J. Math. Soc. Japan
Vol. 30, No. 3, 1978

On representations of finite groups over skewfields

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(Received March 23, 1977)
(Revised Oct. 20, 1977)

Let G be a finite group and D a skewfield. A D-representation of the group G is a homomorphism of G into $GL(n, D)$ where $GL(n, D)$ is the group of all nonsingular $n\times n$ matrices over D. Equivalence, irreducibility, etc. of such representations are defined in the usual manner.

The following question arises:
What is the number of equivalence classes of irreducible D-representations of G? The answer to this question for the case when D is a skewfield of real quaternions was given by J. E. Houle [4] who showed that if r and r' are respectively the number of conjugacy classes and the number of selfinverse conjugacy classes of a finite group G, then the number of equivalence classes of irreducible representations of G over the real quaternions is equal to $\frac{r+r'}{2}$. The aim of this note is to find the group theoretical characterisation of the number of equivalence classes of irreducible D-representations of a finite group where D is finite dimensional over its centre.

1. Notation and definitions. D is a skewfield with characteristic $p\geqq 0$. K is the centre of D. D_n is the ring of all $n\times n$ matrices over D. $\overset{K}{\sim}$ (respectively $\overset{D}{\sim}$) is the K-equivalence (respectively D-equivalence). Let A and B be K-algebras. We shall call two (A, B)-modules M_1 and M_2 isomorphic if and only if M_1 and M_2 are isomorphic regarded as left A-modules and right B-modules.

Finally, let n be the least common multiple of the orders of the p'-elements in G and let ε be a primitive n-th root of unity over K. Let I_n be the multiplicative group consisting of those integers r, taken modulo n, for which $\varepsilon\rightarrow\varepsilon^r$ defines an automorphism of $K(\varepsilon)$ over K. Two p'-elements $a, b, \in G$ are called K-conjugate if $x^{-1}bx=a^r$ for some $x\in G$ and some $r\in I_n$.

2. The number of equivalence classes of irreducible representations of a finite group over a skewfield. If D is a field we may treat the terms matrix representation and DG-module as interchangeable. Slight modification is needed for the case when D is a skewfield. Namely, the following lemma holds.

LEMMA. *There is a one-to-one correspondence between the set of all D-representations of G and the set of all (KG, D)-modules. Moreover, any two D-representations are irreducible, completely reducible, equivalent if and only if the corresponding (KG, D)-modules are irreducible, completely reducible, isomorphic.*

PROOF. If M is a (KG, D)-module then M is a left G-module and since $(gm)\lambda=g(m\lambda)$ for any $g\in G$, $m\in M$, $\lambda\in D$ the module M defines a D-representation of G of degree n, where n is the dimension of M as a right vector space over D. On the other hand, if $g\to\Gamma(g)$ is a D-representation of G then $\sum_i \alpha_i g_i \to \sum_i \alpha_i \Gamma(g_i)$ $(\alpha_i\in K,\ g_i\in G)$ is the homomorphism of the group algebra KG into D_n and thus Γ defines a (KG, D)-module M where M is the right vector space of all $n\times 1$ matrices over D. The proof of the second part of the lemma is exactly the same as in the case of representations over fields (see, for example [**3**]).

THEOREM. *Suppose $(D: K)<\infty$. Then the number of equivalence classes of irreducible D-representations of G is equal to the number of K-conjugacy classes of p'-elements of G. Moreover, if $\Gamma_1, \cdots, \Gamma_s$ are all nonequivalent K-representations of G, then $\Gamma_i \overset{D}{\sim} n_i\Gamma_i'$ $(i=1, \cdots, s)$ where $\{\Gamma_1', \cdots, \Gamma_s'\}$ are all nonequivalent D-representations of G.*

PROOF. In view of lemma it suffices to consider (KG, D)-modules. Let M be a (KG, D)-module. For any $a\in KG$ the mapping $a\to a_L\,(m\to am)$ is a homomorphism of the group algebra KG into $L=\operatorname{Hom}_K(M, M)$ and $d\to d_R\,(m\to md)$ is an anti-homomorphism of D into L $(d\in D,\ m\in M)$.

Let D' be a skewfield anti-isomorphic to D under $d'\to d$. The mapping $\sum_{i=1}^n a_i\otimes d_i' \to \sum_{i=1}^n (a_i)_L(d_i')_R$ is a homomorphism of $KG\underset{K}{\otimes} D'$ into L. Thus M can be regarded as a unitary $KG\underset{K}{\otimes} D'$-module relative to the composition $\left(\sum_{i=1}^n a_i\otimes d_i'\right)m=\sum_{i=1}^n a_i m d_i$. This implies that M is irreducible, completely reducible, etc. as a (KG, D)-module if and only if it is irreducible, completely reducible etc. as a $KG\underset{K}{\otimes} D'$-module. Isomorphisms, homomorphisms, etc. for two (KG, D)-modules yield isomorphisms, homomorphisms, etc. for the corresponding $KG\underset{K}{\otimes} D'$-modules. It is clear if M is a $KG\underset{K}{\otimes} D'$-module then by setting $am=(a\otimes 1_{D'})m$, $md=(1_A\otimes d')m$, $a\in KG$, $m\in M$ we can regard M as a (KG, D)-module. Thus to prove the theorem it is sufficient to consider all $KG\underset{K}{\otimes} D'$-modules. First we observe that

$\operatorname{Rad}(KG\underset{K}{\otimes} D')=\operatorname{Rad} KG\underset{K}{\otimes} D'$ ([**1**], Chapter VIII, p. 7) and hence $KG\underset{K}{\otimes} D'/\operatorname{Rad}(KG\underset{K}{\otimes} D')=(KG\underset{K}{\otimes} D')/\operatorname{Rad} KG\underset{K}{\otimes} D' \cong KG/\operatorname{Rad} KG\underset{K}{\otimes} D'$. Let $\bar{A}=KG/\operatorname{Rad} KG$, $\bar{B}=\bar{A}\underset{K}{\otimes} D'$. Suppose that $\bar{A}=\bar{A}e_1+\cdots+\bar{A}e_s$ is the decomposition of the semisimple algebra $\bar{A}$ into the direct sum of minimal two-sided ideals. Then $e_1, \cdots, e_s$ are all minimal central idempotents of $\bar{B}$ ([**5**] p. 121), i.e. $\bar{B}=\bar{B}e_1+\cdots+\bar{B}e_s$ is the decomposition of the semisimple algebra B into the direct sum of minimal two-sided ideals. This shows that the number of equivalence classes of irreducible D-representations of G is the same as the number of nonequivalent irreducible representations of the algebra $KG/\mathrm{Rad}\,KG$. But the last number coincides with the number of K-conjugacy classes of p'-elements of G ([**2**]). This proves the first part of the theorem. Finally, let $e_i=e_{i1}+\cdots+e_{ik_i}(i=1, \cdots, s)$ be the decomposition of the minimal central idempotents of $\bar{A}$ into the sum of mutually orthogonal minimal idempotents of $\bar{A}$. Then $\bar{B}e_i=\bar{B}e_{i1}+\cdots+\bar{B}e_{ik_i}$ is the decomposition of the simple component of the algebra $\bar{B}$ into the direct sum of left ideals of $\bar{B}$ (not necessarily minimal). The fact that all minimal left ideals of the simple algebra $\bar{B}e_i$ are isomorphic implies that $\bar{B}e_i$ is the direct sum of minimal isomorphic left ideals. This shows that a minimal left ideal of $\bar{A}e_i$ regarded as a left ideal of $\bar{B}$ is the direct sum of isomorphic minimal left ideals of $\bar{B}$, proving the theorem.

COROLLARY 1. [**4**]. *Let r and r' be respectively the number of conjugacy classes and the number of self inverse conjugacy classes of the group G. Then the number of equivalence classes of irreducible representations of G over the skewfield of real quaternions is equal to* $\dfrac{r+r'}{2}$.

PROOF. Let $K=R$ be the real number field, then G splits into R-conjugacy classes as follows:

$$G=C_{g_1}\cup C_{g_2}\cup\cdots\cup C_{g_{r'}}\cup[C_{h_1}\cup C_{h_1^{-1}}]\cup\cdots\cup[C_{h_k}\cup C_{h_k^{-1}}]$$

where C_{g_i} is a self-inverse conjugacy class with representative g_i $(i=1, 2, \cdots, r')$. Hence the number of R-conjugacy classes is equal to $r'+\dfrac{r-r'}{2}=\dfrac{r+r'}{2}$. Now apply the theorem.

COROLLARY 2. *Let T_1 and T_2 be irreducible K-representations of G. If T_1 and T_2 are D-equivalent then they are K-equivalent.*

PROOF. It follows from theorem that $T_1\overset{K}{\sim}\Gamma_i$, $T_2\overset{K}{\sim}\Gamma_j$ and $T_1\overset{D}{\sim}n_i\Gamma'_i$, $T_2\overset{D}{\sim}n_j\Gamma'_j$ for some $1\leqq i, j\leqq s$. Since T_1 and T_2 are D-equivalent, $i=j$, proving the corollary.

ACKNOWLEDGEMENT. I wish to thank the referee for helpful suggestions.

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