J. Math. Soc. Japan
Vol. 29, No. 4, 1977

# Malformed subregions of Riemann surfaces

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(Received Dec. 15, 1976)

The purpose of this note is to improve the results obtained in our former paper: *Extremizations and Dirichlet integrals on Riemann surfaces*, J. Math. Soc. Japan, **28** (1976), 581-603, which we quote as [$E$] throughout this note. We will also quote as [$Ei$] the reference [$i$] in [$E$], and terminologies as well as notations in [$E$] are occasionally used without explanations. We denote by $\mu_D$ and $\mu_{BD}$ the restrictions of *extremization* operator $\mu$ on the relative classes $HD(W;\partial W)$ and $HBD(W;\partial W)$. For convenience, a subregion $W$ of an open Riemann surface $R$ will be referred to in this note as a *malformed subregion* if $\mu_{BD}$ is surjective and yet $\mu_D$ is not. The main result in [$E$] is that there exists a malformed subregion $W$ of $R$ if (i) $HD(R)>HBD(R)$ (proper inclusion), and (ii) $R$ belongs to $CU_{HD^{\sim}}$. The primary purpose of this note is to remark that the condition (ii) is redundant in the above assertion. Therefore the best statement in this aspect is as follows:

THEOREM. *An open Riemann surface $R$ contains malformed subregions if and only if $R$ tolerates unbounded Dirichlet finite harmonic functions.*

The result shows, contrary to the intuition at the first sight, that the existence of malformed subregions is rather usual and it is exceptional that $R$ contains no malformed subregions. The result can be restated as a kind of *subdomain criterion* of degeneracies in the classification theory of Riemann surfaces (cf. [$E7$]):

COROLLARY 1. *An open Riemann surface $R$ belongs to the degenerate class $\bigcup_{0\leqq n<\infty} O_{HD}^n$ if and only if $R$ does not contain any malformed subregion.*

Another application of the above theorem is to the classification of densities $P$ on $R$, i.e. nonnegative Hölder continuous differentials $P(z)dxdy$. Again just for convenience we call a pair $(P, Q)$ of densities a *bad pair* on $R$ if $PBX(R)$ and $QBX(R)$ are canonically isomorphic and yet $PX(R)$ and $QX(R)$ are not for $X=D$ and $E$. The reason for we are tempted to call it bad is motivated by the fact that $PBX(R)$ is dense in $PX(R)$ even in the lattice sense $(X=D, E)$. The term malformed above is also used by the same feeling: $HBD$ is dense in $HD$ even in the lattice sense for $R$ and $(W;\partial W)$. We then have

COROLLARY 2. *There exists a bad pair of densities on a Riemann surface $R$ if and only if $R$ tolerates unbounded Dirichlet finite harmonic functions.*

Although there is no comparison in the completeness between the main result in [$E$] and the present theorem, the essence of the proof of the present theorem is already all there in [$E$] and only minor modifications are in order. We will describe in the sequel how to modify the proof in [$E$] to give the proof to the present theorem.

Since the necessity of the condition in the above theorem is clear, we only have to prove its sufficiency. The proof starts from no. 1 in [$E$] and proceeds to no. 10 in [$E$] without any change. Then we come to the spots where modifications are in order: nos. 11 and 12 in [$E$].

**11** (Revision of **11** in [$E$]). For a regular subregion $F$ we denote by $b(F)$ the *Harnack constant* of the set $\{z_0\}\cup\bar{F}$, i.e. the smallest number $\lambda$ such that $\lambda^{-1}u(z_1)\leqq u(z_2)\leqq \lambda u(z_1)$ for every pair of points $z_1$ and $z_2$ in $\{z_0\}\cup\bar{F}$ and for every nonnegative harmonic function $u$ on $R$. Here $z_0$ is a fixed point in $R$. Let $Y=\bigcup_{j=1}^{\iota} Y_j$ be a stuffed regular open subset of $R$ such that $Y_j$ $(j=1,\cdots \iota)$ are closure disjoint subregions of $R$, and let $F$ be a regular subregion of $R$ such that $\bar{Y}\subset F$. The constant $a(F, Y)=b(F)c(F, Y)$ will play an important role. We take the harmonic measure $\mu$ on $\Delta$ with its center $z_0$ and also the associated harmonic kernel $P(z, \zeta^*)$ (cf. [$E$7, p. 171]). Fix a point $z^*$ in $\Delta$. We shall prove that for any positive number $\varepsilon$ there exists a stuffed normal neighborhood $U^*$ of $z^*$ with $\bar{F}\cap\bar{U}^*=\emptyset$ and

$$\sum_{j=1}^{\iota}\left|\int_{-\partial Y_j} {}^*dw(\cdot, Y\cup X)-\int_{-\partial Y_j} {}^*dw(\cdot, Y)\right|<\varepsilon+a(F, Y)\mu(z^*) \tag{25}$$

for every stuffed regular open subset $X$ with $\bar{X}\subset U=U^*\cap R$. In 11 of [$E$] this is obtained only for the case $\mu(z^*)=0$. Here $\mu(z^*)>0$ may be the case. By the same observation as in 11 in [$E$], we find a decreasing sequence $\{U_n^*\}$ of stuffed normal neighborhoods $U_n^*$ of $z^*$ such that $\bar{U}_n^*\cap\bar{Y}=\emptyset$ and

$$\lim_{n\to\infty}\mu(U_n^*\cap\Delta)=\mu(z^*)\,.$$

Using the same $k_{n,m}$ defined in 11 of [$E$], we have

$$\lim_{m\to\infty}k_{n,m}=\int_{\Delta\cap U_n^*}P(\cdot, \zeta^*)d\mu(\zeta^*)$$

uniformly on each compact subset of $R$. Considering as $P(\cdot, z^*)=0$ for $\mu(z^*)=0$, we have

$$\lim_{n\to\infty}(\lim_{m\to\infty}k_{n,m})=P(\cdot, z^*)$$

again uniformly on each compact subset of $R$. We can thus find an increasing sequence $\{m(n)\}$ of positive integers such that

$$\lim_{n\to\infty}k_n=P(\cdot, z^*)\,,\qquad k_n=k_{n,m(n)}\,,$$

uniformly on each compact subset of $R$. Then as in 11 of [$E$]

$$\sum_{j=1}^{\iota}\left|\int_{-\partial Y_j}{}^*dw(\cdot, Y\cup X)-\int_{-\partial Y_j}{}^*dw(\cdot, Y)\right|\leqq c(F, Y)\sup_{\partial F} k_n$$

for every stuffed regular open subset $X$ with $\bar{X}\subset U_n^*\cap(R-\bar{R}_{m(n)})$. For sufficiently large $n$, $k_n\leqq\varepsilon/c(F, Y)+P(\cdot, z^*)$ on $\partial F$ and $P(\cdot, z^*)\leqq b(F)P(z_0, z^*)=b(F)\mu(z^*)$ on $\bar{F}$. We deduce (25) by taking $U^*$ as $U_n^*\cap(R-\bar{R}_{m(n)})$ with an above $n$.

**12** (Revision of **12** in [$E$]). Since $HD(R)$ forms a lattice, the existence of functions in $HD(R)-HBD(R)$ implies the existence of an $h$ in $HD(R)-HBD(R)$ with $h>0$ on $R$. This function $h$ will be fixed throughout the proof. We shall construct a sequence $\{\alpha_n\}$ of positive numbers $\alpha_n>n$ $(n=1, 2, \cdots)$ such that $3\alpha_n<\alpha_{n+1}$ $(n=1, 2, \cdots)$ and a sequence $\{X_n\}$ of stuffed regular open subsets $X_n$ of $R$ such that $X_n$ does not contain any nonzero dividing cycle of $R$, $\bar{X}_n\cap\bar{X}_m=\emptyset$ $(n\neq m)$,

$$\alpha_j^{-2}<\int_{-\partial X_j}{}^*dw(\cdot, \bigcup_{n=1}^{k} X_n)<2\alpha_j^{-2} \qquad (j=1, \cdots, k) \tag{26}$$

for every $k=1, 2, \cdots$, and

$$h \mid \bar{X}_n>\alpha_n \qquad (n=1, 2, \cdots). \tag{27}$$

The construction goes as follows. By the maximum principle there exists a sequence $\{\zeta_n^*\}$ of distinct points in $\Delta$ such that $\{h(\zeta_n^*)\}$ is an increasing sequence of real numbers with $h(\zeta_n^*)>3n$ $(n=1, 2, \cdots)$. Observe that

$$\sum_{n=1}^{\infty}\mu(\zeta_n^*)\leqq\mu(\Delta)=1\,.$$

First let $z_1^*=\zeta_1^*$, $2\alpha_1=h(z_1^*)$, and $V_1^*$ be a stuffed normal open neighborhood of $z_1^*$ such that $\alpha_1<h<3\alpha_1$ on $V_1^*$. By (24) there exists a stuffed regular open subset $X_1$ of $R$ with $\bar{X}_1\subset V_1^*\cap R$ such that

$$\alpha_1^{-2}<\int_{-\partial X_1}{}^*dw(\cdot, X_1)<2\alpha_1^{-2}\,.$$

Here by adding cuts in $X_1$ and deforming cut $X_1$ slightly, if necessary, we can assume that $R-\bar{X}_1$ is connected. Suppose $\alpha_1, \cdots, \alpha_k$ and $X_1, \cdots, X_k$ have already been chosen as required. We set $Y=\bigcup_{j=1}^{k} X_j$ and take a regular subregion $F$ with $\bar{Y}\subset F$. Let $\delta=\min_{1\leqq j\leqq k}(\min(2\alpha_j^{-2}-\beta_j, \beta_j-\alpha_j^{-2}))$ with

$$\beta_j=\int_{-\partial X_j}{}^*dw(\cdot, \bigcup_{n=1}^{k} X_n) \qquad (j=1, \cdots, k)\,.$$

Since $\sum_n\mu(\zeta_n^*)\leqq1$ and $h(\zeta_n^*)>3n$, we can find a $\zeta_n^*$ such that $\mu(\zeta_n^*)<\delta/2a(F, Y)$ and $h(\zeta_n^*)/2>k+1, 3\alpha_k$. We set $z_{k+1}^*=\zeta_n^*$ and $\alpha_{k+1}=h(z_{k+1}^*)/2$. By (25) with

$\varepsilon=\delta/2$ and $z^*=z^*_{k+1}$, we can find a stuffed normal open neighborhood $V^*_{k+1}$ of $z^*_{k+1}$ such that $\bar{V}^*_{k+1}\cap\bar{F}=\emptyset$, and $\alpha_{k+1}<h<3\alpha_{k+1}$ on $\bar{V}^*_{k+1}$, and (25) with $\varepsilon=\delta/2$ and $z^*=z^*_{k+1}$ is valid for every stuffed regular open subset $X$ with $\bar{X}\subset V^*_n\cap R$ so that

$$\alpha_j^{-2}<\int_{-\partial X_j} {}^*dw(\cdot, (\bigcup_{n=1}^{k} X_n)\cup X)<2\alpha_j^{-2} \qquad (j=1, \cdots, k).$$

By (24) in [$E$] we can choose a stuffed regular open subset $X=X_{k+1}$ with $\bar{X}_{k+1}\subset V^*_n$ such that

$$\alpha_{k+1}^{-2}<\int_{-\partial X_{k+1}} {}^*dw(\cdot, (\bigcup_{n=1}^{k} X_n)\cup X_{k+1})<2\alpha_{k+1}^{-2}.$$

Here by deforming slightly the set $X_{k+1}$ as in the first step we can assume that $R-\bar{X}_{k+1}$ and hence $R-(\bigcup_{n=1}^{k}\bar{X}_n)\cup\bar{X}_{k+1}$ is connected. This completes the induction for the proofs of (26) and (27).

Although the revisions are needed, after all, we obtained (26) and (27) which are identical with those in [$E$]. Then the rest of the proof proceeds as nos. 13 and 14 in [$E$] without any change. The proof of the present theorem is thus complete.

Just a rewording the theorem gives Corollary 1. The necessity of the condition of Corollary 2 is clear, and the proof for the sufficiency of the condition of Corollary 2 proceeds as in nos. 16 to 19 in [$E$] without any change.

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