes_x V$ is a right $U\otimes V$-module with structure determined by $(u\otimes v)(u'\otimes v')=uu'\otimes vv'$, u, $u'\in U$, v, $v'\in V$.*

i) *There is a module isomorphism*

$$N\colon U\times_A V \longrightarrow \mathrm{End}_{\text{right } U\otimes V}\left(\int_x {}_{\bar{x}}U\otimes_x V\right)$$

determined by $N(\sum_i u_i\otimes v_i)(u\otimes v)=\sum_i u_i u\otimes v_i v$, $\sum_i u_i\otimes v_i\in U\times_A V$, $u\in U$, $v\in V$.

ii) *$U\times_A V$ has an algebra structure determined by*

$$(\sum_i u_i\otimes v_i)(\sum_j u'_j\otimes v'_j)=\sum_{i,j} u_i u'_j\otimes v_i v'_j\,,$$

$\sum_i u_i\otimes v_i$, $\sum_j u'_j\otimes v'_j\in U\times_A V$, with unit $1\otimes 1$.

iii) *N is an algebra isomorphism.*

If $f\colon U\to U'$ is a map of algebras over $\bar{A}$ and $g\colon V\to V'$ a map of algebras over A, then $f\times g\colon U\times_A V\to U'\times_A V'$ is a map of algebras.

Corresponding to Remark 1.3, if (U, i) (resp. (V, j)) is an algebra over

$A\otimes\bar{A}$, $U\times_A V$ is an algebra over A (resp. $\bar{A}$) with respect to the algebra map

$$h: A \longrightarrow U\times_A V, \quad h(a)=i(a)\otimes 1$$

$$(\text{resp. } h: \bar{A} \longrightarrow U\times_A V, \quad h(\bar{a})=1\otimes j(\bar{a}))\,.$$

If both U and V are algebras over $A\otimes\bar{A}$, then $U\times_A V$ is an algebra over $A\otimes\bar{A}$ with respect to

$$h: A\otimes\bar{A} \longrightarrow U\times_A V, \quad h(a\otimes\bar{b})=i(a)\otimes j(\bar{b})\,.$$

The underlying bimodule structures are the same as described in (1.3).

If f (resp. g) is a map of algebras over $A\otimes\bar{A}$, then $f\times g$ is a map of algebras over A (resp. $\bar{A}$). Thus "$\times_A$" determines a product on the category of algebras over $A\otimes\bar{A}$.

Similarly we have:

3.2. PROPOSITION. *Let U be an algebra over $\bar{A}$, W an algebra over $A\otimes\bar{A}$ and V an algebra over A. $\int_{x,a} {}_{\bar{x}}U\otimes_{x,\bar{a}}W\otimes_a V$ is a right $U\otimes W\otimes V$-module as in* (3.1).

i) *There is a module isomorphism*

$$N: U\times_A W\times_A V \longrightarrow \mathrm{End}_{\mathrm{right}\ U\otimes W\otimes V}\Big(\int_{x,a} {}_{\bar{x}}U\otimes_{x,\bar{a}}W\otimes_a V\Big)$$

determined by $N(\sum_i u_i\otimes w_i\otimes v_i)(u\otimes w\otimes v)=\sum_i u_i u\otimes w_i w\otimes v_i v$, $\sum_i u_i\otimes w_i\otimes v_i\in U\times_A W\times_A V$, $u\in U$, $w\in W$, $v\in V$.

ii) *$U\times_A W\times_A V$ has an algebra structure determined by*

$$(\sum_i u_i\otimes w_i\otimes v_i)(\sum_j u'_j\otimes w'_j\otimes v'_j)=\sum_{i,j} u_i u'_j\otimes w_i w'_j\otimes v_i v'_j\,,$$

$\sum_i u_i\otimes w_i\otimes v_i$, $\sum_j u'_j\otimes w'_j\otimes v'_j\in U\times_A W\times_A V$, *with unit* $1\otimes 1\otimes 1$.

iii) *N is an algebra isomorphism.*

Corresponding to Remark 1.6, if U (resp. V) is an algebra over $A\otimes\bar{A}$, then $U\times_A W\times_A V$ has an algebra structure over A (resp. $\bar{A}$). If U and V are algebras over $A\otimes\bar{A}$, then $U\times_A W\times_A V$ has a natural algebra structure over $A\otimes\bar{A}$. Description of the structure maps is left to the reader.

3.3. PROPOSITION. *Let U be an algebra over $\bar{A}$, W an algebra over $A\otimes\bar{A}$ and V an algebra over A. The following are all algebra maps:*

$$\alpha: (U\times_A W)\times_A V \longrightarrow U\times_A W\times_A V$$

$$\alpha': U\times_A(W\times_A V) \longrightarrow U\times_A W\times_A V$$

$$\theta: U\times_A \mathrm{End}\, A \longrightarrow U$$

$$\theta': \mathrm{End}\, A\times_A V \longrightarrow V$$

$$\theta'' : U\times_A \operatorname{End} A\times_A V \longrightarrow U\times_A V .$$

Here θ (resp. θ') is a map of algebras over $\bar{A}$ (resp. A).

If U is an algebra over $A\otimes\bar{A}$, then α, α' and θ'' are maps of algebras over A and θ a map of algebras over $A\otimes\bar{A}$.

If V is an algebra over $A\otimes\bar{A}$, then α, α' and θ'' are maps of algebras over $\bar{A}$ and θ' a map of algebras over $A\otimes\bar{A}$.

Hence if U and V are algebras over $A\otimes\bar{A}$, then the above are all maps of algebras over $A\otimes\bar{A}$.

PROOF. Straightforward and left to the reader. Q. E. D.

3.4. COROLLARY. *Let U, W and V be as above. If the triple (U, W, V) associates as bimodules* (1.8) *then the association isomorphism $(U\times_A W)\times_A V\cong U\times_A(W\times_A V)$ is an isomorphism of algebras.*

Let (U, i) be an algebra over $A\otimes\bar{A}$. Suppose that i sends $\bar{A}$ isomorphically onto $\int^x{}_x U_x=$the centralizer of $i(A)$ in U. It follows from [**1**, (3.4)] that there is an algebra map

$$3.5. \qquad \zeta : (U^0\times_A U)^0 \longrightarrow \operatorname{End} A$$

determined by $i(\overline{\zeta((\sum_i u_i{}^0\otimes v_i)^0)(a)})=\sum_i u_i i(\bar{a})v_i$, $(\sum_i u_i{}^0\otimes v_i)^0\in(U^0\times_A U)^0$, $a\in A$.

It is easy to see that ζ is a map of algebras *over* $A\otimes\bar{A}$.

If U is a subalgebra of End A over $A\otimes\bar{A}$, then the above condition holds true. We have then a commutative diagram by [**1**, (3.8)]:

$$\begin{array}{ccc} (U^0\times_A U)^0 & \xrightarrow{(1\times\iota)^0} & (U^0\times_A \operatorname{End} A)^0 \\ \downarrow \zeta & & \downarrow \theta^0 \\ \operatorname{End} A & \xleftarrow{\ \iota\ } & U=U^{00} \end{array}$$

consisting of maps of algebras over $A\otimes\bar{A}$.

Since A is not commutative, the lemma [**1**, (3.10)] must be rewritten as:

3.6. LEMMA. *Let A be a division algebra, C a left A-subspace of* End A *and M an $\bar{A}$-bimodule.*

i) *If $\{c_1, \cdots, c_s\}\subset C$ is a finite left A-linearly independent set, then there exists $\{a_{ij}\}\cup\{b_{ij}\}\subset A$ satisfying $\sum_j c_k(b_{ij})a_{ij}=\delta_{ik}$ with i, $k=1, \cdots, s$.*

ii) *If C is a sub-A-bimodule of* End$A$, *then $m\in\theta(\bar{A}m\bar{A}\times_A C)$ if $m\in\theta(M\times_A C)$.*

Similar results hold, for θ', an A-bimodule N and a left sub-$\bar{A}$-module D of End A.

Just as [**1**, (3.12)] we have:

3.7. THEOREM. *Let A be a division algebra and E a subalgebra of* End A *over $A\otimes\bar{A}$. If $\theta : E^0\times_A E\rightarrow E^0$ is surjective, then E is a simple algebra. The* *center of E is*

$$\{a^l \mid a\in A,\ u(a)=au(1),\ \forall u\in E\}.$$

3.8. REMARK. If (U, i) is an algebra over $A\otimes\bar{A}$, we denote by $\langle U\rangle$ the class of algebras over $A\otimes\bar{A}$ which are isomorphic to U as *algebras over* $A\otimes\bar{A}$. If U and V are algebras over $A\otimes\bar{A}$, then the product $\langle U\rangle\langle V\rangle=\langle U\times_A V\rangle$ is well-defined. This is the canonical product on isomorphism classes of algebras over $A\otimes\bar{A}$. The product is neither commutative nor associative. If U, V and W are algebras over $A\otimes\bar{A}$, where (U, V, W) associates, then $(\langle U\rangle\langle V\rangle)\langle W\rangle=\langle U\rangle(\langle V\rangle\langle W\rangle)$ in view of (3.4).

For an $A\otimes\bar{A}$-bimodule M, define $\mathcal{E}_M$ by

3.9. $\mathcal{E}_M=\{$isomorphism classes $\langle U\rangle$ of algebras over $A\otimes\bar{A}$ where $U\cong M$ as an $A\otimes\bar{A}$-bimodule$\}$.

If $e\in\mathcal{E}_M$, $f\in\mathcal{E}_N$ where M and N are $A\otimes\bar{A}$-bimodules, then $ef\in\mathcal{E}_{M\times_A N}$.

3.10. DEFINITION. An $A\otimes\bar{A}$-bimodule M is idempotent as a bimodule if $M\cong M\times_A M$ as $A\otimes\bar{A}$-bimodules. An algebras (U, i) over $A\otimes\bar{A}$ is idempotent as an algebra over $A\otimes\bar{A}$ if $U\cong U\times_A U$ as an algebra over $A\otimes\bar{A}$, i.e., $\langle U\rangle=\langle U\rangle\langle U\rangle$.

If M is an idempotent $A\otimes\bar{A}$-bimodule, then, for $e, f\in\mathcal{E}_M$, we have $ef\in\mathcal{E}_M$. If M is also an associative bimodule, then the product in $\mathcal{E}_M$ is associative.

3.11. DEFINITION. If (U, i) is an algebra over $A\otimes\bar{A}$ which is idempotent as an algebra over $A\otimes\bar{A}$ and associative as an $A\otimes\bar{A}$-bimodule, let $\mathcal{E}\langle U\rangle$ denote the monoid of equivalence classes $C\in\mathcal{E}_U$ where $C\langle U\rangle=C=\langle U\rangle C$. Let $\mathcal{G}\langle U\rangle$ denote the group of invertible elements in $\mathcal{E}\langle U\rangle$.

Similarly to [**1**, (4.9)] we have

3.12. PROPOSITION. *Let E be a subalgebra of* End A *over $A\otimes\bar{A}$, where $\theta=\theta'$: $E\times_A E\to E$ is an isomorphism of algebras over $A\otimes\bar{A}$. Assume E is associative as an $A\otimes\bar{A}$-bimodule. If U is an algebra over $A\otimes\bar{A}$ with $U\cong E$ as an $A\otimes\bar{A}$-bimodule, then*

$$\theta: U\times_A E\to U \quad \textit{and} \quad \theta': E\times_A U\to U$$

are isomorphisms of algebras over $A\otimes\bar{A}$. Hence we have $\mathcal{E}\langle E\rangle=\mathcal{E}_E$.

§ 4. $\times_A$-Coalgebras and $\times_A$-bialgebras.

4.1. DEFINITION. *A $\times_A$-coalgebra* is a triple $(C, \Delta, \mathcal{J})$ where C is an $A\otimes\bar{A}$-bimodule and $\Delta: C\to C\times_A C$ and $\mathcal{J}: C\to$End A are $A\otimes\bar{A}$-bilinear maps such that the following diagrams commute:

$$\begin{array}{ccccccc} & & C\times_A C & \xrightarrow{\Delta\times 1} & (C\times_A C)\times_A C & & \\ & \overset{\Delta}{\nearrow} & & & & \overset{\alpha}{\searrow} & \\ C & & & & & & C\times_A C\times_A C \\ & \underset{\Delta}{\searrow} & & & & \underset{\alpha'}{\nearrow} & \\ & & C\times_A C & \xrightarrow{1\times\Delta} & C\times_A(C\times_A C) & & \end{array}$$

$$\begin{array}{ccccc} C\times_A C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C\times_A C \\ {\scriptstyle \mathcal{J}\times 1}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1\times\mathcal{J}} \\ \mathrm{End}\, A\times_A C & \xrightarrow{\theta'} & C & \xrightarrow{\theta} & C\times_A \mathrm{End}\, A\,. \end{array}$$

We do not assume the associativity (1.9) of C.

4.2. DEFINITION. When $(C, \Delta, \mathcal{J})$ is a $\times_A$-coalgebra, we put $c[a]=\mathcal{J}(c)(a)$, $c\in C$, $a\in A$.

A morphism of $\times_A$-coalgebras from $(C, \Delta, \mathcal{J})$ to $(C', \Delta', \mathcal{J}')$ is an $A\otimes\bar{A}$-bilinear map $u: C\to C'$ such that $\Delta'\circ u=(u\times u)\circ\Delta$ and $\mathcal{J}'\circ u=\mathcal{J}$. They form a category.

4.3. PROPOSITION. *Let $(C, \Delta, \mathcal{J})$ be a $\times_A$-coalgebra except that co-associativity of Δ is not assumed. If $d\in C$ and $\Delta(d)=\sum_i d_i\otimes d'_i\in C\times_A C\subset\int_x {}_{\bar{x}}C\otimes_x C$, then, for a, $b\in A$,*

i) $da=\sum_i d_i[a]d'_i$, $d\bar{a}=\sum_i \overline{d'_i[a]}d_i$,

ii) $d[ab]=\sum_i d_i[a]d'_i[b]$.

PROOF. i) Since $\Delta(da)=\sum_i d_i a\otimes d'_i$, $da=\theta'(\sum_i d_i a\otimes d'_i)=\sum_i (d_i a)[1]d'_i=\sum_i d_i[a]d'_i$. Similar for $d\bar{a}$.

ii) $d[ab]=(da)[b]=\sum_i d_i[a]d'_i[b]$ by i).

4.4. PROPOSITION. *Let $(C, \Delta, \mathcal{J})$ be a $\times_A$-coalgebra. If $\Delta: C\to C\times_A C$ is an isomorphism (or equivalently if $\theta: C\times_A C\to C$ is or equivalently if $\theta': C\times_A C\to C$ is) then the $A\otimes\bar{A}$-bimodule C is associative.*

PROOF. By definition, $\alpha\circ(\Delta\times 1)\circ\Delta=\alpha'\circ(1\times\Delta)\circ\Delta: C\to C\times_A C\times_A C$ where $(\Delta\times 1)\circ\Delta$ and $(1\times\Delta)\circ\Delta$ are isomorphisms. This map is injective having the retract

$$C\times_A C\times_A C\xrightarrow{\theta''}C\times_A C\xrightarrow{\theta}C\,.$$

Hence α and α' are injective and have the same image.

4.5. DEFINITION. A $\times_A$-*bialgebra* is a $\times_A$-coalgebra $(B, \Delta, \mathcal{J})$ where B is an algebra over $A\otimes\bar{A}$ and $\Delta: B\to B\times_A B$ and $\mathcal{J}: B\to\mathrm{End}\, A$ are maps of algebras over $A\otimes\bar{A}$.

If $(B, \Delta, \mathcal{J})$ is a $\times_A$-bialgebra, then A is a left B-module via (4.2).

Morphisms of $\times_A$-bialgebras are morphisms of $\times_A$-coalgebras which are at the same time maps of algebras over $A\otimes\bar{A}$. They also make a category.

4.6. EXAMPLE. Let H be a bialgebra over R and A an H-module algebra [2]. Cocommutativity is not needed. Let $B=A\otimes\bar{A}\otimes H$. This is an algebra with unit $1\otimes\bar{1}\otimes 1$ and multiplication determined by

$$(a\otimes\bar{b}\otimes g)(c\otimes\bar{d}\otimes h)=\sum_{(g)} ag_{(1)}(c)\otimes\overline{g_{(3)}(d)b}\otimes g_{(2)}h\,,$$

$a, b, c, d\in A, g, h\in H$, where we used the sigma notation of [2]. B is an algebra over $A\otimes\bar{A}$ with respect to

$$A\otimes\bar{A}\to B,\quad a\otimes\bar{b}\mapsto a\otimes\bar{b}\otimes 1\,.$$

The map $\mathcal{J}: B\to\operatorname{End} A$, $\mathcal{J}(a\otimes\bar{b}\otimes g)(c)=ag(c)b$, a, b, $c\in A$, $g\in H$, is a map of algebras over $A\otimes\bar{A}$. The map

$$\Delta: B\longrightarrow\int_x {}_{\bar{x}}B\otimes_x B$$

$$\Delta(a\otimes\bar{b}\otimes g)=\sum_{(g)}(a\otimes\bar{1}\otimes g_{(1)})\otimes(1\otimes\bar{b}\otimes g_{(2)})\,,$$

$a, b\in A$, $g\in H$, is left $A\otimes\bar{A}$-linear with image contained in $B\times_A B$. The induced map $\Delta: B\to B\times_A B$ is a map of algebras over $A\otimes\bar{A}$.

The reader can easily check that $(B=A\otimes\bar{A}\otimes H, \Delta, \mathcal{J})$ is a $\times_A$-bialgebra.

Taking $H=R$ (acting trivially on A) in particular, we know that $A\otimes\bar{A}$ has a canonical $\times_A$-bialgebra structure $(\Delta, \mathcal{J})$ where $\Delta: A\otimes\bar{A}\to(A\otimes\bar{A})\times_A(A\otimes\bar{A})$ and $\mathcal{J}: \bar{A}\otimes A\to\operatorname{End} A$ are the *unique* maps of algebras over $A\otimes\bar{A}$.

4.7. PROPOSITION. i) *Let* C *and* D *be* $\times_A$*-coalgebras.* $C\otimes_{A\otimes\bar{A}}D$ *is an* $A\otimes\bar{A}$*-bimodule by the above of* (1.12). *Let*

$$\Delta: C\otimes_{A\otimes\bar{A}}D\xrightarrow{\Delta\otimes\Delta}(C\times_A C)\otimes_{A\otimes\bar{A}}(D\times_A D)\xrightarrow{\xi}(C\otimes_{A\otimes\bar{A}}D)\times_A(C\otimes_{A\otimes\bar{A}}D)$$

$$\mathcal{J}: C\otimes_{A\otimes\bar{A}}D\xrightarrow{\mathcal{J}\otimes\mathcal{J}}\operatorname{End} A\otimes_{A\otimes\bar{A}}\operatorname{End} A\xrightarrow{\text{product}}\operatorname{End} A$$

where ξ *is defined at* (1.12). *Then* $(C\otimes_{A\otimes\bar{A}}D, \Delta, \mathcal{J})$ *is a* $\times_A$*-coalgebra.*

ii) *Let* C, D *and* E *be* $\times_A$*-coalgebras. The canonical isomorphism* $(C\otimes_{A\otimes\bar{A}}D)\otimes_{A\otimes\bar{A}}E\cong C\otimes_{A\otimes\bar{A}}(D\otimes_{A\otimes\bar{A}}E)$ *is an isomorphism of* $\times_A$*-coalgebras.*

PROOF. Left to the reader. Q.E.D.

In general an algebra over A can be identified with a triple (M, p, u) where M is an A-bimodule, $p: M\otimes_A M\to M$ and $u: A\to M$ are A-bilinear maps such that the product p is associative with unit u.

4.8. PROPOSITION. *A $\times_A$-bialgebra can be identified with a triple (B, p, u) where B is a $\times_A$-coalgebra and $p: B\otimes_{A\otimes\bar{A}}B\to B$ and $u: A\otimes\bar{A}\to B$ are maps of $\times_A$-coalgebras such that (M, p, u) is an algebra over $A\otimes\bar{A}$, where M denotes the underlying $A\otimes\bar{A}$-bimodule of B. (Note that $A\otimes\bar{A}$ has a canonical $\times_A$-coalgebra structure by* (4.6)).

PROOF. Follows immediately from the definition.

4.9. LEMMA. *Let C be a $\times_A$-coalgebra and $f: C\to \mathrm{End}\,A$ an $A\otimes\bar{A}$-bilinear map. Put*

$$F: C \longrightarrow C\times_A C \xrightarrow{1\times f} C\times_A \mathrm{End}\,A \xrightarrow{\theta} C\,.$$

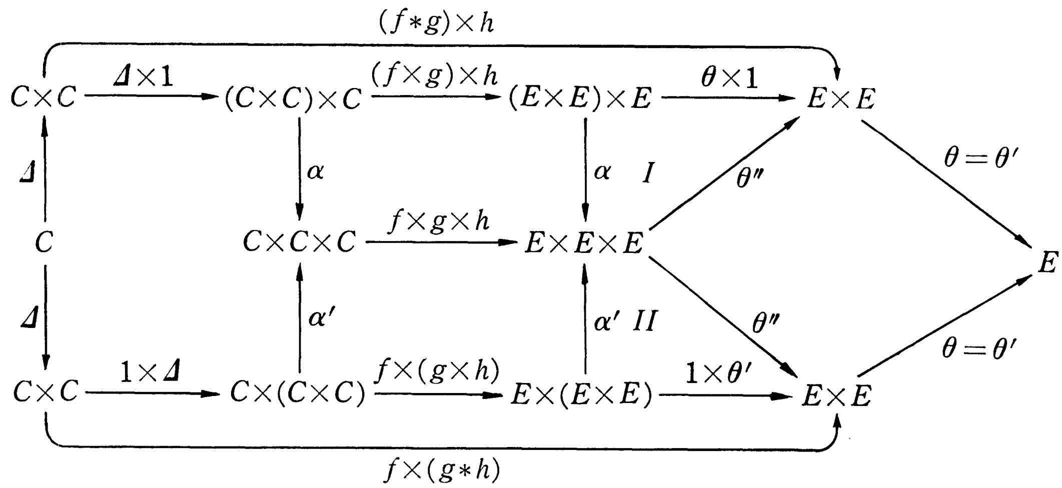
Then F is $A\otimes\bar{A}$-bilinear and $\mathcal{J}F=f$.

PROOF. Let $c\in C$ and $\Delta(c)=\sum_i c_i\otimes c_i'$. Then $F(c)=\sum_i \overline{f(c_i')(1)}c_i$ and $F(c)[1]=\sum_i \overline{f(c_i')(1)}c_i[1]=\sum_i c_i[1]f(c_i')(1)=f(\sum_i c_i[1]c_i')(1)=f(c)(1)$ using (4.3), i). Hence $\partial\mathcal{J}F=\partial f$. Since f and $\mathcal{J}F$ are $A\otimes\bar{A}$-bilinear, $f=\mathcal{J}F$ by (2.6).

4.10. PROPOSITION. *Let C be a $\times_A$-coalgebra. Let $C^\#$ denote the module of all $A\otimes\bar{A}$-bilinear maps from C into* End A. *$C^\#$ is an algebra with unit $\mathcal{J}$ and with product determined by*

$$f*g: C \longrightarrow C\times_A C \xrightarrow{f\times g} \mathrm{End}\,A\times_A \mathrm{End}\,A \xrightarrow{\theta=\theta'} \mathrm{End}\,A,\ f,\ g\in C^\#\,.$$

PROOF. Let f, g, $h\in C^\#$. That $f*\mathcal{J}=f$ follows from (4.9). Similarly $\mathcal{J}*g=g$. The associativity $(f*g)*h=f*(g*h)$ follows from the following commutative diagram:



where we write $\times$ and E to denote $\times_A$ and End A. The commutativity of the regions I and II follows from (2.3).

4.11. COROLLARY. *Let C be a $\times_A$-coalgebra. Suppose $\mathcal{J}: C\to \mathrm{End}\,A$ is* *injective.*

i) *The maps* θ, θ': $C\times_A C\to C$ *coincide.*

ii) *Let* $\mathrm{End}_{A\otimes\bar{A}\text{-}bi}(C)$ *denote the R-module of* $A\otimes\bar{A}$*-bilinear endomorphisms of C. This is an algebra with unit* 1 *and with product determined by*

$$f*g: C\xrightarrow{f\times g} C\times_A C\xrightarrow{\theta=\theta'} C\times_A C\longrightarrow C,\quad f,g\in\mathrm{End}_{A\otimes\bar{A}\text{-}bi}(C).$$

iii) *We have* $f*g=f\circ g=g\circ f$, f, $g\in\mathrm{End}_{A\otimes\bar{A}\text{-}bi}(C)$. *Hence the algebra* $(\mathrm{End}_{A\otimes\bar{A}\text{-}bi}(C),*)$ *equals the endomorphism ring of the* $A\otimes\bar{A}$*-bimodule under composite. This is commutative.*

iv) *The map* $\mathcal{G}$ *induces an isomorphism of algebras* $\mathrm{End}_{A\otimes\bar{A}\text{-}bi}(C)\cong C^{\#}$.

PROOF. i) is obvious. ii) and iv) follow from (4.9) and (4.10). Let $c\in C$ and $\Delta(c)=\sum_i c_i\otimes c'_i$. Then $(f*g)(c)=\theta(\sum_i f(c_i)\otimes g(c'_i))=\sum_i \overline{g(c'_i)[1]}f(c_i)=f(\sum_i \overline{g(c'_i)[1]}c_i)=f(g(c))$, where we used $1*g=g$. Similarly $(f*g)(c)=\theta'(\sum_i f(c_i)\otimes g(c'_i))=\sum_i f(c_i)[1]g(c'_i)=g(\sum_i f(c_i)[1]c'_i)=g(f(c))$ by $f*1=f$. This proves iii).

4.12. PROPOSITION. *Let C be a* $\times_A$*-coalgebra and* $E=\mathcal{G}(C)$. *Assume* θ: $E\times_A E\to E$ *is injective. Then* θ *is an* $A\otimes\bar{A}$*-bimodule isomorphism and* (E,θ^{-1},ι) *gives E the structure of a* $\times_A$*-coalgebra, where* $\iota: E\to\mathrm{End}\,A$ *is the inclusion. The map* $\mathcal{G}: C\to E$ *is a* $\times_A$*-coalgebra map.*

PROOF. The same as [**1**, (6.3)].

4.13. THEOREM. *Let* $D\subset\mathrm{End}\,A$ *be a sub-*$A\otimes\bar{A}$*-bimodule where* $\theta: D\times_A D\to D$ *is an isomorphism and* $\Lambda:\int_{x,y}{}_{\bar{x}}D\otimes_{x,\bar{y}}D\otimes_y D\to\mathrm{Hom}(A,\int_x{}_{\bar{x}}D\otimes_x D)$ (*see* (2.1)) *is injective. Then* (D,θ^{-1},ι) *is an associative* $\times_A$*-coalgebra.*

PROOF. The associativity follows from (4.4). We have only to show $\alpha(\Delta\times 1)\Delta=\alpha'(1\times\Delta)\Delta$: $D\to D\times_A D\times_A D$ where $\Delta=\theta^{-1}$. Let $d\in D$ and $\Delta(d)=\sum_i d_i\otimes d'_i$. Put $u=\sum_i\Delta(d_i)\otimes d'_i$, $v=\sum_i d_i\otimes\Delta(d'_i)=\sum_k e_k\otimes e'_k\otimes e''_k$ in $\int_{x,y}{}_{\bar{x}}D\otimes_{x,\bar{y}}D\otimes_y D$. Then, for $a\in A$, $\Lambda(u)(a)=\sum_i\overline{d'_i[a]}\Delta(d_i)=\Delta(\sum_i\overline{d'_i[a]}d_i)=\Delta(d\bar{a})$ by (4.3). $\Lambda(v)(a)=\sum_i e_k\otimes\overline{e''_k[a]}e'_k=\sum_i d_i\otimes d'_i\bar{a}$ by (4.3) again. Since $\Delta(d\bar{a})=\Delta(d)\bar{a}=\sum_i d_i\otimes d'_i\bar{a}$, we have $\Lambda(u)(a)=\Lambda(v)(a)$. Hence $u=v$ by assumption. Q.E.D.

If in addition D is a subalgebra over $A\otimes\bar{A}$ in the above, then (D,θ^{-1},ι) is clearly a $\times_A$-bialgebra.

4.14. COROLLARY. *Suppose A is a finite projective R-module. Then for all* $\bar{A}$*-bimodule M and A-bimodule N, the maps*

$$\theta: M\times_A\mathrm{End}\,A\longrightarrow M\quad\textit{and}\quad\theta':\mathrm{End}\,A\times_A N\longrightarrow N$$

are isomorphisms. In particular there is a unique $A\otimes\bar{A}$*-bilinear map* $\Delta:\mathrm{End}\,A\to\mathrm{End}\,A\times_A\mathrm{End}\,A$ *making* $(\mathrm{End}\,A,\Delta,\iota)$ *into a* $\times_A$*-bialgebra.*

PROOF. Since all Λ maps (2.1) are isomorphisms, the θ maps are isomorphisms. Similarly the θ' maps are too. Hence the latter half follows from (4.13).

§ 5. The case where A is a division algebra.

Suppose that A is a division algebra. But R is arbitrary. We know that the θ, θ' and θ'' maps are injective by [**1**, (1.5)]. The maps α and α' are isomorphisms and hence any triple (M, P, N) of $A\otimes\bar{A}$-bimodules associates (1.8).

Let B be the image of

$$\theta : \mathrm{End}\, A\times_A \mathrm{End}\, A \longrightarrow \mathrm{End}\, A$$

which is a subalgebra of End A over $A\otimes\bar{A}$.

5.1. THEOREM. *$\theta : B\times_A B\rightarrow B$ is bijective and (B, θ^{-1}, ι) is the unique maximal $\times_A$-coalgebra in* End A *with co-unit ι. B is actually a $\times_A$-bialgebra.*

PROOF. Using (3.6) instead of [**1**, (3.10)] the proof is similar to [**1**, (7.1)].

5.2. LEMMA. *Let M be an $\bar{A}$-bimodule and B as above.*

i) *The inclusion $M\times_A B \xrightarrow{1\times\iota} M\times_A \mathrm{End}\, A$ is an isomorphism of $\bar{A}$-bimodules.*

ii) *The map $\theta : M\times_A \mathrm{End}\, A\rightarrow M$ has the image $M'=\{m\in M \mid \bar{A}m\bar{A}$ is left $\bar{A}$-finite dimensional$\}$.*

iii) *M' is a sub-$\bar{A}$-bimodule of M and the inclusion $M'\times_A B \xrightarrow{\iota\times\iota} M\times_A \mathrm{End}\, A$ is an isomorphism of $\bar{A}$-bimodules.*

iv) *$\theta : M'\times_A B\rightarrow M'$ is an isomorphism.*

PROOF. M' is clearly a sub-$\bar{A}$-bimodule of M. Let $m=\theta(\sum_i m_i\otimes c_i)$ with $\sum_i m_i\otimes c_i\in M\times_A \mathrm{End}\, A$. Then $m\bar{a}=\theta(\sum_i m_i\otimes c_i\bar{a})=\sum_i \overline{c_i(a)}m_i$. Hence $\bar{A}m\bar{A}\subset\sum_i \bar{A}m_i$ is left $\bar{A}$-finite dimensional and so $\theta(M\times_A \mathrm{End}\, A)\subset M'$.

Conversely if $m\in M'$, then $A : \int_x {}_{\bar{x}}\bar{A}m\bar{A}\otimes_x \mathrm{End}\, A\rightarrow \mathrm{Hom}\,(A, \bar{A}m\bar{A})$ is an isomorphism. Hence so is $\theta : \bar{A}m\bar{A}\times_A \mathrm{End}\, A\rightarrow\bar{A}m\bar{A}$. This means that $\theta : M'\times_A \mathrm{End}\, A\rightarrow M'$ is an isomorphism. In particular $M'=\theta(M\times_A \mathrm{End}\, A)$.

Consider the following diagram:

$$\begin{array}{ccccc}
(M\times_A \mathrm{End}\, A)\times_A \mathrm{End}\, A & \xrightarrow[\sim]{\theta\times 1} & M'\times_A \mathrm{End}\, A & \xrightarrow[\sim]{\theta} & M' \\
\downarrow \alpha\ \wr & & \cap\downarrow & & \cap\downarrow \\
M\times_A \mathrm{End}\, A\times_A \mathrm{End}\, A & \xrightarrow{\theta''} & M\times_A \mathrm{End}\, A & \xrightarrow{\theta} & M \\
\uparrow \alpha'\ \wr & & \cup\uparrow & \nearrow \theta & \\
M\times_A(\mathrm{End}\, A\times_A \mathrm{End}\, A) & \xrightarrow[\sim]{1\times\theta'} & M\times_A B\, . & &
\end{array}$$

It follows that $\theta(M\times_A B)=\theta(M'\times_A \mathrm{End}\,A)=M'$ in M. This proves the lemma.
Q. E. D.

As a corollary we have

$$B=\{c\in\mathrm{End}\,A \mid \bar{A}c\bar{A} \text{ is left } \bar{A}\text{-finite dimensional}\}.$$

On the other hand we also have

$$B=\{c\in\mathrm{End}\,A \mid AcA \text{ is left } A\text{-finite dimensional}\}$$

since $B=\theta'(\mathrm{End}\,A\times_A \mathrm{End}\,A)$ and the dual statement of (5.2) holds.

Let $D=\{f\in\mathrm{End}\,A \mid AfA$ is right A-finite dimensional$\}$. Then $D^0=\mathrm{Im}\,(\theta: (\mathrm{End}\,A)^0\times_A \mathrm{End}\,A\to(\mathrm{End}\,A)^0)$ and $\theta: D^0\times_A B\to D^0$ is an isomorphism by (5.2).

Let $\delta: D^0\to D^0\times_A B$ and $\varDelta: B\to B\times_A B$ be the inverses of the θ maps.

Let E denote the sum of all $A\otimes\bar{A}$-bimodules $X\subset\mathrm{End}\,A$ which satisfy (i) $X\subset B\cap D$, (ii) $\varDelta(X)\subset X\times_A X\subset B\times_A B$ and (iii) $\delta(X^0)\subset X^0\times_A X\subset D^0\times_A B$. Then just as [**1**, (7.3)] we have

5.3. THEOREM. *E is the unique maximal $\times_A$-bialgebra (with $\mathcal{J}=\iota$) in* End A *which satisfies: $E^0\times_A E\to E^0$ is surjective (or bijective).*

§ 6. The Ess map and simplicity.

So far in the generalization (over commutative $A\to$over $A\otimes\bar{A}$) we have encountered no difficulties. To obtain the theorem [**1**, (10.2), (10.3)] we also need the *Ess* map. Its definition must be changed when we work over $A\otimes\bar{A}$, since the maps $\mathcal{B}$ and C in [**1**, (9.1)] make no sense unless A is commutative.

The definition of an ess map of a $\times_A$-bialgebra is given in (6.8) after the following rather long chain of definitions and lemmas. The analogies of [**1**, (10.2), (10.3)] are established in (6.13) and (6.14).

6.1. DEFINITION. Let U, V, W and X be $A\otimes\bar{A}$-bimodules. Let

$$\Phi(U, W, V)=\int^{y,b}\int_{x,a} {}_{b,\bar{x}}U_{a,\bar{y}}\otimes_a W_b\otimes_x V_y\,,$$

$$\Psi(U, W, V, X)=\int^{b,q}\int_p\int^y\int_{x,a} {}_{b,\bar{x}}U_{a,\bar{y}}\otimes_{a,\bar{p}} W_{b,\bar{q}}\otimes_x V_y\otimes_p X_q\,.$$

We make the above modules into $A\otimes\bar{A}$-bimodules with structure determined by

$${}_{l,\bar{u}}\Phi(U, W, V)_{r,\bar{v}}=\Phi(U, {}_{\bar{u}}W_{\bar{v}}, {}_{\bar{r}}V_{\bar{l}})$$

$${}_{l,\bar{u}}\Psi(U, W, V, X)_{r,\bar{v}}=\Psi(U, W, {}_{\bar{u}}V_{\bar{v}}, {}_{\bar{r}}X_{\bar{l}})\,.$$

6.2. LEMMA. *Let U, V, W and X be $A\otimes\bar{A}$-bimodules.*

a) *The image of the composite*

$$(U\times_A V)^0\times_A W \subsetneq \int_a (U_a\times_A V)\otimes_a W \xrightarrow{\iota\otimes 1} \int_{x,a} {}_{\bar{x}}U_a\otimes_x V\otimes_a W \cong \int_{x,a} {}_{\bar{x}}U_a\otimes_a W\otimes_x V$$

is contained in $\Phi(U, W, V)$. *Let*

$$\mathcal{B}:\ (U\times_A V)^0\times_A W \longrightarrow \Phi(U, W, V)$$

denote the induced map. This is $A\otimes\bar{A}$*-bilinear.*

b) *The image of the composite*

$$((U^0\times_A W)^0\times_A V)^0 \subsetneq \int_x (({}_{\bar{x}}U)^0\times_A W)\otimes_x V \xrightarrow{\iota\otimes 1} \int_{x,a} {}_{\bar{x}}U_a\otimes_a W\otimes_x V$$

is contained in $\Phi(U, W, V)$. *Let*

$$\mathcal{C}:\ ((U^0\times_A W)^0\times_A V)^0 \longrightarrow \Phi(U, W, V)$$

denote the induced map. This is $A\otimes\bar{A}$*-bilinear.*

c) *The map*

$$\Phi(U, W, V)\otimes X \xrightarrow{\iota\otimes 1} \int_{x,a} {}_{\bar{x}}U_a\otimes_a W\otimes_x V\otimes X$$

induces a canonical map

$$\Phi(U, W, V)\otimes X \longrightarrow \int^{y,b}\int_{x,a} {}_{b,\bar{x}}U_{a,\bar{y}}\otimes_a W_b\otimes_x V_y\otimes X\,.$$

This induces a canonical map

$$\Phi(U, W, V)\times_A X \longrightarrow \int^q\int_p\int^{y,b}\int_{x,a} {}_{b,\bar{x}}U_{a,\bar{y}}\otimes_{a,\bar{p}} W_{b,\bar{q}}\otimes_x V_y\otimes_p X_q\,.$$

Applying the exchange map: $\int_p\int^b \longrightarrow \int^b\int_p$ *(see Conventions) we obtain the map*

$$\lambda:\ (\Phi(U, W, V)\times_A X)^0 \longrightarrow \Psi(U, W, V, X)\,.$$

This is $A\otimes\bar{A}$*-bilinear.*

PROOF. Straightforward.

6.3. LEMMA. *Let* (U, i) *and* (V, j) *be algebras over* $A\otimes\bar{A}$.

a) *If* j *induces an isomorphism:* $\bar{A}\cong\int^x {}_xV_x$ *and there is an* $A\otimes\bar{A}$*-bilinear isomorphism* $\sigma: U\to V$, *then*

i) *There is a unique invertible element* $b\in A$ *where* $\sigma(i(\bar{a}))=j(\bar{a}\bar{b})=j(\bar{b}\bar{a})$ *for all* $a\in A$. *In particular* b *belongs to the center of* A.

ii) i *induces an isomorphism:* $\bar{A}\cong\int^x {}_xU_x$.

iii) *We have* $\bar{T}=\{\bar{a}\in\bar{A}\mid \bar{a}u=u\bar{a}$ *for all* $u\in U\}=\{\bar{a}\in\bar{A}\mid \bar{a}v=v\bar{a}$ *for all* $v\in V\}$ *and* $i:\bar{T}\to$ center (U), $j:\bar{T}\to$ center (V) *are isomorphisms.*

b) *If $h:\bar{A}\to U\times_A V$ is injective, so is $j:\bar{A}\to V$.*

PROOF. This can be proved in the same way as [1, (9.3)]. Q.E.D.

Let (U, i) and (V, j) be algebras over $A\otimes\bar{A}$. Fix an element $d\in A$ for a moment. The map

$$f(d):\int_{x,a}{}_{\bar{x}}U_a\otimes_a U\otimes_x V\otimes V\longrightarrow\int_x{}_{\bar{x}}U\otimes_x V,\ f(d)(u\otimes u'\otimes v\otimes v')=uu'\otimes v\bar{d}v',$$

$u,\ u'\in U,\ v,\ v'\in V$ restricted to $\int^y\int_{x,a}{}_{\bar{x}}U_{a,\bar{y}}\otimes_a U\otimes_x V_y\otimes V$ induces

$$f(d):\int_p\int^y\int_{x,a}{}_{\bar{x}}U_{a,\bar{y}}\otimes_{a,\bar{p}}U\otimes_x V_y\otimes_p V\longrightarrow\int_x{}_{\bar{x}}U\otimes_x V.$$

Indeed if $\sum_i u_i\otimes u_i'\otimes v_i\otimes v_i'\in\int^y\int_{x,a}{}_{\bar{x}}U_{a,\bar{y}}\otimes_a U\otimes_x V_y\otimes V$, then $f(d)(\sum_i u_i\otimes\bar{p}u_i'\otimes v_i\otimes v_i')=\sum_i u_i\bar{p}u_i'\otimes v_i\bar{d}v_i'=f(d)(\sum_i u_i\bar{p}\otimes u_i'\otimes v_i\otimes v_i')=f(d)(\sum_i u_i\otimes u_i'\otimes v_ip\otimes v_i')=\sum_i u_iu_i'\otimes v_ip\bar{d}v_i'=\sum_i u_iu_i'\otimes v_i\bar{d}pv_i'=f(d)(\sum_i u_i\otimes u_i'\otimes v_i\otimes pv_i')$ for all $p\in A$.

This map induces an R-linear map

$$f(d):\Psi(U, U, V, V)\longrightarrow\int^{b,q}\int_x{}_{b,\bar{x}}U_{b,\bar{q}}\otimes_x V_q=\int^b{}_b(U\times_A V)_b.$$

It can be easily verified that this map satisfies:

$$6.4.\qquad\begin{cases} f(d)(ax)=f(d)(x)\bar{a}\\ f(d)(\bar{a}x)=\bar{a}f(d)(x)\\ f(d)(xa)=f(ad)(x)\\ f(d)(x\bar{a})=f(da)(x)\end{cases}$$

$a,\ d\in A,\ x\in\Psi(U, U, V, V)$.

Recall that $\int_x{}_{\bar{x}}U\otimes_x V$ is a left $U\times_A V$-module (3.1). Hence if $d\in A$ is fixed, the map

$$g(d):\int_{x,a}(U_a\times_A V)\otimes_{a,\bar{x}}U\otimes_x V\longrightarrow\int_x{}_{\bar{x}}U\otimes_x V,$$

$$g(d)[(\sum_i u_i\otimes v_i)\otimes u'\otimes v']=\sum_i u_iu'\otimes v_i\bar{d}v',$$

$\sum_i u_i\otimes v_i\in U\times_A V,\ u'\in U,\ v'\in V$, is well-defined. This induces a map

$$\int^{y,b}\int_{x,a}({}_bU_a\times_A V)\otimes_{a,\bar{x}}U_{b,\bar{y}}\otimes_x V_y\longrightarrow\int^{y,b}\int_x{}_{b,\bar{x}}U_{b,\bar{y}}\otimes_x V_y$$

or equivalently a map

$$g(d):((U\times_A V)^0\times_A U\times_A V)^0\longrightarrow\int^b{}_b(U\times_A V)_b.$$

This also satisfies:

6.5. $$\begin{cases} g(d)(ax)=g(d)(x)\bar{a} \\ g(d)(\bar{a}x)=\bar{a}g(d)(x) \\ g(d)(xa)=g(ad)(x) \\ g(d)(x\bar{a})=g(da)(x) \end{cases}$$

$a,\ d\in A,\ x\in((U\times_A V)^0\times_A U\times_A V)^0$.

6.6. LEMMA. *Let (U, i) and (V, j) be algebras over $A\otimes\bar{A}$. Suppose $h:\bar{A}\to\int^b{}_b(U\times_A V)_b$ is an isomorphism.*

a) *The linear maps*

$$\mathcal{D}:\ \Psi(U, U, V, V)\longrightarrow \mathrm{End}\, A$$

$$\mathcal{E}:\ ((U\times_A V)^0\times_A U\times_A V)^0\longrightarrow \mathrm{End}\, A$$

are defined by

$$h(\overline{\mathcal{D}(x)(a)})=f(a)(x)\,, \qquad h(\overline{\mathcal{E}(y)(a)})=g(a)(y)\,,$$

$a\in A$, $x\in\Psi(U, U, V, V)$, $y\in((U\times_A V)^0\times_A U\times_A V)^0$. Then $\mathcal{D}$ and $\mathcal{E}$ are $A\otimes\bar{A}$-bilinear.

b) *Suppose further $i:\bar{A}\to\int^b{}_bU_b$ and $j:\bar{A}\to\int^b{}_bV_b$ are both isomorphisms. We have then the following commutative diagram:*

$$\begin{array}{ccccc}
(((U^0\times U)^0\times V)^0\times V)^0 & \xrightarrow{(\mathcal{C}\times 1)^0} & (\Phi(U, U, V)\times V)^0 & \xleftarrow{(\mathcal{B}\times 1)^0} & (((U\times V)^0\times U)^0\times V)^0 \\
\downarrow{\scriptstyle ((\zeta\times 1)^0\times 1)^0} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \alpha^0} \\
((\mathrm{End}\, A\times V)^0\times V)^0 & & \Psi(U, U, V, V) & & ((U\times V)^0\times U\times V)^0 \\
\downarrow{\scriptstyle (\theta'^0\times 1)^0} & & \downarrow{\scriptstyle \mathcal{D}} & \swarrow{\scriptstyle \mathcal{E}} & \uparrow{\scriptstyle \alpha'^0} \\
(V^0\times V)^0 & \xrightarrow{\zeta} & \mathrm{End}\, A & \xleftarrow{\zeta} & ((U\times V)^0\times(U\times V))^0\,.
\end{array}$$

Here the ζ maps are defined in (3.5) *and $\times$ denotes $\times_A$. This diagram consists of $A\otimes\bar{A}$-bilinear maps.*

PROOF. The existence of $\mathcal{D}$ and $\mathcal{E}$ is clear. That they are $A\otimes\bar{A}$-bilinear follows from (6.4) and (6.5). To check the commutativity of the diagram is left to the reader.

6.7. LEMMA. *Let (U, i) be an algebra over $A\otimes\bar{A}$ where $i:\bar{A}\to\int^x{}_xU_x$ is isomorphic. If $\mathcal{S}: U\to\mathrm{End}\, A$ is a map of algebras over $A\otimes\bar{A}$, the following diagram commutes:*

$$\begin{array}{ccc} (U^0\times_A U)^0 & \xrightarrow{\quad\zeta\quad} & \operatorname{End} A \\ \big\downarrow{\scriptstyle (1^0\times\mathcal{J})^0} & & \big\uparrow{\scriptstyle \mathcal{J}} \\ (U^0\times_A \operatorname{End} A)^0 & \xrightarrow{\ \theta^0\ } & U^{00}=U\,. \end{array}$$

PROOF. If $x=\sum_i u_i^0\otimes u_i'\in U^0\times_A U$, then $\mathcal{J}(\sum_i u_i\mathcal{J}(u_i')(1))(d)=\sum_i \mathcal{J}(u_i)(\mathcal{J}(u_i')(1)d)$ $=\sum_i \mathcal{J}(u_i)(\mathcal{J}(\bar{d}u_i')(1))=\mathcal{J}(\sum_i u_i\bar{d}u_i')(1)$, $d\in A$. Since $\sum_i u_i\bar{d}u_i'=i(\overline{\zeta(X^0)(d)})$, we have $\mathcal{J}(\sum_i u_i\bar{d}u_i')(1)=\zeta(x^0)(d)$.

6.8. DEFINITION. Suppose $(B,\Delta,\mathcal{J})$ is a $\times_A$-bialgebra. *An Ess* is a map of algebras over $A\otimes\bar{A}$

$$\mathcal{S}:\ B\longrightarrow (B^0\times_A B)^0$$

which makes the following diagrams commute:

$$\begin{array}{ccc} B^0 & \xrightarrow{\ \mathcal{S}^0\ } & B^0\times_A B \\ \big\downarrow{\scriptstyle 1^0} & & \big\downarrow{\scriptstyle 1^0\times\mathcal{J}} \\ B^0 & \xleftarrow{\ \theta\ } & B^0\times_A \operatorname{End} A \end{array}$$

$$\begin{array}{ccccccc} & & B^0\times_A B & \xrightarrow{\ \Delta^0\times 1\ } & (B\times_A B)^0\times_A B & & \\ & \nearrow{\scriptstyle \mathcal{S}^0} & & & & \searrow{\scriptstyle \mathcal{B}} & \\ B^0 & & & & & & \Phi(B,B,B) \\ & \searrow{\scriptstyle \Delta^0} & & & & \nearrow{\scriptstyle \mathcal{C}} & \\ & & (B\times_A B)^0 & \xrightarrow{\ (\mathcal{S}\times 1)^0\ } & ((B^0\times_A B)^0\times_A B)^0 & & \end{array}$$

$$\begin{array}{ccccccc} & & B^0\times_A B & \xrightarrow{\ 1^0\times\Delta\ } & B^0\times_A(B\times_A B) & & \\ & \nearrow{\scriptstyle \mathcal{S}^0} & & & & \searrow{\scriptstyle \alpha'} & \\ B^0 & & & & & & B^0\times_A B\times_A B \\ & \searrow{\scriptstyle \mathcal{S}^0} & & & & \nearrow{\scriptstyle \alpha} & \\ & & B^0\times_A B & \xrightarrow{\ \mathcal{S}^0\times 1\ } & (B^0\times_A B)\times_A B & & \end{array}$$

We see in the next section (§7) when the inverse of $\theta: B^0\times_A B\to B^0$ which is assumed to be bijective, satisfies the above conditions. In particular if A is a finite projective R-algebra, then $\theta: (\operatorname{End} A)^0\times_A \operatorname{End} A\to(\operatorname{End} A)^0$ is an isomorphism by (4.14) and we see that the inverse θ^{-1} gives the unique Ess map of the $\times_A$-bialgebra $\operatorname{End} A$ (ibid.).

6.9. PROPOSITION. *Let $(B,\Delta,\mathcal{J},\mathcal{S})$ be a $\times_A$-bialgebra with Ess where*

$\Delta: B \rightarrow B\times_A B$ and $\mathcal{S}: B\rightarrow (B^0\times_A B)^0$ are isomorphisms and $\mathcal{J}$ is injective. Let (U, i) and (V, j) be algebras over $A\otimes\bar{A}$ which are $A\otimes\bar{A}$-bimodule isomorphic to B.

a) *We have isomorphisms:*

$$i: \bar{A} \longrightarrow \int^x {}_x U_x, \quad j: \bar{A} \longrightarrow \int^x {}_x V_x, \quad h: \bar{A} \longrightarrow \int^x {}_x(U\times_A V)_x.$$

b) *We have*

$$\mathcal{J}: B \xrightarrow{\mathcal{S}} (B^0\times_A B)^0 \xrightarrow{\zeta} \operatorname{End} A$$

or equivalently

$$\zeta: (B^0\times_A B)^0 \xrightarrow{\theta^0} B^{00}=B \xrightarrow{\mathcal{J}} \operatorname{End} A.$$

c) *The triple $((U\times_A V)^0, U, V)$ associates* (1.8). *The maps $\mathcal{B}: (U\times_A V)^0\times_A U \rightarrow \Phi(U, U, V)$ and $\mathcal{C}: ((U^0\times_A U)\times_A V)^0\rightarrow \Phi(U, U, V)$ are injective and have the same image. Let $\tau: (U\times_A V)^0\times_A U \cong ((U^0\times_A U)^0\times_A V)^0$ denote the induced isomorphism. We have then the following commutative diagram:*

$$\begin{array}{ccccc}
(((U^0\times U)^0\times V)^0\times V)^0 & \overset{(\tau\times 1)^0}{\cong} & (((U\times V)^0\times U)\times V)^0 & \overset{ass}{\cong} & ((U\times V)^0\times(U\times V))^0 \\
\downarrow ((\zeta\times 1)^0\times 1)^0 & & & & \downarrow \zeta \\
((\operatorname{End} A\times V)^0\times V)^0 & \xrightarrow{(\theta'^0\times 1)^0} & (V^0\times V)^0 & \xrightarrow{\zeta} & \operatorname{End} A
\end{array}$$

where $\times$ denotes $\times_A$.

PROOF. a) Since $U\cong V\cong U\times_A V\cong B$ as $A\otimes\bar{A}$-bimodules and $\mathcal{J}: B\rightarrow \operatorname{End} A$ is injective, this follows from (6.3). b) follows from (6.7).

c) Since $\mathcal{S}$ and Δ are isomorphisms, the third diagram in (6.8) shows that $\alpha': B^0\times_A(B\times_A B)\rightarrow B^0\times_A B\times_A B$ and $\alpha: (B^0\times_A B)\times_A B\rightarrow B^0\times_A B\times_A B$ have the same image. The composite $\alpha'(1^0\times\Delta)\mathcal{S}^0: B^0\rightarrow B^0\times_A B\times_A B$ is injective having as retract the composite $B^0\times_A B\times_A B \xrightarrow{\theta''} B^0\times_A B \xrightarrow{\theta} B^0$. Hence (B^0, B, B) associates and $((U\times_A V)^0, U, V)$ does too. Since the composite

$$(B^0\times_A B)^0\times_A B \xrightarrow{\zeta\times 1} \operatorname{End} A\times_A B \xrightarrow{\theta'} B$$

which equals by b)

$$(B^0\times_A B)^0\times_A B \xrightarrow{\theta^0\times 1} B^{00}\times_A B=B\times_A B \xrightarrow{\theta'} B$$

is an isomorphism, the proposition (6.6), b) applied to $U=V=B$ implies that $\mathcal{D}\lambda(\mathcal{C}\times 1)^0$ there is injective. Hence $\mathcal{C}\times 1$ is injective. Then the injectivity of $\mathcal{C}$ follows from the next lemma (6.10). It follows from the second diagram in (6.8) that $\mathcal{B}$ and $\mathcal{C}$ for (B, B, B) are both injective and have the same image in $\Phi(B, B, B)$. Since $U\cong V\cong B$ as $A\otimes\bar{A}$-bimodules, it follows from the functoriality that $\mathcal{B}: (U\times_A V)^0\times_A U\to\Phi(U, U, V)$ and $\mathcal{C}: ((U^0\times_A U)^0\times_A V)^0\to\Phi(U, U, V)$ are injective having the same image. The commutativity of the diagram is an immediate consequence of (6.6), b).

6.10. LEMMA. *Let $(B, \Delta, \mathcal{I})$ be a $\times_A$-bialgebra where $\Delta: B\to B\times_A B$ is an isomorphism. If M, N and P are $A\otimes\bar{A}$-bimodules isomorphic to B and $f: M\to N$ an $A\otimes\bar{A}$-bilinear map, then the map $f\times 1: M\times_A P\to N\times_A P$ is injective (resp. surjective) if and only if so is f.*

PROOF. We can assume $P=B$. Then $\theta: M\times_A B\to M$ and $\theta: N\times_A B\to N$ are isomorphisms, since so is $\theta: B\times_A B\to B$ and we have a commutative diagram

$$\begin{array}{ccc} M\times_A B & \xrightarrow{\theta} & M \\ \downarrow{\scriptstyle f\times 1} & & \downarrow{\scriptstyle f} \\ N\times_A B & \xrightarrow{\theta} & N. \end{array}$$

This proves the lemma.

6.11. LEMMA. *Let $(B, \Delta, \mathcal{I}, \mathcal{S})$ be a $\times_A$-bialgebra with ess where $\mathcal{S}: B\to (B^0\times_A B)^0$ is an isomorphism. If M, N and P are $A\otimes\bar{A}$-bimodules isomorphic to B and $f: M\to N$ an $A\otimes\bar{A}$-bilinear map, then the map $f^0\times 1: M^0\times_A P\to N^0\times_A P$ is injective (resp. surjective) if and only if so is f.*

PROOF. The same as (6.10).

6.12. THEOREM. *Let $(B, \Delta, \mathcal{I}, \mathcal{S})$ be a $\times_A$-bialgebra with ess where $\mathcal{I}$ is injective and Δ and $\mathcal{S}$ are isomorphisms.*

a) *Suppose U is an algebra over $A\otimes\bar{A}$ which is $A\otimes\bar{A}$-bimodule isomorphic to B. Then $(U^0\times_A U)^0$ is $A\otimes\bar{A}$-bimodule isomorphic to B. There is a unique map of algebras over $A\otimes\bar{A}$*

$$\mathcal{Z}: (U^0\times_A U)^0 \longrightarrow B$$

such that $\mathcal{I}\mathcal{Z}=\zeta$ (3.5).

b) *If U and V are algebras over $A\otimes\bar{A}$ which are $A\otimes\bar{A}$-bimodule isomorphic to B and where $U\times_A V\cong B$ as an algebra over $A\otimes\bar{A}$, then $\mathcal{Z}: (U^0\times_A U)^0\to B$ is injective and $\mathcal{Z}: (V^0\times_A V)^0\to B$ is surjective.*

c) *If in addition $V\times_A U\cong B$ in* b), *then both the $\mathcal{Z}$ maps there are isomorphisms.*

PROOF. a) $(U^0\times_A U)^0\cong(B^0\times_A B)^0\cong B$ as $A\otimes\bar{A}$-bimodules. Hence $\zeta: (U^0\times_A U)^0\to\operatorname{End} A$ factors as $\zeta=\mathcal{I}\mathcal{Z}$ uniquely by (4.9). c) follows from b).

b) Let $\gamma: U\times_A V\cong B$ be an isomorphism of algebras over $A\otimes\bar{A}$. The map

$$\zeta: ((U\times_A V)^0\times_A(U\times_A V))^0 \longrightarrow \operatorname{End} A$$

which equals the composite

$$((U\times_A V)^0\times_A(U\times_A V))^0 \xrightarrow{(\gamma^0\times\gamma)^0} (B^0\times_A B)^0 \xrightarrow{\zeta} \operatorname{End} A$$

is injective having $\mathcal{J}(B)$ as its image. Then the commutative diagrom of (6.9), c) tells us that $\mathcal{Z}:(V^0\times_A V)^0\to B$ is surjective and $(\mathcal{Z}\times 1)^0\times 1:(M\times_A V)^0\times_A V\to(B\times_A V)^0\times_A V$ is injective, where we put $M=(U^0\times_A U)^0$. Applying the lemmas (6.10) and (6.11) we conclude that $\mathcal{Z}:(U^0\times_A U)\to B$ is injective.

6.13. COROLLARY. *Let $(B, \Delta, \mathcal{J}, \mathcal{S})$ be a $\times_A$-bialgebra with ess where $\mathcal{J}$ is injective and Δ and $\mathcal{S}$ are isomorphisms.*

i) *The triple (B^0, B, B) associates.*

ii) *If U is an algebra over $A\otimes\bar{A}$ with $\langle U\rangle\in\mathcal{G}\langle B\rangle$, then $U^0\cong B^0\times_A U^{-1}$ as an algebra over $A\otimes\bar{A}$.*

iii) *If $B^0\cong B$ as an algebra over $A\otimes\bar{A}$, then for each $\langle U\rangle\in\mathcal{G}\langle B\rangle$, $\langle U^0\rangle$ belongs to $\mathcal{G}\langle B\rangle$ and $\langle U^0\rangle=\langle U\rangle^{-1}$.*

PROOF. i) is shown in (6.9), c). If $\langle U\rangle\in\mathcal{G}\langle B\rangle$, then $U^0\times_A U\cong B^0$ as an algebra over $A\otimes\bar{A}$ by (6.12), c). Since $\theta: U^0\times_A B\to U^0$ is an isomorphism, we have $U^0\cong U^0\times_A B\cong U^0\times_A(U\times_A U^{-1})\cong(U^0\times_A U)\times_A U^{-1}\cong B^0\times_A U^{-1}$ as algebras over $A\otimes\bar{A}$. This proves ii) and iii). Q.E.D.

Since we have established the analogy of [**1**, (10.2)] the following theorem which is similar to [**1**, (10.3)] follows from [**1**, (3.7), (3.9)].

6.14. THEOREM. *Let $(B, \Delta, \mathcal{J}, \mathcal{S})$ be a $\times_A$-bialgebra with ess where $\mathcal{J}$ is injective and Δ and $\mathcal{S}$ are isomorphisms. Furthermore assume that B is flat as a left (right) A-module and $0\neq M^0\times_A B$ $(B^0\times_A M)$ for any A-bimodule $0\neq M\subset B$. The following are equivalent:*

a) *A is a simple B-module,*

b) *B is a simple algebra,*

c) *If U is any algebra over $A\otimes\bar{A}$ with $\langle U\rangle\in\mathcal{G}\langle B\rangle$, then U is a simple algebra.*

§ 7. Existence of the ess.

Let $(C, \Delta, \mathcal{J})$ be a $\times_A$-coalgebra. We give a sufficient condition for some section of $\theta: C^0\times_A C\to C^0$ (assumed to be surjective) to satisfy the conditions of (6.8).

Define the maps

$$\Omega_1:\int_{x,a} {}_{\bar{x}}C_a\otimes_a C\otimes_x C \longrightarrow \mathrm{Hom}(A\otimes A\otimes A, A)$$

$$\Omega_2:\int_{x,a} C_x\otimes_{x,\bar{a}} C\otimes_a C \longrightarrow \mathrm{Hom}(A\otimes A\otimes A, A)$$

to be the composites

$$\Omega_1:\int_{x,a}{}_xC_a\otimes_a C\otimes_x C\xrightarrow{\Lambda_1}\mathrm{Hom}\,(A,\int_a C_a\otimes_a C)\xrightarrow{\mathrm{Hom}\,(A,\Lambda_2)}$$

$$\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,C))\xrightarrow{\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,\mathcal{J}))}\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,A)))$$

$$\cong\mathrm{Hom}\,(A\otimes A\otimes A,A)\,,$$

$$\Omega_2:\int_{x,a}C_x\otimes_{x,\bar{a}}C\otimes_a C\xrightarrow{\Lambda_3}\mathrm{Hom}\,(A,\int_x C_x\otimes_x C)\xrightarrow{\mathrm{Hom}\,(A,\Lambda_4)}$$

$$\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,C))\xrightarrow{\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,\mathcal{J}))}\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,\mathrm{Hom}\,(A,A)))$$

$$\cong\mathrm{Hom}\,(A\otimes A\otimes A,A)\,.$$

In the above the map Λ_1 (resp. Λ_2, Λ_3, Λ_4) denotes the Λ-map (2.1) with respect to the left $\bar{u}$ $\bar{A}$-module $\int_a{}_{\bar{u}}C_a\otimes C_a$ (resp. C_u, $\int_x C_x\otimes_{x,\bar{u}}C$, C_u).

Explicitly we have

$$\Omega_1(c_1\otimes c_2\otimes c_3)(a_1\otimes a_2\otimes a_3)=c_1[c_2[a_2]a_1]c_3[a_3]$$

$$\Omega_2(c_1\otimes c_2\otimes c_3)(a_1\otimes a_2\otimes a_3)=c_1[c_2[a_2]c_3[a_3]a_1]$$

$a_i\in A$, $c_i\in C$. (Recall (4.2).)

If the map $\mathcal{J}$ and all the Λ-maps for C are injective, then Ω_1 and Ω_2 are injective.

7.1. LEMMA. *If $\theta^0:(C^0\times_A C)^0\to C$ is surjective and has an A-bilinear section $\mathcal{S}:C\to(C^0\times_A C)^0$, then*

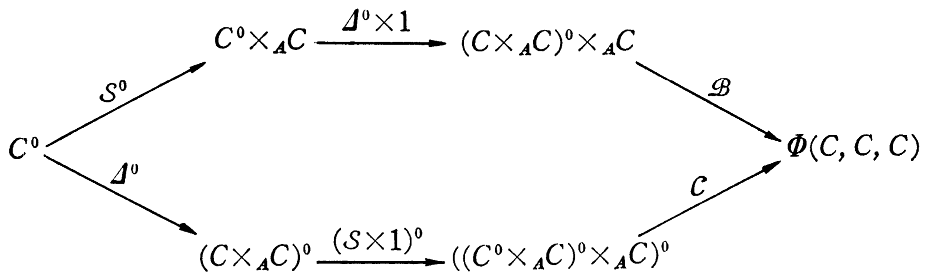
$$axb=\sum_i x_i y_i[a]b$$

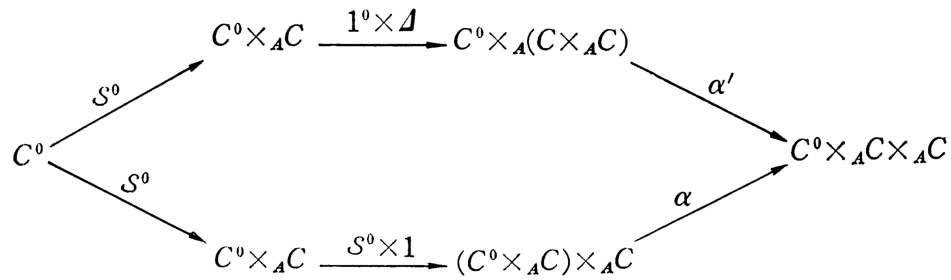
where a, $b\in A$ and $x\in C$ with $\mathcal{S}(x)=\sum_i x_i\otimes y_i$ in $\int_a C_a\otimes_a C$. In particular we have

$$ax[b]=\sum_i x_i[y_i[a]b]\,.$$

PROOF. Since $\mathcal{S}$ is A-bilinear, we have $\mathcal{S}(axb)=\sum_i x_i\otimes\bar{b}y_i\bar{a}$ in $\int_a C_a\otimes_a C$. Applying the θ map, we have $axb=\sum_i x_i(\bar{b}y_i\bar{a})[1]=\sum_i x_i y_i[a]b$. Evaluating at 1, we have $ax[b]=(axb)[1]=\sum_i x_i[y_i[a]b]$.

7.2. PROPOSITION. *Let $(C,\Delta,\mathcal{J})$ be a $\times_A$-coalgebra where there is an $A\otimes\bar{A}$-bilinear map $\mathcal{S}:C\to(C^0\times_A C)^0$ such that $1=\theta^0\circ\mathcal{S}$. If the maps Ω_1 and Ω_2 are injective, then the following diagrams commute:*





PROOF. Let b, $d\in C$, $\mathcal{S}(b)=\sum b_i\otimes c_i$ in $\int_a C_a\otimes_a C$ and $\Delta(d)=\sum d_j\otimes d'_j$ in $\int_x \bar{x}C\otimes_x C$. Then $\mathcal{B}(\Delta^0\times 1)\mathcal{S}^0(b)=\sum b_{ik}\otimes c_i\otimes b'_{ik}$ and $\mathcal{C}(\mathcal{S}\times 1)^0\Delta^0(d)=\sum d_{jh}\otimes e_{jh}\otimes d'_j$ in $\int_{x,a}\bar{x}C_a\otimes_a C\otimes_x C$, where $\Delta(b_i)=\sum b_{ik}\otimes b'_{ik}$ and $\mathcal{S}(d_j)=\sum d_{jh}\otimes e_{jh}$. Let $f_1=\Omega_1\mathcal{B}(\Delta^0\times 1)\mathcal{S}^0(b)$ and $f_2=\Omega_1\mathcal{C}(\mathcal{S}\times 1)^0\Delta^0(d)$. Then $f_1(a_1\otimes a_2\otimes a_3)=\sum b_{ik}[c_i[a_2]a_1]b'_{ik}[a_3]=\sum b_i[c_i[a_2]a_1a_3]=a_2b[a_1a_3]$ by (4.3) and (7.1) and $f_2(a_1\otimes a_2\otimes a_3)=d_{jh}[e_{jh}[a_2]a_1]d'_j[a_3]=\sum a_2d_j[a_1]d'_j[a_3]=a_2d[a_1a_3]$ similarly, for a_1, a_2, $a_3\in A$. Hence if $b=d$, then $f_1=f_2$, since Ω_1 is injective.

Let $g_1=\Omega_2\alpha'(1\times\Delta)\mathcal{S}^0(b)$ and $g_2=\Omega_2\alpha(\mathcal{S}^0\times 1)\mathcal{S}^0(d)$. Then for $a_i\in A$, $g_1(a_1\otimes a_2\otimes a_3)=\sum b_i[c_{ih}[a_2]c'_{ih}[a_3]a_1]=\sum b_i[c_i[a_2a_3]a_1]=a_2a_3b[a_1]$ by (4.3) and (7.1) and $g_2(a_1\otimes a_2\otimes a_3)=\sum d_{jk}[f_{jk}[a_2]e_j[a_3]a_1]=\sum a_2d_j[e_j[a_3]a_1]=a_2a_3d[a_1]$ by (7.1), where $\mathcal{S}(b)=\sum b_i\otimes c_i$, $\mathcal{S}(d)=d_j\otimes e_j$, $\Delta(c_i)=\sum c_{ih}\otimes c'_{ih}$ and $\mathcal{S}(d_j)=\sum d_{jk}\otimes f_{jk}$. It follows from the injectivity of Ω_2 that $g_1=g_2$ if $b=d$.

7.3. COROLLARY. *Let $(B, \Delta, \mathcal{J})$ be a $\times_A$-bialgebra, Suppose $\mathcal{J}$ is injective and the map $\Lambda: \int_x \bar{x}M\otimes_x B\to\mathrm{Hom}\,(A, M)$ is injective for each left $\bar{A}$-module M. If $\theta^0: (B^0\times_A B)^0\to B^{00}=B$ is an isomorphism, then $\mathcal{S}=\theta^{0-1}$ is an Ess map.*

7.4. COROLLARY. *Let A be a finite projective R-algebra. Then the $\times_A$-bialgebra* End A (4.14) *has a unique Ess map.*

7.5. COROLLARY. *If A is a division algebra, then the $\times_A$-bialgebra E of* (5.3) *has a unique Ess map.*

§ 8. Cohomology of a $\times_A$-bialgebra.

Let $(B, \Delta, \mathcal{J})$ be a $\times_A$-bialgebra, where $\mathcal{J}$ is *injective*. Let $\overset{0}{B}=A\otimes\bar{A}$ and

$$\overset{n}{B}=\overbrace{B\otimes_{A\otimes\bar{A}}\cdots\otimes_{A\otimes\bar{A}}B}^{n}, \qquad n>0.$$

These are $\times_A$-coalgebras by (4.6) and (4.7). Hence we can form algebras $(\overset{n}{B})^{\#}$ by (4.10).

Define the $A\otimes\bar{A}$-bilinear maps $i_n: \overset{n}{B}\to B$ by $i_0(1)=1$, $i_n(b_1\otimes\cdots\otimes b_n)=b_1\cdots b_n$, $b_i\in B$. The co-unit for $\overset{n}{B}$ is $\overset{n}{B}\xrightarrow{i_n}B\xrightarrow{\mathcal{J}}\operatorname{End}A$.

Let M_n be the module of $A\otimes\bar{A}$-bilinear maps from $\overset{n}{B}$ to B. It follows from (4.9) that M_n is an algebra with unit i_n and with product determined by

$$f*g: \overset{n}{B}\longrightarrow \overset{n}{B}\times_A\overset{n}{B}\xrightarrow{f\times g}B\times_A B\xrightarrow{\theta=\theta'}B$$

and the injection $\mathcal{J}: B\to\operatorname{End}A$ induces the algebra isomorphisms $M_n\cong(\overset{n}{B})^{\#}$.

In view of (4.11), M_1 is identified with the endomorphism algebra of the $A\otimes\bar{A}$-bimodule B. It is *commutative*.

For an algebra M, $M^{\times}$ denotes the group of units in M.

$M_1^{\times}$ is hence identified with the group of automorphisms of the $A\otimes\bar{A}$-bimodule B.

8.1. LEMMA. *For each* $n\geqq 0$, *define the linear maps*

$$e_i: M_n\longrightarrow M_{n+1}, \qquad i=0, 1, \cdots, n+1,$$

by

$$e_0(f): \overset{n+1}{B}\cong B\otimes_{A\otimes\bar{A}}\overset{n}{B}\xrightarrow{1\otimes f}B\otimes_{A\otimes\bar{A}}B\xrightarrow{i_2}B$$

$$e_i(f): \overset{n+1}{B}\cong\overset{i-1}{B}\otimes_{A\otimes\bar{A}}\overset{2}{B}\otimes_{A\otimes\bar{A}}\overset{n-i}{B}\xrightarrow{1\otimes i_2\otimes 1}\overset{i-1}{B}\otimes_{A\otimes\bar{A}}B\otimes_{A\otimes\bar{A}}\overset{n-i}{B}\xrightarrow{f}B$$

$$0<i<n+1$$

$$e_{n+1}(f): \overset{n+1}{B}\cong\overset{n}{B}\otimes_{A\otimes\bar{A}}B\xrightarrow{f\otimes 1}B\otimes_{A\otimes\bar{A}}B\xrightarrow{i_2}B$$

for $f\in M_n$.

a) *These are algebra maps.*

b) $\{M_n, e_0, \cdots, e_{n+1}\}_{n=0}^{\infty}$ *forms a semi-co-simplicial complex.*

PROOF. This is left to the reader.

In the following we consider the partial complex $\{M_n, e_i\}_{n=0}^{3}$ and form the cohomology groups $H^n(B)$, $n=0, 1, 2$ with respect to the "units" functor $(?)^{\times}$.

8.2. H^0 *theorem.* The map $M_0\to B$, $f\mapsto f(1)$ is an injective algebra map with image

$$\int^{x,y}{}_{x,\bar{y}}B_{x,\bar{y}}=\text{the centralizer of } A\otimes\bar{A} \text{ in } B\,.$$

Since $\bar{A}\cong\int^{x}{}_{x}B_{x}$, we have

$$\int^{x,y}{}_{x,\bar{y}}B_{x,\bar{y}}\cong\int^{y}{}_{\bar{y}}\bar{A}_{\bar{y}}=\text{center}\,(\bar{A})\,.$$

In particular M_0 is commutative. .We identify M_0 with the centralizer of $A\otimes\bar{A}$ in B. Then $e_0(m)(b)=bm$, $e_1(m)(b)=mb$, $m\in M_0$, $b\in B$. Hence

$$\text{Ker}\,(e_0, e_1: M_0 \rightrightarrows M_1)=\text{center}\,(B)\,.$$

If we define $H^0(B)=\text{Ker}\,(e_0, e_1: M_0^\times\rightrightarrows M_1^\times)$, then $H^0(B)\cong\text{center}\,(B)^\times$.

8.3. LEMMA. *Let* $\sigma, \gamma\in M_1$ *and* $f\in M_2$.

a) $e_0(\gamma)*e_2(\sigma)=e_2(\sigma)*e_0(\gamma): \overset{2}{B} \xrightarrow{\sigma\otimes\gamma} \overset{2}{B} \xrightarrow{i_2} B.$

b) $e_0(\gamma)*e_2(\sigma)*f=f*e_0(\gamma)*e_2(\sigma): \overset{2}{B} \xrightarrow{\sigma\otimes\gamma} \overset{2}{B} \xrightarrow{f} B.$

c) $e_1(\sigma)*f=f*e_1(\sigma): \overset{2}{B} \xrightarrow{f} B \xrightarrow{\sigma} B.$

d) *The images of the algebra maps* $e_i: M_1\to M_2$, $i=0, 1, 2$, *are contained in the center of* M_2.

PROOF. Let $b, c\in B$ and $\Delta(b)=\sum b_i\otimes b'_i$, $\Delta(c)=\sum c_j\otimes c'_j$. Then $\Delta(b\otimes c)=\sum b_i\otimes c_j\otimes b'_i\otimes c'_j$.

a) $[e_0(\gamma)*e_2(\sigma)](b\otimes c)=\theta'(\sum b_i\gamma(c_j)\otimes\sigma(b'_i)c'_j)=\sum b_i[\gamma(c_i)[1]]\,\sigma(b'_i)c'_j=\sum\sigma(b_i[\gamma(c_j)[1]]b'_i)c'_j=\sum\sigma(b\gamma(c_j)[1])c'_j$ (by (4.3))$=\sum\sigma(b)\gamma(c_j)[1]c'_j=\sigma(b)\gamma(c)$, since $\gamma*i_1=\gamma$. That $e_0(\gamma)*e_2(\sigma)=e_2(\sigma)*e_0(\gamma)$ is proved in the following.

b) Let $g=e_0(\gamma)*e_2(\sigma)$. $(g*f)(b\otimes c)=\theta'(\sum\sigma(b_i)\gamma(c_j)\otimes f(b'_i\otimes c'_j))=\sum f(\sigma(b_i)[\gamma(c_j)[1]]b'_i\otimes c'_j)=\sum f(\sigma(b\gamma(c_j)[1])\otimes c'_j)$ (since $\sigma*i_1=\sigma$ and $\Delta(b\gamma(c_j)[1])=\sum b_i\gamma(c_j)[1]\otimes b'_i)=\sum f(\sigma(b)\otimes\gamma(c_j)[1]c'_j)=f(\sigma(b)\otimes\gamma(c))$, since $\gamma*i_1=\gamma$.

$\underline{(f*g)(b\otimes c)}=\theta(\sum f(b_i\otimes c_j)\otimes\sigma(b'_i)\gamma(c'_j))=\sum\overline{\sigma(b'_i)[\gamma(c'_j)[1]]}f(b_i\otimes c_j)=\sum f(\sigma(b'_i\overline{\gamma(c'_j)[1]})[1]b_i\otimes c_j)=\sum f(\sigma(b\overline{\gamma(c'_j)[1]})\otimes c_j$ (since $i_1*\sigma=\sigma$ and $\Delta(b\overline{\gamma(c'_j)[1]})=\sum b_i\otimes b'_i\overline{\gamma(c'_j)[1]})=\sum f(\sigma(b)\otimes\overline{\gamma(c'_j)[1]}c_j)=f(\sigma(b)\otimes\gamma(c))$ since $i_1*\gamma=\gamma$.

Hence g belongs to the center of M_2. Taking $\sigma=i_1$ or $\gamma=i_1$ we see that the images $e_0(M_1)$ and $e_2(M_1)$ are also contained in the center of M_2. In particular we have $e_0(\gamma)*e_2(\sigma)=e_2(\sigma)*e_0(\gamma)$.

c) $(f*e_1(\sigma))(b\otimes c)=\theta'(\sum f(b_i\otimes c_j)\otimes\sigma(b'_ic'_j))=\sum\sigma(f(b_i\otimes c_j)[1]b'_ic'_j)=\sigma(f(b\otimes c))$ since $f*i_2=f$.

$(e_1(\sigma)*f)(b\otimes c)=\theta(\sum\sigma(b_ic_j)\otimes f(b'_i\otimes c'_j))=\sum\sigma(\overline{f(b'_i\otimes c'_j)[1]}b_ic_j)=\sigma(f(b\otimes c))$ since $i_2*f=f$.

d) follows from the above.

8.4. As a corollary we have the following complex of abelian groups

$$M_0^\times \xrightarrow{\delta_0} M_1^\times \xrightarrow{\delta_1} \text{center}(M_2)^\times$$

where $\delta_0(x)=e_0(x)*e_1(x)^{-1}$, $\delta_1(y)=e_0(y)*e_1(y)^{-1}*e_2(y)$, $x\in M_0^\times$, $y\in M_1^\times$, and can form the cohomology groups $H^0(B)=\text{Ker}(\delta_0)$ and $H^1(B)=\text{Ker}(\delta_1)/\text{Im}(\delta_0)$.

8.5. *H^1 theorem.* An element $f\in M_1$ is a *1-cocycle* if $e_1(f)=e_0(f)*e_2(f)$ *or* equivalently if $f(bc)=f(b)f(c)$, $b, c\in B$. If a 1-cocycle is invertible, the inverse is also a 1-cocycle.

Hence $\text{Ker}(\delta_1: M_1^\times\to\text{center}(M_2)^\times)$ consists of all $A\otimes\bar{A}$-bilinear automorphisms $f: B\to B$ such that $f(bc)=f(b)f(c)$, $b, c\in B$. Then $f(1)=1$ clearly. Hence

$$\text{Ker}(\delta_1)\cong\text{Aut}_{\text{alg}/A\otimes\bar{A}}(B)$$

as groups. If $x\in M_0^\times\subset B^\times$, then

$$\delta_0(x)(b)=x^{-1}bx\,, \qquad b\in B\,.$$

Hence $\delta_0(M_0^\times)$ consists of all inner automorphisms by elements of the centralizer of $A\otimes\bar{A}$ in B.

Therefore the group $H^1(B)$ is isomorphic to the group of automorphisms of B as an algebra over $A\otimes\bar{A}$ modulo the subgroup of inner automorphisms of B induced by invertible elements of center $(\bar{A})$.

8.6. LEMMA. *Let $f, g\in M_2$. Then*

$$e_0(f)*e_2(g)=e_2(g)*e_0(f): \overset{3}{B} \xrightarrow{1\otimes f} \overset{2}{B} \xrightarrow{g} B\,,$$

$$e_1(f)*e_3(g)=e_3(g)*e_1(f): \overset{3}{B} \xrightarrow{g\otimes 1} \overset{2}{B} \xrightarrow{f} B\,.$$

PROOF. The computation is similar to (8.3) and left to the reader.

8.7. DEFINITION. Let $f, g\in M_2$ and $\sigma\in M_1^\times$.

a) f is a *2-cocycle* if $e_0(f)*e_2(f)=e_1(f)*e_3(f)$.

b) $f\underset{\sigma}{\sim}g$ if $f*\delta_2(\sigma)=g$ where $\delta_2(\sigma)=e_0(\sigma)*e_1(\sigma)^{-1}*e_3(\sigma)$.

8.8. LEMMA. *Let $f, f', g, g'\in M_2$ and $\sigma, \tau\in M_1^\times$.*

a) *If $f\underset{\sigma}{\sim}g$, $f'\underset{\tau}{\sim}g'$, then $f*f'\underset{\sigma*\tau}{\sim}g*g'$.*

b) *$f\underset{\sigma}{\sim}g$ if and only if $f(\sigma(b)\otimes\sigma(c))=\sigma(g(b\otimes c))$, $b, c\in B$.*

c) *If $f\underset{\sigma}{\sim}g$ then f is a 2-cocycle if and only if so is g,*

d) *If f, g are 2-cocycles, then so is $f*g$.*

e) *f is a 2-cocycle if and only if $f(b\otimes f(c\otimes d))=f(f(b\otimes c)\otimes d)$, $b, c, d\in B$.*

f) *If f is an invertible 2-cocycle then so is f^{-1}.*

g) *$\delta_2(\sigma)$ is an invertible 2-cocycle.*

PROOF. Easy.

8.9. DEFINITION. $H^2(B)=\{\text{invertible 2-cocycles}\}/\delta_2(M_1^\times)$.

8.10. REMARK. An $A\otimes\bar{A}$-bilinear map $f: \overset{2}{B}\to B$ is a 2-cocycle if and only if f gives on B a structure of an associative, non-unitary algebra over $A\otimes\bar{A}$.

If M and N are associative non-unitary algebras over $A\otimes\bar{A}$, then $M\times_A N$ is too, in the same way as (3.1).

8.11. LEMMA. *Let f be an invertible 2-cocycle. Then the associative product $f:\overset{2}{B}\to B$ has the unit in $\int^{x,y}{}_{x,\bar{y}}B_{x,\bar{y}}$. Hence (B,f) is an algebra over $A\otimes\bar{A}$.*

PROOF. $g=f^{-1}$ is also a 2-cocycle. Since $f*g=i_2$, $\sum f(b_i\otimes c_j)[1]g(b_i'\otimes c_j')=bc$, $b, c\in B$. In particular $c=\sum g(f(1\otimes c_j)[1]\otimes c_j')=\sum g(1\otimes f(1\otimes c_j)[1]c_j')$. Since the map $f':B\to B$, $f'(b)=f(1\otimes b)$ is $A\otimes\bar{A}$-bilinear, $f'*i_1=f'$. Hence $\sum f(1\otimes c_j)[1]c_j'=f(1\otimes c)$. Therefore $g(1\otimes f(1\otimes c))=c$. If we write

$$b\underset{f}{\circ}c=f(b\otimes c))\,,\qquad b\underset{g}{\circ}c=g(b\otimes c)$$

then the map $1\underset{f}{\circ}?:B\to B$ is injective and $1\underset{g}{\circ}?:B\to B$ is surjective. Interchanging f and g or the left and the right, we conclude that the maps $1\underset{f}{\circ}?$ and $?\underset{f}{\circ}1$ are bijective. Since $1\in\int^{x,y}{}_{x,\bar{y}}B_{x,\bar{y}}$, we conclude that the project $f:B\to B$ has the unit in the centralizer of $A\otimes\bar{A}$ in B just as [**1**, (16.4)].

8.12. H^2 *theorem.* Suppose $\Delta:B\to B\times_A B$ is an isomorphism. Then the $A\otimes\bar{A}$-bimodule B is associative (4.4) and $\mathcal{E}_B=\mathcal{E}\langle B\rangle$ by (3.12).

Let X be the set of 2-cocycles f such that the product $f:\overset{2}{B}\to B$ has the unit in the centralizer of $A\otimes\bar{A}$ in B. X contains the invertible 2-cocycles by (8.11).

If f, $g\in X$, then we have an isomorphism of algebras over $A\otimes\bar{A}$

$$\Delta:(B,f*g)\cong(B,f)\times_A(B,g)\,.$$

This means $f*g\in X$ and the map $X\to\mathcal{E}\langle B\rangle$, $f\mapsto\langle B,f\rangle$ which is clearly surjective, is a monoid homomorphism.

It follows from (8.8), b) that $\langle B,f\rangle=\langle B,g\rangle$ where $f,g\in X$ if and only if $f\underset{\sigma}{\sim}g$ for some $\sigma\in M_1^\times$. Hence we have a monoid isomorphism

$$X/\delta_2(M_1^\times)\cong\mathcal{E}\langle B\rangle\,.$$

Taking the invertible elements we have a group isomorphism

$$H^2(B)\cong\mathcal{G}\langle B\rangle\,.$$

References

[1] M.E. Sweedler, Groups of Simple Algebras, I.H.E.S. Publ., $n^0$44 (1975), 79-189.
[2] M.E. Sweedler, Hopf Algebras, W.A. Benjamin Inc., New York, 1969.

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