J. Math. Soc. Japan
Vol. 30, No. 4, 1978

A finiteness theorem for foliated manifolds

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(Received March 4, 1977)

§ 1. Introduction.

Let M be a smooth closed manifold which carries a smooth foliation $\mathfrak{F}$. A smooth form ω of degree p is said to be of *filtration* $\geqq i$ if it vanishes whenever $p-i+1$ of the vectors are tangent to the foliation. In this way the deRham complex of smooth forms becomes a filtered complex and we have the spectral sequence $E_k(\mathfrak{F})$ which converges after a finite number of steps to the (finite-dimensional) real cohomology of M.

Let us denote by $I(\mathfrak{F})$ the Lie algebra of all infinitesimal transformations of the foliation. At each point $x \in M$, we get a subspace $I(\mathfrak{F})(x)$ of the tangent space T_x (by evaluating the vector fields at x). The foliation is called *transitive* if $I(\mathfrak{F})(x)=T_x$ for all x.

THEOREM. *If a smooth closed manifold M carries a transitive foliation $\mathfrak{F}$ then $E_2(\mathfrak{F})$ is finite dimensional.*

This theorem is proved in no. 6) below: the method of proof, which is of independent interest, consists in making a parametrix for the exterior derivative by averaging over flows and then employing the theory of compact operators.

In no. 1) we define the filtration and in no. 2) by using the theory of exact couples reduce the finiteness problem to an equivalent simpler form (Lemma 3). In no. 3) the relevant results from functional analysis are recalled, a *k-parametrix* is defined and it is related to the finite dimensionality of $E_k(\mathfrak{F})$. In nos. 5) and 6) we construct a 2-parametrix when $\mathfrak{F}$ is transitive and thereby prove the theorem. Beyond some elementary differential geometry (chapter I of [2]), functional analysis ([7]) and homological algebra (definition of spectral sequence as in [3]) the treatment below is self-contained. Further comments about the various aspects of this construction and the literature are made in no. 7) below.

§ 2. Lie subalgebras of $\mathfrak{H}_M$.

If M is a smooth manifold, the set of all its smooth tangent vector fields is a Lie algebra with the usual bracket operation. We shall denote this Lie algebra by $\mathfrak{H}_M$. We also denote by $\Lambda(M)$ the algebra of all smooth forms on

M; as usual we will equip this algebra with the (skew) derivations d (exterior derivative), τ_X (interior product) and L_X (Lie derivative). Here $X \in \mathfrak{H}_M$. We have the following well-known formulas:

$$\begin{array}{llll} \text{(a)} \ d^2=0 & \text{(b)} \ \tau_X\tau_Y+\tau_Y\tau_X=0 & \text{(c)} \ L_XL_Y-L_YL_X=L_{[X,Y]} & \\ \text{(d)} \ d\tau_X+\tau_Xd=L_X & \text{(e)} \ \tau_XL_Y-L_Y\tau_X=\tau_{[Y,X]} & \text{(f)} \ L_Xd-dL_X=0. & \end{array} \quad (1)$$

(So one cannot make any more (skew) derivations out of d, L_X and τ_X by using the 'bracket' operations).

Let $\mathfrak{F}$ be any subalgebra of $\mathfrak{H}_M$. A degree p form $\omega \in \Lambda^p(M)$ is said to be of *filtration* $\geqq i$ if

$$(\tau_{X_1}\tau_{X_2}\cdots\tau_{X_{p-i+1}})(\omega)=0 \quad (2)$$

whenever $X_1, \cdots, X_{p-i+1} \in \mathfrak{F}$. In other words $\omega(X_1, \cdots, X_p)=0$ whenever $p-i+1$ of the vectors are in $\mathfrak{F}$. We shall denote the subalgebra of all forms of filtration $\geqq i$ by $\Lambda_i(\mathfrak{F})$ whereas $\Lambda_i^p(\mathfrak{F})$ will denote those which are also of degree p.

LEMMA 1. *$d(\Lambda_i(\mathfrak{F})) \subseteq \Lambda_i(\mathfrak{F})$. Also if $X \in \mathfrak{F}$, $L_X(\Lambda_i(\mathfrak{F})) \subseteq \Lambda_i(\mathfrak{F})$.*

PROOF. The second inclusion follows by (1) (e).

After that the first inclusion follows from (1) (d). Q. E. D.

A subalgebra $\mathfrak{F} \subseteq \mathfrak{H}_M$ is called a *foliation-with-singularities* if it is a module over the ring $\Lambda^0(M)$ of all smooth functions. By tensoring any subalgebra $\mathfrak{F} \subseteq \mathfrak{H}_M$ with $\Lambda^0(M)$ we get a foliation $\overline{\mathfrak{F}}$ with singularities which contains $\mathfrak{F}$. For each $x \in M$ we have the evaluation map $e_x: \mathfrak{H}_M \to T_x$. If the subalgebra $\mathfrak{F}$ is a module over $\Lambda^0(M)$ and furthermore if the vector spaces $e_x(\mathfrak{F})=D_x$ are of the same dimension l we say that $\mathfrak{F}$ is a *foliation* of codimension $c=m-l$, (here $m=\dim M$). Now $\bigcup_{x \in M} D_x$ is a subbundle D of the tangent bundle T.

Conversely given any subbundle D of T one could define $\Lambda_i(D)$ exactly as above (by using (2) and $C^\infty(D)$ in place of $\mathfrak{F}$).

LEMMA 2. *$d(\Lambda_i(D)) \subseteq \Lambda_i(D)$ for all i if and only if D is tangent to a foliation.*

PROOF. Take any two vector fields X, $Y \in C^\infty(D)$ and let ω be any 1-form that vanishes on D. Then $\omega([X, Y])=2d\omega(X, Y)=0$ since $d\omega \in \Lambda_1^2(D)$. Hence $[X, Y] \in C^\infty(D)$ Q. E. D.

Henceforth we will concentrate on the case when $\mathfrak{F}$ is a foliation though almost everything holds, with some occasional modifications, if $\mathfrak{F}$ is any subalgebra of $\mathfrak{H}_M$.

$X \in \mathfrak{H}_M$ is called an *infinitesimal transformation* of $\mathfrak{F}$ if for all $Y \in \mathfrak{F}$, $[X, Y] \in \mathfrak{F}$. By Jacobi's identity all such vector fields form a subalgebra $I(\mathfrak{F}) \supseteq \mathfrak{F}$. However by repeating this process one gets nothing new i.e. $I(I(\mathfrak{F}))=I(\mathfrak{F})$.

(Proof. Over an open cell $U \subseteq M$ choose local coordinates so that the vector fields $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^l}$ span D. Choose any $Y \in I(I(\mathfrak{F}))$. We then see that

$$-\left[Y, \frac{\partial}{\partial x^i}\right]=-\left[Y, \left[\frac{\partial}{\partial x^j}, x^j\frac{\partial}{\partial x^i}\right]\right]$$

$$=\left[\frac{\partial}{\partial x^j}, \left[x^j\frac{\partial}{\partial x^i}, Y\right]\right]+\left[x^j\frac{\partial}{\partial x^i}, \left[Y, \frac{\partial}{\partial x^j}\right]\right]$$

$$\in C^\infty(D)=\mathfrak{F}.)$$

We shall say that a foliation $\mathfrak{F}$ is *transitive* if $\overline{I(\mathfrak{F})}$ is the one-leaf foliation of M. This is clearly same as the definition above.

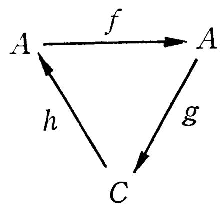
§ 3. Spectral sequences.

Lemma 2 makes it quite clear that given a foliated manifold $(M, \mathfrak{F})$ a natural object of study is the *filtered complex*

$$\Lambda=\Lambda_0(\mathfrak{F}) \longleftarrow \Lambda_1(\mathfrak{F}) \longleftarrow \cdots \longleftarrow \Lambda_c(\mathfrak{F}) \longleftarrow 0\,. \tag{3}$$

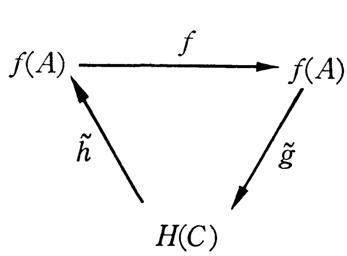
(It will be understood that $\Lambda_n(\mathfrak{F})=\Lambda$ if $n<0$ and $=0$ if $n>c$). To study this we recall some terminology from [3].

An *exact couple* is an exact sequence



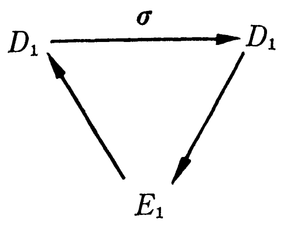
of vector spaces and linear maps. One always equips C with the differential gh (Note that $ghgh=0$).

Then the *derived exact couple* is the induced exact sequence



where the definition of $\tilde{g}$ alone could be mysterious; but we don't need this.

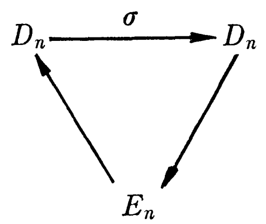
Corresponding to the filtered complex (3) one has the exact couple



(4)

Here $D_1=\sum_i H(\Lambda_i)$, $E_1=\sum_i H\left(\frac{\Lambda_i}{\Lambda_{i+1}}\right)$ and these spaces shall be given the usual grading and bigrading $\left(\text{e.g. } E_1^{p,q}=H^{p+q}\left(\frac{\Lambda_p}{\Lambda_{p+1}}\right),\ D_1^p=H(\Lambda_p) \text{ etc.}\right)$. The maps are made up of the maps of the long exact homology sequences of $0\to\Lambda_{i+1}\to\Lambda_i\to\Lambda_i/\Lambda_{i+1}\to 0$. The important thing for us is to note that σ is induced by the inclusions $\Lambda_{i+1}\to\Lambda_i$.

Now by repeatedly forming the derived of this exact couple we get our *spectral sequence*:



(5)

Note that each space E_n is equipped with a differential d_n and is the homology of the previous. It is also clear that $D_n=\sigma^{n-1}(D_1)$, so it is made up of the images D_n^i of the maps $H(\Lambda_i)\to H(\Lambda_{i-n+1})$ induced by inclusion.

LEMMA 3. *Let* $\mathfrak{F}$ *be a foliation of the closed manifold* M. *Then* $E_k(\mathfrak{F})$ *is finite dimensional if and only if the images of all the induced maps* $H(\Lambda_i)\to H(\Lambda_{i-k+1})$ *are finite dimensional.*

PROOF. First, if D_k^i is finite dimensional for all i the exactness of (5) shows that all the E_k^i are also finite dimensional. Conversely suppose in (5) E_n is finite dimensional and $D_{n+1}=\sigma(D_n)$ is of finite type; then by exactness it follows that D_n is also of finite type. So by downward induction we get required result. (The induction is permissible since for n large enough $E_n=H(\Lambda)$ and $D_n^i\subseteq H(\Lambda)$ or $=0$). Q.E.D.

COROLLARY 1. *Let* $\mathfrak{F}_1$, $\mathfrak{F}_2$ *be two foliations of* M *such that* $\mathfrak{F}_1\subseteq\mathfrak{F}_2$ *and* $c(\mathfrak{F}_1)-c(\mathfrak{F}_2)=\xi$. *Then if* $E_k(\mathfrak{F}_a)$ *is finite dimensional so is* $E_{k+\xi}(\mathfrak{F}_b)$.

PROOF. Only the case $a\neq b$ is non-trivial. For definiteness let us assume $a=1$, $b=2$. We note first that

$$\begin{aligned} &\Lambda_i(\mathfrak{F}_2) \subseteq \Lambda_i(\mathfrak{F}_1) \\ &\Lambda_i(\mathfrak{F}_1) \subseteq \Lambda_{i-\xi}(\mathfrak{F}_2) \end{aligned} \tag{6}$$

Now the map $H(\Lambda_i(\mathfrak{F}_2)) \rightarrow H(\Lambda_{i-k-\xi-1}(\mathfrak{F}_2))$ is seen to factor through the map $H(\Lambda_i(\mathfrak{F}_1)) \rightarrow H(\Lambda_{i-k-1}(\mathfrak{F}_1))$ which is of finite rank. Q. E. D.

COROLLARY 2. *For any foliation* $\mathfrak{F} \subseteq \mathfrak{H}_M$, $E_{c+1}(\mathfrak{F})$ *and* $E_{m-c+2}(\mathfrak{F})$ *are finite dimensional.*

PROOF. {foliation of M by pts} $\subseteq \mathfrak{F} \subseteq$ {foliation of M by one leaf}. Also one can easily check that

$$E_2\{\text{pt. foliation}\} \cong H(\Lambda) \cong E_1\{\text{one-leaf foliation}\}.$$

Now we use Corollary 1. Q. E. D.

§ 4. Compact operators.

Let us denote by Z_i all the closed forms that lie in Λ_i. Then the image of $H(\Lambda_i) \rightarrow H(\Lambda_{i-k+1})$ is isomorphic to the vector space

$$\frac{Z_i}{d(\Lambda_{i-k+1}) \cap Z_i}.$$

To prove the finite dimensionality of these quotients we use the theory of topological vector spaces.

We shall topologise Λ with the usual C^∞ topology. In this way it becomes a Hausdorff locally convex topological vector space. (Infact it is also complete and metrisable: briefly, a *Frechet space*. One can characterize this topology of Λ as the unique Frechet space topology in which all differential operators are continuous). Further it is clear that each subspace Λ_i is also a closed subspace.

If E and F are HLCTVSes a linear map $s: E \rightarrow F$ is called *compact* if it maps some nhbd $U \ni 0$ to a set $s(U)$ with compact closure. (We note that such a map can be a homeomorphism only if E and F are finite dimensional; for now $s(U)$ is a relatively compact neighbourhood of $0 \in F$, and we can use a theorem of Riesz. The same theorem, coupled with the open mapping theorem, shows that if E and F are Frechet spaces, $s(E)$ can be closed only if it is finite dimensional).

We are interested in using the following

LEMMA 4. *Let E be a HLCTVS and let $s: E \rightarrow E$ be compact. Then* $(1-s)(E)$ *is closed and of finite codimension.*

PROOF. See p. 197 of [7]. Q. E. D.

(Note that only the closedness is non trivial. Granting this we conclude that $E/(1-s)(E)$ is also Hausdorff-besides being locally convex. Since s induces the identity map in this quotient we can use the above mentioned theorem of

Riesz.)

COROLLARY 3. *Let* $\mathfrak{F}\subseteqq\mathfrak{H}_M$ *be a foliation on* M. *Also suppose that we have available two linear maps* s, p *in* Λ *such that*

$$\left.\begin{aligned} &\text{a)}\quad s \text{ is compact} \\ &\text{b)}\quad 1-s=dp+pd \\ &\text{c)}\quad s(\Lambda_i)\subseteqq\Lambda_i \quad \text{for all } i \\ &\text{d)}\quad p(\Lambda_i)\subseteqq\Lambda_{i-k+1} \quad \text{for all } i\,. \end{aligned}\right\} \tag{7}$$

Then $E_k(\mathfrak{F})$ *is finite dimensional.*

PROOF. By (b), (c) and (d) it follows that $(1-s)Z_i\subseteqq d(\Lambda_{i-k+1})\cap Z_i$. But by (a) and Lemma 4, $\dfrac{Z_i}{(1-s)Z_i}$ is finite dimensional. So $Z_i/d(\Lambda_{i-k+1})\cap Z_i$ i.e. the image of $H(\Lambda_i)\rightarrow H(\Lambda_{i-k+1})$ is also finite dimensional. Now use Lemma 3.

Q. E. D.

REMARK. If one omits condition (7) (c) we only get $(1-s)Z_i\subseteqq d(\Lambda_{i-k+1})$. So the question arises as to when

$$\frac{Z_i}{(1-s)(Z_i)\cap Z_i} \quad \text{is finite dimensional?}$$

Unfortunately this happens only if (c) holds at least mod finite dimensional spaces. To see this one can use the following result.

LEMMA 5. *Let* F *be a closed subspace of a Frechet space* E *and let* $s\colon E\rightarrow E$ *be a compact map. Then* $\dfrac{F}{(1-s)(F)\cap F}$ *is finite dimensional if and only if* $\dfrac{s(F)}{F\cap s(F)}$ *is finite dimensional.*

(Its proof employs only the functional analysis mentioned above.)

A pair of linear maps $s, p\colon \Lambda\rightarrow\Lambda$ satisfying (7) will be called a *k-parametrix*.

§ 5. Averaging over flows.

Extending the terminology above we shall say that a vector space $V\subseteqq\mathfrak{H}_M$ is transitive if $V(x)=T_x$ for all $x\in M$. If M is compact it is clear that we can always extract a finite dimensional transitive subspace out of V. (For example if a foliation $\mathfrak{F}$ of M is transitive there exists a finite dimensional subspace $V\subseteqq I(\mathfrak{F})$ which is transitive).

We now fix such a finite dimensional and transitive vector space $V\subseteqq\mathfrak{H}_M$, and, furthermore choose a Riemannian metric g on V. Let $|g|$ denote the volume element. Finally let us choose a smooth non-negative real valued function ϕ on V supported in a compact neighbourhood of o. Using these ingredients we define a map $s: \Lambda \to \Lambda$ by

$$(s\omega)(x)=\int_{V}(\nu_1^*\omega)(x)\phi(\nu)\,|g|. \tag{8}$$

(Let $\nu_t: M \to M$ denote the flow of the vector field $\nu \in V$. Then $\nu_1: M \to M$ is the diffeomorphism corresponding to $t=1$. It induces the map $\nu_1^*: \Lambda \to \Lambda$.)

We can assume, without loss of generality, that $\dim V > m$. ($\dim V = m$ is possible only if M is parallelisable). For each $x \in M$ we denote the kernel of $e_x: V \to T_x$ by V_x and its orthogonal complement by V^x. The induced metrics on these two subspaces are called g_x and g^x respectively. If $\nu \in V$ we have a unique decomposition $\nu = u + w$ where $u \in V_x$ and $w \in V^x$. Using Fubini's theorem about repeated integrations we can write above expression as

$$(s\omega)(x)=\int_{V_x}|g_x|\int_{V^x}(u+w)_1^*(\omega)(x)\phi_u(w)\,|g^x|. \tag{9}$$

Here $\phi_u(w)=\phi(u+w)$.

LEMMA 6. *We can choose* supp ϕ *so small that the map* $F_x^u: V^x \to M$ *given by* $F_x^u(w)=(u+w)_1(x)$ *maps* supp ϕ_u *diffeomorphically into* M *for all* x *and* u.

PROOF. Let N denote the bundle over M which is the union of all the spaces V_x, $x \in M$. Now consider the map $M \times V \xrightarrow{F} N \times M$ given by $(x, \nu) \to ((x, u), \nu_1 x)$: here $\nu = u + w$, $u \in V_x$, $w \in V^x$. It is a smooth map of maximum rank on $M \times \{0\}$ (due to transitivity of V); also it maps the compact set $M \times \{0\}$ diffeomorphically into $N \times M$. So one can find a neighbourhood $M \times U$ of this on which F is a diffeomorphism. If we take supp $\phi \subseteqq U$ the Lemma follows. Q. E. D.

Let us denote the 1-dimensional bundle of volumes by $\Omega \to M$. Also let λT^* denote the bundle of covectors. (So $\Lambda(M)=C^\infty(\lambda T^*)$). For each triple $x \in M$, $y \in M$ and $u \in V_x$ we define a linear map $K_u(x, y): \lambda T_y^* \to \lambda T_x^* \otimes \Omega_y$ by

$$K_u(x, y)=0 \quad \text{if} \quad y \notin F_x^u(\text{supp}\,\phi_u)\,.$$

If $y=F_x^u(w)$, $w \in \text{supp}\,\phi_u$ and $\omega_y \in \lambda T_y^*$ then (10)

$$K_u(x, y)\omega_y=(u+w)_1^*(\omega_y)\otimes((F_x^u)^{-1})^*(\phi_u\cdot|g^x|)(y)\,.$$

Substituting this definition in the above formula (9) we get

$$(s\omega)(x)=\int_{V_x}|g_x|\int_M K_u(x, y)\omega(y)\,. \tag{11}$$

If we furthermore define $K(x, y): \lambda T_y^* \to \lambda T_x^* \otimes \Omega_y$ by

$$K(x, y)=\int_{V_x}K_u(x, y)\,|g_x| \tag{12}$$

we see that

$$(s\omega)(x)=\int_M K(x, y)\omega(y)\,. \tag{13}$$

Operators of this type are called *integral operators* (see [1]). They are in particular compact. We thus have

LEMMA 7. *The operator* $s(V, \phi, g): \Lambda(M)\to\Lambda(M)$ *defined by* (8) *is compact if* M *is compact,* V *is transitive, and* $\operatorname{supp}\phi$ *is small enough.*

REMARK. We have explicitly constructed the smooth kernel $K(x, y)$ of s. The operator s is in fact of *trace class,* the trace being defined by $\operatorname{Tr}(s)=\int_M \operatorname{Tr} K(x, x)$: see [1]. This fact is important when one proves index formulas by using the parametrix.

§ 6. The parametrix.

We define a linear map $p: \Lambda\to\Lambda$ by

$$(p\omega)(x)=\int_V\int_0^1(\tau_\nu\nu_t^*\omega)(x)\phi(\nu)dt\,|g|\,. \tag{14}$$

Furthermore we normalise the function

$$\int_V \phi(\nu)\,|g|=1\,. \tag{15}$$

LEMMA 8. *The map* s *is chain homotopic to the identity map,* $1-s=dp+pd$.

PROOF.

$$(dp+pd)(\omega)=\int_V\int_0^1(d\tau_\nu+\tau_\nu d)\nu_t^*\omega\phi(\nu)dt\,|g|$$

since induced maps commute with exterior derivative. Now by 1(d) this equals

$$\int_V\int_0^1 L_\nu\nu_t^*\omega\phi(\nu)dt\,|g|\,.$$

But the Lie derivative L_ν is the negative derivative along the flow. So this equals

$$\int_V(\omega-\nu_1^*\omega)\phi(\nu)\,|g|=\omega-s\omega$$

by using (15). Q. E. D.

THEOREM. *Let* $\mathfrak{F}$ *be a transitive foliation of the closed manifold* M. *Then* $E_2(\mathfrak{F})$ *is finite dimensional.*

PROOF. Choose a finite dimensional transitive $V\subseteq I(\mathfrak{F})$ and construct s and p as above, taking care that $\operatorname{supp}\phi$ is small enough. So s is compact and $1-s$

$=dp+pd$ by Lemmas 7 and 8. Since each $\nu\in V$ is an infinitesimal transformation of $\mathfrak{F}$, $\nu_t: M\to M$ maps leafs onto leafs. So $\nu_t^*(\Lambda_i)\subseteqq\Lambda_i$ for all i. Now (8) and (14) tell us that $s(\Lambda_i)\subseteqq\Lambda_i$, $p(\Lambda_i)\subseteqq\Lambda_{i-1}$ for all i. So (s, p) is a 2-parametrix and, by Corollary 3, $E_2(\mathfrak{F})$ is finite dimensional. Q. E. D.

§ 7. Concluding remarks.

(a) If $\mathfrak{F}$ is a fibration (i. e. the leaves are preimages $\pi^{-1}(b)$, $b\in B$ where $\pi: M\to B$ is of max. rank) then it is transitive. Again many foliations arising out of Lie group actions are transitive.

(b) In general $E_2(\mathfrak{F})$ is infinite dimensional. The cohomology of [8] is $E_2^{*,0}(\mathfrak{F})$ and examples are given there when it is not finite dimensional.

(c) '$\mathfrak{F}$ transitive' is a very restrictive hypothesis e. g. all leafs have to be diffeomorphic. However by using Corollary 1 above one can construct instances when $E_3(\mathfrak{F})$ is finite dimensional (by finding a foliation $\mathfrak{F}'$ of one dimension more which is transitive and contains $\mathfrak{F}$ etc.). Again for some non-transitive foliations the kernel $K(x, y)$ of s——which is now smooth only on an open subset of $M\times M$——satisfies estimates which allow s to be compact.

(d) Let s^p denote the map $\Lambda^p(M)\overset{s}{\to}\Lambda^p(M)$. Then $\sum_p(-1)^p\mathrm{Tr}\ s^p=e(M)$, the Euler characteristic. One can compute the left side from (10) and (12) and thus get a semi-local Gauss-Bonnet theorem. The reader should compare with [1] which is the source of many ideas of this paper.

(e) If $\mathfrak{F}$ admits a bundle——like metric then $E_2^{*,0}(\mathfrak{F})$ are finite dimensional. This follows from [4]; the result is proved there by using Hodge theory.

(f) If a parametrix is sought for some other skew derivation ∂ (say, the $\bar\partial$ of complex manifolds) then instead of Lie derivatives L_ν one has to deal with the derivations $\tau_\nu\partial+\partial\tau_\nu$ and instead of $\nu_t: M\to M$ with the flow on $T(M)$ given by the vector field on $T(M)$ corresponding to this. This remark is important to give canonical constructions of parametrices which are well behaved with respect to some extra structure on M (say, a connection). We will deal with these topics in a subsequent paper.

(g) Take the foliation $\mathfrak{F}$ on a torus T^2 with irrational slope α. Then if α can be well-approximated by rationals $E_1(\mathfrak{F})$ is infinite dimensional; otherwise finite dimensional. So even for an ergodic, minimal flow $E_1(\mathfrak{F})$ can be infinite dimensional. (One must be able to construct time averages of forms in the C^1-topology for $E_1(\mathfrak{F})$ to be finite dimensional). We also remark that if M is orientable and $E_1(\mathfrak{F})$ is finite dimensional then $E_\infty^{p,q}(\mathfrak{F})\cong E_\infty^{c-p,l-q}(\mathfrak{F})$, (and so, if $c(\mathfrak{F})$ is odd, that signature $(M)=0$). This duality theorem, which is the analog of Serre's duality for complex manifolds, holds in other instances too. All this can be found in [6].

(h) The programme of studying the filtration (3) of the deRham complex was commenced in [5], a thesis written under the guidance of Professor Anthony Phillips at Stony Brook.

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