J. Math. Soc. Japan
Vol. 29, No. 3, 1977

On the singular spectrum of boundary values of real analytic solutions

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(Received April 16, 1975)
(Revised Nov. 12, 1976)

In this article we give an estimation of the singular spectrum of boundary values of real analytic solutions of linear partial differential equations with constant coefficients. The result has been expected from the study of continuation of real analytic solutions. It gives a unified aspect to many problems on continuation of regular solutions. (See [5]. But the present result (Theorem 2.1) is a little weaker than the conjecture in [5].) Our estimate is sharp for the wave equations or for the ultrahyperbolic equations (Example 2.5).

Before my work, Professor Komatsu has given a vast conjecture (unpublished) on precise determination of the singular spectrum of the boundary values of general hyperfunction solutions from the standpoint of purely boundary-value-theoretical origin. Corollary 2.4 is a partial answer to his conjecture. Also in Theorem 2.1 we have added the distinction of the signs in the conormal direction taking account of his conjecture. I am very grateful to Professor Komatsu for his kind advices.

In § 1, we paraphrase the definition of boundary values by way of the Fourier transform. It was originally given for the equations with analytic coefficients in [9] employing the Cauchy-Kowalevsky theorem. In § 2 we prove the main theorem. In § 3, we give an application to continuation of real analytic solutions. In some sense it is an extension of the results in [4] or [5].

§ 1. Boundary values of hyperfunction solutions.

First we prepare the notation. $p(D)$ denotes a linear partial differential operator with constant coefficients, where $D=(D_1, \cdots, D_n)$ and $D_i=\sqrt{-1}\,\partial/\partial x_i$, $i=1, \cdots, n$. By the Fourier transform

$$\tilde{u}=\mathcal{F}[u]=\int u(x)e^{\sqrt{-1}x\xi}dx\,, \qquad x\xi=x_1\xi_1+\cdots+x_n\xi_n\,,$$

it corresponds to the multiplication operator $p(\xi)$. ζ denotes the complexifica-

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tion of ξ. Let U be a bounded convex open set in $\boldsymbol{R}^n$. Put $K=U\cap\{x_1=0\}$ and $U^+=U\cap\{x_1>0\}$. Let L be the closure $\bar{K}$ of K in $\boldsymbol{R}^n$. L is compact. Later, we separate the variables x in the way (x_1, x') with $x'=(x_2, \cdots, x_n)$ and further $x'=(x'', x_n)$ with $x''=(x_2, \cdots, x_{n-1})$. We employ a similar notation for the dual variables ξ or ζ. $\mathcal{B}(U)$ resp. $\mathcal{A}(U)$ denotes the hyperfunctions resp. real analytic functions on U. $\mathcal{B}_p(U)$ resp. $\mathcal{A}_p(U)$ denotes the hyperfunction solutions resp. real analytic solutions of $p(D)u=0$ on U. $'\mathcal{B}(K)$ resp. $'\mathcal{A}(K)$ denotes the hyperfunctions resp. real analytic functions of $n-1$ variables x' in the $n-1$ dimensional open set K. $\mathcal{B}[L]$ resp. $'\mathcal{B}[L]$ denotes the hyperfunctions of n resp. $n-1$ variables with support in L.

Assume that the hyperplane $x_1=0$ is non-characteristic with respect to $p(D)$. Further, for the sake of simplicity assume that $p(\zeta)$ is an irreducible polynomial of degree m. This is not an essential restriction to the argument (see Remark 1.4). For $u\in\mathcal{B}_p(U^+)$ we define its boundary values $b_j^+(u)$ to the hyperplane $x_1=0$ in the following way. Let $[u]\in\mathcal{B}(U)$ be an extension of u satisfying $\operatorname{supp}[u]\subset\{x_1\geqq 0\}$. Then $\operatorname{supp} p(D)[u]\subset\{x_1=0\}$. Let $[[p(D)[u]]]\in\mathcal{B}[L]$ be an extension of $p(D)[u]$. The Fourier transform $F(\zeta)=\widehat{[[p(D)[u]]]}$ is an entire function of ζ satisfying the growth condition

$$|F(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|+H_L(\operatorname{Im}\zeta)),$$

where $H_L(\operatorname{Im}(\zeta)=\sup_{x\in L}\operatorname{Re}\sqrt{-1}x\zeta$ is the supporting function of L. Then the restriction $F(\zeta)|_{N(p)}$ of $F(\zeta)$ to the variety $N(p)=\{p(\zeta)=0\}$ is obviously determined by u with the ambiguity in $\widehat{\mathcal{B}[L\backslash K]}|_{N(p)}$. The elements of the latter space cannot be characterized by a growth condition.

Next we seek the entire function $\tilde{f}(\zeta)$ of the form

$$\tilde{f}(\zeta)=\zeta_1^{m-1}\tilde{f}_0(\zeta')+\zeta_1^{m-2}\tilde{f}_1(\zeta')+\cdots+\tilde{f}_{m-1}(\zeta'), \tag{1.1}$$

satisfying $\tilde{f}(\zeta)|_{N(p)}=F(\zeta)|_{N(p)}$. We can obtain an example of such $\tilde{f}(\zeta)$ from $F(\zeta)$ by reducing ζ_1^N, $N\geqq m$ employing the equation $p(\zeta)=0$. On the other hand, $\tilde{f}(\zeta)$ is uniquely determined by the interpolation formula: Let $\tau_j(\zeta')$, $j=1, \cdots, m$ be the roots of the equation $p(\zeta)=0$ with respect to ζ_1. Then

$$\tilde{f}_j(\zeta')=\begin{vmatrix}\tau_1(\zeta')^{m-1}, & \cdots, & \overset{j+1}{F(\tau_1(\zeta'), \zeta')}, & \cdots, & 1\\ \vdots & & \vdots & & \vdots\\ \tau_m(\zeta')^{m-1}, & \cdots, & F(\tau_m(\zeta'), \zeta'), & \cdots, & 1\end{vmatrix}\div\begin{vmatrix}\tau_1(\zeta')^{m-1}, & \cdots, & 1\\ \vdots & & \vdots\\ \tau_m(\zeta')^{m-1}, & \cdots, & 1\end{vmatrix}. \tag{1.2}$$

Since $p(\zeta)$ is irreducible, the discriminant $\Delta(\zeta')$ of the equation $p(\zeta)=0$ in ζ_1 does not vanish identically. Therefore the above formula gives an expression of $\tilde{f}_j(\zeta')$ as the quotient of an entire function $F_j(\zeta')$ by the polynomial $\Delta(\zeta')^2$, where $F_j(\zeta')$ is defined replacing $\div$ by $\times$ in (1.2). We denote this correspondence by $B[F|_{N(p)}]=(\tilde{f}_0, \cdots, \tilde{f}_{m-1})$. Since B is linear, $\tilde{f}_j$ have the same estimate as $F|_{N(p)}$:

$$|\tilde{f}_j(\zeta')|\leqq C_\varepsilon \exp(\varepsilon|\zeta'|+H_L(\operatorname{Im}\zeta')) .$$

(Note that $H_L(\operatorname{Im}\zeta)$ in fact depends only on ζ'.) In fact, this estimate can be directly obtained for the numerator $F_j(\zeta')$. Hence by Malgrange's inequality it holds also for the quotient $\tilde{f}_j(\zeta')$ (see [3], Theorem 3.8). Thus $\tilde{f}_j(\zeta')$ are the Fourier images of some elements $f_j(x')\in{}'\mathcal{B}[L]$. The ambiguity in $\widetilde{\mathcal{B}[L\backslash K]}|_{N(p)}$ goes into $'\mathcal{B}[L\backslash K]$ by B. In fact, assume that $F|_{N(p)}=G|_{N(p)}$ with some $G\in\widetilde{\mathcal{B}[L\backslash K]}$. Then we can decompose $G=\sum_k G^k$ in accordance with the decomposition of the support $L\backslash K$ into small pieces L_k. Then $B[F|_{N(p)}]=\sum_k B[G^k|_{N(p)}]$. From the growth condition we see that $B[G^k|_{N(p)}]$ is a Fourier image of some element in $({}'\mathcal{B}[\operatorname{ch} L_k])^m$, where ch denotes the convex hull. Thus the support of $B[F|_{N(p)}]$ is contained in any small neighborhood of $L\backslash K$, hence in $L\backslash K$ itself. Thus we can define with no ambiguity that

$$b_j^\dagger(u)=f_j(x')|_K, \qquad j=0,\cdots,m-1 .$$

LEMMA 1.1. *$b_j^\dagger(u)$ agree with the boundary values $C_j(-D)u|_{x_1\to+0}$ defined in* [**9**], *where $\{C_j(D)\}$ denotes the dual system of the boundary condition $\{D_1^j\}$ with respect to $p(D)$. We assume that C_j is of order j, $j=0,\cdots,m-1$.*

PROOF. The boundary values $C_j(-D)u|_{x_1\to+0}$ defined in [**9**] are characterized by the formula

$$p(D)[u]=\sum_{j=0}^{m-1} D_1^{m-j-1}(C_j(-D)u|_{x_1\to+0}\delta(x_1)) ,$$

where $[u]\in\mathcal{B}(U)$ is some (in fact unique) extension of u satisfying $\operatorname{supp}[u]\subset\{x_1\geqq 0\}$. Our construction shows that for an arbitrary such extension $[u]$ we have

$$\widetilde{[[p(D)[u]]]}|_{N(p)}=\{\zeta_1^{m-1}\tilde{f}_0(\zeta')+\cdots+\tilde{f}_{m-1}(\zeta')\}|_{N(p)} .$$

Hence, by the Fundamental Principle ([3], Theorem 3.8) we can find $v\in\mathcal{B}[L]$ such that

$$\widetilde{[[p(D)[u]]]}=\zeta_1^{m-1}\tilde{f}_0(\zeta')+\cdots+\tilde{f}_{m-1}(\zeta')+p(\zeta)\tilde{v}(\zeta) .$$

Thus we have in U

$$p(D)([u]-v)=\sum_{j=0}^{m-1} D_1^{m-j-1}(f_j(x')\delta(x_1)) . \qquad \text{q. e. d.}$$

REMARK 1.2. Because the boundary values in [**9**] are given as a sheaf homomorphism, we see from this lemma that $b_j^\dagger(u)$ are locally determined on K.

REMARK 1.3. Although $D_1^j u|_{x_1\to+0}$ may be more natural as the boundary values, we treat $b_j^\dagger(u)$ for the sake of simplicity. Since $\{C_j(-D)\}$ is a normal system of boundary operators, these two sets of boundary values are easily translated to each other.

REMARK 1.4. When $p(\zeta)$ is not irreducible, we treat as follows. Let $p = q_1^{l_1}\cdots q_k^{l_k}$ be the irreducible decomposition of p; q_j of order m_j, $j=1,\cdots,k$. We have $m=\sum l_j m_j$. Let $\tau_{j1}(\zeta'),\cdots,\tau_{jm_j}(\zeta')$ be the roots of $q_j(\zeta_1,\zeta')=0$ with respect to ζ_1. For a given entire function $F(\zeta)$, seek an entire function

$$\tilde{f}(\zeta)=\zeta_1^{m-1}\tilde{f}_0(\zeta')+\cdots+\tilde{f}_{m-1}(\zeta')$$

such that the difference $\tilde{f}(\zeta)-F(\zeta)$ is divisible by $p(\zeta)$. $\tilde{f}$ is obtained from $F(\zeta)$ by reducing ζ_1^N $(N\geqq m)$, employing the equation $p(\zeta)=0$. Put $F_{ji}(\zeta)=(\partial/\partial\zeta_1)^{i-1}F(\zeta)|_{N(q_j)}$, $i=1,\cdots,l_j$, $j=1,\cdots,k$. Then the restriction to the multiplicity variety $N(p)$: $F(\zeta)\to\partial F=\{F_{ji}\}$ determines $F(\zeta)$ modulo the multiple of $p(\zeta)$. (See, e.g., [11], Chapter IV, § 4, Corollary 2 and Proposition 3.) We claim that $\tilde{f}_j(\zeta')$ is determined from $F_{ji}(\zeta)$ by a formula similar to (1.2). In fact, the identity $\partial\tilde{f}=\partial F$ gives a system of linear equations for $\tilde{f}_j(\zeta')$:

$$\begin{cases} \tau_{11}(\zeta')^{m-1}\tilde{f}_0(\zeta')+\cdots\cdots\cdots\cdots+\tilde{f}_{m-1}(\zeta')=F_{11}(\tau_{11}(\zeta'),\zeta'), \\ \quad\vdots \\ \tau_{1m_1}(\zeta')^{m-1}\tilde{f}_0(\zeta')+\cdots\cdots\cdots\cdots+\tilde{f}_{m-1}(\zeta')=F_{11}(\tau_{1m_1}(\zeta'),\zeta'), \\ (m-1)\tau_{11}(\zeta')^{m-2}\tilde{f}_0(\zeta')+\cdots\cdots\cdots+\tilde{f}_{m-2}(\zeta')=F_{12}(\tau_{11}(\zeta'),\zeta'), \\ \quad\vdots \\ \dfrac{(m-1)!}{(m-l_k)!}\tau_{km_k}(\zeta')^{m-l_k}\tilde{f}_0(\zeta')+\cdots+l_k!\,\tilde{f}_{m-l_k}(\zeta')=F_{kl_k}(\tau_{km_k}(\zeta'),\zeta'). \end{cases}$$

The determinant of the coefficient matrix is obtained from the Van der Monde determinant of the m-th order by successive limit process. Hence it is different from zero when no pairs from $\{\tau_{ji}(\zeta')\}$ agree. Since

$$\{\zeta'\in \boldsymbol{C}^{n-1};\ \tau_{ji}(\zeta')=\tau_{j'i'}(\zeta')\quad \text{for some } (j,i)\neq(j',i')\}$$

is a proper algebraic subvariety of $\boldsymbol{C}^{n-1}$, the above argument (and the argument in the next section) goes with no essential modification in this general case.

§ 2. Estimation of singular spectrum of boundary values of real analytic solutions.

We inherit the situation in the preceding section. For the sake of simplicity, we also inherit the assumption that $p(\zeta)$ is an irreducible polynomial of order m. This is not an essential restriction as mensioned in Remark 1.4. First we give another expression of the Fourier image $\tilde{f}_j$ of the boundary values when u belongs to $\mathcal{A}_p(U^+)$. Let $\chi(x)$ be a function of Gevrey class on U^+ such that supp χ is contained in an ε neighborhood of K, $\overline{\operatorname{supp}\chi}\cap\partial U\subset L\backslash K$ and $\chi\equiv1$ on a smaller neighborhood of K. Let $[(1-\chi)u]_0\in C^\infty(U)$ be the extension of $(1-\chi)u\in C^\infty(U^+)$ by zero. Then we have obviously

$$p(D)[[[(1-\chi)u]_0-[u]]]\equiv[[p(D)([(1-\chi)u]_0)]]-[[p(D)[u]]] \bmod \mathcal{B}[L\backslash K].$$

where [[ ]] denotes in general one of the extension with the smallest support. Thus on $N(p)$ we have

$$F(\zeta)\equiv[[\widetilde{p(D)[(1-\chi)u]_0}]] \quad \bmod \widetilde{\mathscr{B}[L\backslash K]}.$$

Moreover, let $J(D)$ be a local operator with constant coefficients. (For the definition see [2], § 1, 2°.) The same representation holds for another element $J(D)u\in\mathcal{A}_p(U^+)$. Let $J(\zeta)$ be the total symbol of $J(D)$. Since $J(\zeta)F(\zeta)$ is clearly one of the representatives of $J(D)u$, we have on $N(p)$,

$$J(\zeta)F(\zeta)\equiv[[\widetilde{p(D)[(1-\chi)J(D)u]_0}]] \quad \bmod \widetilde{\mathscr{B}[L\backslash K]}.$$

Next, let $\varphi(x)$ be a function of Gevrey class such that $\operatorname{supp}\varphi$ is contained in the ε neighborhood of $L\backslash K$ and $\varphi\equiv 1$ on a smaller neighborhood. Put

$$v=(1-\varphi)p(D)[(1-\chi)J(D)u]_0$$

and

$$w=[[p(D)[(1-\chi)J(D)u]_0]]-v.$$

We can choose χ and φ such that v is in Gevrey class satisfying

$$|\tilde{v}(\zeta)|\leqq C\exp(-A|\operatorname{Re}\zeta|^q+\varepsilon\max\{-\operatorname{Im}\zeta_1, 0\}+H_L(\operatorname{Im}\zeta')),$$

where $q<1$ and A are arbitrarily prescribed positive constants. On the other hand, $\operatorname{supp} w$ is contained in the part $x_1\geqq 0$ of the ε-neighborhood of $L\backslash K$. Employing the Fundamental Principle as before, we can include the ambiguity in $\widetilde{\mathscr{B}[L\backslash K]}|_{N(p)}$ into $\tilde{w}$. Then we have

$$J(\zeta)F(\zeta)=\tilde{v}(\zeta)+\tilde{w}(\zeta).$$

Put $B[v|_{N(p)}]=(\tilde{g}_0(\zeta'),\cdots,\tilde{g}_{m-1}(\zeta'))$ and $B[w|_{N(p)}]=(\tilde{h}_0(\zeta'),\cdots,\tilde{h}_{m-1}(\zeta'))$. If J contains only ζ', then by the linearity of B we have

$$J(\zeta')\tilde{f}_j(\zeta')=\tilde{g}_j(\zeta')+\tilde{h}_j(\zeta').$$

In this formula, $\tilde{g}_j$, $\tilde{h}_j$ depends on J, ε etc. The estimates for $\tilde{g}_j$, $\tilde{h}_j$ provide the necessary information. It depends on the condition on $\tau_j(\zeta')$.

THEOREM 2.1. *Assume that the roots $\tau_j(\zeta')$ of the equation $p(\zeta)=0$ in ζ_1 satisfy*

$$(2.1)\qquad -\operatorname{Im}\tau_j(\zeta')\leqq a|\operatorname{Re}\zeta'|^q+b|\operatorname{Im}\zeta'|+C,\qquad \text{if } \operatorname{Re}\zeta_n\leqq -c|\operatorname{Re}\zeta''|,$$

where $q<1$, a, b, c, C are positive constants. Then, for $u\in\mathcal{A}_p(U^+)$, $S.S.b_j^+(u)$ do not contain the direction $+\sqrt{-1}dx_n\infty$.

PROOF. From the condition on $\tau_j(\zeta')$, we have for real ξ'

$$(2.2)\qquad |\tilde{g}_j(\xi')|\leqq C\exp(-a'|\xi'|^q),\qquad \text{if } \xi_n\leqq -c|\xi''|,$$

where we have assumed $A-\varepsilon a>a'>0$. On the other hand, because $x_1=0$ is non-characteristic with respect to $p(D)$, we have $|\tau_j(\zeta')|\leqq M|\zeta'|$, hence,

$$|\tilde{g}_j(\xi')|\leqq C\exp(M\varepsilon|\xi'|-a'|\xi'|^q), \quad \text{if } \xi_n\geqq -c|\xi''|. \tag{2.3}$$

As for $\tilde{h}_j(\zeta')$, we can decompose it into $\sum_k \tilde{h}_j^k(\zeta')$ in correspondence with the decomposition $w=\sum_k w^k$, where $\operatorname{supp} w^k$ is contained in the ε-neighborhood of $\operatorname{ch} L_k$; $\bigcup_k L_k=L\backslash K$. Then, each $\tilde{h}_j^k$ satisfies: given $\gamma>0$, there exists $C_\gamma>0$ such that

$$|\tilde{h}_j^k(\zeta')|\leqq C_\gamma\exp(\gamma|\zeta'|+(1+b)\varepsilon|\operatorname{Im}\zeta'|+H_{\operatorname{ch}L_k}(\operatorname{Im}\zeta')), \tag{2.4}$$
$$\text{if } \operatorname{Re}\zeta_n\leqq -c|\operatorname{Re}\zeta''|,$$

$$|\tilde{h}_j^k(\zeta')|\leqq C_\gamma\exp(\gamma|\zeta'|+M\varepsilon|\operatorname{Re}\zeta'|+(1+M)\varepsilon|\operatorname{Im}\zeta'|+H_{\operatorname{ch}L_k}(\operatorname{Im}\zeta')), \tag{2.5}$$
$$\text{if } \operatorname{Re}\zeta_n\geqq -c|\operatorname{Re}\zeta''|.$$

Anyway, the appearance of $M\varepsilon|\operatorname{Re}\zeta'|$ in the estimates does not allow us to consider $\tilde{g}_j$, $\tilde{h}_j$ as Fourier image of Fourier hyperfunctions. Therefore, we must make another device. We prepare a tool for the local estimation of the singular spectrum by way of the Fourier image.

LEMMA 2.2. *We have the following decomposition of delta function:* $\delta(x)=\int_{|\omega|=1}W(x,\omega)d\omega$, *where*

$$W(x,\omega)=\frac{(n-1)!\,\psi(x,\omega)e^{-x^2}}{(-2\pi\sqrt{-1})^n(x\omega+\sqrt{-1}(x^2-(x\omega)^2)/\sqrt{1+x^2}+\sqrt{-1}0)^n}$$

with

$$\psi(x,\omega)=\det\left\{\frac{\partial}{\partial\omega_j}\left[\omega_i+\sqrt{-1}\left(x_i|\omega|-(x\omega)\frac{\omega_i}{|\omega|}\right)\Big/\sqrt{1+x^2}\right]\right\}\Big|_{|\omega|=1},$$

$$x^2=x_1^2+\cdots+x_n^2.$$

Let f be a hyperfunction with compact support. Then $S.S.f$ does not contain $(0,\sqrt{-1}dx_1\infty)$, if for each ω in a neighborhood $\Omega\subset\{|\omega|=1\}$ of $(1,0,\cdots,0)$ the function $\int f(y)W(x-y,\omega)dy$ can be extended as a holomorphic function of x to a fixed complex neighborhood of $x=0$ depending continuously on ω.

PROOF. The above formula is obtained from a special case of the curved wave decomposition given by Kashiwara ([**12**], Chapter III, Example 1.2.5) by multiplication by e^{-x^2}. We have

$$f(x)=\int f(y)\delta(x-y)dy=\int_{|\omega|=1}d\omega\int f(y)W(x-y,\omega)dy$$

$$=\int_{\Omega}d\omega\int f(y)W(x-y,\omega)dy+\int_{\complement\Omega}d\omega\int f(y)W(x-y,\omega)dy.$$

Obviously the second term does not contain $(0, \sqrt{-1}dx_1\infty)$ in its singular spectrum. On the other hand, the integral in the first term converges in the space of holomorphic functions of x on a complex neighborhood of $x=0$. Hence it does not contain the same point in its singular spectrum. q. e. d.

LEMMA 2.3. *The Fourier transform* $\tilde{W}(\zeta, \omega)$ *of* $W(x, \omega)$ *with respect to the variables* x *is holomorphic in* ζ, ω *on* $C^n\times\{|\omega|=1\}$. *It satisfies the following condition: For any* $\delta>0$ $(\delta<1/2)$, *there exists* $C_\delta>0$ *and* $C'_\delta>0$ *which are independent of* ω *such that*

$$|\tilde{W}(\zeta,\omega)|\leqq C'_\delta|\zeta|^n\exp(\delta|\operatorname{Im}\zeta|),\qquad \text{if}\quad C_\delta|\operatorname{Im}\zeta|\leqq|\operatorname{Re}\zeta|,$$

moreover,

$$|\tilde{W}(\zeta,\omega)|\leqq C'_\delta\exp(\delta|\operatorname{Im}\zeta|-|\operatorname{Re}\zeta|/C'_\delta),$$

$$\text{if}\quad C_\delta|\operatorname{Im}\zeta|\leqq|\operatorname{Re}\zeta|\quad\text{and}\quad \delta\operatorname{Re}\zeta\cdot\omega\geqq-\sqrt{(\operatorname{Re}\zeta)^2-(\operatorname{Re}\zeta\cdot\omega)^2}.$$

RROOF. The analyticity is obvious. (See, e. g., [**12**], Chapter I, Theorem 2.3.1.) Without loss of generality we can assume that $\omega=(1,0,\cdots,0)$, thus $x\omega=x_1$ and $\operatorname{Re}\zeta\cdot\omega=\operatorname{Re}\zeta_1$. We deform the path of integration to the complex domain with $z=x+\sqrt{-1}y(x)$:

$$\tilde{W}(\zeta,\omega)=\int_{y=y(x)}\frac{\phi(z,\omega)e^{-z^2+\sqrt{-1}z\zeta}}{(z_1+\sqrt{-1}z'^2/\sqrt{1+z^2})^n}dz.$$

This deformation is regitimate if $|y|\leqq(|x|+1)/2$ and if the denominator does not vanish. Assume that

$$|z_1+\sqrt{-1}z'^2/\sqrt{1+z^2}|\geqq r.$$

Then we have obviously

$$|\tilde{W}(\zeta,\omega)|\leqq\frac{C}{r^n}\int_{y=y(x)}e^{-x^2/2+y^2-x\operatorname{Im}\zeta-y\operatorname{Re}\zeta}|dz|.$$

We specify the path $y=y(x)$. Note that the denominator vanishes if and only if

$$z_1^2(1+z^2)+z'^4=0. \tag{2.6}$$

This is an elliptic polynomial of z and has no real roots other than $x=0$. Hence, given any $\delta>0$ there exists $C_\delta>0$ such that (2.6) does not hold if $|x|\geqq\delta$ and $C_\delta|y|\leqq|x|$. Moreover we can simultaneously assume that $|z_1+\sqrt{-1}z'^2/\sqrt{1+z^2}|\geqq r\geqq1/C'_\delta$ there. Without loss of generality we can assume that $C_\delta\geqq8$. Then we choose

$$y=\frac{|x|\operatorname{Re}\zeta}{C_\delta|\operatorname{Re}\zeta|}\qquad\text{for}\quad|x|\geqq\delta. \tag{2.7}$$

Then, for the integral $\widetilde{W}_0(\zeta, \omega)$ on such part of the path we have

$$|\widetilde{W}_0(\zeta, \omega)| \leqq C'_\delta \int e^{-x^2/4-|\mathrm{Re}\,\zeta|/2C_\delta} dx \qquad \text{if} \quad 2C_\delta |\mathrm{Im}\,\zeta| \leqq |\mathrm{Re}\,\zeta|.$$

For $|x| \leqq \delta$, we take the path to the real domain. By integration by parts we have

$$\widetilde{W}(\zeta, \omega) = \int \frac{(-1)^n}{(n-1)!} \Big(\frac{\partial}{\partial x_1}\Big)^n [\psi(x, \omega) e^{-x^2+\sqrt{-1}x\zeta}]$$
$$\times \log (x_1 + \sqrt{-1}x'^2/\sqrt{1+x^2} + \sqrt{-1}0) dx.$$

This integral converges absolutely at the origin. Thus, if we join these two paths by the cylinder $|x| = \delta$, y_i from zero to $|x|\,\mathrm{Re}\,\zeta_i / C_\delta |\mathrm{Re}\,\zeta|$, we obtain the first estimate with another constant $C'_\delta > 0$.

To show the second estimate, we choose the path for $|x| \leqq \delta$ more carefully. First, if $\mathrm{Re}\,\zeta_1 \geqq |\mathrm{Re}\,\zeta'|$ we can take

$$y_1 = \frac{\delta\,\mathrm{Re}\,\zeta_1}{C_\delta |\mathrm{Re}\,\zeta|}, \qquad y' = 0 \qquad \text{for} \quad |x| \leqq \delta.$$

Note that in general $|x| \leqq \delta$ and $|y| \leqq \delta$ implies $|z| \leqq \sqrt{2}\delta$, hence, assuming $\delta \leqq 1/2$ without loss of generality we have

$$\Big|\frac{1}{\sqrt{1+z^2}} - 1\Big| \leqq \delta.$$

Thus we have

(2.8)
$$\begin{aligned} &\mathrm{Im}\,(z_1 + \sqrt{-1}z'^2/\sqrt{1+z^2}) \\ &\geqq y + x'^2 - y'^2 - \delta |z'|^2 \\ &\geqq y_1 + (1-\delta)x'^2 - (1+\delta)y'^2 \\ &\geqq y_1 - 2y'^2. \end{aligned}$$

For our special value of y, we have

$$\mathrm{Im}\,(z_1 + \sqrt{-1}z'^2/\sqrt{1+z^2}) \geqq \frac{\delta}{\sqrt{2}C_\delta} = r,$$

$$-\frac{1}{2}x^2 + y^2 - x\,\mathrm{Im}\,\zeta - y\,\mathrm{Re}\,\zeta \leqq -\frac{1}{2}x^2 + 1 + \delta |\mathrm{Im}\,\zeta| - \frac{\delta}{2C_\delta} |\mathrm{Re}\,\zeta|.$$

On the cylinder $|x| = \delta$, we join this path with (2.7) in a natural way. Then obviously we obtain the second estimate.

Next, if $|\mathrm{Re}\,\zeta_1| \leqq |\mathrm{Re}\,\zeta'|$, we put

$$y_1 = \frac{4\delta^2}{C_\delta^2} \frac{|\mathrm{Re}\,\zeta'|^2}{|\mathrm{Re}\,\zeta|^2}, \qquad y' = \frac{\delta}{C_\delta} \frac{\mathrm{Re}\,\zeta'}{|\mathrm{Re}\,\zeta|}, \qquad \text{for} \quad |x| \leqq \delta.$$

Then we have

$$y_1-2y'^2\geqq\frac{2\delta^2}{C_\delta^2}\frac{|\operatorname{Re}\zeta'|^2}{|\operatorname{Re}\zeta|^2}\geqq\frac{\delta^2}{C_\delta^2}=r\,,$$

$$-\frac{1}{2}x^2+y^2-x\operatorname{Im}\zeta-y\operatorname{Re}\zeta$$

$$\leqq-\frac{1}{2}x^2+1+\delta|\operatorname{Im}\zeta|+\frac{4\delta^2}{C_\delta^2}\frac{|\operatorname{Re}\zeta'|^2}{|\operatorname{Re}\zeta|^2}|\operatorname{Re}\zeta_1|-\frac{\delta}{C_\delta}\frac{|\operatorname{Re}\zeta'|^2}{|\operatorname{Re}\zeta|}$$

$$\leqq-\frac{1}{2}x^2+1+\delta|\operatorname{Im}\zeta|-\frac{\delta}{4C_\delta}|\operatorname{Re}\zeta|\,.$$

On the cylinder $|x|=\delta$, we join this path with (2.7) in a natural way. Obviously the latter estimate at least does not break on this joint.

Finally, if $-|\operatorname{Re}\zeta'|/\delta\leqq\operatorname{Re}\zeta_1\leqq-|\operatorname{Re}\zeta'|$, we put

$$y_1=\frac{4\delta^2}{C_\delta^2}\frac{(\operatorname{Re}\zeta_1)^2}{|\operatorname{Re}\zeta|^2}\,,\qquad y'=\frac{\delta}{C_\delta}\frac{\operatorname{Re}\zeta'}{|\operatorname{Re}\zeta|}\qquad\text{for}\quad|x|\leqq\delta\,.$$

Then we have

$$y_1-2y'^2\geqq\frac{\delta^2}{C_\delta^2}\frac{4|\operatorname{Re}\zeta_1|^2-2|\operatorname{Re}\zeta'|^2}{|\operatorname{Re}\zeta|^2}\geqq\frac{\delta^2}{C_\delta^2}\,,$$

$$-\frac{1}{2}x^2+y^2-x\operatorname{Im}\zeta-y\operatorname{Re}\zeta$$

$$\leqq-\frac{1}{2}x^2+1+\delta|\operatorname{Im}\zeta|-\frac{4\delta^2}{C_\delta^2}\frac{(\operatorname{Re}\zeta_1)^2}{|\operatorname{Re}\zeta|^2}\operatorname{Re}\zeta_1-\frac{\delta|\operatorname{Re}\zeta'|^2}{C_\delta|\operatorname{Re}\zeta}$$

$$\leqq-\frac{1}{2}x^2+1+\delta|\operatorname{Im}\zeta|+\Big(\frac{4}{C_\delta^2}-\frac{\delta}{C_\delta}\Big)\frac{|\operatorname{Re}\zeta'|^2}{|\operatorname{Re}\zeta|}\,.$$

Thus, if we further assume $C_\delta\geqq 8/\delta$, we have

$$-\frac{1}{2}x^2+y^2-x\operatorname{Im}\zeta-y\operatorname{Re}\zeta\leqq-\frac{1}{2}x^2+1+\delta|\operatorname{Im}\zeta|-\frac{\delta^2}{4C_\delta}|\operatorname{Re}\zeta|\,.$$

When we join this path with (2.7), the estimate does not become worse on the joint. Thus we have established the second estimate. q. e. d.

END OF PROOF OF THEOREM 2.1. Now we denote by $W(x',\omega')$ the component of the decomposition of $(n-1)$-dimensional delta function $\delta(x')$ given in Lemma 2.2. We have

$$J(\zeta')\tilde{f}_j(\zeta')\widetilde{W}(\zeta',\omega')=\tilde{g}_j(\zeta')\widetilde{W}(\zeta',\omega')+\tilde{h}_j(\zeta')\widetilde{W}(\zeta',\omega')\,.$$

From (2.2), (2.3) and Lemma 2.3 we have

$$|\tilde{g}_j(\xi')\widetilde{W}(\xi',\omega')|\leqq C_\delta'|\xi'|^n\exp(-a'|\xi'|^q)$$

provided that $M\varepsilon\leqq 1/C_\delta'$ and

$$\delta\xi'\cdot\omega'\geqq-\sqrt{|\xi'|^2-|\xi'\omega'|^2}\qquad\text{for}\quad\zeta_n\geqq-c|\zeta''|\,.$$

The last condition holds if $\omega_n\geqq 4c|\omega''|$. In fact, it suffices to consider only the case $\xi'\omega'\leqq 0$. Then we have

$$\begin{aligned}|\xi'\omega'|^2&\leqq|\xi''|^2|\omega''|^2+2\xi''\omega''\xi_n\omega_n+|\zeta_n|^2|\omega_n|^2\\&\leqq|\zeta''|^2(|\omega''|^2+2c|\omega''||\omega_n|)+|\xi_n|^2|\omega_n|^2\,.\end{aligned}$$

Hence

$$\begin{aligned}\sqrt{|\xi'|^2-|\xi'\omega'|^2}&\geqq\sqrt{|\xi''|^2(|\omega_n|^2-2c|\omega''||\omega_n|)}\\&\geqq\frac{1}{\sqrt{2}}|\xi''||\omega_n|\geqq\frac{4c}{\sqrt{2}}|\xi''||\omega''|\,.\end{aligned}$$

Thus if $\xi_n\geqq 0$, we have $\xi'\omega'\geqq-|\xi''||\omega''|$. If $\xi_n<0$, we have

$$|\xi'\omega'|\leqq\Big(\frac{1}{4c}+c\Big)|\xi''||\omega_n|\,.$$

Therefore the assertion holds if $\delta\leqq\min\{4c/\sqrt{2}(1+4c^2),4c/\sqrt{2}\}$. The obtained estimate shows that $\tilde{g}_j(\xi')\widetilde{W}(\xi',\omega')$ can be considered as the Fourier image of a function in Gevrey class, depending continuously on ω, if $\omega_n\geqq 4c|\omega''|$.

Next we decompose $\tilde{h}_j(\zeta')=\sum_k\tilde{h}_j^k(\zeta')$ as before. Choose δ and ε as above. Then $\tilde{h}_j^k(\zeta')\widetilde{W}(\zeta',\omega')$ satisfy

$$\begin{aligned}(2.9)\qquad&|\tilde{h}_j^k(\zeta')\widetilde{W}(\zeta',\omega')|\\&\leqq C_{\delta,\gamma}\exp(\gamma|\zeta'|+\delta|\mathrm{Im}\,\zeta'|+(1+b+M)\varepsilon|\mathrm{Im}\,\zeta'|+H_{\mathrm{ch}L_k}(\mathrm{Im}\,\zeta'))\,,\\&\qquad\text{if}\quad C_\delta|\mathrm{Im}\,\zeta'|\leqq|\mathrm{Re}\,\zeta'|\quad\text{and}\quad\omega_n\geqq 4c|\omega''|\,.\end{aligned}$$

We fix a small δ and an ε corresponding to it, and apply [**6**], Lemma 5.1.2 (see also the remark after Lemma 2.3 in [**5**bis]). Thus we conclude that each $\tilde{h}_j^k(\zeta')\widetilde{W}(\zeta',\omega')$ is the Fourier image of a Fourier hyperfunction whose analytic singular support is contained in the $[(1+b+M)\varepsilon+\delta]$-neighborhood of $\mathrm{ch}\,L_k$. Choosing L_k small enough, we thus conclude that $\tilde{h}_j(\zeta')\widetilde{W}(\zeta',\omega')$ is the Fourier image of a Fourier hyperfunction whose analytic singular support is contained in the $[(2+b+M)\varepsilon+\delta]$-neighborhood of $L\backslash K$. We can see more precisely that the family of functions $\{\mathcal{F}^{-1}[\tilde{h}_j(\zeta')\widetilde{W}(\zeta',\omega')]\,;\ \omega_n\geqq 4c|\omega''|\}$ in x' is holomorphic on a fixed complex neighborhood of

$$K_{\delta,\varepsilon}=\{x'\in K\,;\ \mathrm{dis}\,(x',L\backslash K)\geqq 2(2+b+M)\varepsilon+2\delta\}$$

depending continuously on ω'. This follows from the method of proof of [**6**], Lemma 5.1.2 estimating the integral after the deformation of the path, because the estimate (2.9) is uniform in ω'. Thus we conclude that $J(D')[f_j(x')*W(x',\omega')]$ is a continuous function of x',ω' for $x'\in K_{\delta,\varepsilon}$, $\omega_n\geqq 4c|\omega''|$. In view of Proposition 2.4 in [2] this implies that $\{f_j(x')*W(x',\omega');\ \omega_n\geqq 4c|\omega''|\}$ is a bounded set of real analytic functions of x' on $K_{\delta,\varepsilon}$. Hence it can be extended holomorphically to a fixed complex neighborhood of $K_{\delta,\varepsilon}$ and depends continuously on ω'. Thus we can apply Lemma 2.2 and conclude that $S.S.f_j(x')\cap K_{\delta,\varepsilon}\times\{\sqrt{-1}dx_n\infty\}=\emptyset$. Since δ and ε are arbitrary, we have proved the theorem. q. e. d.

We present the intrinsic form of our theorem.

COROLLARY 2.4. *Let $p(D)$ be an operator of order m and $p_m(D)$ be its principal part. Assume that $x_1=0$ is non-characteristic with respect to p. Let $V_{(1,0,\cdots,0),A}(p)$ be the sets of points $\xi'\in \boldsymbol{S}^{n-2}$ such that the equation $p_m(\zeta_1,\xi')=0$ in ζ_1 has a root with positive imaginary part. Then for every real analytic solution u of $p(D)u=0$ in $x_1>0$, we have*

$$S.S.b_j^+(u)\subset \boldsymbol{R}^{n-1}\times\sqrt{-1}\,\overline{V_{(1,0,\cdots,0),A}(p)}dx'\infty\,,$$

where the upper bar denotes the closure operation in $\boldsymbol{S}^{n-2}$.

PROOF. Assuming that

$$\{\xi'\in\boldsymbol{S}^{n-2};\ \xi_n\geqq c|\xi''|\}\subset\boldsymbol{S}^{n-2}\backslash\overline{V_{(1,0,\cdots,0),A}(p)}\,,$$

we will show that the roots $\zeta_1=\tau_j(\zeta')$ of $p(\zeta_1,\zeta')=0$ satisfy (2.1) if $\operatorname{Re}\zeta_n\leqq -c|\operatorname{Re}\zeta''|$. Then Theorem 2.1 can be applied to the point $(0,\cdots,0,1)$. Since we have $|\tau_j(\zeta')|\leqq M|\zeta'|$, the estimate (2.1) trivially holds for $|\operatorname{Im}\zeta'|\geqq\varepsilon|\operatorname{Re}\zeta'|$. Therefore we only have to discuss for $|\operatorname{Im}\zeta'|\leqq\varepsilon|\operatorname{Re}\zeta'|$ for some $\varepsilon>0$. Replacing ζ_1, ξ' by $-\zeta_1$, $-\xi'$, we see from the assumption that the roots $\zeta_1=\tau_j^0(\xi')$ of $p_m(\zeta_1,\xi')=0$ satisfy

$$\operatorname{Im}\tau_j^0(\xi')\geqq 0\quad\text{if }\xi'\text{ real and }\ \xi_n\leqq-(c-\delta)|\xi''|\,, \tag{2.10}$$

for some $\delta>0$. Therefore the function $1/p_m(\zeta_1,\zeta')$ is holomorphic on

$$\{\zeta=\xi+\sqrt{-1}\eta\in\boldsymbol{C}^n;\ \xi_n\leqq-(c-\delta)|\xi''|,\ \eta_1<0,\ \eta'=0\}\,.$$

The local version of Bochner's tube theorem shows that then $1/p_m(\zeta_1,\zeta')$ can be extended holomorphically to a domain containing

$$\{\zeta=\xi+\sqrt{-1}\eta\in\boldsymbol{C}^n;\ \xi_n\leqq-c|\xi''|,\ \ |\xi'|=1,\ \ |\eta'|\leqq-\varepsilon\eta_1\leqq 2\varepsilon M\}\,,$$

where $\varepsilon>0$ is a constant. (See, e. g., [8], Theorem 5. Here we follow the argument of Bony-Schapira [1] on non-strict hyperbolic operators.) This implies that the roots $\zeta_1=\tau_j^0(\zeta')$ of $p_m(\zeta_1,\zeta')=0$ satisfy

$$\operatorname{Im}\tau_j^0(\zeta')>-\frac{1}{\varepsilon}|\eta'|\quad\text{if}\ \ \xi_n\leqq-c|\xi''|\,,\ \ |\xi'|=1\,,\ \ |\eta'|\leqq\varepsilon\,.$$

By the homogeneity we conclude that there exist positive constants b, ε such that

$$\operatorname{Im}\tau_j^0(\zeta')\geqq -b|\operatorname{Im}\zeta'| \quad \text{if} \quad \operatorname{Re}\zeta_n\leqq -c|\operatorname{Re}\zeta''|\,, \quad |\operatorname{Im}\zeta'|\leqq\varepsilon|\operatorname{Re}\zeta'|\,.$$

Now we compare $\tau_j^0(\zeta')$ with $\tau_j(\zeta')$. Since the difference of the coefficients of ζ_1^{m-k} in these two equations is bounded by $c_k|\zeta'|^{k-1}$, we can apply Lemma 2.4 in Chapter IV in [**10**], after dividing the equations by $|\zeta'|_1^m$. Thus we have

$$|\tau_j^0(\zeta')-\tau_j(\zeta')|\leqq c'|\zeta'|^{(m-1)/m}+c''\,.$$

Thus we have proved (2.1) with $q=(m-1)/m<1$. q. e. d.

We leave to examine the points in $\overline{V_{(1,0,\cdots,0),A}(p)}\setminus V_{(1,0,\cdots,0),A}(p)$. It needs a more strong analytical tool. Instead we give an example of operators to which our theorem gives a sharp answer.

EXAMPLE 2.5. Consider the wave equation: $p(D)=D_1^2+\cdots+D_{n-1}^2-D_n^2$. Then we have

$$\overline{V_{(1,0,\cdots,0,A}(p)}=\{\xi'\in \boldsymbol{S}^{n-2}\,;\ \xi_n^2\leqq\xi''^2\}\,.$$

By Corollary 2.4 we conclude that for a real analytic solution u of $p(D)u=0$ on $\{x_1>0\}$,

$$S.S.b_j^+(u)\subset \boldsymbol{R}^{n-1}\times\{\sqrt{-1}\xi'dx'\infty\,;\ \xi_n^2\leqq\xi''^2\}\,,\quad j=0,1\,.$$

This estimate cannot be improved in general. In fact, for $\xi'\in\overline{V_{(1,0,\cdots,0),A}(p)}$ we can give a solution u whose boundary values $b_j^+(u)$ contain the direction ξ' in their singular spectrum. First assume that $\xi_n^2<\xi''^2$. Let $E(x_1,\cdots,x_{n-1})$ be the fundamental solution of $D_1^2+\cdots+D_{n-1}^2$. Let $E(x,\xi')$ be the function obtained by a Lorentz transform which brings the x_n axis to a line in $\{x_1=0\}$ perpendicular to ξ'. Then obviously $S.S.b_j^+(E(x,\xi'))$ contains the direction ξ'. Next assume that $\xi_n^2=\xi''^2$. Then we can take as u the solution whose singular spectrum contains only one bicharacteristic strip corresponding to the direction $(0,\xi')$ (see [7], Theorem 2.8). Then $S.S.b_j^+(u)$ obviously contains the direction ξ'. Similar argument holds for ultrahyperbolic equations $D_1^2+\cdots+D_k^2-D_{k+1}^2-\cdots-D_n^2$.

§ 3. Continuation of real analytic solutions.

The following result is a localization of those of [4] or [5] and extends them in some aspect.

THEOREM 3.1. *Let K be a closed set in $\{x_1=0\}$. Assume that K is contained in one side of an analytic hypersurface in $\boldsymbol{R}^{n-1}$ passing through the origin. Let ξ' be its unit normal vector at the origin. Assume that both ξ' and $-\xi'$ do not belong to $\overline{V_{(1,0,\cdots,0),A}(p)}$. Then every real analytic solution u of $p(D)u=0$ defined* *on a neighborhood of the origin except on K, can be continued as a hyperfunction solution to K in a smaller neighborhood.*

PROOF. We consider the difference $b_j(u)=b_j^+(u)-b_j^-(u)$ of the boundary values to $x_1=0$ from the two sides. Obviously this is a hyperfunction of $n-1$ variables with support in K. Thus by the Holmgren type theorem (see [**12**], Chapter III, Proposition 2.1.3), each $b_j(u)$ either vanishes on a neighborhood of the origin or contains the direction $\pm\xi'$ in its singular spectrum. The latter does not hold due to Corollary 2.5. Thus $b_j(u)\equiv 0$, and by Theorem 4 in [**9**] we conclude that u can be continued as a hyperfunction solution. q. e. d.

COROLLARY 3.2. *In addition to Theorem* 3.1, *assume that the principal part of $p(D)$ is of principal type and with real coefficients. Assume further that every bicharacteristic line of $p(D)$ flows out of K on any small neighborhood of the origin. Then, under the same situation u can be continued as a real analytic solution.*

PROOF. If the origin belongs to the analytic singular support, then by [**7**], Theorem 3.3′, the singularity flows out along one of the bicharacteristic line passing through the origin. Thus by the assumption u must be real analytic at the origin. q. e. d.

Note that in case $n=3$, every bicharacteristic line flows out of K automatically if the equation $p_m(\zeta_1, \xi')=0$ in ζ_1 has no multiple roots. In fact, the only possible counterexample is the bicharacteristic line $dx/dt=d_\eta p_m(\eta)$ contained in $x_1=0$ and perpendicular to ξ'. This implies

$$p_m(\eta)=0\,, \qquad \frac{\partial}{\partial\eta_1}p_m(\eta)=0\,,$$

$$\xi_2\frac{\partial}{\partial\eta_2}p_m(\eta)+\xi_3\frac{\partial}{\partial\eta_3}p_m(\eta)=0\,.$$

Euler's identity gives

$$\eta_2\frac{\partial}{\partial\eta_2}p_m(\eta)+\eta_3\frac{\partial}{\partial\eta_3}p_m(\eta)=0\,.$$

Since p_m is of principal type, either $(\partial/\partial\eta_2)p_m(\eta)$ or $(\partial/\partial\eta_3)p_m(\eta)$ is different of zero. Thus we conclude that $\xi'=k\eta'$ with a non-zero constant k. Then the equation $p_m(\zeta_1, \xi')=0$ in ζ_1 has real double root $\zeta_1=k\eta_1$. This contradicts to the assumption.

Note that in case K is convex the propagation of regularity holds with no assumption on $p(D)$ ([**6**], Theorem 5.1.1). But the corresponding result is weaker than our earlier one which was obtained by the direct application of convex Fourier analysis (see [**4**], Theorem 2.7, 2).

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