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L^p-spaces and maximal unbounded Hilbert algebras

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§ 0. Introduction.

Inductive and projective limits of the L^p-spaces with respect to a Hilbert algebra are studied. By using their spaces we give necessary and sufficient conditions under which a maximal unbounded Hilbert algebra defined in [11] is pure.

In this paper $\mathcal{D}_0$ denotes a Hilbert algebra, $\mathfrak{h}$ the completion of $\mathcal{D}_0$, $\mathcal{U}_0(\mathcal{D}_0)$ the left von Neumann algebra of $\mathcal{D}_0$, ϕ_0 the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$ and π_0 the left regular representation of $\mathfrak{h}$.

In [11~12], we have studied unbounded Hilbert algebra which is a generalization of the notion of Hilbert algebra to unbounded case. Let $L^p(\phi_0)$ be the L^p-space with respect to ϕ_0 and let $\|T\|_p$ be the L^p-norm of $T \in L^p(\phi_0)$. The space $L_2^\omega(\mathcal{D}_0)$ defined by:

$$L_2^\omega(\mathcal{D}_0) = \bigcap_{2 \leq p < \infty} L_2^p(\mathcal{D}_0) \qquad (\text{where} \quad L_2^p(\mathcal{D}_0) := \{x \in \mathfrak{h}\,;\, \overline{\pi_0(x)} \in L^p(\phi_0)\})$$

is maximal among unbounded Hilbert algebras containing $\mathcal{D}_0$ and it plays an important role for our study of unbounded Hilbert algebras.

In this paper we shall investigate the space $L_2^\omega(\mathcal{D}_0)$ by using the L_2^p-spaces and inductive, projective limits of L_2^p-spaces.

Under the norm $\|x\|_{(2,p)} := \max(\|x\|_2, \|x\|_p)$ (where $\|x\|_p := \|\overline{\pi_0(x)}\|_p$), $L_2^p(\mathcal{D}_0)$ is a Banach space. Furthermore,

$$\mathfrak{h} \supset L_2^p(\mathcal{D}_0) \supset L_2^q(\mathcal{D}_0) \supset L_2^\omega(\mathcal{D}_0) \supset L_2^\infty(\mathcal{D}_0) \quad (2 < p < q < \infty).$$

We define

$$L_2^{p-}(\mathcal{D}_0) = \bigcap_{2 \leqq t < p} L_2^t(\mathcal{D}_0) \quad (2 < p \leqq \infty),$$

$$L_2^{p+}(\mathcal{D}_0) = \bigcup_{t > p} L_2^t(\mathcal{D}_0) \quad (2 \leqq p < \infty)$$

and give $L_2^{p-}(\mathcal{D}_0)$ (resp. $L_2^{p+}(\mathcal{D}_0)$) the projective limit topology τ_2^{p-} (resp. the inductive limit topology τ_2^{p+}) for the Banach spaces $(L_2^t(\mathcal{D}_0)\,;\, \|\ \|_{(2,t)})$. Then it is proved that $(L_2^{p-}(\mathcal{D}_0)\,;\, \tau_2^{p-})$ is a Fréchet space, $L_2^{\infty-}(\mathcal{D}_0) = L_2^\omega(\mathcal{D}_0)$ and $(L_2^{p+}(\mathcal{D}_0)\,;\, \tau_2^{p+})$ is a separated barrelled space.

We shall investigate the dual spaces of the Banach space $L_2^p(\mathscr{D}_0)$ and locally convex spaces $L_2^{p-}(\mathscr{D}_0)$, $L_2^{p+}(\mathscr{D}_0)$. We set

$$_2L_p(\phi_0)=\{T_0\dot{+}T_1\,;\ T_0\in L^2(\phi_0),\ T_1\in L^p(\phi_0)\},$$

$$_2\|T\|_p=\inf\{\|T_0\|_2+\|T_1\|_p\,;\ T=T_0\dot{+}T_1,\ T_0\in L^2(\phi_0),\ T_1\in L^p(\phi_0)\},\qquad T\in {}_2L_p(\phi_0)$$

where $T_0\dot{+}T_1$ denotes the strong sum of closed operators T_0, T_1. Then $_2L_p(\phi_0)$ is a Banach space under the operations of strong sum and strong scalar multiplication and the norm $_2\|\ \ \|_p$. Furthermore,

$$_2L_1(\phi_0)\supset {}_2L_p(\phi_0)\supset {}_2L_q(\phi_0)\supset L^2(\phi_0),\quad 1<p<q<2.$$

We define

$$_2L_p^-(\phi_0)=\bigcap_{1\leqq t<p} {}_2L_t(\phi_0),\quad 1<p\leqq 2,$$

$$_2L_p^+(\phi_0)=\bigcup_{p<t\leqq 2} {}_2L_t(\phi_0),\quad 1\leqq p<2$$

and give $_2L_p^-(\phi_0)$ (resp. $_2L_p^+(\phi_0)$) the projective limit topology $_2\tau_p^-$ (resp. the inductive limit topology $_2\tau_p^+$) for the Banach spaces $({}_2L_t(\phi_0)\,;\,{}_2\|\ \ \|_t)$. It is proved that the Banach spaces $L_2^p(\mathscr{D}_0)$ is dual of the Banach space $_2L_{p'}(\phi_0)$ and the spaces $L_2^{p\pm}(\mathscr{D}_0)$ are dual of the locally convex spaces $_2L_{p'}^{\mp}(\phi_0)$, (where $1<p\leqq\infty$ and $1/p+1/p'=1$, $p=\infty$ if $p'=1$).

By using these spaces we shall give the necessary and sufficient conditions under which the maximal unbounded Hilbert algebra $L_2^\omega(\mathscr{D}_0)$ is pure. That is, the following conditions are equivalent:

(1) $L_2^\omega(\mathscr{D}_0)$ is pure, i.e., $L_2^\omega(\mathscr{D}_0)\neq L_2^\infty(\mathscr{D}_0)$;

(2) $\mathfrak{h}\neq L_2^\infty(\mathscr{D}_0)$, i.e., $\mathfrak{h}$ is not a Hilbert algebra;

(3) $L_2^p(\mathscr{D}_0)\supsetneqq L_2^q(\mathscr{D}_0)$ for each $2\leqq p<q\leqq\infty$;

(4) $L_2^{p-}(\mathscr{D}_0)\supsetneqq L_2^p(\mathscr{D}_0)$ for each $2<p\leqq\infty$;

(5) $L_2^{p+}(\mathscr{D}_0)\subsetneqq L_2^p(\mathscr{D}_0)$ for each $2\leqq p<\infty$;

(6) $_2L_p(\phi_0)\supsetneqq {}_2L_q(\phi_0)$ for each $1\leqq p<q\leqq 2$;

(7) $_2L_p^+(\phi_0)\subsetneqq {}_2L_p(\phi_0)$ for each $1\leqq p<2$;

(8) $_2L_p^-(\phi_0)\supsetneqq {}_2L_p(\phi_0)$ for each $1<p\leqq 2$.

§ 1. Preliminaries.

We give here only the basic definitions and facts needed. Let S and T are linear operators on a Hilbert space $\mathfrak{K}$ with domains $\mathcal{D}(S)$ and $\mathcal{D}(T)$. We say S is an extension of T and we denote $S\supset T$, if $\mathcal{D}(S)\supset\mathcal{D}(T)$ and $S\xi=T\xi$ for all $\xi\in\mathcal{D}(T)$. If S is a closable operator we denote by $\bar{S}$ the smallest closed extension of S. Let $\mathfrak{A}$ be a set of closable operators on $\mathfrak{K}$. Then we set $\bar{\mathfrak{A}}=\{\bar{S}\,;\,S\in\mathfrak{A}\}$. If S is a linear operator with dense domain, then we denote by S^* the hermitian adjoint of S. Let $\mathcal{B}(\mathfrak{K})$ denote the set of all bounded linear operators on $\mathfrak{K}$. Let S, T be closed operators on $\mathfrak{K}$. If $S+T$ is closable, then $\overline{S+T}$ is called the strong sum of S and T, and is denoted by $S \boldsymbol{+} T$. The strong product is likewise defined to be $\overline{ST}$ if it exists, and is denoted by $S\cdot T$. The strong scalar multiplication $\lambda\in\boldsymbol{C}$ (: the field of complex numbers) and S is defined by $\lambda\cdot S=\lambda S$ if $\lambda\neq 0$, and $\lambda\cdot S=0$ if $\lambda=0$.

Let π_0 (resp. π_0') be the left (resp. right) regular representation of $\mathcal{D}_0$. For each $x\in\mathfrak{h}$ we define $\pi_0(x)$ and $\pi_0'(x)$ by:

$$\pi_0(x)\xi=\overline{\pi_0'(\xi)}\,x,\quad \pi_0'(x)\xi=\overline{\pi_0(\xi)}\,x,\quad \xi\in\mathcal{D}_0.$$

Then $\pi_0(x)$ and $\pi_0'(x)$ are linear operators on $\mathfrak{h}$ with the domain $\mathcal{D}_0$ and π_0 (resp. π_0') is called the left (resp. right) regular representation of $\mathfrak{h}$. The involution on $\mathcal{D}_0$ is extended to an involution on $\mathfrak{h}$, which is also denoted by $*$. Then we have $\overline{\pi_0(x^*)}=\pi_0(x)^*$ and $\overline{\pi_0'(x^*)}=\pi_0'(x)^*$. Putting $(\mathcal{D}_0)_b=\{x\in\mathfrak{h}\,;\,\overline{\pi_0(x)}\in\mathcal{B}(\mathfrak{h})\}$, $(\mathcal{D}_0)_b$ is a Hilbert algebra containing $\mathcal{D}_0$. If $\mathcal{D}_0=(\mathcal{D}_0)_{b'}$ then it is called a maximal Hilbert algebra in $\mathfrak{h}$. Let $\mathfrak{M}$ (resp. $\mathfrak{M}^+$) be the set of all measurable (resp. positive measurable) operators on $\mathfrak{h}$ with respect to $\mathcal{U}_0(\mathcal{D}_0)$. For every $T\in\mathfrak{M}^+$ we set

$$\mu_0(T)=\sup\left[\phi_0(\overline{\pi_0(\xi)})\,;\,0\leqq\overline{\pi_0(\xi)}\leqq T,\ \xi\in(\mathcal{D}_0)_b^2\right],$$

$$L^p(\phi_0)=\{T\in\mathfrak{M}\,;\,\|T\|_p:=\mu_0(|T|^p)^{1/p}<\infty\},\quad 1\leqq p<\infty.$$

Then $\|T\|_p$ is called the L^p-norm of $T\in L^p(\phi_0)$ and μ_0 is called the integral on $L^1(\phi_0)$. If $p=\infty$, we shall identify $\mathcal{U}_0(\mathcal{D}_0)$ with $L^\infty(\phi_0)$ and we denote by $\|T\|_\infty$ the operator norm of $T\in\mathcal{U}_0(\mathcal{D}_0)$. We define L_2^ω-spaces with respect to ϕ_0 and $\mathcal{D}_0$ as follows:

$$L_2^\omega(\phi_0)=\bigcap_{2\leqq p<\infty}L^p(\phi_0),\quad L_2^\omega(\mathcal{D}_0)=\{x\in\mathfrak{h}\,;\,\overline{\pi_0(x)}\in L_2^\omega(\phi_0)\},$$

respectively. Then $L_2^\omega(\mathcal{D}_0)$ is maximal among unbounded Hilbert algebras containing $\mathcal{D}_0$. For the definitions and basic properties of unbounded Hilbert algebras the reader is referred to [**11**~**12**].

§ 2. The spaces $L_2^{p-}(\mathscr{D}_0)$ and $L_2^{p+}(\mathscr{D}_0)$.

In this section we define the inductive and projective limits of the L_2^p-spaces and by using their spaces we give necessary and sufficient conditions under which the maximal unbounded Hilbert algebra $L_2^\omega(\mathscr{D}_0)$ is pure.

NOTATION. For $1\leqq p\leqq\infty$ we set

$$L_2^p(\mathscr{D}_0)=\{x\in\mathfrak{h}\,;\,\overline{\pi_0(x)}\in L^p(\phi_0)\},$$

$$\|x\|_{(2,p)}=\max(\|x\|_2,\ \|x\|_p),\quad x\in L_2^p(\mathscr{D}_0).$$

It is immediately showed that $L_2^2(\mathscr{D}_0)=\mathfrak{h}$, $\|x\|_{(2,2)}=\|x\|_2$ and $L_2^\infty(\mathscr{D}_0)=(\mathscr{D}_0)_b$.

LEMMA 2.1. *For* $1\leqq p\leqq\infty$ $\|\ \|_{(2,p)}$ *is a norm on* $L_2^p(\mathscr{D}_0)$, *which makes* $L_2^p(\mathscr{D}_0)$ *a Banach space.*

PROOF. It is easy to show that $\|\ \|_{(2,p)}$ is a norm on $L_2^p(\mathscr{D}_0)$. We shall show that $L_2^p(\mathscr{D}_0)$ is complete. Suppose that $\{x_n\}$ is a Cauchy sequence of $L_2^p(\mathscr{D}_0)$. From the completeness of $\mathfrak{h}$ and $L^p(\phi_0)$ there exist $x\in\mathfrak{h}$ and $T\in L^p(\phi_0)$ such that $\lim_{n\to\infty}\|x_n-x\|_2=0$ and $\lim_{n\to\infty}\|\overline{\pi_0(x_n)}-T\|_p=0$. We have only to show $T=\overline{\pi_0(x)}$. For each $\xi,\ \eta\in\mathscr{D}(T)\cap\mathscr{D}_0$ we have

$$\begin{aligned}\lim_{n\to\infty}|((\overline{\pi_0(x_n)}-T)\xi|\eta)|&=\lim_{n\to\infty}\mu_0(\overline{\pi_0(\eta)}^*\cdot(\overline{\pi_0(x_n)}-T)\cdot\overline{\pi_0(\xi)})\\&=\lim_{n\to\infty}\mu_0(\pi_0(\eta\xi^*)^*\cdot(\overline{\pi_0(x_n)}-T))\\&\leqq\lim_{n\to\infty}\|\overline{\pi_0(x_n)}-T\|_p\,\|\eta\xi^*\|_{p'}=0\quad(1/p+1/p'=1)\end{aligned}$$

and

$$\lim_{n\to\infty}|((\overline{\pi_0(x_n)}-\overline{\pi_0(x)})\xi|\eta)|\leqq\lim_{n\to\infty}\|x_n-x\|_2\,\|\xi\|_2\,\|\eta\|_2=0.$$

It follows that $T\xi=\overline{\pi_0(x)}\xi$ for all $\xi\in\mathscr{D}(T)\cap\mathscr{D}_0$. Since T and $\pi_0(x)$ is essentially measurable, $T+\pi_0(x)$ is essentially measurable ([**16**] Theorem 4). Hence, $\mathscr{D}(T)\cap\mathscr{D}_0$ is dense in $\mathfrak{h}$ and it follows from ([**16**] Lemma 1.2) that $T=\overline{\pi_0(x)}$.

LEMMA 2.2. (1) *For* $1\leqq p<2$ $(\mathscr{D}_0)_b^2$ *is dense in* $(L_2^p(\mathscr{D}_0)\,;\,\|\ \|_{(2,p)})$.

(2) *For* $2\leqq p\leqq\infty$ $(\mathscr{D}_0)_b$ *is dense in* $(L_2^p(\mathscr{D}_0)\,;\,\|\ \|_{(2,p)})$.

PROOF. (2) If $p=\infty$, then this follows from $L_2^\infty(\mathscr{D}_0)=(\mathscr{D}_0)_b$. Suppose $x\in L_2^p(\mathscr{D}_0)$ $(2\leqq p<\infty)$. Let $\overline{\pi_0(x)}=U|\overline{\pi_0(x)}|$ be the polar decomposition of $\overline{\pi_0(x)}$ and let $|\overline{\pi_0(x)}|=\int_0^\infty\lambda dE(\lambda)$ be the spectral resolution of $|\overline{\pi_0(x)}|$. Then, $\overline{\pi_0(E(n)x)}$ $=\int_0^n\lambda dE(\lambda)\in\mathcal{U}_0(\mathscr{D}_0)$. Hence, we have

$$E(n)x\in(\mathscr{D}_0)_b,\quad(n=1,2,\cdots),$$

$$\|x-E(n)x\|_2^2=-\int_n^\infty \lambda^2\, d\phi_0(E(\lambda)^\perp),$$

$$\|x-E(n)x\|_p^p=-\int_n^\infty \lambda^p\, d\phi_0(E(\lambda)^\perp).$$

Since $x\in L_2^p(\mathcal{D}_0)$, i.e., $-\int_0^\infty \lambda^2\, d\phi_0(E(\lambda)^\perp)<\infty$ and $-\int_0^\infty \lambda^p\, d\phi_0(E(\lambda)^\perp)<\infty$, we get that $\lim_{n\to\infty}\|x-E(n)x\|_{(2,p)}=0$. Thus $(\mathcal{D}_0)_b$ is dense in $L_2^p(\mathcal{D}_0)$.

(1) After a slight modification of (2) we can prove the assertion (1).

LEMMA 2.3. (1) *For* $1\leqq p<q\leqq 2$ *we have*

$$L_2^1(\mathcal{D}_0)\subset L_2^p(\mathcal{D}_0)\subset L_2^q(\mathcal{D}_0)\subset \mathfrak{h},$$

$$\|x\|_{(2,q)}^q\leqq \|x\|_{(2,p)}^2+\|x\|_{(2,p)}^p,\quad x\in L_2^p(\mathcal{D}_0).$$

(2) *For* $2\leqq p<q<\infty$ *we have*

$$\mathfrak{h}\supset L_2^p(\mathcal{D}_0)\supset L_2^q(\mathcal{D}_0)\supset L_2^\infty(\mathcal{D}_0),$$

$$\|x\|_{(2,p)}\leqq \|x\|_{(2,q)}^2+\|x\|_{(2,q)}^q,\quad x\in L_2^q(\mathcal{D}_0),$$

$$\|x\|_{(2,p)}\leqq \|x\|_{(2,\infty)},\quad x\in L_2^\infty(\mathcal{D}_0).$$

PROOF. (2) Suppose $x\in L_2^q(\mathcal{D}_0)$ $(2\leqq p<q<\infty)$. Let $\overline{\pi_0(x)}=U|\overline{\pi_0(x)}|$ be the polar decomposition of $\overline{\pi_0(x)}$ and let $|\overline{\pi_0(x)}|=\int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|\overline{\pi_0(x)}|$. Then,

$$\begin{aligned}\|x\|_p^p&=-\int_0^\infty \lambda^p\, d\phi_0(E(\lambda)^\perp)\\ &=-\int_0^1 \lambda^p\, d\phi_0(E(\lambda)^\perp)-\int_1^\infty \lambda^p\, d\phi_0(E(\lambda)^\perp)\\ &\leqq -\int_0^1 \lambda^2\, d\phi_0(E(\lambda)^\perp)-\int_1^\infty \lambda^q\, d\phi_0(E(\lambda)^\perp)\\ &\leqq \|x\|_2^2+\|x\|_q^q<\infty.\end{aligned}$$

Hence, $x\in L_2^p(\mathcal{D}_0)$ and it is also showed that $\|x\|_{(2,p)}^p\leqq \|x\|_{(2,q)}^2+\|x\|_{(2,q)}^q$, $x\in L_2^q(\mathcal{D}_0)$. Suppose that $2<p<\infty$ and $x\in L_2^\infty(\mathcal{D}_0)$. Then,

$$\begin{aligned}\|x\|_p^p&=\mu_0(|\overline{\pi_0(x)}|^p)=\mu_0(|\overline{\pi_0(x)}|^{p-2}|\overline{\pi_0(x)}|^2)\\ &\leqq \|x\|_\infty^{p-2}\|x\|_2^2\leqq \|x\|_{(2,\infty)}^p.\end{aligned}$$

Hence, $\|x\|_{(2,p)}\leqq \|x\|_{(2,\infty)}$.

(1) This follows after a slight modification of (2).

DEFINITION 2.4. We set

$$L_2^{p-}(\mathscr{D}_0)=\bigcap_{2\leqq t<p} L_2^t(\mathscr{D}_0), \quad 2<p\leqq\infty,$$

$$L_2^{p+}(\mathscr{D}_0)=\bigcup_{p<t\leqq\infty} L_2^t(\mathscr{D}_0), \quad 2\leqq p<\infty.$$

Hereafter we shall treat only the spaces $L_2^{p-}(\mathscr{D}_0)$ and $L_2^{p+}(\mathscr{D}_0)$ $(2\leqq p\leqq\infty)$, though we can similarly treat the spaces $L_2^{p-}(\mathscr{D}_0)$ and $L_2^{p+}(\mathscr{D}_0)$ $(1\leqq p\leqq 2)$.

$L_2^{p-}(\mathscr{D}_0)$ and $L_2^{p+}(\mathscr{D}_0)$ are vector subspaces of the Hilbert space $\mathfrak{h}$. We define topologies $\tau_2^{p\pm}$ on $L_2^{p\pm}(\mathscr{D}_0)$ as follows: Take $p\in(2, \infty]$. For $2\leqq t<p$ let

$$v_t\,;\; L_2^{p-}(\mathscr{D}_0)\longrightarrow L_2^t(\mathscr{D}_0)$$

be the identity map. The topology τ_2^{p-} is defined to be the coarsest vector space topology for $L_2^{p-}(\mathscr{D}_0)$ such that all the maps v_t $(2\leqq t<p)$ are continuous when $L_2^t(\mathscr{D}_0)$ is given the norm topology $\|\ \|_{(2,t)}$. This topology is locally convex. $(L_2^{p-}(\mathscr{D}_0)\,;\,\tau_2^{p-})$ is called the projective limit of the Banach spaces $L_2^t(\mathscr{D}_0)$ $(2\leqq t<p)$.

Take $p\in[2, \infty)$. For $p<t<\infty$ let

$$u_t\,;\; L_2^t(\mathscr{D}_0)\longrightarrow L_2^{p+}(\mathscr{D}_0)$$

be the identity map. The topology τ_2^{p+} is defined to be the finest locally convex topology on $L_2^{p+}(\mathscr{D}_0)$ such that all the maps u_t $(p<t<\infty)$ are continuous when $L_2^t(\mathscr{D}_0)$ is given in the norm topology $\|\ \|_{(2,t)}$. This topology exists and is locally convex. $(L_2^{p+}(\mathscr{D}_0)\,;\,\tau_2^{p+})$ is called the inductive limit of the Banach spaces $L_2^t(\mathscr{D}_0)$ $(p<t<\infty)$.

The locally convex space $(L_2^{\infty-}(\mathscr{D}_0)\,;\,\tau_2^{\infty-})$ coincides with the locally convex space $(L_2^{\omega}(\mathscr{D}_0)\,;\,\tau_2^{\omega})$ defined in [**14**].

THEOREM 2.5. (1) *For* $2<p\leqq\infty$ $(L_2^{p-}(\mathscr{D}_0)\,;\,\tau_2^{p-})$ *is a Fréchet space.*

(2) $(L_2^{\infty-}(\mathscr{D}_0)\,;\,\tau_2^{\infty-})$ *is a Fréchet *-algebra* (i.e., *complete metrizable locally convex *-algebra*).

(3) *For* $2\leqq p<\infty$ $(L_2^{p+}(\mathscr{D}_0)\,;\,\tau_2^{p+})$ *is a separated barrelled space.*

(4) $(\mathscr{D}_0)_b$ *is dense in* $(L_2^{p-}(\mathscr{D}_0)\,;\,\tau_2^{p-})$ $(2<p\leqq\infty)$.

(5) $(\mathscr{D}_0)_b$ *is dense in* $(L_2^{p+}(\mathscr{D}_0)\,;\,\tau_2^{p+})$ $(2\leqq p<\infty)$.

PROOF. (1) It is easily showed that $(L_2^{p-}(\mathscr{D}_0)\,;\,\tau_2^{p-})$ is a metrizable locally convex space. We shall prove that $(L_2^{p-}(\mathscr{D}_0)\,;\,\tau_2^{p-})$ is complete. Suppose that $\{x_n\}$ is a Cauchy sequence of $L_2^{p-}(\mathscr{D}_0)$. For each $t\in[2, p)$ $\{x_n\}$ is a Cauchy sequence of $(L_2^t(\mathscr{D}_0)\,;\,\|\ \|_{(2,t)})$ and it follows that there exists an element

$x^{(t)}$ of $L_2^t(\mathcal{D}_0)$ such that $\lim_{n\to\infty}\|x_n-x^{(t)}\|_{(2,t)}=0$. The element $x^{(t)}$ of $L_2^t(\mathcal{D}_0)$ is independent of t. In fact, for each $t'\in[2, p)$ put $t''=\max(t, t')$. Then, from Lemma 2.3, $\lim_{n\to\infty}\|x_n-x^{(t'')}\|_{(2,t)}=\lim_{n\to\infty}\|x_n-x^{(t'')}\|_{(2,t')}=0$. Hence, $x^{(t)}=x^{(t'')}=x^{(t')}$. Putting $x=x^{(t)}$, $x\in\bigcap_{2\leq t<p}L_2^t(\mathcal{D}_0)=L_2^{p-}(\mathcal{D}_0)$ and $\lim_{n\to\infty}\|x_n-x\|_{(2,t)}=0$ for all $t\in[2, p)$. Thus $(L_2^{p-}(\mathcal{D}_0); \tau_2^{p-})$ is complete.

(2) This follows from ([**14**] Theorem 3.2).

(3) It is obvious that $(L_2^{p+}(\mathcal{D}_0); \tau_2^{p+})$ is barrelled. We shall show that $(L_2^{p+}(\mathcal{D}_0); \tau_2^{p+})$ is separated. Let $L_2^p(\mathcal{D}_0)^*$ denote the dual space of the Banach space $(L_2^p(\mathcal{D}_0); \|\ \|_{(2,p)})$. Suppose $F\in L_2^p(\mathcal{D}_0)^*$. Then, for each $t\in(p, \infty]$ the restriction $F/L_2^t(\mathcal{D}_0)$ belongs to $L_2^t(\mathcal{D}_0)^*$. Furthermore, $F/L_2^{p+}(\mathcal{D}_0)\circ u_t=F/L_2^t(\mathcal{D}_0)\in L_2^t(\mathcal{D}_0)^*$ for all $t\in(p, \infty]$. Hence $F/L_2^{p+}(\mathcal{D}_0)$ is a continuous linear functional on $(L_2^{p+}(\mathcal{D}_0); \tau_2^{p+})$. As the set of all such F separates the points of $L_2^p(\mathcal{D}_0)$, it certainly separates the points of $L_2^{p+}(\mathcal{D}_0)$. Thus, $(L_2^{p+}(\mathcal{D}_0); \tau_2^{p+})$ is separated.

(4) We can prove (4) in the same way as in the proof of Lemma 2.2 (2).

(5) Suppose that $x\in L_2^{p+}(\mathcal{D}_0)$ and V is a neighbourhood of zero in $(L_2^{p+}(\mathcal{D}_0); \tau_2^{p+})$. Then, $x\in L_2^t(\mathcal{D}_0)$ for some $t\in(p, \infty]$. Since u_t is continuous, $x+u_t^{-1}(V)$ is a neighbourhood of x in $L_2^t(\mathcal{D}_0)$. From Lemma 2.2. $(\mathcal{D}_0)_b$ is dense in $L_2^t(\mathcal{D}_0)$, and so there exists an element ξ of $(\mathcal{D}_0)_b$ such that $\xi\in x+u_t^{-1}(V)$. That is, $\xi\in x+V$. It follows that $(\mathcal{D}_0)_b$ is dense in $L_2^{p+}(\mathcal{D}_0)$.

We shall give necessary and sufficient conditions under which the maximal unbounded Hilbert algebra $L_2^\omega(\mathcal{D}_0)$ of $\mathcal{D}_0$ is pure. The conditions (1)~(4) of the following theorem have been given in ([**12**] Theorem 3.4.).

THEOREM 2.6. *The following conditions are equivalent.*

(1) $L_2^\omega(\mathcal{D}_0)$ *is a pure unbounded Hilbert algebra,* i.e., $L_2^\omega(\mathcal{D}_0)\neq(\mathcal{D}_0)_b$.

(2) *There exists a sequence* $\{e_n\}$ *of non-zero mutually orthogonal projections in* $(\mathcal{D}_0)_b$ *such that* $\sum_{n=1}^{\infty}\|e_n\|_2^2<\infty$.

(3) $\mathfrak{h}$ *is not a Hilbert algebra,* i.e., $(\mathcal{D}_0)_b\neq\mathfrak{h}$.

(4) $L_2^\omega(\mathcal{D}_0)\neq\mathfrak{h}$.

(5) $L_2^p(\mathcal{D}_0)\supsetneqq L_2^q(\mathcal{D}_0)$ *for each* $q>p\geqq 2$.

(6) $L_2^{p-}(\mathcal{D}_0)\supsetneqq L_2^p(\mathcal{D}_0)$ *for each* $p\in(2, \infty]$.

(7) $L_2^p(\mathcal{D}_0)\supsetneqq L^{p+}(\mathcal{D}_0)$ *for each* $p\in[2, \infty)$.

PROOF. It follows from ([**12**] Theorem 3.4) that the conditions (1)~(4) are equivalent.

(2) ⇒ (5) For each $q>p\geqq 2$ take $r\in(p, q)$. Then, since

$$\lim_{t\to 0}\frac{1/\sqrt[r]{t}}{1/\sqrt[p]{t}}=0 \quad \text{and} \quad \sum_{n=1}^{\infty}\|e_n\|_2^2<\infty,$$

there exists a positive integer k_0 such that

$$a_1:=\sum_{n=1}^{\infty}\|e_{k_0+n}\|_2^2<1, \qquad a_2:=\sum_{n=2}^{\infty}\|e_{k_0+n}\|_2^2, \cdots,$$

$$b_1:=\frac{1}{\sqrt[r]{a_1}}>\frac{1}{\sqrt[q]{a_2}}, \qquad b_2:=\frac{1}{\sqrt[r]{a_2}}>\frac{1}{\sqrt[q]{a_3}}, \cdots.$$

We set

$$x=\sum_{n=1}^{\infty}b_n e_{k_0+n}.$$

Then we have

$$\|x\|_2^2=\sum_{n=1}^{\infty}b_n^2\|e_{k_0+n}\|_2^2=\sum_{n=1}^{\infty}b_n^2(a_n-a_{n+1})$$

$$<\int_0^{a_1}\left(\frac{1}{\sqrt[r]{t}}\right)^2 dt<\infty,$$

$$\|x\|_p^p=\sum_{n=1}^{\infty}b_n^p\|e_{k_0+n}\|_2^2=\sum_{n=1}^{\infty}b_n^p(a_n-a_{n+1})$$

$$<\int_0^{a_1}\left(\frac{1}{\sqrt[r]{t}}\right)^p dt<\infty.$$

Hence, $x\in L_2^p(\mathcal{D}_0)$. On the other hand,

$$\|x\|_q^q=\sum_{n=1}^{\infty}b_n^q\|e_{k_0+n}\|_2^2>\int_0^{a_1}\left(\frac{1}{\sqrt[q]{t}}\right)^q dt=\int_0^{a_1}\frac{1}{t}dt=\infty.$$

Hence, $x\notin L_2^q(\mathcal{D}_0)$. Thus, $L_2^p(\mathcal{D}_0)\supsetneqq L_2^q(\mathcal{D}_0)$.

(5) ⇒ (6) Suppose that $L_2^{p-}(\mathcal{D}_0)=L_2^p(\mathcal{D}_0)$ for some $p\in(2, \infty]$. The identity map ι of the Banach space $(L_2^p(\mathcal{D}_0);\ \|\ \|_{(2,p)})$ onto the Fréchet space $(L_2^{p-}(\mathcal{D}_0);\ \tau_2^{p-})$ is continuous. By the open mapping theorem ι is an isomorphism. Hence there exist an element p_0 of $(2, p)$ and a positive number γ such that

$$\|x\|_{(2,p)}\leqq\gamma\|x\|_{(2,p_0)}, \quad x\in L_2^p(\mathcal{D}_0).$$

From Lemma 2.2 $(\mathcal{D}_0)_b$ is dense in $L_2^{p_0}(\mathcal{D}_0)$, and so $L_2^{p_0}(\mathcal{D}_0)\subset L_2^p(\mathcal{D}_0)$. Hence, $L_2^{p_0}(\mathcal{D}_0)=L_2^p(\mathcal{D}_0)$. This contradicts the assumption (5).

(6) $\Rightarrow$ (3) By the assumption (6),

$$\mathfrak{h} \supset L_2^{p-}(\mathscr{D}_0) \supsetneqq L_2^p(\mathscr{D}_0) \supset (\mathscr{D}_0)_b.$$

(2) $\Rightarrow$ (7) For each $p \in [2, \infty)$ we set $t_n = p + 1/n$ $(n=1, 2, \cdots)$. By the proof of (2) $\Rightarrow$ (5) there exists a non-zero element x_n of $L_2^p(\mathscr{D}_0) - L_2^{t_n}(\mathscr{D}_0)$ such that $\overline{\pi_0(x_n)} \geqq 0$. We set

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n \|x_n\|_{(2,p)}}.$$

Then, $\|x\|_{(2,p)} \leqq \sum_{n=1}^{\infty} 1/2^n < \infty$. For each $t > p$ there is an integer n such that $p < t_n < t$. Then, $x \geqq x_n / 2^n \|x_n\|_{(2,p)}$ and it follows that

$$\|x\|_t \geqq \frac{\|x_n\|_t}{2^n \|x_n\|_{(2,p)}} = \infty.$$

Hence, $x \notin L_2^t(\mathscr{D}_0)$ for each $t > p$. Thus, $L_2^p(\mathscr{D}_0) \supsetneqq L_2^{p+}(\mathscr{D}_0)$.

(7) $\Rightarrow$ (3) By the assumption (7),

$$\mathfrak{h} \supset L_2^p(\mathscr{D}_0) \supsetneqq L_2^{p+}(\mathscr{D}_0) \supset (\mathscr{D}_0)_b.$$

COROLLARY 2.7. *The following conditions are equivalent.*

(1) *The Hilbert space* $\mathfrak{h}$ *is a Hilbert algebra,* i.e., $\mathfrak{h} = (\mathscr{D}_0)_b$.

(2) $L_2^{\omega}(\mathscr{D}_0)$ *is a Hilbert algebra,* i.e., $L_2^{\omega}(\mathscr{D}_0) = (\mathscr{D}_0)_b$.

(3) $\mathfrak{h} = L_2^{\omega}(\mathscr{D}_0) = (\mathscr{D}_0)_b$.

(4) *Either* $E((\mathscr{D}_0)_b)$ (*the set of all non-zero projections in* $(\mathscr{D}_0)_b$) *is a finite set or* $\sum_{n=1}^{\infty} \|e_n\|_2^2 = \infty$ *for each sequence* $\{e_n\}$ *of mutually orthogonal projections in* $(\mathscr{D}_0)_b$.

(5) *There exists* $c > 0$ *such that* $\|e\|_2 \geqq c$ *for all* $e \in E((\mathscr{D}_0)_b)$.

(6) $L_2^p(\mathscr{D}_0) = L_2^q(\mathscr{D}_0)$ *for some* $q > p \geqq 2$.

(7) $L_2^{p-}(\mathscr{D}_0) = L_2^p(\mathscr{D}_0)$ *for some* $p \in (2, \infty]$.

(8) $L_2^{p+}(\mathscr{D}_0) = L_2^p(\mathscr{D}_0)$ *for some* $p \in [2, \infty)$.

PROOF. This follows from ([**12**] Corollary 3.5) and Theorem 2.6.

§ 3. The spaces ${}_2L_p(\phi_0)$, ${}_2L_p^-(\phi_0)$ and ${}_2L_p^+(\phi_0)$.

In this section we shall define the spaces ${}_2L_p(\phi_0)$, ${}_2L_p^-(\phi_0)$ and ${}_2L_p^+(\phi_0)$ and investigate their properties.

NOTATION. For $1\leqq p\leqq\infty$ we set

$$ {}_2L_p(\phi_0)=\{T_0+T_1\,;\ T_0\in L^2(\phi_0),\ T_1\in L^p(\phi_0)\}, $$

$$ {}_2\|T\|_p=\inf\{\|T_0\|_2+\|T_1\|_p\,;\ T=T_0+T_1,\ T_0\in L^2(\phi_0),\ T_1\in L^p(\phi_0)\}, $$

$$ T\in{}_2L_p(\phi_0). $$

It is clear that ${}_2L_p(\phi_0)$ is a vector space under the operations of strong sum and strong scalar multiplication.

THEOREM 3.1. *For* $1\leqq p\leqq\infty$ ${}_2\|\ \|_p$ *is a norm on* ${}_2L_p(\phi_0)$, *which makes* ${}_2L_p(\phi_0)$ *a Banach space.*

PROOF. We shall show that ${}_2\|\ \|_p$ is a norm on ${}_2L_p(\phi_0)$. Suppose that $T\in{}_2L_p(\phi_0)$ and ${}_2\|T\|_p=0$. For each positive integer n there exist a sequence $\{T_0^{(n)}\}$ of $L^2(\phi_0)$ and a sequence $\{T_1^{(n)}\}$ of $L^p(\phi_0)$ such that

$$ T=T_0^{(n)}+T_1^{(n)},\quad \|T_0^{(n)}\|_2+\|T_1^{(n)}\|_p<1/n. $$

Hence, $\lim_{n\to\infty}\|T_0^{(n)}\|_2=0$ and $\lim_{n\to\infty}\|T_1^{(n)}\|_p=0$. For each $\xi,\ \eta\in\mathscr{D}(T)\cap\mathscr{D}_0$ we have

$$ \lim_{n\to\infty}|(T_0^{(n)}\xi|\eta)|\leqq\lim_{n\to\infty}\|T_0^{(n)}\|_2\,\|\xi\|_\infty\|\eta\|_2=0, $$

$$ \lim_{n\to\infty}|(T_1^{(n)}\xi|\eta)|=\lim_{n\to\infty}\mu_0(T_1^{(n)}\cdot\overline{\pi_0(\xi\eta^*)}) $$

$$ \leqq\lim_{n\to\infty}\|T_1^{(n)}\|_p\,\|\xi\eta^*\|_{p'}=0\quad(1/p+1/p'=1). $$

Hence, $(T\xi|\eta)=0$ for each $\xi,\ \eta\in\mathscr{D}(T)\cap\mathscr{D}_0$. In the same way as the proof of Lemma 2.1, $\mathscr{D}(T)\cap\mathscr{D}_0$ is dense in $\mathfrak{h}$ and it follows that $T=0$. Suppose that $S,\ T\in{}_2L_p(\phi_0)$. Let $S=S_0+S_1$; $S_0\in L^2(\phi_0)$, $S_1\in L^p(\phi_0)$ and $T=T_0+T_1$; $T_0\in L^2(\phi_0)$, $T_1\in L^p(\phi_0)$ be each decompositions of S and T, respectively. Then, $S+T=(S_0+T_0)+(S_1+T_1)$ is a decomposition of $S+T$. Hence,

$$ {}_2\|S+T\|_p\leqq\|S_0+T_0\|_2+\|S_1+T_1\|_p $$

$$ \leqq(\|S_0\|_2+\|S_1\|_p)+(\|T_0\|_2+\|T_1\|_p). $$

It follows that ${}_2\|S+T\|_p\leqq{}_2\|S\|_p+{}_2\|T\|_p$. It is easily proved that ${}_2\|\ \|_p$ satisfies the other conditions of norm. Thus $({}_2L_p(\phi_0)\,;\ {}_2\|\ \|_p)$ is a normed space.

We shall show that $({}_2L_p(\phi_0)\,;\ {}_2\|\ \|_p)$ is complete. Suppose that $\{T_n\}$ is a Cauchy sequence of ${}_2L_p(\phi_0)$. Then there exists a subsequence $\{T_{n(k)}\}$ of $\{T_n\}$ such that

$$ {}_2\|T_{n(k+1)}-T_{n(k)}\|_p\leqq1/2^{k+1},\quad k=1,2,\cdots. $$

From the definition of the norm ${}_2\| \ \|_p$, for each k there exists a decomposition of $T_{n(k+1)}-T_{n(k)}$ such that

$$T_{n(k+1)}-T_{n(k)}=(T_{n(k+1)}-T_{n(k)})_0+(T_{n(k+1)}-T_{n(k)})_1,$$
$$(T_{n(k+1)}-T_{n(k)})_0\in L^2(\phi_0),\quad (T_{n(k+1)}-T_{n(k)})_1\in L^p(\phi_0),$$
$$\|(T_{n(k+1)}-T_{n(k)})_0\|_2\leqq 1/2^k,\quad \|(T_{n(k+1)}-T_{n(k)})_1\|_p\leqq 1/2^k.$$

Let $T_{n(1)}=(T_{n(1)})_0+(T_{n(1)})_1$; $(T_{n(1)})_0\in L^2(\phi_0)$, $(T_{n(1)})_1\in L^p(\phi_0)$ be a decomposition of $T_{n(1)}$. We set

$$(T_{n(2)})_0=(T_{n(1)})_0+(T_{n(2)}-T_{n(1)})_0,\quad (T_{n(2)})_1=(T_{n(1)})_1+(T_{n(2)}-T_{n(1)})_1,$$
$$\cdots\cdots$$
$$(T_{n(k)})_0=(T_{n(k-1)})_0+(T_{n(k)}-T_{n(k-1)})_0,\quad (T_{n(k)})_1=(T_{n(k-1)})_1+(T_{n(k)}-T_{n(k-1)})_1,$$
$$\cdots\cdots.$$

Then, for each k

$$T_{n(k)}=(T_{n(k)})_0+(T_{n(k)})_1,\quad (T_{n(k)})_0\in L^2(\phi_0),\quad (T_{n(k)})_1\in L^p(\phi_0)$$

and it is a decomposition of $T_{n(k)}$. Furthermore, for each $k>r$,

$$\|(T_{n(k)})_0-(T_{n(r)})_0\|=\|(T_{n(k)}-T_{n(k-1)})_0+\cdots+(T_{n(r+1)}-T_{n(r)})_0\|_2$$
$$\leqq 1/2^{k-1}+\cdots+1/2^r,$$
$$\|(T_{n(k)})_1-(T_{n(r)})_1\|_p=\|(T_{n(k)}-T_{n(k-1)})_1+\cdots+(T_{n(r+1)}-T_{n(r)})_1\|_p$$
$$\leqq 1/2^{k-1}+\cdots+1/2^r.$$

Hence $\{(T_{n(k)})_0\}$ and $\{(T_{n(k)})_1\}$ are Cauchy sequences of $L^2(\phi_0)$ and $L^p(\phi_0)$ respectively, and so there exist $T_0\in L^2(\phi_0)$ and $T_1\in L^p(\phi_0)$ such that $\lim_{k\to\infty}\|(T_{n(k)})_0-T_0\|_2=0$ and $\lim_{k\to\infty}\|(T_{n(k)})_1-T_1\|_p=0$. We set

$$T=T_0+T_1.$$

Then, $T\in {}_2L_p(\phi_0)$ and $\lim_{k\to\infty} {}_2\|T_{n(k)}-T\|_p=0$. Furthermore, we have

$$\lim_{k\to\infty} {}_2\|T_k-T\|_p\leqq \lim_{k\to\infty}\{{}_2\|T_k-T_{n(k)}\|_p+{}_2\|T_{n(k)}-T\|_p\}=0.$$

Thus, $({}_2L_p(\phi_0);{}_2\| \ \|_p)$ is complete.

It is easy to prove that the Banach space $({}_2L_2(\phi_0);{}_2\| \ \|_2)$ equals the Banach space $(L^2(\phi_0);\| \ \|_2)$.

LEMMA 3.2. (1) *For* $1 \leqq p < q \leqq 2$,

$$ {}_2L_1(\phi_0) \supset {}_2L_p(\phi_0) \supset {}_2L_q(\phi_0) \supset L^2(\phi_0), $$

$$ {}_2\|T\|_p \leqq \max\{{}_2\|T\|_q + ({}_2\|T\|_q)^{q/p},\ {}_2\|T\|_q + ({}_2\|T\|_q)^{q/2}\}, \quad T \in {}_2L_q(\phi_0). $$

(2) *For* $2 \leqq p < q < \infty$,

$$ L^2(\phi_0) \subset {}_2L_p(\phi_0) \subset {}_2L_q(\phi_0) \subset {}_2L_\infty(\phi_0), $$

$$ {}_2\|T\|_q \leqq \max\{{}_2\|T\|_p + ({}_2\|T\|_p)^{q/p},\ {}_2\|T\|_p + ({}_2\|T\|_p)^{q/2}\}, \quad T \in {}_2L_p(\phi_0). $$

PROOF. (1) Suppose $T \in {}_2L_q(\phi_0)$. Let $T = T_0 + T_1$; $T_0 \in L^2(\phi_0)$, $T_1 \in L^q(\phi_0)$ be each decomposition of T. Let $T_1 = U|T_1|$ be the polar decomposition of T_1 and let $|T_1| = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|T_1|$. Then,

$$
\begin{aligned}
\|T_1\|_q^q &= -\int_0^\infty \lambda^q \, d\phi_0(E(\lambda)^\perp) \\
&= -\int_0^1 \lambda^q \, d\phi_0(E(\lambda)^\perp) - \int_1^\infty \lambda^q \, d\phi_0(E(\lambda)^\perp) \\
&\geqq -\int_0^1 \lambda^2 \, d\phi_0(E(\lambda)^\perp) - \int_1^\infty \lambda^p \, d\phi_0(E(\lambda)^\perp) \\
&= \|UE(1)|T_1|\,\|_2^2 + \|UE(1)^\perp|T_1|\,\|_p^p.
\end{aligned}
$$

Hence, $UE(1)|T_1| \in L^2(\phi_0)$, $UE(1)^\perp|T_1| \in L^p(\phi_0)$ and $T_1 = UE(1)|T_1| + UE(1)^\perp|T_1|$. It follows that $T = (T_0 + UE(1)|T_1|) + (UE(1)^\perp|T_1|) \in {}_2L_p(\phi_0)$. Furthermore, we have

$$
\begin{aligned}
{}_2\|T\|_p &\leqq \|T_0 + UE(1)|T_1|\,\|_2 + \|UE(1)^\perp|T_1|\,\|_p \\
&\leqq \|T_0\|_2 + \|E(1)|T_1|\,\|_2 + \|E(1)^\perp|T_1|\,\|_p \\
&= \|T_0\|_2 - \Big[\int_0^1 \lambda^2 \, d\phi_0(E(\lambda)^\perp)\Big]^{1/2} - \Big[\int_1^\infty \lambda^p \, d\phi_0(E(\lambda)^\perp)\Big]^{1/p} \\
&\leqq \|T_0\|_2 - \Big[\int_0^1 \lambda^q \, d\phi_0(E(\lambda)^\perp)\Big]^{1/2} - \Big[\int_1^\infty \lambda^q \, d\phi_0(E(\lambda)^\perp)\Big]^{1/p} \\
&\leqq \|T_0\|_2 + \|T_1\|_q^{q/2} + \|T_1\|_q^{q/p}.
\end{aligned}
$$

If $\|T_1\|_q \geqq 1$, then ${}_2\|T\|_p \leqq (\|T_0\|_2 + \|T_1\|_q) + (\|T_0\|_2 + \|T_1\|_q)^{q/p}$ and if $\|T_1\|_q \leqq 1$ then ${}_2\|T\|_p \leqq (\|T_0\|_2 + \|T_1\|_q) + (\|T_0\|_2 + \|T_1\|_q)^{q/2}$. Hence, we have

$$ {}_2\|T\|_p \leqq \max\{{}_2\|T\|_q + ({}_2\|T\|_q)^{q/p},\ {}_2\|T\|_q + (\|T\|_q)^{q/2}\}. $$

(2) This follows after a slight modification of (1).

DEFINITION 3.3. We set

$$ {}_2L_p^-(\phi_0)=\bigcap_{1\leqq t<p} {}_2L_t(\phi_0),\quad 1<p\leqq 2, $$

$$ {}_2L_p^+(\phi_0)=\bigcup_{p<t\leqq 2} {}_2L_t(\phi_0),\quad 1\leqq p<2. $$

Hereafter we shall treat only the spaces ${}_2L_p^-(\phi_0)$ and ${}_2L_p^+(\phi_0)$ $(1\leqq p\leqq 2)$, though we can similarly treat the spaces ${}_2L_p^-(\phi_0)$ and ${}_2L_p^+(\phi_0)$ $(p>2)$.

Under the operations of strong sum and strong scalar multiplication ${}_2L_p^-(\phi_0)$ and ${}_2L_p^+(\phi_0)$ are vector spaces. We define the projective limit topology ${}_2\tau_p^-$ on ${}_2L_p^-(\phi_0)$ and the inductive limit topology ${}_2\tau_p^+$ on ${}_2L_p^+(\phi_0)$ for the Banach spaces $({}_2L_t(\phi_0);\ {}_2\|\ \|_t)$. Then the following theorem is proved after a slight modification of the proof of Theorem 2.5.

THEOREM 3.4. (1) *For* $1<p\leqq 2$ $({}_2L_p^-(\phi_0);\ {}_2\tau_p^-)$ *is a Fréchet space.*

(2) *For* $1\leqq p<2$ $({}_2L_p^+(\phi_0);\ {}_2\tau_p^+)$ *is a separated barrelled space.*

(3) $(\mathcal{D}_0)_b$ *is dense in* $({}_2L_p(\phi_0);\ {}_2\|\ \|_p)$ $(1\leqq p\leqq 2)$.

(4) $(\mathcal{D}_0)_b$ *is dense in* $({}_2L_p^-(\phi_0);\ {}_2\tau_p^-)$ $(1<p\leqq 2)$.

(5) $(\mathcal{D}_0)_b$ *is dense in* $({}_2L_p^+(\phi_0);\ {}_2\tau_p^+)$ $(1\leqq p<2)$.

§ 4. Duality and some consequences.

In this section we shall investigate the dual spaces of the Banach spaces $(L_2^p(\mathcal{D}_0);\ \|\ \|_{(2,p)})$, $({}_2L_p(\phi_0);\ {}_2\|\ \|_p)$ and the locally convex spaces $(L_2^{p\pm}(\mathcal{D}_0);\ \tau_2^{p\pm})$, $({}_2L_p^\pm(\phi_0);\ {}_2\tau_p^\pm)$.

Let $L_2^p(\mathcal{D}_0)^*$ (resp. ${}_2L_p(\phi_0)^*$) denote the dual space of the Banach space $(L_2^p(\mathcal{D}_0);\ \|\ \|_{(2,p)})$ (resp. $({}_2L_p(\phi_0);\ {}_2\|\ \|_p)$). Then $L_2^p(\mathcal{D}_0)^*$ and ${}_2L_p(\phi_0)^*$ are Banach spaces under the norms:

$$ \|f\|_{(2,p)}=\sup[|f(x)|;\ x\in L_2^p(\mathcal{D}_0),\ \|x\|_{(2,p)}\leqq 1],\quad f\in L_2^p(\mathcal{D}_0)^*, $$

$$ {}_2\|f\|_p=\sup[|f(T)|;\ T\in {}_2L_p(\phi_0),\ {}_2\|T\|_p\leqq 1],\quad f\in {}_2L_p(\phi_0)^*, $$

respectively. For each $t\in[1,\infty]$ we set $t'=t/t-1$, i.e., $1/t+1/t'=1$ (where $t'=\infty$ if $t=1$ and $t'=1$ if $t=\infty$).

THEOREM 4.1. (1) ${}_2L_p(\phi_0)^*=L_2^{p'}(\mathcal{D}_0)$ $(1\leqq p<\infty)$. *That is, for each* $x\in L_2^{p'}(\mathcal{D}_0)$ *putting*

$$ [\Phi(x)](T)=\mu_0(\overline{\pi_0(x)}\cdot T),\quad T\in {}_2L_p(\phi_0), $$

Φ *is an isometric isomorphism of the Banach space* $(L_2^{p'}(\mathcal{D}_0);\ \|\ \|_{(2,p')})$ *onto the Banach space* $({}_2L_p(\phi_0)^*;\ {}_2\|\ \|_p)$.

(2) *Let* $1<p\leqq\infty$. *For each* $T\in {}_2L_{p'}(\phi_0)$ *putting*

$$[\Psi(T)](x)=\mu_0(\overline{\pi_0(x)}\cdot T),\quad x\in L_2^p(\mathcal{D}_0),$$

Ψ *is an isometric isomorphism of the Banach space* ${}_2L_{p'}(\phi_0)$ *into the Banach space* $L_2^p(\mathcal{D}_0)^*$.

PROOF. (1) For each $x\in L_2^{p'}(\mathcal{D}_0)$ we have $\Phi(x)\in {}_2L_p(\phi_0)^*$. In fact, it is easily showed that $\Phi(x)$ is a well-defined linear functional on ${}_2L_p(\phi_0)$. Furthermore, for each decomposition $T=T_0+T_1$; $T_0\in L^2(\phi_0)$, $T_1\in L^p(\phi_0)$ we get

$$\begin{aligned}|[\Phi(x)](T)|&=|\mu_0(\overline{\pi_0(x)}\cdot(T_0+T_1))|\\&\leqq|\mu_0(\overline{\pi_0(x)}\cdot T_0)|+|\mu_0(\overline{\pi_0(x)}\cdot T_1)|\\&\leqq\|x\|_2\,\|T_0\|_2+\|x\|_{p'}\,\|T_1\|_p\\&\leqq\|x\|_{(2,p')}(\|T_0\|_2+\|T_1\|_p).\end{aligned}$$

Hence, $\Phi(x)\in {}_2L_p(\phi_0)^*$ and ${}_2\|\Phi(x)\|_p\leqq\|x\|_{(2,p')}$. It follows that Φ is a map of $L_2^{p'}(\mathcal{D}_0)$ into ${}_2L_p(\phi_0)^*$. We shall show that ${}_2\|\Phi(x)\|_p=\|x\|_{(2,p')}$. Suppose that $\|x\|_2\geqq\|x\|_{p'}$. Since $\|x\|_2=\sup[|\mu_0(\overline{\pi_0(x)}\cdot T_0)|\,;\,T_0\in L^2(\phi_0),\ \|T_0\|_2\leqq 1]$, for each $\varepsilon>0$ there exists an element T_0 of $L^2(\phi_0)$ ($\subset {}_2L_p(\phi_0)$) such that

$${}_2\|T_0\|_p\leqq\|T_0\|_2\leqq 1\quad\text{and}\quad|\mu_0(\overline{\pi_0(x)}\cdot T_0)|+\varepsilon\geqq\|x\|_2=\|x\|_{(2,p')}.$$

On the other hand, suppose that $\|x\|_2\leqq\|x\|_{p'}$. Since $\|x\|_{p'}=\sup[|\mu_0(\overline{\pi_0(x)}\cdot T_1)|\,;\,T_1\in L^p(\phi_0),\ \|T_1\|_p\leqq 1]$, there exists an element T_1 of $L^p(\phi_0)$ ($\subset {}_2L_p(\phi_0)$) such that

$${}_2\|T_1\|_p\leqq\|T_1\|_p\leqq 1\quad\text{and}\quad|\mu_0(\overline{\pi_0(x)}\cdot T_1)|+\varepsilon\geqq\|x\|_{p'}=\|x\|_{(2,p')}.$$

Thus, for each $\varepsilon>0$ there exists an element T of ${}_2L_p(\phi_0)$ such that

$${}_2\|T\|_p\leqq 1\quad\text{and}\quad|\mu_0(\overline{\pi_0(x)}\cdot T)|+\varepsilon\geqq\|x\|_{(2,p')}.$$

Hence, $\|x\|_{(2,p')}={}_2\|\Phi(x)\|_p$. Next we shall show that Φ is onto. Suppose that $f\in {}_2L_p(\phi_0)^*$, that is, there exists a positive constant γ such that

$$|f(T)|\leqq\gamma({}_2\|T\|_p)$$

for all $T\in {}_2L_p(\phi_0)$. In particular,

$$|f(T_0)|\leqq\gamma({}_2\|T_0\|_p)\leqq\gamma\|T_0\|_2,\quad T_0\in L^2(\phi_0),$$

$$|f(T_1)|\leqq\gamma({}_2\|T_1\|_p)\leqq\gamma\|T_1\|_p,\quad T_1\in L^p(\phi_0).$$

Hence, $f/L^2(\phi_0)$ (the restriction of f to $L^2(\phi_0)$) $\in L^2(\phi_0)^*$ and $f/L^p(\phi_0)\in L^p(\phi_0)^*$.

Since $L^2(\phi_0)^*=L^2(\phi_0)$ and $L^p(\phi_0)^*=L^{p'}(\phi_0)$, there exist $a\in\mathfrak{h}$ and $B\in L^{p'}(\phi_0)$ such that

$$f(T_0)=\mu_0(\overline{\pi_0(a)}\cdot T_0),\quad T_0\in L^2(\phi_0),$$

$$f(T_1)=\mu_0(B\cdot T_1),\qquad T_1\in L^p(\phi_0).$$

Then we have $\overline{\pi_0(a)}=B$. In fact, for each $x,\ y\in\mathcal{D}_0\cap\mathcal{D}(B)$

$$f(\overline{\pi_0(x)}\cdot\overline{\pi_0(y)}^*)=\mu_0(\overline{\pi_0(a)}\cdot\overline{\pi_0(x)}\cdot\overline{\pi_0(y)}^*)$$

$$=(\overline{\pi_0(a)}x|y),$$

$$f(\overline{\pi_0(x)}\cdot\overline{\pi_0(y)}^*)=\mu_0(B\cdot\overline{\pi_0(x)}\cdot\overline{\pi_0(y)}^*)$$

$$=(Bx|y).$$

Hence, $\overline{\pi_0(a)}x=Bx$ for all $x\in\mathcal{D}_0\cap\mathcal{D}(B)$. Since $\pi_0(a)+B$ is essentially measurable, $\mathcal{D}_0\cap\mathcal{D}(B)$ is dense in $\mathfrak{h}$, and so $B=\overline{\pi_0(a)}$. Hence, $a\in L_2^{p'}(\mathcal{D}_0)$ and

$$f(T)=\mu_0(\overline{\pi_0(a)}\cdot T)=[\Phi(a)](T),\quad T\in{}_2L_p(\phi_0).$$

Hence, Φ is onto. It is clear that Φ is a linear map of $L_2^{p'}(\mathcal{D}_0)$ onto ${}_2L_p(\phi_0)^*$. Thus Φ is an isometric isomorphism of $L_2^{p'}(\mathcal{D}_0)$ onto ${}_2L_p(\phi_0)^*$.

(2) In the same way as (1) we can prove that Ψ is a continuous linear map of ${}_2L_{p'}(\phi_0)$ into $L_2^p(\phi_0)^*$. By (1), ${}_2L_{p'}(\phi_0)^*=L_2^p(\mathcal{D}_0)$ and it follows that

$${}_2\|T\|_{p'}=\sup_{x\in L_2^p(\mathcal{D}_0);\|x\|_{(2,p)}\leqq 1}|[\Psi(T)](x)|,\quad T\in{}_2L_{p'}(\phi_0).$$

From the completeness of ${}_2L_{p'}(\phi_0)$, $\Psi({}_2L_{p'}(\phi_0))$ is a closed subspace of $L_2^p(\mathcal{D}_0)^*$. Hence, Ψ is an isometric isomorphism of ${}_2L_{p'}(\phi_0)$ into $L_2^p(\mathcal{D}_0)^*$.

QUESTION. We don't know whether the isomorphism Ψ is into (that is, the Banach space ${}_2L_{p'}(\phi_0)$ is dual of the Banach space $L_2^p(\mathcal{D}_0)$), or not.

QUESTION. ${}_2L_\infty(\phi_0)^*=L_2^1(\mathcal{D}_0)$?

In order to solve the above problem, we shall introduce a topology on ${}_2L_\infty(\phi_0)$ as follows: for each $x,\ y\in\mathfrak{h}$ and $T\in{}_2L_\infty(\phi_0)$ we set

$${}_2\|T\|_{(x,y)}=\inf\{\|T_0\|_2+|(T_1x|y)|\,;\ T=T_0+T_1,\ T_0\in L^2(\phi_0),\ T_1\in L^\infty(\phi_0)\}.$$

Then it is easily proved that ${}_2\|\ \ \|_{(x,y)}$ is a seminorm on ${}_2L_\infty(\phi_0)$. The topology induced by the family $\{{}_2\|\ \ \|_{(x,y)}\,;\ x,\ y\in\mathfrak{h}\}$ of the seminorms is called the

${}_2L_\infty$-weak topology on ${}_2L_\infty(\phi_0)$ and is denoted by ${}_2\tau_\infty(\omega)$. It is easily showed that the topology ${}_2\tau_\infty(\omega)$ is coarser than the topology ${}_2\| \ \|_\infty$ (denoted by ${}_2\tau_\infty(\omega) \succ {}_2\| \ \|_\infty$) and $({}_2L_\infty(\phi_0); {}_2\tau_\infty(\omega))$ is a separated locally convex space. Let $({}_2L_\infty(\phi_0); {}_2\tau_\infty(\omega))^*$ denote the dual space of a locally convex space $({}_2L_\infty(\phi_0); {}_2\tau_\infty(\omega))$. Since ${}_2\tau_\infty(\omega) \succ {}_2\| \ \|_\infty$, we have $({}_2L_\infty(\phi_0); {}_2\tau_\infty(\omega))^* \subset {}_2L_\infty(\phi_0)^*$. When we regard $({}_2L_\infty(\phi_0); {}_2\tau_\infty(\omega))^*$ as a normed subspace of the Banach space ${}_2L_\infty(\phi_0)^*$, we denote it by ${}_2L_\infty(\phi_0)_*$.

THEOREM 4.3. ${}_2L_\infty(\phi_0)_* = L^1_2(\mathcal{D}_0)$. *That is, for each* $x \in L^1_2(\mathcal{D}_0)$ *putting*

$$[\Phi(x)](T) = \mu_0(\overline{\pi_0(x)} \cdot T), \quad T \in {}_2L_\infty(\phi_0),$$

Φ *is an isometric isomorphism of the Banach space* $(L^1_2(\mathcal{D}_0); \| \ \|_{(2,1)})$ *onto the Banach space* $({}_2L_\infty(\phi_0)_*; {}_2\| \ \|_\infty)$.

PROOF. In the same way as in Theorem 4.1 we can prove that Φ is an isometric isomorphism of the Banach space $L^1_2(\mathcal{D}_0)$ into the Banach space ${}_2L_\infty(\phi_0)^*$. We shall show that $\Phi(L^1_2(\mathcal{D}_0)) = {}_2L_\infty(\phi_0)_*$. Suppose $x \in L^1_2(\mathcal{D}_0)$. Then, $\overline{\pi_0(x)} = \overline{\pi_0(x_1)} \cdot \overline{\pi_0(x_2)}^*$ for some $x_1, x_2 \in \mathfrak{h}$. Let $T = T_0 + T_1$; $T_0 \in L^2(\phi_0)$, $T_1 \in L^\infty(\phi_0)$ be each decomposition of $T \in {}_2L_\infty(\phi_0)$. Then,

$$\begin{aligned}
|[\Phi(x)](T)| &= |\mu_0(\overline{\pi_0(x)} \cdot (T_0 + T_1))| \\
&\leqq |\mu_0(\overline{\pi_0(x)} \cdot T_0)| + |\mu_0(\overline{\pi_0(x_1)} \cdot \overline{\pi_0(x_2)}^* \cdot T_1)| \\
&\leqq \|x\|_2 \|T_0\|_2 + |(T_1 x_1 | x_2)| \\
&\leqq (\|x\|_2 + 1)(\|T_0\|_2 + |(T_1 x_1 | x_2)|).
\end{aligned}$$

Hence, for all $T \in {}_2L_\infty(\phi_0)$

$$|[\Phi(x)](T)| \leqq (\|x\|_2 + 1)_2 \|T\|_{(x_1, x_2)}.$$

Therefore, $\Phi(x) \in {}_2L_\infty(\phi_0)_*$. Conversely suppose that $f \in {}_2L_\infty(\phi_0)_*$. Then, $f/L^2(\phi_0) \in L^2(\phi_0)^*$ and $f/L^\infty(\phi_0) \in L^\infty(\phi_0)_*$ (, where $L^\infty(\phi_0)_*$ denote the predual of the von Neumann algebra $L^\infty(\phi_0)$) are easily showed. Since $L^2(\phi_0)^* = L^2(\phi_0)$ and $L^\infty(\phi_0)_* = L^1(\phi_0)$, there exist $a \in L^2(\phi_0)$ and $B \in L^1(\phi_0)$ such that

$$f(T_0) = \mu_0(\overline{\pi_0(a)} \cdot T_0), \quad T_0 \in L^2(\phi_0),$$

$$f(T_1) = \mu_0(B \cdot T_1), \qquad T_1 \in L^\infty(\phi_0).$$

In the same way as in the proof of Theorem 4.1, we can prove $\overline{\pi_0(a)} = B$. Hence, $a \in L^1_2(\mathcal{D}_0)$ and for all $T \in {}_2L_\infty(\phi_0)$

$$f(T) = \mu_0(\overline{\pi_0(a)} \cdot T) = [\Phi(a)](T).$$

Hence Φ is a map of $L^1_2(\mathcal{D}_0)$ onto ${}_2L_\infty(\phi_0)_*$. Thus Φ is an isometric isomorphism of the Banach space $L^1_2(\mathcal{D}_0)$ onto the Banach space ${}_2L_\infty(\phi_0)_*$.

Let X be a locally convex space with a topology τ and let X^* be the dual space of $(X;\tau)$. We denote by $\beta(X^*, X)$ (resp. $\tau(X^*, X)$) the strong topology (resp. Mackey topology) on X^*.

THEOREM 4.4. *Let* $1\leqq p\leqq 2$.

(1) *The dual space* ${}_2L_p^+(\phi_0)^*$ *of the locally convex space* $({}_2L_p^+(\phi_0);{}_2\tau_p^+)$ *consists of the maps*

$$\Phi(x);\ T \longrightarrow \mu_0(\overline{\pi_0(x)}\cdot T)$$

where $x\in L_2^{p'-}(\mathcal{D}_0)$ *and*

$${}_2\tau_p^+=\beta({}_2L_p^+(\phi_0),\ L_2^{p'-}(\mathcal{D}_0))=\tau({}_2L_p^+(\phi_0),\ L_2^{p'-}(\mathcal{D}_0)).$$

(2) *The dual space* $({}_2L_p^-(\phi_0))^*$ *of the locally convex space* $({}_2L_p^-(\phi_0);{}_2\tau_p^-)$ *consists of the maps*

$$\Psi(x);\ T \longrightarrow \mu_0(\overline{\pi_0(x)}\cdot T)$$

where $x\in L_2^{p'+}(\mathcal{D}_0)$ *and*

$${}_2\tau_p^-=\beta({}_2L_p^-(\phi_0),\ L_2^{p'+}(\mathcal{D}_0))=\tau({}_2L_p^-(\phi_0),\ L_2^{p'+}(\mathcal{D}_0)).$$

PROOF. It is not difficult to show that $\Phi(x)$ ($x\in L_2^{p'-}(\mathcal{D}_0)$) is a well-defined linear functional on ${}_2L_p^+(\phi_0)$. Let $T\in{}_2L_t(\phi_0)$ ($p<t\leqq 2$). Let $T=T_0+T_1$; $T_0\in L^2(\phi_0)$, $T_1\in L^t(\phi_0)$ be each decomposition of T. Then,

$$\begin{aligned}|[\Phi(x)](T)|&=|\mu_0(\overline{\pi_0(x)}\cdot(T_0+T_1))|\\&\leqq|\mu_0(\overline{\pi_0(x)}\cdot T_0)|+|\mu_0(\overline{\pi_0(x)}\cdot T_1)|\\&\leqq\|x\|_2\|T_0\|_2+\|x\|_{t'}\|T_1\|_t\\&\leqq\|x\|_{(2,t')}(\|T_0\|_2+\|T_1\|_t).\end{aligned}$$

Hence, $|[\Phi(x)](T)|\leqq\|x\|_{(2,t')_2}\|T\|_t$ for all $T\in{}_2L_t(\phi_0)$. It follows that $[\Phi(x)]/{}_2L_t(\phi_0)\in{}_2L_t(\phi_0)^*$ for all $t\in(p, 2]$. Hence, $\Phi(x)\in({}_2L_p^+(\phi_0);{}_2\tau_p^+)^*$. Next we shall show that the map Φ is onto. Suppose $f\in({}_2L_p^+(\phi_0);{}_2\tau_p^+)^*$. Then, $f\circ u_t\in{}_2L_t(\phi_0)^*$ for all $t\in(p, 2]$. From Theorem 4.1, for each $t\in(p, 2]$ there exists an element $x^{(t)}$ of $L_2^{t'}(\mathcal{D}_0)$ such that

$$(f\circ u_t)(T)=\mu_0(\overline{\pi_0(x^{(t)})}\cdot T),\quad T\in{}_2L_t(\phi_0).$$

The element $x^{(t)}$ is independent of t. In fact, for each $t\in(p, 2]$ and $T\in L^2(\phi_0)$ we have

$$\begin{aligned}f(T)&=(f\circ u_2)(T)=\mu_0(\overline{\pi_0(x^{(2)})}\cdot T)\\&=(f\circ u_t)(T)=\mu_0(\overline{\pi_0(x^{(t)})}\cdot T).\end{aligned}$$

Hence, $\mu_0(\overline{\pi_0(x^{(2)}-x^{(t)})}\cdot T)=0$ for all $T\in L^2(\phi_0)$ and it follows that $x^{(t)}=x^{(2)}$. Putting $x=x^{(t)}$, $x\in\bigcap_{2\leq t'<p'} L_2^{t'}(\mathcal{D}_0)=L_2^{p'-}(\mathcal{D}_0)$ and $f(T)=\mu_0(\overline{\pi_0(x)}\cdot T)=[\Phi(x)](T)$ for all $T\in{}_2L_p^+(\phi_0)$. Thus Φ is onto. Similarly we can prove that ${}_2L_p^-(\phi_0)^*=\Psi(L_2^{p'+}(\phi_0))$. From Theorem 2.5, 3.4, $L_2^{p'\pm}(\mathcal{D}_0)$ and ${}_2L_p^{\pm}(\phi_0)$ are separated barrelled spaces. From ([**19**] Corollary 4.1.1) we have

$$ {}_2\tau_p^+=\beta({}_2L_p^+(\phi_0),\ L_2^{p'-}(\mathcal{D}_0))=\tau({}_2L_p^+(\phi_0),\ L_2^{p'-}(\mathcal{D}_0)), $$

$$ {}_2\tau_p^-=\beta({}_2L_p^-(\phi_0),\ L_2^{p'+}(\mathcal{D}_0))=\tau({}_2L_p^-(\phi_0),\ L_2^{p'+}(\mathcal{D}_0)). $$

Corollary 4.5. (1) ${}_2L_1^+(\phi_0)^*=L_2^{\omega}(\mathcal{D}_0)$.

(2) ${}_2\tau_1^+=\beta({}_2L_1^+(\phi_0),\ L_2^{\omega}(\mathcal{D}_0))=\tau({}_2L_1^+(\phi_0),\ L_2^{\omega}(\mathcal{D}_0))$.

Theorem 4.7. *The following conditions are equivalent.*

(1) $L_2^{\omega}(\mathcal{D}_0)$ *is a pure unbounded Hilbert algebra.*

(2) ${}_2L_p(\phi_0)\neq{}_2L_q(\phi_0)$ *for each* $1\leqq p<q\leqq 2$.

(3) ${}_2L_p^+(\phi_0)\neq{}_2L_p(\phi_0)$ *for each* $1\leqq p<2$.

(4) ${}_2L_p^-(\phi_0)\neq{}_2L_p(\phi_0)$ *for each* $1<p\leqq 2$.

Proof. This follows from Theorem 2.2 and Theorem 4.4.

§ 5. The L^p-spaces with respect to a Hilbert algebra with an identity.

Suppose that a Hilbert algebra $\mathcal{D}_0$ has an identity e and $\|e\|_2=1$. Then, for $1<p<q<\infty$ we have

$$ L^1(\phi_0)\supset L^p(\phi_0)\supset L^q(\phi_0)\supset L^{\infty}(\phi_0), $$

$$ \|T\|_q\geqq\|T\|_p,\quad T\in L^q(\phi_0). $$

We define

$$ L^{p-}(\phi_0)=\bigcap_{1\leq t<p} L^t(\phi_0),\quad 1<p\leqq\infty, $$

$$ L^{p+}(\phi_0)=\bigcup_{t>p} L^t(\phi_0),\quad 1\leqq p<\infty $$

and give $L^{p-}(\phi_0)$ (resp. $L^{p+}(\phi_0)$) the projective limit topology τ^{p-} (resp. inductive limit topology τ^{p+}) for the Banach spaces $(L^t(\phi_0);\ \|\ \|_t)$. Then we have

$$ (L^p(\phi_0);\ \|\ \|_p)=(L_2^p(\phi_0);\ \|\ \|_{(2,p)}),\quad 2\leqq p\leqq\infty, $$

$$ (L^{p-}(\phi_0);\ \tau^{p-})=(L_2^{p-}(\phi_0);\ \tau_2^{p-}),\quad 2<p\leqq\infty, $$

$$ (L^p(\phi_0);\ \|\ \|_p)=({}_2L_p(\phi_0);\ {}_2\|\ \|_p),\quad 1\leqq p\leqq 2, $$

$$ (L^{p+}(\phi_0);\ \tau^{p+})=({}_2L_p^+(\phi_0);\ {}_2\tau_p^+),\quad 1\leqq p<2. $$

THEOREM 5.1. (1) *For* $1<p\leqq\infty$ $(L^{p-}(\phi_0);\tau^{p-})$ *is a Fréchet space.*

(2) $(L^{\infty-}(\phi_0);\tau^{\infty-})=(L_2^{\omega}(\phi_0);\tau_2^{\omega})$. *Hence* $(L^{\infty-}(\phi_0);\tau^{\infty-})$ *is a complete metrizable GB*-algebra defined by* Allan [**1**].

(3) *For* $1\leqq p<\infty$ $(L^{p+}(\phi_0);\tau^{p+})$ *is a separated barrelled space.*

(4) *For* $1<p\leqq\infty$ *we have*

$$(L^{p-}(\phi_0);\tau^{p-})^*=L^{p'+}(\phi_0),$$

$$\tau^{p-}=\beta(L^{p-}(\phi_0), L^{p'+}(\phi_0))=\tau(L^{p-}(\phi_0), L^{p'+}(\phi_0)).$$

(5) *For* $1\leqq p<\infty$ *we have*

$$(L^{p+}(\phi_0);\tau^{p+})^*=L^{p'-}(\phi_0),$$

$$\tau^{p+}=\beta(L^{p+}(\phi_0), L^{p'-}(\phi_0))=\tau(L^{p+}(\phi_0), L^{p'-}(\phi_0)).$$

THEOREM 5.2. *The following conditions are equivalent.*

(1) $L_2^{\omega}(\phi_0)$ *is a pure unbounded Hilbert algebra.*

(2) $L^p(\phi_0)\neq L^q(\phi_0)$ *for each* $1\leqq p<q\leqq\infty$.

(3) $L^{p-}(\phi_0)\neq L^p(\phi_0)$ *for each* $1<p\leqq\infty$.

(4) $L^{p+}(\phi_0)\neq L^p(\phi_0)$ *for each* $1\leqq p<\infty$.

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