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The maximal ideal space of certain algebra $H^{\infty}(m)$

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§1. Introduction.

Let A be a uniform algebra on a compact Hausdorff space X and m a complex homomorphism of A. We suppose that m has a unique representing measure μ_m on X and that the Gleason part $P(m)$ containing m consists of more than one point. We denote by $H^{\infty}(m)$ the w^* closure of A in $L^{\infty}(d\mu_m)$, and by I^{∞} the ideal $\{f\in H^{\infty}(m): \varphi(f)=0$ for all $\varphi\in P(m)\}$ of $H^{\infty}(m)$. In [**10**], Merrill proved that $H^{\infty}(m)$ is maximal as a w^* closed subalgebra of $L^{\infty}(d\mu_m)$ if and only if $I^{\infty}=\{0\}$. In this paper we shall deal with the case when $I^{\infty}\neq\{0\}$.

In §2 we shall state some preliminaries and two lemmas. In §3 we shall study some properties of the maximal ideal space of the Banach algebra $H^{\infty}(m)$ with $I^{\infty}\neq\{0\}$. In §4 we shall study some properties of a Gleason part $P(m)$ such that $A|P(m)=H^{\infty}(D)$ (for the precise meaning see §4). In §5 we shall give some examples relating to §3 and §4.

§2. Preliminaries and lemmas.

For a complex commutative Banach algebra B, let B^{-1} be the set of all invertible elements of B. Let $M(B)$ be the maximal ideal space of B endowed with the Gelfand topology, let $\hat{f}$ and $\hat{B}$ be the Gelfand transforms of f $(\in B)$ and B respectively, and let $\Gamma(B)$ be the Šilov boundary of B.

Let X be a compact Hausdorff space, and let $C(X)$ $(C_R(X))$ be the complex (real) Banach algebra of all complex (real) valued continuous functions on X. Let A be a *uniform algebra* on X, i.e., A is a uniformly closed subalgebra of $C(X)$ which contains the function 1 and separates the points of X. A representing measure for $\varphi\in M(A)$ is a probability measure μ on X such that $\varphi(f)=\int fd\mu$ for all $f\in A$. We denote by supp μ the closed support of a measure μ. When $\varphi\in M(A)$ has a unique representing measure, sometimes we use the same symbol φ to denote its representing measure. Given φ and ψ in $M(A)$, we set

$$d(\varphi, \psi)=\sup\{|\varphi(f)| : f\in A,\ \|f\|=\sup|f|\leqq 1,\ \psi(f)=0\}$$

and

$$G(\varphi, \psi)=\sup \{|\varphi(f)-\psi(f)| : f\in A,\ \|f\|\leqq 1\},$$

and write $\varphi\sim\psi$ if and only if $d(\varphi, \psi)<1$ (or, equivalently, $G(\varphi, \psi)<2$). Then $\sim$is an equivalence relation in $M(A)$, and an equivalence class $P(m)=\{\varphi\in M(A): m\sim\varphi\}(\supsetneqq\{m\})$ is called the *(nontrivial) Gleason part* for A which contains m.

Henceforth we suppose that m $(\in M(A))$ *has a unique representing measure* m *and that the Gleason part* $P=P(m)$ *containing* m *is nontrivial.* Then it is known that $\varphi\in P(m)$ has a unique representing measure φ and that representing measures m and φ are mutually absolutely continuous.

We denote by A_m the kernel of a complex homomorphism $m\in M(A)$. Let $H^\infty(m)$ and H_m^∞ be the w^* closures in $L^\infty(dm)$ of A and A_m respectively, and for $1\leqq p<\infty$ let $H^p(m)$ and H_m^p be the closures in $L^p(dm)$ norm of A and A_m respectively. If we denote by $\tilde{H}^\infty$ the restriction of $\hat{H}^\infty(m)$ to $\tilde{X}$ $(=M(L^\infty(dm))$, then $\tilde{H}^\infty$ is a logmodular algebra on $\tilde{X}$, i.e., $\log|(\tilde{H}^\infty)^{-1}|=C_R(\tilde{X})$ (cf. Hoffman [**5**]). Sometimes we shall identify $H^\infty(m)$ with $\tilde{H}^\infty$. A function $h\in H^\infty(m)$ with $|h|=1$ a.e. (dm) is called an *inner function.*

Theorem 2.1 (Wermer's Embedding Theorem). *Let* A *be a uniform algebra on a compact space* X. *Suppose that* $m\in M(A)$ *has a unique representing measure* m *on* X, *and that the Gleason part* $P(m)$ *containing* m *is nontrivial. Then there is an inner function* Z *known as Wermer's embedding function such that* $ZH^\infty(m)=H_m^\infty$ *and* $\varphi\mapsto\hat{Z}(\varphi)=\int Z\,d\varphi$ *is a one-to-one map of the part* $P(m)$ *onto the open unit disk* D. *The inverse map* τ *of* $\hat{Z}$ *is a one-to-one continuous map of* D *onto* $P(m)$, *and for every* f *in* A, *the composite function* $\hat{f}\circ\tau$ *is analytic on* D. (Cf. Leibowitz [**9**], p. 143).

Given $\varphi\in P(m)$, we define $\tilde{\varphi}$ by $\tilde{\varphi}(f)=\int fd\varphi$ for $f\in H^\infty(m)$, and set

$$\mathcal{P}=\{\tilde{\varphi} : \varphi\in P(m)\}. \tag{2.1}$$

Then $\mathcal{P}$ is the nontrivial Gleason part for $H^\infty(m)$ which contains $\tilde{m}$, and we have $\mathcal{P}=\{\varphi\in M(H^\infty(m)): |\varphi(Z)|<1\}$ for the Wermer's embedding function Z. Thus $\mathcal{P}$ is an open set in the space $M(H^\infty(m))$, which is homeomorphic to the open unit disk D (cf. Kishi [**7**], [**8**]).

Let $\mathcal{H}^p$ be the closure in $L^p(dm)$ norm of the polynomials in Z, and let $\mathcal{L}^p$ be the closure in $L^p(dm)$ norm of the polynomials in Z and $\bar{Z}$. (For $p=\infty$, the closure is taken in the w^* topology.) Let σ be the normalized Lebesgue measure on the unit circle C in the complex plane, and let $H^p(d\sigma)=H^p(D)$ be the classical Hardy space. For $1\leqq p\leqq\infty$, the correspondence

$$T: Z\longmapsto e^{i\theta} \tag{2.2}$$

induces an isometric $*$-isomorphism (i. e., taking complex conjugates into complex conjugates) of $\mathcal{L}^p$ onto $L^p(d\sigma)=L^p(C)$. This map is also an isometric isomorphism of $\mathcal{H}^\infty$ onto $H^\infty(D)$. Therefore the adjoint T^* of T is a homeomorphism of $M(L^\infty(C))$ and $M(H^\infty(D))$ onto $M(\mathcal{L}^\infty)$ and $M(\mathcal{H}^\infty)$ respectively. It is easily seen that $\log|(\mathcal{H}^\infty)^{-1}|=\mathcal{L}_R^\infty$, where $\mathcal{L}_R^\infty$ is the set of all real valued functions in $\mathcal{L}^\infty$. By Fatou's theorem, $H^\infty(D)$ is identified with the Banach algebra of all bounded analytic functions on D. (Cf. Merrill-Lal [11].)

For $1\leqq p\leqq\infty$, if we set

$$I^p=\{f\in H^p(m): \int \bar{Z}^n f\, dm=0,\ n=0, 1, 2, \cdots\}$$

and

$$M^p=\{f\in L^p(dm): \int Z^n f\, dm=0 \quad \text{for all integers } n\},$$

then we have

$$H^p(m)=\mathcal{H}^p\oplus I^p \quad \text{and} \quad L^p(dm)=\mathcal{L}^p\oplus M^p, \tag{2.3}$$

where $\oplus$ denotes algebraic direct sum. It is known that $f\in I^p$ if and only if $\hat{f}(\varphi)=\int f\, d\varphi=0$ for all $\varphi\in P(m)$. Further it is known that $I^2\cap L^\infty(dm)=I^\infty$ and I^∞ is dense in I^2. (Cf. Merrill-Lal [11].)

We prove here two lemmas which will be needed in § 3.

LEMMA 2.2. $\mathcal{L}^\infty I^\infty=I^\infty$.

PROOF. If $f\in I^\infty$, then we have $Z^n f\in I^\infty$ for all integers n (cf. Merrill-Lal [11], Lemma 1). For $f\in I^\infty$ and $g=\sum_{n=0}^\infty a_n Z^n\in\mathcal{H}^2$ we have

$$\int\Big|f\bar{g}-\sum_{n=0}^k \bar{a}_n\bar{Z}^n f\Big|\, dm\leqq\|f\|\Big[\int\Big|g-\sum_{n=0}^k a_n Z^n\Big|^2 dm\Big]^{1/2}\to 0$$

as $k\to\infty$. Hence we obtain $\bar{\mathcal{H}}^2 I^\infty\subset I^2$. We obtain similarly $\mathcal{H}_m^2 I^\infty\subset I^2$, where $\mathcal{H}_m^2=\{f\in\mathcal{H}^2: \int f\, dm=0\}$. Since $\mathcal{L}^2=\bar{\mathcal{H}}^2\oplus\mathcal{H}_m^2$ we have $\mathcal{L}^2 I^\infty\subset I^2$, and therefore $\mathcal{L}^\infty I^\infty\subset I^\infty$. q. e. d.

LEMMA 2.3. *$\Gamma(\mathcal{H}^\infty)$ can be identified with $M(\mathcal{L}^\infty)$, and a complex homomorphism φ of $\mathcal{H}^\infty$ belongs to $\Gamma(\mathcal{H}^\infty)$ if and only if $|\varphi(h)|=1$ for every inner function h in $\mathcal{H}^\infty$.*

PROOF. Let T^* be the adjoint of the map T defined in (2.2). Since $T^*(\Gamma(H^\infty(D))=\Gamma(\mathcal{H}^\infty)$ and $\Gamma(H^\infty(D))=M(L^\infty(C))$, we have $\Gamma(\mathcal{H}^\infty)=M(\mathcal{L}^\infty)$. It is easily seen that T carries all inner functions in $\mathcal{H}^\infty$ onto all inner functions in $H^\infty(D)$. Therefore we see that φ belongs to $\Gamma(\mathcal{H}^\infty)$ if and only if $|(T^*)^{-1}(\varphi)(h)|=1$ for every inner function h in $H^\infty(D)$ (cf. Hoffman [4], p. 179) if and only if $|\varphi(h)|=1$ for every inner function h in $\mathcal{H}^\infty$. q. e. d.

§ 3. The maximal ideal space of $H^\infty(m)$.

For an ideal I in a complex commutative Banach algebra B we denote the hull of I by hull(I) i.e., hull$(I)=\{\varphi\in M(B):\varphi(f)=0$ for all $f\in I\}$. We denote the closure of a subset E of $M(B)$ by $\bar{E}$.

THEOREM 3.1. (i) *The quotient Banach algebra* $H^\infty(m)/I^\infty$ *is isometrically isomorphic to* $\mathcal{H}^\infty$, *and under certain identification* we have hull$(I^\infty)=M(\mathcal{H}^\infty)=\bar{\mathcal{P}}$ *and* $M(I^\infty)=M(H^\infty(m))\backslash\bar{\mathcal{P}}$.

(ii) $H^\infty(m)$ *is maximal as a* w^* *closed subalgebra of* $L^\infty(dm)$ *if and only if* $M(H^\infty(m))=\bar{\mathcal{P}}$.

PROOF. (i) By (2.3) and Lemma 2.2, for every function $f=g+h\in H^\infty(m)$, $g\in\mathcal{H}^\infty$, $h\in I^\infty$ and for every positive integer n, we have $f^n=g^n+h_n$, $g^n\in\mathcal{H}^\infty$, $h_n\in I^\infty$. Hence we have

$$\int|f|^{2n}dm=\int|g|^{2n}dm+\int|h_n|^2dm\geqq\int|g|^{2n}dm\,,$$

so we obtain $\|f\|=\lim_{n\to\infty}\left(\int|f|^{2n}dm\right)^{1/2n}\geqq\lim_{n\to\infty}\left(\int|g|^{2n}dm\right)^{1/2n}=\|g\|$. Hence we have $\|g+I^\infty\|=\inf\{\|g+h\|:h\in I^\infty\}=\|g\|$ for $g\in\mathcal{H}^\infty$ and $g+I^\infty\in H^\infty(m)/I^\infty$. Therefore, by (2.3), the quotient algebra $H^\infty(m)/I^\infty$ is isometrically isomorphic to $\mathcal{H}^\infty$. Thus, under natural identification, we have hull$(I^\infty)=M(H^\infty(m)/I^\infty)=M(\mathcal{H}^\infty)$ and $M(I^\infty)=M(H^\infty(m))\backslash$hull$(I^\infty)$ (cf. Stout [**12**], pp. 27–28).

Let T^* be the adjoint of the map $T:\mathcal{H}^\infty\to H^\infty(D)$ defined in (2.2). Since the open unit disk D is dense in $M(H^\infty(D))$ (cf. Carleson [**2**]) and $T^*(D)=\mathcal{P}$, we have $T^*(M(H^\infty(D)))=\bar{\mathcal{P}}$. Therefore we have $M(\mathcal{H}^\infty)=\bar{\mathcal{P}}$.

(ii) $H^\infty(m)$ is maximal as a w^* closed subalgebra of $L^\infty(dm)$ if and only if $I^\infty=\{0\}$ (cf. Merrill [**10**], Theorem 1) if and only if $M(H^\infty(m))=\bar{\mathcal{P}}$. q. e. d.

THEOREM 3.2. (i) *If* $\varphi\in M(I^\infty)\,(=M(H^\infty(m))\backslash\bar{\mathcal{P}})$, *then* φ *is extendable to a complex homomorphism of* $\mathcal{L}^\infty$.

(ii) $\overline{M(I^\infty)}\cap\bar{\mathcal{P}}$ *is contained in* $M(\mathcal{L}^\infty)\,(=\Gamma(\mathcal{H}^\infty))$.

(iii) *If* $\varphi\in M(I^\infty)\cup M(\mathcal{L}^\infty)$, *then for every* f *in* $\mathcal{L}^\infty$, $\hat{f}$ *is a constant* $(=\varphi(f))$ *on the closed support* $(=\text{supp}\,\varphi)$ *of the representing measure for* φ.

PROOF. (i) If $\varphi\in M(I^\infty)$, there is $h\in I^\infty$ with $\varphi(h)=1$. Define Φ on $H^\infty(m)$ by $\Phi(f)=\varphi(hf)$ for all $f\in H^\infty(m)$. Then, as we have stated already in Theorem 3.1, the map $\varphi\mapsto\Phi$ is a homeomorphism of $M(I^\infty)$ onto $M(H^\infty(m))\backslash\bar{\mathcal{P}}$, and by this homeomorphism $M(I^\infty)$ may be identified with $M(H^\infty(m))\backslash\bar{\mathcal{P}}$. On the other hand we define Φ' on $\mathcal{L}^\infty$ by $\Phi'(f)=\varphi(hf)$ for all $f\in\mathcal{L}^\infty$. By Lemma 2.2, Φ' is multiplicative on $\mathcal{L}^\infty$, and we have $\Phi'|\mathcal{H}^\infty=\Phi|\mathcal{H}^\infty$. Therefore φ is extendable to a complex homomorphism on $\mathcal{L}^\infty$.

(ii) If $\varphi\in\overline{M(I^\infty)}\cap\bar{\mathcal{P}}$, there is a net $\{\varphi_\alpha\}$ in $M(I^\infty)$ converging to φ. By (i), we have $|\varphi_\alpha(h)|=1$ for every inner function h in $\mathcal{H}^\infty$, so we have $|\varphi(h)|=1$. Therefore, by Lemma 2.3, φ belongs to $M(\mathcal{L}^\infty)$.

(iii) Let Q be the set of all functions of the form $h_1\bar{h}_2$, where h_1 is a finite linear combination of inner functions in $H^\infty(D)$ and h_2 is an inner function in $H^\infty(D)$. Then Q is norm-dense in $L^\infty(C)$ (cf. Douglas-Rudin [3], Theorem 2). By using the map T defined in (2.2), we see that the same holds for $\mathcal{H}^\infty$ and $\mathcal{L}^\infty$.

If $\varphi\in M(I^\infty)\cup M(\mathcal{L}^\infty)$ then, by (i) and Lemma 2.3, we have $|\varphi(h)|=1$ for every inner function h in $\mathcal{H}^\infty$. Thus we have $\int|\hat{h}-\varphi(h)|^2d\varphi=0$, so $\hat{h}=\varphi(h)$ a. e. $(d\varphi)$. Given $f\in\mathcal{L}^\infty$ and any positive number ε, there are g_1 and g_2 such that $\|f-g_1/g_2\|<\varepsilon/2$, where g_1 is a finite linear combination of inner functions in $\mathcal{H}^\infty$ and g_2 is an inner function in $\mathcal{H}^\infty$. Then we have

$$\int|\hat{f}-\varphi(f)|\,d\varphi\leqq\int\left|\hat{f}-\frac{\hat{g}_1}{\hat{g}_2}\right|d\varphi+\int\left|\frac{\hat{g}_1}{\hat{g}_2}-\varphi(f)\right|d\varphi<\varepsilon\,,$$

so we have $\hat{f}=\varphi(f)$ a. e. $(d\varphi)$. Since $\hat{f}$ belongs to $C(\tilde{X})$ and $\varphi(f)$ is constant, we obtain $\hat{f}=\varphi(f)$ on $\operatorname{supp}\varphi$. q. e. d.

It is known that $\varphi\in M(H^\infty(m))$ belongs to $\tilde{X}=M(L^\infty(dm))$ if and only if $|\varphi(h)|=1$ for every inner function h in $H^\infty(m)$ (cf. Douglas-Rudin [3], Theorem 4). Hence, if $\tilde{x}\in\tilde{X}$ then, by Lemma 2.3, $\tilde{x}|\mathcal{H}^\infty$ belongs to $M(\mathcal{L}^\infty)$.

We define a continuous map $\tilde{\pi}$ of $\tilde{X}=M(L^\infty(dm))$ into $M(\mathcal{L}^\infty)$ by

$$\tilde{\pi}(\tilde{x})=\tilde{x}|\mathcal{H}^\infty,\ \tilde{x}\in\tilde{X}\,, \tag{3.1}$$

and for every $\varphi\in M(\mathcal{L}^\infty)$ we set

$$K(\varphi)=\{\tilde{x}\in\tilde{X}:\ \tilde{\pi}(\tilde{x})=\varphi\}\,. \tag{3.2}$$

Then, by Theorem 3.2, (iii), we see that $\tilde{\pi}(\tilde{X})=M(\mathcal{L}^\infty)$ and $\operatorname{supp}\varphi\subset K(\varphi)$ for every $\varphi\in M(\mathcal{L}^\infty)$. It is not known whether $\operatorname{supp}\varphi=K(\varphi)$ holds for $\varphi\in M(\mathcal{L}^\infty)$.

For a set E in the maximal ideal space $M(H^\infty(m))$ of $H^\infty(m)$ the $H^\infty(m)$-convex hull of E is the closed set $\hat{E}=\{\varphi\in M(H^\infty(m)):\ |\varphi(f)|\leqq\sup_E|f|$ for all $f\in H^\infty(m)\}$. It is easy to see that, for a compact subset E of $\tilde{X}$, $\varphi\in\hat{E}$ if and only if φ has a (unique) representing measure which is supported on E.

THEOREM 3.3. (i) *If $\varphi,\psi\in M(\mathcal{L}^\infty)$ and $\varphi\neq\psi$, then $\hat{K}(\varphi)$ and $\hat{K}(\psi)$ are disjoint.*

(ii) $M(I^\infty)=\cup\{\hat{K}(\varphi)\setminus\{\varphi\}:\ \varphi\in M(\mathcal{L}^\infty)\}$.

(iii) *The map $\tilde{\pi}$ of $M(L^\infty(dm))$ onto $M(\mathcal{L}^\infty)$ has a continuous cross section, i.e., there is a homeomorphism S from $M(\mathcal{L}^\infty)$ into $M(L^\infty(dm))$ such that $\tilde{\pi}\circ S$ is the identity.*

PROOF. (i) Since $\mathcal{H}^\infty$ separates the points of $M(\mathcal{L}^\infty)$, there is a function f in $\mathcal{H}^\infty$ such that $\varphi(f)\neq\psi(f)$. Hence $K(\varphi)$ and $K(\psi)$ are disjoint, and therefore $\hat{K}(\varphi)$ and $\hat{K}(\psi)$ are disjoint.

(ii) Let θ be any element in $M(I^\infty)$. Then, by Theorem 3.2, (iii), there is a unique $\varphi\in M(\mathcal{L}^\infty)$ such that $\operatorname{supp}\theta\subset K(\varphi)$. Thus we have $M(I^\infty)\subset\cup\{\hat{K}(\varphi)\setminus\{\varphi\}:\varphi\in M(\mathcal{L}^\infty)\}$.

Conversely if $\theta\in\cup\{\hat{K}(\varphi)\setminus\{\varphi\}:\varphi\in M(\mathcal{L}^\infty)\}$, then there is a unique $\varphi\in M(\mathcal{L}^\infty)$ such that $\theta\in\hat{K}(\varphi)\setminus\{\varphi\}$. For every inner function h in $\mathcal{H}^\infty$ we have $\hat{h}=\varphi(h)$ on $K(\varphi)$, so we have also $\hat{h}=\varphi(h)$ on $\operatorname{supp}\theta$. By Lemma 2.3, we have $|\theta(h)|=|\varphi(h)|=1$, so θ cannot belong to $\bar{\mathcal{P}}\setminus M(\mathcal{L}^\infty)$. Thus θ belongs to $M(I^\infty)\cup M(\mathcal{L}^\infty)$. But, by (i), θ does not belong to $M(\mathcal{L}^\infty)$. Thus θ belongs to $M(I^\infty)$.

(iii) Since $M(\mathcal{L}^\infty)$ is extremely disconnected and $\tilde{\pi}$ is the continuous map of a compact Hausdorff space $M(L^\infty(dm))$ onto $M(\mathcal{L}^\infty)$, so $\tilde{\pi}$ has a continuous cross section (cf. Bade [1], Theorem 7.4). q. e. d.

We define a continuous map π of $M(H^\infty(m))$ into $M(A)$ by

$$\pi(\Phi)=\Phi|A\,,\qquad \Phi\in M(H^\infty(m))\,. \tag{3.3}$$

COROLLARY 3.4. *If $\varphi\in\pi(M(I^\infty)\cup M(\mathcal{L}^\infty))$, then there is a $\Psi\in M(\mathcal{L}^\infty)$ such that* $\operatorname{supp}\varphi\subset\pi(K(\Psi))$.

PROOF. Suppose that $\varphi=\pi(\Phi)$ for some Φ in $M(I^\infty)\cup M(\mathcal{L}^\infty)$. Then, by Theorem 3.3, there is an element Ψ of $M(\mathcal{L}^\infty)$ such that $\operatorname{supp}\Phi\subset K(\Psi)$. Therefore we obtain $\operatorname{supp}\varphi=\pi(\operatorname{supp}\Phi)\subset\pi(K(\Psi))$. q. e. d.

THEOREM 3.5. *Let $B=\mathcal{L}^\infty\oplus I^\infty$ be the algebraic direct sum of $\mathcal{L}^\infty$ and I^∞. Then we have the following.*

(i) *B is a w^* closed subalgebra of $L^\infty(dm)$.*

(ii) *The quotient Banach algebra B/I^∞ is isometrically isomorphic to $\mathcal{L}^\infty$.*

(iii) *$M(B)=\cup\{\hat{K}(\varphi):\varphi\in M(\mathcal{L}^\infty)\}$, where $\hat{K}(\varphi)$ is the $H^\infty(m)$-convex hull of $K(\varphi)$. If $\varphi,\psi\in M(\mathcal{L}^\infty)$ and $\varphi\neq\psi$, then $\hat{K}(\varphi)$ and $\hat{K}(\psi)$ are disjoint.*

PROOF. (i) For every function $f=g+h$ in B, where $g\in\mathcal{L}^\infty$ and $h\in I^\infty$, we have $\|g\|+\|h\|\leqq 3\|g+h\|$ (see the proof of Theorem 3.1 and Lemma 2.2). Then, since $\mathcal{L}^\infty$ and I^∞ are w^* closed subspaces of $L^\infty(dm)$, B is a w^* closed subalgebra of $L^\infty(dm)$ (cf. Leibowitz [9], p. 203).

(ii) This is proved by the same argument as in the proof of Theorem 3.1.

(iii) By (ii), we obtain $\operatorname{hull}(I^\infty)=\{\varphi\in M(B):\varphi(f)=0 \text{ for all } f\in I^\infty\}=M(\mathcal{L}^\infty)$ and $M(I^\infty)=M(B)\setminus\operatorname{hull}(I^\infty)$. Therefore, by Theorem 3.3, we have $M(B)=\cup\{\hat{K}(\varphi):\varphi\in M(\mathcal{L}^\infty)\}$. q. e. d.

THEOREM 3.6. (i) *$\bar{\mathcal{P}}\setminus M(\mathcal{L}^\infty)$ is a union of Gleason parts for $H^\infty(m)$.*

(ii) *If $\varphi,\theta\in\bar{\mathcal{P}}$, then we have* $\sup\{|\varphi(f)-\theta(f)|:f\in H^\infty(m),\|f\|\leqq 1\}=\sup\{|\varphi(f)-\theta(f)|:f\in\mathcal{H}^\infty,\|f\|\leqq 1\}$.

(iii) *If $\varphi\in\bar{\mathcal{P}}\setminus M(\mathcal{L}^\infty)$ and $\theta\in M(I^\infty)\cup M(\mathcal{L}^\infty)$, then we have $\mu_\varphi(\operatorname{supp}\theta)=0$ for a (unique) representing measure μ_φ on $\tilde{X}$ for φ.*

PROOF. (i) If $\psi\in M(\mathcal{L}^\infty)\cup M(I^\infty)$ then, by Lemma 2.3 and Theorem 3.2, (i), we have $|\psi(f)|=1$ for any inner function f in $\mathcal{H}^\infty$. If $\varphi\in\bar{\mathcal{P}}\setminus M(\mathcal{L}^\infty)$ then, by Lemma 2.3, we have $|\varphi(f_0)|<1$ for some inner function f_0 in $\mathcal{H}^\infty$. Hence, for an inner function $F=\frac{f_0-\varphi(f_0)}{1-\overline{\varphi(f_0)}f_0}$ in $\mathcal{H}^\infty$, we have $|\psi(F)|=1$ and $\varphi(F)=0$. Thus we obtain $d(\varphi, \psi)=\sup\{|\psi(f)| : f\in H^\infty(m),\ \|f\|\leqq 1,\ \varphi(f)=0\}=1$, so $\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$ is a union of Gleason parts for $H^\infty(m)$.

(ii) If $f=g+h\in H^\infty(m)$, $g\in\mathcal{H}^\infty$, $h\in I^\infty$ and $\|f\|\leqq 1$, then we obtain $\|g\|\leqq\|f\|\leqq 1$ (see the proof of Theorem 3.1). And, for any $\varphi, \theta\in\bar{\mathcal{P}}$, we have $|\varphi(f)-\theta(f)| =|\varphi(g)-\theta(g)|$. Thus we have $|\varphi(f)-\theta(f)|\leqq\sup\{|\varphi(g)-\theta(g)| : g\in\mathcal{H}^\infty,\ \|g\|\leqq 1\}$, so we obtain $\sup\{|\varphi(f)-\theta(f)| : f\in H^\infty(m),\ \|f\|\leqq 1\}=\sup\{|\varphi(g)-\theta(g)| : g\in\mathcal{H}^\infty,\ \|g\|\leqq 1\}$.

(iii) Since $\hat{\mathcal{H}}^\infty$ is a logmodular algebra on $Y=M(\mathcal{L}^\infty)$, $\varphi\in\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$ also has a unique representing measure on Y for $\mathcal{H}^\infty$. The map $\tilde{\pi}$ defined in (3.1) is a continuous map of $\tilde{X}$ onto Y. Thus there is a natural linear transformation $\tilde{\sigma}$ of the dual space $\mathcal{M}(\tilde{X})$ of $C(\tilde{X})$ onto the dual space $\mathcal{M}(Y)$ of $C(Y)$ which takes $\mu\in\mathcal{M}(\tilde{X})$ onto the measure $\tilde{\sigma}(\mu)$ defined on the Borel set E of Y by

$$\tilde{\sigma}(\mu)(E)=\mu(\tilde{\pi}^{-1}(E)),$$

or, equivalently, on the function $g\in C(Y)$ by

$$\int_Y g\, d(\tilde{\sigma}(\mu))=\int_{\tilde{X}} g\circ\tilde{\pi}\, d\mu .$$

Then, if μ_φ is the representing measure on $\tilde{X}$ of $\varphi\in\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$ for $H^\infty(m)$, then $\tilde{\sigma}(\mu_\varphi)$ is the representing measure on Y of φ for $\mathcal{H}^\infty$. Thus, for any $\theta\in M(\mathcal{L}^\infty)$, we have $0=\tilde{\sigma}(\mu_\varphi)(\{\theta\})=\mu_\varphi(K(\theta))$. If $\theta\in M(\mathcal{L}^\infty)$, we have $\operatorname{supp}\theta\subset K(\theta)$, so we obtain $\mu_\varphi(\operatorname{supp}\theta)=0$. If $\theta\in M(I^\infty)$, then, by Theorem 3.3, (ii), we have $\operatorname{supp}\theta\subset K(\psi)$ for some $\psi\in M(\mathcal{L}^\infty)$, so we obtain $\mu_\varphi(\operatorname{supp}\theta)=0$. q.e.d.

COROLLARY 3.7. *If $\varphi\in\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$, then the closure of the Gleason part $P(\varphi)$ for $H^\infty(m)$ which contains φ does not meet $M(I^\infty)\cup M(\mathcal{L}^\infty)$. The union G of all nontrivial Gleason parts for $H^\infty(m)$ which are contained in $\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$ is open in the space $M(H^\infty(m))$.*

PROOF. The map T defined in (2.2) is an isometric isomorphism of $\mathcal{H}^\infty$ onto $H^\infty(D)$, and thus the adjoint T^* of T is a homeomorphism of $M(H^\infty(D))$ onto $M(\mathcal{H}^\infty)$. If P is a nontrivial (trivial) Gleason part for $H^\infty(D)$ which is contained in $M(H^\infty(D))\backslash M(L^\infty(C))$, then, by Theorem 3.6, $T^*(P)$ is also a nontrivial (trivial) Gleason part for $H^\infty(m)$ which is contained in $\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)$. Hence, for $P(\varphi)$ there is a Gleason part P for $H^\infty(D)$ such that $P\cap M(L^\infty(C))=\emptyset$ and $T^*(P)=P(\varphi)$. Since $\bar{P}$ does not meet $M(L^\infty(C))$ (cf. Hoffman [6], p. 102), $\overline{P(\varphi)}$ does not meet $M(\mathcal{L}^\infty)$. Hence, by Theorem 3.1, we have $\overline{P(\varphi)}\cap(M(I^\infty)\cup M(\mathcal{L}^\infty))=\emptyset$.

The union G_1 of all nontrivial Gleason parts for $H^\infty(D)$ is open in the subspace $M(H^\infty(D))\backslash M(L^\infty(C))$ (cf. Hoffman [6], p. 89), so that $G=T^*(G_1)$ is open in the subspace $\bar{\mathcal{P}}\backslash M(\mathcal{L}^\infty)=M(H^\infty(m))\backslash(M(I^\infty)\cup M(\mathcal{L}^\infty))$. But, by Theorem 3.2, (ii), $M(I^\infty)\cup M(\mathcal{L}^\infty)$ is closed in $M(H^\infty(m))$, so G is open in $M(H^\infty(m))$. q.e.d.

§ 4. A Gleason part P satisfying $A|P=H^{\infty}(D)$.

By Theorem 2.1, there is a one-to-one continuous map τ of the open unit disk D onto a nontrivial Gleason part $P=P(m)$ containing $m\in M(A)$ such that, for every f in A, the composition $\hat{f}\circ\tau$ belongs to $H^{\infty}(D)$. If we set

$$A|P=\{\hat{f}\circ\tau:\ f\in A\},$$

then we have $A|P\subset H^{\infty}(D)$. When $\{\hat{f}\circ\tau:\ f\in A\}=H^{\infty}(D)$ holds, we denote it by $A|P=H^{\infty}(D)$. Note that $H^{\infty}(m)|\mathcal{P}=\{\hat{f}\circ\tau:\ f\in H^{\infty}(m)\}$ is contained in $H^{\infty}(D)$ (cf. Leibowitz [**9**], p. 142).

THEOREM 4.1. *Let A be a uniform algebra on a compact space X. Suppose that $m\in M$ has a unique representing measure m on X, and that the part P containing m consists of more than one point. If $A|P=H^{\infty}(D)$, then the map π defined in* (3.3) *is a homeomorphism of the closure $\bar{\mathcal{P}}$ of $\mathcal{P}$ in $M(H^{\infty}(m))$ onto the closure $\bar{P}$ of P in $M(A)$. Therefore $\bar{P}$ is homeomorphic to the maximal ideal space of $H^{\infty}(D)$.*

PROOF. By the continuity of π we have $\pi\bar{\mathcal{P}}\subset\overline{\pi\mathcal{P}}=\bar{P}$, and clearly we have $\pi\bar{\mathcal{P}}\supset\overline{\pi\mathcal{P}}$, so that we have $\pi\bar{\mathcal{P}}=\bar{P}$. From $H^{\infty}(D)=A|P=A|\mathcal{P}\subset H^{\infty}(m)|\mathcal{P}=\mathcal{H}^{\infty}|\mathcal{P}\subset H^{\infty}(D)$ we obtain

$$A|P=\mathcal{H}^{\infty}|\mathcal{P}=H^{\infty}(m)|\mathcal{P}\,. \tag{3.4}$$

If $\varphi_1,\ \varphi_2\in\bar{\mathcal{P}}$ and $\varphi_1\neq\varphi_2$, then there is a function $f\in\mathcal{H}^{\infty}$ satisfying $\hat{f}(\varphi_1)\neq\hat{f}(\varphi_2)$. There is a function $g\in A$ such that $\hat{f}=\hat{g}$ on $\mathcal{P}$ and thus $\hat{f}=\hat{g}$ on $\bar{\mathcal{P}}$. For such a function g we have $\hat{g}(\varphi_1)\neq\hat{g}(\varphi_2)$. Thus π is a homeomorphism of $\bar{\mathcal{P}}$ onto $\bar{P}$.

In the proof of Theorem 3.1 we have shown that $\bar{\mathcal{P}}$ is homeomorphic to $M(H^{\infty}(D))$, so that $\bar{P}$ is homeomorphic to $M(H^{\infty}(D))$. q. e. d.

COROLLARY 4.2. *If $A|P=H^{\infty}(D)$, then P is homeomorphic to the open unit disk D.*

A complex valued function f on P is called a bounded analytic function on P if $f\circ\tau$ is analytic on D for τ in Theorem 2.1 and $\sup\{|(f\circ\tau)(\lambda)|:\ \lambda\in D\}$ is finite. We denote by $H^{\infty}(P)$ the set of all bounded analytic functions on P, then it is known that $H^{\infty}(P)=\{\hat{f}\circ\tau:\ f\in H^{\infty}(m)\}$ (cf. Leibowitz [**9**], p. 155). By (3.4), we obtain the following corollary.

COROLLARY 4.3. *If $A|P=H^{\infty}(D)$, then $A|P=H^{\infty}(P)$.*

THEOREM 4.4. *Let A, m and P be as in Theorem* 4.1, *and let $I=\{f\in A:\varphi(f)=0$ for all $\varphi\in P\}$. Then, if $A|P=H^{\infty}(D)$, we have $M(A/I)=$*hull (I)*$=\bar{P}$. Therefore $M(A)\supsetneqq\bar{P}$ if and only if $I\supsetneqq\{0\}$.*

PROOF. As we have shown in the proof of Theorem 4.1, if $A|P=H^{\infty}(D)$ then $A|\mathcal{P}=H^{\infty}(m)|\mathcal{P}$. Hence if we consider A as a subset of $H^{\infty}(m)$, we have

$\{f+I^{\infty}: f\in H^{\infty}(m)\}=\{f+I^{\infty}: f\in A\}$. Therefore, if we set $A/I^{\infty}=\{f+I^{\infty}: f\in A\}$, then we have $A/I^{\infty}=H^{\infty}(m)/I^{\infty}$. Since $I=I^{\infty}\cap A$, we have $(f+I^{\infty})\cap A=f+I$ for every $f\in A$. The correspondence $f+I^{\infty}\mapsto f+I$, which is defined for $f\in A$, induces an algebra isomorphism Σ of $H^{\infty}(m)/I^{\infty}=A/I^{\infty}$ onto A/I. Therefore the adjoint Σ^* of Σ is a homeomorphism of $M(A/I)$ onto $M(H^{\infty}(m)/I^{\infty})$. We set $\sigma=(\Sigma^*)^{-1}$. If $\sigma(\Phi)=\varphi$ for $\Phi\in M(H^{\infty}(m)/I^{\infty})$, then we have

$$\varphi(f+I)=(\sigma(\Phi))(f+I)=\Phi(f+I^{\infty})$$

for every $f\in A$.

If ρ is the natural homomorphism of $H^{\infty}(m)$ onto $H^{\infty}(m)/I^{\infty}$, then the adjoint ρ^* of ρ is a homeomorphism of $M(H^{\infty}(m)/I^{\infty})$ onto hull(I^{∞}). Similarly if ρ_A is the natural homomorphism of A onto A/I, then the adjoint ρ_A^* of ρ_A is a homeomorphism of $M(A/I)$ onto hull(I). We then have

$$M(A/I)=\{(\sigma\circ(\rho^*)^{-1})(\Phi): \Phi\in \text{hull}(I^{\infty})=\bar{\mathcal{P}}\}$$

and, by Theorem 4.1, we have

$$\bar{P}=\{\pi(\Phi): \Phi\in\bar{\mathcal{P}}\}.$$

And, for every $f\in A$ and $\Phi\in\bar{\mathcal{P}}$, we have

$$(\pi\circ\Phi)(f)=\Phi(f)=(\sigma\circ(\rho^*)^{-1}\circ\Phi)(f+I).$$

Therefore, by using the identification map ρ_A^*, we have

$$M(A/I)=\bar{P},$$

and thus we have hull$(I)=\bar{P}$. q. e. d.

COROLLARY 4.5. *Suppose that* $A|P=H^{\infty}(D)$ *and the closure* $\bar{P}$ *of* P *in* $M(A)$ *is disjoint from* X. *If we put* $A_1=\{f\in A: \hat{f}\circ\tau$ *is a constant*$\}$ *for* τ *in Theorem* 2.1, *then* A_1 *is a uniform algebra on* X.

§ 5. Examples.

1. Let $A=A(T^2)$ be the Dirichlet algebra of continuous functions on the torus $T^2=\{(z, w): |z|=|w|=1\}$ which are uniform limits of the polynomials in $z^i w^j$, where

$$(i, j)\in S=\{(i, j): j>0\}\cup\{(i, 0): i\geqq 0\}.$$

Then the maximal ideal space of A can be identified with

$$\{(z, w): |z|=1, |w|\leqq 1\}\cup\{(z, 0): |z|<1\}$$

with the normalized Haar measure m identified with $z=w=0$. The Gleason part $P(m)$ containing m is $\{(z, 0): |z|<1\}$ and the closure of $P(m)$ does not meet T^2. The Wermer's embedding function for $P(m)$ is given by $Z=z$ (cf. Merrill-Lal [**11**].)

Let $H^{\infty}(m)$ be the w^* closure in $L^{\infty}(dm)$ of A. Then $H^{\infty}(m)$ is not maximal as a w^* closed subalgebra of $L^{\infty}(dm)$. $\mathcal{H}^{\infty}$ and I^{∞} are the w^* closure in $L^{\infty}(dm)$ of the polynomials in z^i, $i=0, 1, 2, \cdots$ and the w^* closure in $L^{\infty}(dm)$ of the polynomials in $z^i w^j$ for $i=0, \pm 1, \pm 2, \cdots$ and $j \geqq 1$ respectively. By using an inner function $f=zw$ we see that the closure of the Gleason part $\mathcal{P}$ for $H^{\infty}(m)$ does not meet the Šilov boundary $M(L^{\infty}(dm))$ for $H^{\infty}(m)$ (see (2.1)). $\mathcal{L}^{\infty}$ is the w^* closure in $L^{\infty}(dm)$ of the polynomials in z^i for $i=0, \pm 1, \pm 2, \cdots$ and the Banach algebra B defined in Theorem 3.5 is the w^* closure in $L^{\infty}(dm)$ of the polynomials in $z^i w^j$ for $i=0, \pm 1, \pm 2 \cdots$ and $j=0, 1, 2, \cdots$.

2. Let $H^{\infty}=H^{\infty}(D)$ be the algebra of all bounded analytic functions on the open unit disk D. For $|\alpha|=1$ let M_{α} be the fiber of $M(H^{\infty})$ over α, i.e., $M_{\alpha} = \{\varphi \in M(H^{\infty}) : \varphi(z)=\alpha\}$ and let $X_{\alpha}=M_{\alpha} \cap M(L^{\infty}(C))$. Then $A_{\alpha}=\hat{H}^{\infty}|M_{\alpha}$ is a uniform algebra on M_{α} and the Šilov boundary of A_{α} is X_{α}. Evidently we have

$$C_R(X_{\alpha}) \supset \log|A_{\alpha}^{-1}| \supset (\log|(\hat{H}^{\infty})^{-1}|)|X_{\alpha}=C_R(M(L^{\infty}(C)))|X_{\alpha}=C_R(X_{\alpha}),$$

so we obtain $\log|A_{\alpha}^{-1}|=C_R(X_{\alpha})$. Therefore A_{α} is a logmodular algebra on X_{α}. As we have stated already in the proof of Corollary 3.7, if P is any nontrivial Gleason part for A_{α}, then there is an inner function f such that $\hat{f}=0$ on P (cf. Hoffman [6], p. 102). Therefore if $m \in P$ and $H^{\infty}(m)$ is the w^* closure of A_{α} in $L^{\infty}(dm)$, then $H^{\infty}(m)$ is not maximal as a w^* closed subalgebra of $L^{\infty}(dm)$.

There is a nontrivial Gleason part P for A_{α} such that $A_{\alpha}|P=H^{\infty}(D)$ (cf. Hoffman [4], p. 106). By Corollary 4.2, we see that if $A_{\alpha}|P=H^{\infty}(D)$, then P is homeomorphic to the open unit disk. But there is an example such that P is not homeomorphic to D (cf. Hoffman [6], p. 109), and for such a Gleason part we have $A_{\alpha}|P \neq H^{\infty}(D)$.

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