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On a mixed problem for $\Box$ with a discontinuous boundary condition (II)

—an example of moving boundary—

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§1. Introduction.

Let Ω be a domain in $\boldsymbol{R}^n$ with smooth *compact*[1] boundary $\partial\Omega$. That is, Ω is the interior or exterior domain of $\partial\Omega$.

Consider the following Initial-Boundary-Value Problem (in short, I. B. V. P. or mixed problem).

$$(1.1)\quad \begin{cases} \Box u(x, t)=f(x, t) & \text{in } Q=\Omega\times(0, T), \\ u(x, 0)=u_0(x), \\ \dfrac{\partial u}{\partial t}(x, 0)=u_1(x), \end{cases}$$

$$(1.2)\quad \begin{cases} u(\tilde{x}, t)^{2)}=0 & \text{on } \Sigma_D=\bigcup_{0\leq t\leq T}\partial_D\Omega(t)\times\{t\} \quad \text{and} \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}}(\tilde{x}, t)=0 & \text{on } \Sigma_N=\bigcup_{0\leq t\leq T}\partial_N\Omega(t)\times\{t\} \end{cases}$$

where $\Box=\dfrac{\partial^2}{\partial t^2}-\sum_{j=1}^n\dfrac{\partial^2}{\partial x_j^2}$, $\boldsymbol{\nu}$: the unit exterior normal of $\partial\Omega$ and $\partial_D\Omega(t)$ and $\partial_N\Omega(t)$ are open sets in $\partial\Omega$ for each $t\in[0, T]$ which change with t and satisfy $\partial_D\Omega(t)\cap\partial_N\Omega(t)=\emptyset$, $\partial_D\Omega(t)\cup\overline{\partial_N\Omega(t)}=\overline{\partial_D\Omega(t)}\cup\partial_N\Omega(t)=\partial\Omega$, $\Gamma(t)=\overline{\partial_D\Omega(t)}\cap\overline{\partial_N\Omega(t)}$: $(n-2)$ dimensional smooth manifold in $\boldsymbol{R}^n$. We write $\Gamma=\bigcup_{0\leq t\leq T}\Gamma(t)\times\{t\}$.

For the future use, we rewrite the boundary condition (1.2) in the following form.

$$(1.2)'\qquad Y(\tilde{x}, t)\frac{\partial u}{\partial \boldsymbol{\nu}}(\tilde{x}, t)+(1-Y(\tilde{x}, t))u(\tilde{x}, t)=0 \quad \text{on} \quad \Sigma=\partial\Omega\times[0, T],$$

---

1) This is assumed only for the sake of simplicity. If $\partial\Omega$ is not compact, we must add some uniformity assumption at ∞ in the following argument.

2) We represent the generic points in Ω and $\partial\Omega$ by x and $\tilde{x}$, respectively.

where $Y(\tilde{x}, t)=1$ on Σ_N and $=0$ on Σ_D.

The purpose of this paper is to investigate the following problems:

(I) Under what condition on Γ, we can construct a 'weak' solution of (1.1) with (1.2)?

(II) Does the solution has the 'dependence domain' or not?

(III) Does the regularity theorem proved in Hayashida [**4**], more explicitly in Ibuki [**5**], hold in this case or not?

We give affirmative answers to the problems (I) and (II), assuming that Γ is *time-like.*

REMARK 1.1. The precise definition of time-like will be given in §2. Roughly speaking, it means that $\Gamma(t)$ changes slowlier than the propagation speed of the wave motion governed by $\square$.

Before stating our result, we give the following definition.

DEFINITION 1.2. A function $u(x, t)$ is called a *weak solution* of the problem (1.1) with (1.2) if it satisfies the followings.

$$(1.3)\qquad \begin{cases} u(x, t)\in L^\infty(0, T: H^1(\Omega))^{3)}, \dfrac{\partial u}{\partial t}(x, t)\in L^\infty(0, T: L^2(\Omega)) \quad \text{and} \\ u(\cdot, t)\in V_0(t) \qquad \text{for a. e. } t. \end{cases}$$

$$(1.4)\qquad -\int_0^T\left(\frac{\partial u}{\partial t}, \frac{\partial \eta}{\partial t}\right)dt+\int_0^T\sum_{j=1}^n\left(\frac{\partial u}{\partial x_j}, \frac{\partial \eta}{\partial x_j}\right)dt=(u_1, \eta(\cdot, 0))+\int_0^T(f, \eta)dt,$$

holds for all *test function* $\eta(x, t)\in L^1(0, T: H^1(\Omega))$ such that

$$(1.5)\qquad \begin{cases} \dfrac{\partial \eta}{\partial t}(x, t)\in L^1(0, T: L^2(\Omega)), \eta(\cdot, T)=0 \quad \text{and} \\ \eta(\cdot, t)\in V_0(t) \qquad \text{for a. e. } t. \end{cases}$$

$$(1.6)\qquad \begin{cases} u(x, 0)=u_0(x), \\ \dfrac{\partial u}{\partial t}(x, 0)=u_1(x). \end{cases}$$

Here, (,) stands for the scalar product in $L^2(\Omega)$ and for each t, we set

$$(1.7)\qquad V_0(t)=\{v\in H^1(\Omega)\,;\, v|_{\partial_D\Omega(t)}=0\}.$$

Then, we have

THEOREM A. *Let Γ be time-like. For any data $\{u_0, u_1, f\}\in V_0(0)\times L^2(\Omega)\times L^2(Q)$, there exists a weak solution $u(x, t)$ of (1.1) with (1.2). Moreover, there exists a constant C depending only on T and Γ such that*

3) $H^l(\Omega)$ stands for the usual Sobolev spacc of order l, with norm denoted by $\|\cdot\|_l$. We write $\|\cdot\|$ instead of $\|\cdot\|_0$. For any Banach space X, the space $L^p(0, T: X)$ $(1\leq p\leq\infty)$ are defined as the space of X-valued p-summable function on (0, T). See Lions-Magenes [**9**].

$$\left\|\frac{\partial u}{\partial t}(t)\right\|^2+\|u(t)\|_1^2\leqq C\{\|u_1\|^2+\|u_0\|_1^2+\int_0^T\|f(s)\|^2ds\} \tag{1.8}$$

$$\text{for a.e. } t\in(0, T).$$

THEOREM B. *Let Γ be time-like. Then, the weak solution of* (1.1) *and* (1.2) *is unique.*

THEOREM C. *Let Γ be time-like. The weak solution of* (1.1) *with* (1.2) *has the same 'dependence domain' as that of the free wave motion in the following sense. For any point $(x^0, t^0)\in Q$, we consider the light cone $C(x^0, t^0)=\{(x, t)\in \boldsymbol{R}^{n+1}: |x-x^0|<t^0-t, t>0\}$. If the initial data $\{u_0, u_1\}$ vanish on $C(x^0, t^0)\cap(\Omega\times\{0\})$ and the exterior force f vanishes on $C(x^0, t^0)\cap Q$, then we have*

$$\int_Q\int \chi(x, t)u(x, t)\phi(x, t)dxdt=0 \qquad \text{for any } \phi\in C^\infty(\bar{Q}) \tag{1.9}$$

where $\chi(x, t)=1$ on $\overline{C(x^0, t^0)\cap Q}$ and $=0$ otherwise.

In [2], Cooper and Bardos consider the same problem with some non-linear term. They impose to Γ some complicated conditions which imply that Γ is 'time-like'. But it seems to the author that their conditions are very difficult to check whether the given Γ satisfies them or not. In this sense, this paper is the completion of the linear problem treated in [2].

On the other hand, there exists an interesting paper of Čehlov [**1**], where the following problem is considered:

$$\begin{cases} \{-(D_t+ia)^2+(AD_x, D_x)\}u(x, t)=0 & \text{for } x_1>0, t>0 \\ u(x, 0)=D_tu(x, 0)=0 & x_1>0 \\ u(x, t)|_{x_1=0}=h^{(0)}(x', t) & x_2>kt, \quad t>0 \\ -iD_1u(x, t)|_{x_1=0}=\dfrac{\partial u}{\partial x_1}(x', 0, t)=h^{(1)}(x', t) & x_2<kt \end{cases} \tag{1.10}$$

where $D_j=\dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, $D_t=\dfrac{1}{i}\dfrac{\partial}{\partial t}$, $x'=(x_2, \cdots, x_n)$ and

$A=\begin{pmatrix} 1 & & 0 \\ & a_{22} & \\ 0 & & A'' \end{pmatrix}$ $A''=(a_{ij})$ symmetric positive definite matrix. He considers the above problem when (i) $|k|<\sqrt{a_{22}}$ and (ii) $|k|>\sqrt{a_{22}}$. The case (i) is time-like in our sense, so the case (i) is the very special case of ours. Using the Laplace and Fourier transformations, he proves the precise results concerning the regularity.

Lastly, we may believe that the method developed in this paper will be useful to solve the problem posed by Duvaut-Lions [3] p. 106: They consider the following linear equation, which governs the deformed state of elastic body, with the discontinuous boundary condition.

$$(1.11)\quad \begin{cases} \dfrac{\partial^2 u_i}{\partial t^2} = \sum_{j,k,h} \dfrac{\partial}{\partial x_j}\left\{a_{ijkh}(x)\dfrac{1}{2}\left(\dfrac{\partial u_k}{\partial x_h}+\dfrac{\partial u_h}{\partial x_k}\right)\right\}+f_i & \text{in } \Omega\times(0, T) \\ u_i=U_i & \text{on } \Gamma_U \\ \sum_{j,k,h} a_{ijkh}(x)\nu_j\dfrac{1}{2}\left(\dfrac{\partial u_k}{\partial x_h}+\dfrac{\partial u_h}{\partial x_k}\right)=F_i & \text{on } \Gamma_F \end{cases}$$

where Ω is the shape of the elastic body at equilibrium, the vector $\{u_i(x, t)\}$ presents the displacement from equilibrium position, $\nu=\{\nu_j\}$ is the outerward unit normal vector on $\partial\Omega$, and Γ_U and Γ_F are disjoint open subsets of $\partial\Omega$ satisfying $\partial\Omega=\bar{\Gamma}_U\cup\Gamma_F=\Gamma_U\cup\bar{\Gamma}_F$ and $\Gamma_U\cap\Gamma_F=0$. The coefficients $a_{ijkh}(x)$ are assumed to be C^∞-functions satisfying the following symmetry and definiteness conditions

$$\begin{cases} a_{ijkh}(x)=a_{khij}(x)=a_{jihk}(x) \\ a_{ijkh}\xi_{ij}\xi_{kh}\geqq\alpha_1\xi_{ij}\xi_{ij} \quad \text{for any } \xi_{ij}\in\boldsymbol{R},\ \alpha_1: \text{positive constant}. \end{cases}$$

They treated this problem assuming that Γ_U and Γ_F are independent of t. And they propose that 'L'abandon de cette hypothèse semble conduire à des problèmes ouvert et fort intéressants.'

We announced Theorem A above in [8] whose proof will be given in § 4.

We owe much to Professors D. Fujiwara and M. Wakimoto whose contributions to the proof of Theorem 2.6 are very essential to this paper.

§ 2. The transformation of class (E).

For the sake of self-containedness of this paper, we begin with citing the definitions in [7].

Definition 2.1. Let Γ be a submanifold of codimension 1 of the lateral boundary $\Sigma=\partial\Omega\times[0, T]$ such that $\Gamma(t)=\Gamma\cap P(t)$ is a submanifold of codimension 1 of the boundary $\partial\Omega$ and $\Gamma(t)$ are diffeomorphic to each other where $P(t^0)=\{(x, t)\in\boldsymbol{R}^{n+1};\ t=t^0\}$. We say that Γ is *time-like* if it satisfies the following: For any $t^0\in[0, T]$, there exists a positive number ε_1 depending on t^0 such that

$$(2.1)\quad \bigcup_{|t^0-\tau|\leq\varepsilon}\Gamma(\tau)\subset\bigcup_{\substack{\tilde{x}^0\in\Gamma(t^0)\\|t^0-\tau|\leq\varepsilon}}\{(x, t)\in\boldsymbol{R}^n;\ |x-\tilde{x}^0|\leqq|\tau-t^0|\}\cap\Sigma,$$

for any ε, $0\leqq\varepsilon<\varepsilon_1$, where we put $\Gamma(t)=\Gamma(0)$ for $t<0$, $\Gamma(t)=\Gamma(T)$ for $t>T$.

To reformulate the condition (2.1), we remember the following: Let $\boldsymbol{\nu}(\tilde{x}^0)$

be the unit exterior normal of $\partial\Omega$ at $\tilde{x}^0\in\partial\Omega$. As $\Gamma(t^0)$ is of codimension 2 in $\boldsymbol{R}^n$, there exists another unit vector $m(\tilde{x}^0, t^0)$ orthogonal to $\boldsymbol{\nu}(\tilde{x}^0)$ such that the space spanned by $\boldsymbol{\nu}(\tilde{x}^0)$ and $m(\tilde{x}^0, t)$ forms the normal bundle of $\Gamma(t^0)$ at $\tilde{x}^0$. Then we have the following.

REMARK 2.2. Let us consider the trajectory of the 'point' of the intersection of $\Gamma(t)$ with the normal bundle of $\Gamma(t^0)$ at $\tilde{x}^0$. Then the condition (2.1) implies the condition (2.2) below.

(2.2) The speed of the trajectory at $t=t^0$ is smaller than 1 for each $\tilde{x}^0\in\Gamma(t^0)$.

Moreover, as $\partial\Omega$ is compact, (2.2) implies (2.1).

By the translation, we may suppose that $(0,0)$ represents the general point in Γ. Rotating the x-coordinates, if necessary, we may suppose that $\partial\Omega$ is represented locally by $x_n=f(x')$, satisfying $0=f(0)$ and $f_{x_j}(0)=0$ $j=1,2,\cdots,n-1$ and Γ is represented locally by $x_{n-1}=g(x'',t)$ satisfying $0=g(0,0)$ and $g_{x_j}(0,0)=0$ $j=1,2,\cdots,n-2$ where $x'=(x_1,x_2,\cdots,x_{n-1})$, $x''=(x_1,x_2,\cdots,x_{n-2})$.

Then we have

PROPOSITION 2.3 [7]. *If* $(0,0)$ *belongs to* Γ *and* $\partial\Omega$ *and* $\Gamma(0)$ *are represented locally as above, then the condition* (2.1) *implies that* $|g_t(0,0)|<1$.

Let us consider a level preserving transformation Φ in $\boldsymbol{R}^{n+1}$, $(y,s)=\Phi(x,t)$. More precisely, Φ is given by $y_j=\phi_j(x,t)$, $j=1,2,\cdots,n$ and $s=t$. By this transformation, □ is transformed to the operator L:

$$(2.3)\qquad \begin{aligned} L&=L(y,s:D_y,D_s)\\ &=\frac{\partial^2}{\partial s^2}+2\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial t}\frac{\partial^2}{\partial y_j\partial s}-\sum_{i\cdot j=1}^{n}\Big(\sum_{k=1}^{n}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial x_k}\\ &\quad-\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\Big)\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{j=1}^{n}\Big(\frac{\partial^2\phi_j}{\partial t^2}-\sum_{k=1}^{n}\frac{\partial^2\phi_j}{\partial x_k^2}\Big)\frac{\partial}{\partial y_j}\,. \end{aligned}$$

DEFINITION 2.4. We say that a transformation Φ belongs to the class (E) if the matrix E_ϕ whose (i,j) element is defined by $\Big(\sum_{k=1}^{n}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial x_k}-\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\Big)$ is positive definite for $(x,t)\in\mathrm{dom}\,\Phi$ (=domain of Φ). We denote it simply by $\Phi\in(E)$.

It is clear that the matrix E_ϕ is represented by

$$(2.4)\qquad E_\phi={}^tS_\phi JS_\phi$$

$$\text{where}\quad S_\phi=\overbrace{\begin{pmatrix} \dfrac{\partial\phi_1}{\partial t} & \cdots\cdots & \dfrac{\partial\phi_n}{\partial t} \\ \dfrac{\partial\phi_1}{\partial x_1} & \cdots\cdots & \dfrac{\partial\phi_n}{\partial x_1} \\ \vdots & & \vdots \\ \dfrac{\partial\phi_1}{\partial x_n} & \cdots\cdots & \dfrac{\partial\phi_n}{\partial x_n} \end{pmatrix}}^{n}\Bigg\} n+1 \qquad J=\overbrace{\begin{pmatrix} -1 & & & \mathbf{0} \\ & 1 & & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix}}^{n+1}\Bigg\} n+1 .$$

Now, we define a family of functions $\alpha_\varepsilon(\tilde{x}, t)$ on Σ which approximate suitably the characteristic function $Y(\tilde{x}, t)$ of Σ_N.

We denote a neighborhood of Γ in $\boldsymbol{R}^{n+1}$ by N_0 such that $N_0\cap P(t)$ forms a tubular neighborhood of $\Gamma(t)$ in $\boldsymbol{R}^n$. Here, we project the set $N_0\cap P(t)$ in $\boldsymbol{R}^{n+1}$ to $\boldsymbol{R}^n$ and identify it with the original one. We use this identification freely if there will occur no confusion. Moreover, we suppose that N_0 is taken sufficiently small such that for any point $(\tilde{x}^0, t^0)\in N_0\cap P(t_0)\cap\Sigma$, there exists a unique geodesic curve from $(\tilde{x}^0, t^0)$ to $\Gamma(t_0)$ in $\partial\Omega$. We define

$$\alpha_\varepsilon(\tilde{x}^0, t^0)=\varepsilon^{-1}\int_0^\infty \rho\Big(\frac{d(\tilde{x}^0, t^0)-s}{\varepsilon}\Big)ds \tag{2.5}$$

where $\rho(s)$ is a function satisfying $\rho(s)=\rho(-s)\in C_0^\infty(\boldsymbol{R})$, $\int_{-\infty}^\infty \rho(s)ds=1$, $\rho(s)\geqq 0$ and the support of $\rho=[-1, 1]$ and $d(\tilde{x}^0, t^0)$ is defined by

$$d(\tilde{x}^0, t^0)=\begin{cases} +\text{geodesical distance from } (\tilde{x}^0, t^0) \text{ to } \Gamma(t^0) \\ \qquad\qquad\qquad\qquad \text{if } (\tilde{x}^0, t^0)\in\Sigma_D \\ -\text{geodesical distance from } (\tilde{x}^0, t^0) \text{ to } \Gamma(t^0) \\ \qquad\qquad\qquad\qquad \text{if } (\tilde{x}^0, t^0)\in\Sigma_N . \end{cases} \tag{2.6}$$

We extend the function $d(\tilde{x}^0, t^0)$ to 1 or to 0 conveniently outside the neighborhood N_0.

Then, clearly we have

PROPOSITION 2.5. *The function defined above satisfies the following:*

(a) $\alpha_\varepsilon(\tilde{x}, t)\in C^\infty(\Sigma)$ *and* Γ_ε *(=the boundary of* D_ε*) forms a submanifold of codimension* 1 *in* Σ *where* $D_\varepsilon=\{(\tilde{x}, t)\in\Sigma\,;\ \alpha_\varepsilon(\tilde{x}, t)=0\}^0$, A^0 *stands for the interior of a set* A.

(b) *The distance from* Γ_ε *to* Γ *is of order* ε *when* ε *tends to* 0.

(c) $\alpha_\varepsilon(\tilde{x}, t)=\dfrac{1}{2}$ *if* $(\tilde{x}, t)\in\Gamma$.

The following theorem is important in this paper.

THEOREM 2.6. *Suppose that* Γ *is time-like. Then, for each* $(\tilde{x}^0, t^0)\in\Gamma$, *there exists a neighborhood* $V_{(\tilde{x}^0,t^0)}$ *and a transformation* $\Phi_{(\tilde{x}^0,t^0)}\in(E)$ *which transforms* $V_{(\tilde{x}^0,t^0)}$ *to a neighborhood* V *of* $(0, 0)$ *such that* (i) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Omega)=\tilde{V}\cap\{(y, t)\,;\ y=(y_1, y_2, \cdots, y_n), y_n>0\}$, (ii) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\partial\Omega)=\tilde{V}\cap\{(y, t)\,;\ y_n=0\}$, (iii) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Gamma(t^0))=\tilde{V}\cap\{(y, t)\,;\ y_n=y_{n-1}=0\}$, (iv) *the Jacobian of the inverse transformation* $\Phi^{-1}_{(\tilde{x}^0,t^0)}$ *is bounded on* V *and* (v) *the function*

$$\tilde{\alpha}_\varepsilon(\tilde{y}, s)\equiv\alpha_\varepsilon(\Phi^{-1}_{(\tilde{x}^0,t^0)}(\tilde{y}, s))$$

is independent of s *for* $(\tilde{y}, s)\in\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Sigma)$.

REMARK 2.7. The above theorem was announced in [7] as Proposition 2.7 with a sketchy proof. But the proof there was not correct. In the following, using the geodesic coordinate, we give the correct proof which is heavily due to D. Fujiwara and M. Wakimoto.

PROOF OF THEOREM 2.6. As mentioned before, we may suppose, without loss of generality, $(\tilde{x}^0, t^0)=(0, 0)$ and $\partial\Omega$ near 0 is represented locally by $x_n=f(x')$ satisfying $f(0)=0$, $f_{x_j}(0)=0$ for $1\leqq j\leqq n-1$ and Γ near $(0, 0)$ is represented locally by $x_n=f(x')$ and $x_{n-1}=g(x'')$ satisfying $g(0, 0)=0$, $g_{x_j}(0, 0)=0$ for $1\leqq j\leqq n-2$ and $|g_t(0, 0)|<1$.

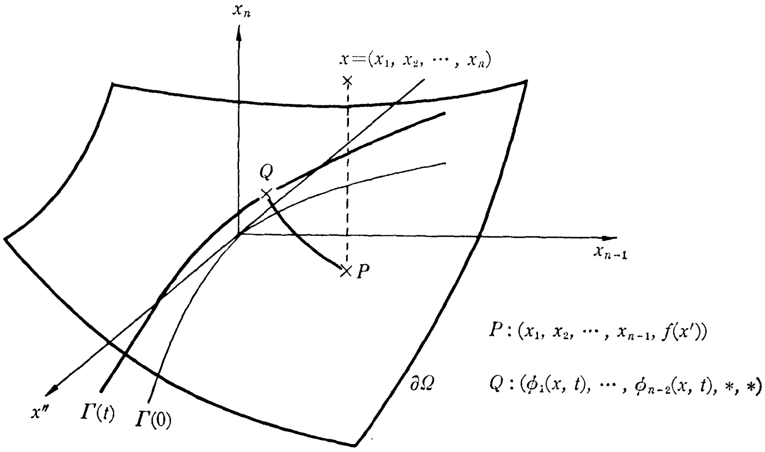

Figure 1

For the point $(x, t)\in V_{(0,0)}$, we define the following coordinate system which gives the desired transformation Φ at $(0, 0)$.

$$(2.7)\qquad \begin{cases} y_j=\phi_j(x, t) & \text{for } 1\leqq j\leqq n-2, \\ y_{n-1}=d(P, \Gamma(t))=\phi_{n-1}(x, t), \\ y_n=x_n-f(x')=\phi_n(x, t), \\ s=t, \end{cases}$$

where P is the point on $\partial\Omega$ represented by $(x_1, x_2, \cdots, x_{n-1}, f(x'))$, Q is the point on $\Gamma(t)$ which gives the geodesical distance from P to $\Gamma(t)$ and $\{\phi_j(x, t)\}_{1\leqq j\leqq n-2}$ are the components of Q in x-coordinate. (See Figure 1).

It is clear from the definition above that the transformation Φ satisfies the conditions (i) (ii) (iii) and (v) in the theorem.

We shall claim that Φ belongs to the class (E) and satisfies (iv). In order to do so, we introduce another coordinate system in $\boldsymbol{R}^n$. Let us define the z-coordinate as follows:

$$
(2.8)\qquad \begin{cases} z_j = x_j & \text{for } 1 \leqq j \leqq n-2, \\ z_{n-1} = x_{n-1} - g(x'', 0), \\ z_n = x_n - f(x'). \end{cases}
$$

From the properties on f and g, we have

$$
(2.9)\qquad \left(\frac{\partial}{\partial x_i}\right)_0 = \left(\frac{\partial}{\partial z_i}\right)_0 \qquad \text{for } 1 \leqq i \leqq n
$$

where $(X)_0$ means the vector field at 0. That is, the coordinate system $(z_1, z_2, \cdots, z_n)$ gives the local coordinate system, orthogonal at 0. Moreover, the part of $\partial\Omega$ near 0 is given by $\{z;\ z_n=0\}$, $\Gamma(0)$ is represented locally by $\{z;\ z_n = z_{n-1} = 0\}$ and $\Gamma(t)$ is represented locally by $\{z;\ z_n = 0,\ z_{n-1} = g(z'', t) - g(z'', 0)\}$. Using this coordinate system, we represent the transformation Φ in (z, t)-coordinate, that is,

$$
(2.10)\qquad \begin{cases} \tilde{y}_j = \psi_j(z, t) \\ \tilde{y}_{n-1} = d(\tilde{P}, \Gamma(t)) \\ \tilde{y}_n = z_n \\ s = t \end{cases}
$$

where $\tilde{P}$ is the point P represented by z-coordinate, and $\{\psi_j\}_{1 \leqq j \leqq n-2}$ are the components of $\tilde{Q}$ $(=Q)$ in z-coordinate, more precisely, ψ_j is defined by

$$
(2.11)\qquad \psi_j(z, t) = \psi_j(z_1, \cdots, z_{n-2}, z_{n-1} + g(z'', 0), \\ z_n + f(z'', z_{n-1} + g(z'', 0)), t) \qquad \text{for } 1 \leqq j \leqq n-2.
$$

Our problem is to prove the matrix E_Φ is positive definite at $(0, 0)$. We define the matrices S_Φ^0 and $\tilde{S}_\Phi^0$ by

$$
(2.12)\qquad S_\Phi^0 = \begin{pmatrix} \frac{\partial y_1}{\partial t} & \cdots & \frac{\partial y_n}{\partial t} \\ \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_n} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}_{(x,t)=(0,0)} \quad \text{and} \quad \tilde{S}_\Phi^0 = \begin{pmatrix} \frac{\partial \tilde{y}_1}{\partial t} & \cdots & \frac{\partial \tilde{y}_n}{\partial t} \\ \frac{\partial \tilde{y}_1}{\partial z_1} & \cdots & \frac{\partial \tilde{y}_n}{\partial z_1} \\ \vdots & & \vdots \\ \frac{\partial \tilde{y}_1}{\partial z_n} & \cdots & \frac{\partial \tilde{y}_n}{\partial z_n} \end{pmatrix}_{(z,t)=(0,0)}.
$$

LEMMA 2.8. *The positivity of* E_{Φ} *at* $(x, t)=(0, 0)$ *is equivalent to that of* $\tilde{E}_{\Phi}$ *at* $(z, t)=(0, 0)$.

PROOF. As noted in (2.4), $E_{\Phi}^0={}^tS_{\Phi}^0JS_{\Phi}^0$. On the other hand, we have easily $S_{\Phi}^0=\tilde{S}_{\Phi}^0$ from (2.9). So $E_{\Phi}^0={}^tS_{\Phi}^0JS_{\Phi}^0={}^t\tilde{S}_{\Phi}^0J\tilde{S}_{\Phi}^0=\tilde{E}_{\Phi}^0$. Q. E. D.

In the following, we use only the (z, t)-coordinte. We have easily,

LEMMA 2.9.

$$(2.13)\qquad \tilde{S}_{\Phi}^0=\begin{pmatrix} \frac{\partial\tilde{y}_1}{\partial t} & \cdots\cdots & \frac{\partial\tilde{y}_{n-1}}{\partial t} & 0 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 \quad 0 & 0 \\ \frac{\partial\tilde{y}_1}{\partial z_{n-1}} & \cdots & \frac{\partial\tilde{y}_{n-1}}{\partial z_{n-1}} & 0 \\ 0 & \cdots\cdots\cdots & 0 & 1 \end{pmatrix}_{(z,t)=(0,0)}.$$

LEMMA 2.10.

$$\left(\frac{\partial\tilde{y}_1}{\partial z_{n-1}}, \cdots, \frac{\partial\tilde{y}_{n-2}}{\partial z_{n-1}}, \frac{\partial\tilde{y}_{n-1}}{\partial z_{n-1}}\right)_{(0,0)}=(0, \cdots, 0, 1).$$

PROOF. Let us consider the point A_τ in $\partial\Omega$ given by A_τ: $(0, \cdots, 0, \tau, 0)$ in z-coordinate. Let γ_τ be the minimal geodesic from A_τ to $\Gamma(0)$. We denote the

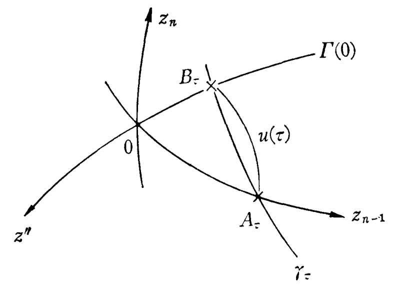

Figure 2

intersection of γ_τ with $\Gamma(0)$ by B_τ: $(w_1(\tau), w_2(\tau), \cdots, w_{n-2}(\tau), 0, 0)$. We parametrize the geodesic curve γ_τ by the arc length σ and the geodesical distance between A_τ and B_τ is denoted by $u(\tau)$ $(=|d(A_\tau, \Gamma(0))|)$. Then

$$(2.14)\qquad \begin{cases} \gamma_\tau(0)=(0, \cdots, 0, \tau, 0), \\ \gamma_\tau(u(\tau))=(w_1(\tau), w_2(\tau), \cdots, w_{n-2}(\tau), 0, 0), \\ \gamma_\tau(\sigma)=(\gamma_1(\tau, \sigma), \gamma_2(\tau, \sigma), \cdots, \gamma_{n-1}(\tau, \sigma), 0) \quad \text{for} \quad 0\leqq\sigma\leqq u(\tau) \\ \text{represents the minimal geodesic from } A_\tau \text{ to } B_\tau. \end{cases}$$

So we have

$$(2.15)\qquad w_j(\tau)=\gamma_j(\tau, u(\tau)) \quad \text{for} \quad 1\leqq j\leqq n-2 \quad \text{and} \quad \gamma_{n-1}(\tau, u(\tau))=0\,.$$

From the definition of $\left(\frac{\partial \tilde{y}_i}{\partial z_{n-1}}\right)_0$, we have

$$(2.16)\qquad \begin{cases} \dfrac{dw_i(0)}{d\tau}=\left(\dfrac{\partial \tilde{y}_i}{\partial z_{n-1}}\right)_{(0,0)} & \text{for} \quad 1\leqq i\leqq n-2\,, \\ \dfrac{du(0)}{d\tau}=\left(\dfrac{\partial \tilde{y}_{n-1}}{\partial z_{n-1}}\right)_{(0,0)}\,. \end{cases}$$

Differentiating the equation (2.15) with respect to τ at $\tau=0$, we have

$$(2.17)\qquad \begin{cases} \dfrac{dw_i(0)}{d\tau}=\dfrac{\partial\gamma_i(0,0)}{\partial\tau}+\dfrac{\partial\gamma_i(0,0)}{\partial\sigma}\dfrac{du(0)}{d\tau} & \text{for} \quad 1\leqq i\leqq n-2 \\ 0=\dfrac{\partial\gamma_{n-1}(0,0)}{\partial\tau}+\dfrac{\partial\gamma_{n-1}(0,0)}{\partial\sigma}\dfrac{du(0)}{d\tau}\,. \end{cases}$$

From $\gamma_\tau(0)=(0, \cdots, 0, \tau, 0)$, we have

$$(2.18)\qquad \frac{\partial\gamma_j(0,0)}{\partial\tau}=\delta_{j,n-1} \quad \text{and} \quad \frac{\partial\gamma_j(0,0)}{\partial\sigma}=-\delta_{j,n-1}$$

$$\text{for} \quad 1\leqq j\leqq n-1\,.$$

Combining the relations (2.16), (2.17) and (2.18), we have the desired relation.

Q. E. D.

By the lemmas 2.8, 2.9 and 2.10, we have from the easy calculation that the matrix S^0_Φ is positive definite if and only if

$$(2.19)\qquad \sum_{i=1}^{n-1}\left(\frac{\partial \tilde{y}_i}{\partial t}\right)^2_{(z,t)=(0,0)}<1\,.$$

Let $\tilde{\gamma}_t$ be the minimal geodesic from 0 to $\Gamma(t)$. We denote the intersection of $\tilde{\gamma}_t$ with $\Gamma(t)$ by $\tilde{B}_t$: $(\tilde{w}_1(t), \tilde{w}_2(t), \cdots, \tilde{w}_{n-2}(t), \tilde{w}_{n-1}(t), 0)$. We parametrize the

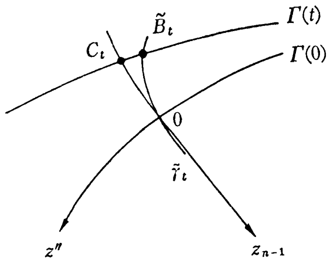

Figure 3

curve $\tilde{\gamma}_t$ by the arc length s and the geodesical distance between 0 and $\tilde{B}_t$ is denoted by $\tilde{u}(t)$ $(=|d(0, \Gamma(t))|)$. Then

$$(2.20)\quad \begin{cases} \tilde{\gamma}_t(0)=(0,\cdots,0), \\ \tilde{\gamma}_t(\tilde{u}(t))=(\tilde{w}_1(t), \tilde{w}_2(t), \cdots, \tilde{w}_{n-1}(t), 0), \\ \tilde{\gamma}_t(s)=(\tilde{\gamma}_1(t,s), \tilde{\gamma}_2(t,s), \cdots, \tilde{\gamma}_{n-1}(t,s), 0) \quad \text{for } 0\leqq s\leqq \tilde{u}(t) \\ \text{represents the minimal geodesic from 0 to } \tilde{B}_t. \end{cases}$$

So we have

$$(2.21)\qquad \tilde{w}_j(t)=\tilde{\gamma}_j(t, \tilde{u}(t)) \qquad \text{for} \quad 1\leqq j\leqq n-1.$$

As $\tilde{\gamma}_j(t,0)=0$ for $1\leqq j\leqq n-1$, differentiating the equation (2.15) with respect to t at $t=0$, we have

$$(2.22)\qquad \frac{d\tilde{w}_j(0)}{dt}=\frac{\partial\tilde{\gamma}_j(0,0)}{\partial s}\frac{d\tilde{u}(0)}{dt}.$$

LEMMA 2.11. $\dfrac{\partial\tilde{\gamma}_j(0,0)}{\partial s}=\delta_{j,n-1}$ *for* $1\leqq j\leqq n-1$.

PROOF. Let R_g denote the Riemann metric on $\partial\Omega$ induced from $\boldsymbol{R}^n$. As $\tilde{\gamma}_t(s)$ is the minimal geodesic from 0 to $\Gamma(t)$, $\tilde{\gamma}_t(s)$ is orthogonal to $\Gamma(t)$ at $s=\tilde{u}(t)$, that is,

$$(2.23)\qquad R_g\Big((\tilde{\gamma}_t)_*\Big(\frac{d}{ds}\Big)_{\tilde{u}(t)},\ T_{\tilde{B}_t}(\Gamma(t))\Big)=0 \qquad \text{for any} \quad t$$

where $(\tilde{\gamma}_t)_*\Big(\dfrac{d}{ds}\Big)_{\tilde{u}(t)}=\sum_{i=1}^{n-1}\dfrac{\partial\tilde{\gamma}_i(t,\tilde{u}(t))}{\partial s}\Big(\dfrac{\partial}{\partial z_i}\Big)_{\tilde{B}_t}$, $T_{\tilde{B}_t}(\Gamma(t))$ represents the tangent space at $\tilde{B}_t$ to $\Gamma(t)$.

Letting t tend to 0 in (2.23), we have

$$(2.24)\qquad R_g\Big(\sum_{i=1}^{n-1}\frac{\partial\tilde{\gamma}_i(0,0)}{\partial s}\Big(\frac{\partial}{\partial z_i}\Big)_0,\ T_0(\Gamma(0))\Big)=0,$$

where

$$T_0(\Gamma(0))=\Big\{\sum_{i=1}^{n-2}a_i\Big(\frac{\partial}{\partial z_i}\Big)_0;\ a_i\in\boldsymbol{R}\Big\}.$$

Remarking that $R_g\Big(\dfrac{\partial}{\partial z_i}, \dfrac{\partial}{\partial z_j}\Big)_0=\delta_{ij}$ and (2.24) holds, we have

$$\frac{\partial\tilde{\gamma}_i(0,0)}{\partial s}=0 \qquad \text{for} \quad 1\leqq i\leqq n-2.$$

On the other hand, $\tilde{\gamma}_t(s)$ is parametrized by the arc length s,

$$\Big\|\frac{d\tilde{\gamma}_t(s)}{ds}\Big\|=1 \qquad \text{for any} \quad s.$$

From this, we have

$$\frac{\partial\tilde{\gamma}_{n-1}(0,0)}{\partial s}=1.$$

Q. E. D.

LEMMA 2.12.

$$\left|\frac{d\tilde{u}(0)}{dt}\right|\leqq|g_t(0,0)|<1.$$

PROOF. (See, Figure 3 described before). Consider the curve $C:[0,t]\to\partial\Omega$ defined by $C(s)=(0,\cdots 0,g(0,s),0)$. Then $C(0)=0$ and $C(t)\in\Gamma(t)$. As $\tilde{\gamma}_t$ gives the minimal geodesic from 0 to $\Gamma(t)$, we have

$$0\leqq\tilde{u}(t)\leqq\int_0^t\left\|\frac{dc(s)}{ds}\right\|ds \quad \text{for any} \quad t. \tag{2.25}$$

As $\tilde{u}(0)=0$ and (2.25) holds for any t, we have the desired result. Q. E. D.

As $\left(\frac{\partial\tilde{y}_i}{\partial t}\right)^2_{(z,t)=(0,0)}=\left(\frac{d\tilde{w}_j(0)}{dt}\right)^2$, combining the lemmas 2.11 and 2.12 with (2.19), we prove that E^0_Φ is positive definite.

Moreover, as the calculation above shows, if we take the neighborhood $V_{(\tilde{x}0,t0)}$ sufficiently small, the transformation $\Phi_{(\tilde{x}0,t0)}$ and its inverse transformation $\Phi^{-1}_{(\tilde{x}0,t0)}$ have bounded derivatives. The end of the proof of Theorem 2.6.

§ 3. An energy inequality for the auxiliary problem.

As the auxiliary problem to the problem (1.1) with (1.2)′, we consider the following I. B. V. P.

$$\begin{cases} \Box u_\varepsilon(x,t)=f_\varepsilon(x,t) & \text{in } Q, \\ u_\varepsilon(x,0)=u_{0\varepsilon}(x) & \\ \frac{\partial u_\varepsilon}{\partial t}(x,0)=u_{1\varepsilon}(x) & \text{on } \Omega, \end{cases} \tag{3.1}$$

$$\alpha_\varepsilon(\tilde{x},t)\frac{\partial u_\varepsilon}{\partial \boldsymbol{\nu}}(\tilde{x},t)+(1-\alpha_\varepsilon(\tilde{x},t))u_\varepsilon(\tilde{x},t)=0 \quad \text{on } \Sigma \tag{3.2}$$

where the function $\alpha_\varepsilon(\tilde{x},t)$ is defined in Proposition 2.5, and the data $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x,t)\}$ are 'suitably' defined. Then we have

THEOREM 3.1[4]. *Let the data* $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x,t)\}$ *of* (3.1) *belong to the space* $C^\infty(\bar{\Omega})\times C^\infty(\bar{\Omega})\times C^\infty(\bar{Q})$. *If they are compatible*[5] *of order* ∞ *at* $t=0$, *then there exists a solution* $u_\varepsilon(x,t)\in C^\infty(\bar{Q})$ *of* (3.1) *with* (3.2). *Moreover, the phenomenon governed by* (3.1) *with* (3.2) *has the same dependence domain as that of*

4) In this theorem, the compactness of $\partial\Omega$ is superfluous.
5) The definition of the compatibility is given in [7].

the Cauchy problem for $\square$ *in the whole space* $\boldsymbol{R}^n$.

As was proved in Theorem 2.6 that the function $\alpha_\varepsilon(\tilde{x}, t)$ constructed in Proposition 2.5 satisfies the assumptions (a) and (b) in [7], we have the above result.

Now we prepare two function spaces to construct the suitable data $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x, t)\}$ from the given data $\{u_0(x), u_1(x), f(x, t)\}$.

$V_{\alpha_\varepsilon(t)}(\Omega)$ is the completion with the norm $\|\cdot\|_{V_{\alpha_\varepsilon(t)}}$ of all functions $u\in C^\infty(\bar{\Omega})$, each of which vanishes in a neighborhood of $\{\tilde{x}\in\partial\Omega:\ \alpha_\varepsilon(\tilde{x}, t)=0\}$ and $\|u\|_{V_{\alpha_\varepsilon(t)}}<\infty$ where

$$\|u\|^2_{V_{\alpha_\varepsilon(t)}}=\|u\|^2_{H^1(\Omega)}+\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x}, t)}{\alpha_\varepsilon(\tilde{x}, t)}|u(\tilde{x})|^2 d\tilde{x}\,.$$

$H^2_{\alpha_\varepsilon(t)}(\Omega)$ is defined through

$$H^2_{\alpha_\varepsilon(t)}(\Omega)=\left\{u\in H^2(\Omega)\,;\ \alpha_\varepsilon(\tilde{x}, t)\frac{\partial u}{\partial\boldsymbol{\nu}}+(1-\alpha_\varepsilon(\tilde{x}, t))u=0 \quad\text{on}\quad \partial\Omega\right\}.$$

PROPOSITION 3.2. *Let the data* $\{u_0(x), u_1(x), f(x, t)\}$ *of the problem* (1.1) *with* (1.2)$'$ *belong to* $V_0(0)\times L^2(\Omega)\times L^2(Q)$. *Then sufficiently small* $\varepsilon>0$, *there exist the data* $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x, t)\}$ *such that*

(i) *they belong to* $(H^2_{\alpha_\varepsilon(0)}(\Omega)\cap C^\infty(\bar{\Omega}))\times(V_{\alpha_\varepsilon(0)}(\Omega)\cap C^\infty(\bar{\Omega}))\times(H^1(Q)\cap C^\infty(\bar{Q}))$,

(ii) *they are compatible of order* ∞ *at* $t=0$ *and*

(iii) $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x, t)\}$ *converges to* $\{u_0(x), u_1(x), f(x, t)\}$ *in* $V_0(0)\times L^2(\Omega)\times L^2(Q)$.

This proposition is easily verified if we take into account the following lemma due to Hayashida [4].

LEMMA 3.3. *For any* $u\in V_0(0)$, *there exists a sequence* $\{u_j\}\subset C^\infty(\bar{\Omega})$ *satisfying* $\frac{\partial}{\partial\boldsymbol{\nu}}u_j=0$ *on* $\partial_N\Omega(0)$ *such that each* u_j *vanishes in a neighborhood of* $\overline{\partial_0\Omega(0)}$ *and* u_j *converges to* u *in* $H^1(\Omega)$.

PROPOSITION 3.4. *There exists a vector field* $X(x, t)=\frac{\partial}{\partial t}+\sum_{j=1}^n\beta_j(x, t)\frac{\partial}{\partial x_j}$ *such that* (a) $\sum_{j=1}^n\beta_j(\tilde{x}, t)\nu_j(\tilde{x})=0$ *on* $(\tilde{x}, t)\in\Sigma$, (b) *by the transformation* $\Phi_{(\tilde{x}^0, t^0)}$, $X(x, t)$ *is transformed to* $\frac{\partial}{\partial s}$ *in* $V_{(\tilde{x}^0, t^0)}$ *for each* $(\tilde{x}^0, t^0)\in\Gamma(t^0)$ *and* (c) $|\beta_j(\tilde{x}, t)|<1$ *for* $1\leqq j\leqq n$.

PROOF. Let $\tilde{X}(x, t)$ be defined by

$$\tilde{X}(x, t)=(\Phi^{-1}_{(\tilde{x}^0, t^0)})_*\left(\frac{\partial}{\partial s}\right)=\sum_{j=1}^n\frac{\partial\psi_j}{\partial s}\frac{\partial}{\partial x_j}+\frac{\partial}{\partial t}\qquad\text{in}\quad V_{(\tilde{x}^0, t^0)}\,, \tag{3.3}$$

where $\Phi^{-1}_{(\tilde{x}^0, t^0)}(y, x)=(\psi_1(y, s), \cdots, \psi_n(y, s), s)$. By easy calculation, or rather by definition, we have that $\sum_{j=1}^{n} \frac{\partial \psi_j}{\partial s} \nu_j = 0$ on $\Sigma \cap V_{(\tilde{x}^0, t^0)}$. In fact, $\boldsymbol{\nu}$ is represented by $\boldsymbol{\nu} = \frac{(f_{x_1}, \cdots, f_{x_{n-1}}, -1)}{\sqrt{1+|\nabla f|^2}}$. On the other hand, differentiating the equality $y_n = \phi_n(\psi_1(y, s), \cdots, \psi_n(y, s), s)$ with respect to s and using $\psi_n(x, t) = x_n - f(x')$ in Theorem 2.6, we have

$$0 = \sum_{j=1}^{n} \frac{\partial \phi_n}{\partial x_j} \frac{\partial \psi_j}{\partial s} = \frac{\partial \psi_n}{\partial s} - \sum_{j=1}^{n-1} f_{x_j} \frac{\partial \psi_j}{\partial s}. \tag{3.4}$$

Differentiating $y_j = \phi_j(\psi_1(y, s), \cdots, \psi_n(y, s), s)$ with respect to s, and remarking (2.13), (2.19) and the proof of Lemma 2.8, we have

$$\sum_{j=1}^{n-1} \left(\frac{\partial \psi_j}{\partial s} \right)^2_{(y,s)=(0,0)} < 1. \tag{3.5}$$

From (3.5), taking a smaller neighborhood $\tilde{V}_{(\tilde{x}^0, t^0)}$ of $(\tilde{x}^0, t^0)$ contained in $V_{(\tilde{x}^0, t^0)}$, we have

$$\sum_{j=1}^{n} \left(\frac{\partial \psi_j}{\partial s} \circ \Phi_{(\tilde{x}^0, t^0)} \right)^2 < 1 \quad \text{in} \quad \tilde{V}_{(\tilde{x}^0, t^0)}. \tag{3.5$'$}$$

As $\partial\Omega$ is compact, there exists a finite family of points $(\tilde{x}^0_\gamma, t^0_\gamma) \in \Gamma$ such that (i) the neighborhood $V_\gamma = \tilde{V}_{(\tilde{x}^0_\gamma, t^0_\gamma)}$ and the transformation $\Phi_\gamma = \Phi_{(\tilde{x}^0_\gamma, t^0_\gamma)} \in (E)$ satisfy the properties enumerated in Theorem 2.6 and (ii) $N_\Gamma = \bigcup_\gamma V_\gamma$ contains the closure of ε_0-neighborhood N_{ε_0} of Γ in $\boldsymbol{R}^{n+1}$. Let $\{\rho_\gamma\}$ be a partition of the unity subordinate to the covering $\{V_\gamma\}$. Then defining $\beta_j(x, t)$ by

$$\beta_j(x, t) = \sum_\gamma \rho_\gamma(x, t) \frac{\partial \psi_j^\gamma}{\partial s}((\Phi_\gamma R_\gamma)(x, t)),$$

we have the desired function, where R_γ is the transformation composed only from the translation in (x, t)-axis and the rotation in x-axis, ψ_j^γ is the component of Φ_γ^{-1}. Q. E. D.

Now, we present our main theorem in this section.

THEOREM 3.5. *Let $u_\varepsilon(x, t)$ be the solution of* (3.1) *with* (3.2). *Then, we have the following energy inequality.*

$$\|u_{\varepsilon t}(\cdot, t)\|^2 + \|u_\varepsilon(\cdot, t)\|^2_{V_{\alpha_\varepsilon(t)}} \leqq e^{CT}\{\|u_{1\varepsilon}(\cdot)\|^2_{V_{\alpha_\varepsilon(0)}} + \int_0^T \|f_\varepsilon(\cdot, s)\|^2 ds\} \tag{3.6}$$

for each $t \in [0, T]$ where C is the constant independent of the data and ε, $0 \leqq \varepsilon \leqq \varepsilon_0$.

PROOF. For the notational convention, we denote $u_\varepsilon(\cdot, t)$ by $v(\cdot, t)$ or $v(t)$.

By integration by parts, we have

$$\text{(3.7)} \qquad \operatorname{Re}(\square v(\cdot, t), X(\cdot, t)v(\cdot, t)) = I_1(t) + I_2(t) + I_3(t)$$

where

$$I_1(t) = \frac{d}{dt}\Big(\frac{1}{2}\|v_t(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2\Big) + \frac{d}{dt}\operatorname{Re}\sum_{j=1}^{n}(v_t(t), \beta_j(t)v_{x_j}(t)),$$

$$I_2(t) = \operatorname{Re}\Big[\sum_{k,j=1}^{n}\Big\{(v_{x_k}(t), \beta_{j,x_k}(t)v_{x_j}(t)) - \frac{1}{2}(\beta_{j,x_j}(t)v_{x_k}(t), v_{x_k}(t))\Big\}$$

$$+ \sum_{j=1}^{n}\{(\beta_{j,x_j}(t)v_t(t), v_t(t)) - (v_t(t), \beta_{j,t}(t)v_{x_j}(t))\}\Big],$$

$$I_3(t) = -\operatorname{Re}\int_{\partial\Omega}\frac{\partial v(\tilde{x}, t)}{\partial \boldsymbol{\nu}}\overline{(v_t(\tilde{x}, t) + \sum_{j=1}^{n}\beta_j(\tilde{x}, t)v_{x_j}(\tilde{x}, t))}d\tilde{x}.$$

Here, we used the fact that $\sum_{j=1}^{n}\beta_j(\tilde{x}, t)\nu_j(\tilde{x}) = 0$ on Σ.

Our problem is to calculate the term $I_3(t)$.

$$\text{(3.8)} \qquad \int_0^t I_3(\tau)d\tau = \int_0^t\int_{\substack{\partial\Omega \\ \alpha_\varepsilon(\tilde{x},\tau)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x}, \tau)}{\alpha_\varepsilon(\tilde{x}, \tau)}\frac{1}{2}X(\tilde{x}, \tau)|v(\tilde{x}, \tau)|^2 d\tilde{x}d\tau$$

$$= \sum_\gamma\int_0^t\int_{\substack{\partial\Omega \\ \alpha_\varepsilon(\tilde{x},\tau)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x}, \tau)}{\alpha_\varepsilon(\tilde{x}, \tau)}\frac{1}{2}X(\tilde{x}, \tau)|\sqrt{\rho_\gamma(\tilde{x}, t)}\, v(\tilde{x}, \tau)|^2 d\tilde{x}d\tau$$

$$= \sum_\gamma\iint_{\substack{\Phi_\gamma R_\gamma(V_\gamma\cap\Sigma) \\ \tilde{\alpha}_\varepsilon(\tilde{y},\sigma)\neq 0}}\frac{1-\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}{\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}\frac{1}{2}\frac{\partial}{\partial\sigma}|\widetilde{\sqrt{\rho_\gamma}\, v}(\tilde{y}, \sigma)|^2 J_{(\Phi_\gamma R_\gamma)^{-1}}d\tilde{y}d\sigma.$$

Here we used the facts that the integral is only effective on N_Γ, $\sum_\gamma \rho_\gamma(\tilde{x}, t) = 1$ near Γ. As $\sqrt{\rho_\gamma}$ has compact support in V_γ and $\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)$ is independent of σ on each $\Phi_\gamma R_\gamma(V_\gamma\cap\Sigma)$, we have, by integration by parts,

$$\text{(3.9)} \qquad \int_0^t I_3(\tau)d\tau = \sum_{\substack{\gamma \\ V_\gamma\cap P(t)\neq\emptyset}}\int_{\substack{\partial\Omega \\ \alpha_\varepsilon(\tilde{x},t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x}, t)}{\alpha_\varepsilon(\tilde{x}, t)}\frac{1}{2}|(\sqrt{\rho_\gamma}\, v)(\tilde{x}, t)|^2 d\tilde{x}$$

$$- \sum_{\substack{\gamma \\ V_\gamma\cap P(0)\neq\emptyset}}\int_{\substack{\partial\Omega \\ \alpha_\varepsilon(\tilde{x},0)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x}, 0)}{\alpha_\varepsilon(\tilde{x}, 0)}\frac{1}{2}|(\sqrt{\rho_\gamma}\, v)(\tilde{x}, 0)|^2 d\tilde{x}$$

$$- \sum_\gamma\iint_{\substack{\Phi_\gamma R_\gamma(V_\gamma\cap\Sigma) \\ \tilde{\alpha}_\varepsilon(\tilde{y},\sigma)\neq 0}}\frac{1-\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}{\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}\frac{1}{2}|\sqrt{\rho_\gamma}\, v(\tilde{y}, \sigma)|^2\frac{\partial}{\partial\sigma}(J_{(\Phi_\gamma R_\gamma)^{-1}})d\tilde{y}d\sigma.$$

The last term in (3.9) is calculated as follows:

$$\sum_\gamma\iint_{\substack{\Phi_\gamma R_\gamma(V_\gamma\cap\Sigma) \\ \tilde{\alpha}_\varepsilon(\tilde{y},\sigma)\neq 0}}\frac{1-\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}{\tilde{\alpha}_\varepsilon(\tilde{y}, \sigma)}\frac{1}{2}|(\widetilde{\sqrt{\rho_\gamma}\, v})(\tilde{y}, \sigma)|^2\Big(\frac{\partial}{\partial\sigma}J_{(\Phi_\gamma R_\gamma)^{-1}}\Big)$$

$$\times J^{-1}_{(\Phi_\gamma R_\gamma)^{-1}}J_{(\Phi_\gamma R_\gamma)^{-1}}d\tilde{y}d\sigma$$

$$=\int_0^t\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},\tau)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},\tau)}{\alpha_\varepsilon(\tilde{x},\tau)}\frac{1}{2}|v(\tilde{x},\tau)|^2\lambda(\tilde{x},\tau)d\tilde{x}d\tau$$

where

$$\lambda(\tilde{x},\tau)=\sum_\gamma\left[\left(\frac{\partial}{\partial\sigma}J_{(\Phi_\gamma R_\gamma)^{-1}}\right)J^{-1}_{(\Phi_\gamma R_\gamma)^{-1}}\right]\circ(\Phi_\gamma R_\gamma)(\tilde{x},\tau)$$

and bounded by M on $N_\Gamma\cap\Sigma$. Remarking that

$$\|v(t)\|^2-\|v(0)\|^2=2\,\mathrm{Re}\int_0^t(v(s),\ v_s(s))ds\,,$$

we integrate (3.7) over $(0, t)$ and we have

$$\frac{1}{2}\|v(t)\|_E^2+\frac{1}{2}\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},t)}{\alpha_\varepsilon(\tilde{x},t)}|v(\tilde{x},t)|^2d\tilde{x} \tag{3.10}$$

$$\leqq\frac{1}{2}\|v(0)\|_E^2+\frac{1}{2}\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},0)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},0)}{\alpha_\varepsilon(\tilde{x},0)}|v(\tilde{x},0)|^2d\tilde{x}$$

$$+\int_0^t|\mathrm{Re}(f_\varepsilon(s),\ Xv(s))|ds+\int_0^t|\mathrm{Re}(v(s),\ v_s(s))|ds$$

$$+\Big|\sum_{j=1}^n\{(v_t(t),\ \beta_jv_{x_j}(t))-(v_t(0),\ \beta_jv_{x_j}(0))\}\Big|$$

$$+\int_0^t|I_2(s)|ds+\int_0^t\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},\tau)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},\tau)}{\alpha_\varepsilon(\tilde{x},\tau)}\frac{1}{2}|v(\tilde{x},\tau)|^2|\lambda(\tilde{x},\tau)|d\tilde{x}d\tau$$

where $\|v(t)\|_E^2=\|v_t(t)\|^2+\|v(t)\|^2+\sum_{j=1}^n\|v_{x_j}(t)\|^2$.

As $|\beta_j|<1$ by Proposition 3.4 and the support of β_j is compact, there exists δ such that $|\beta_j|\leqq 1-\delta$ for $1\leqq j\leqq n$. As all coefficients in (3.10) are bounded, we have the desired inequality immediately from (3.10) by applying Gronwall's lemma.

Q. E. D.

§ 4. Proof of Theorem A.

Let the data $\{u_0(x), u_1(x), f(x, t)\}$ of the problem (1.1) belong to $V_0(0)\times L^2(\Omega)\times L^2(Q)$. Then, by Proposition 3.2, there exist the data $\{u_{0\varepsilon}(x), u_{1\varepsilon}(x), f_\varepsilon(x, t)\}$, for sufficiently small $\varepsilon>0$, which satisfy the conditions (i)~(iii). Moreover, as $u_0(x)\in V_0(0)$ and $\alpha_\varepsilon(\tilde{x}, t)=\frac{1}{2}$ on $\Gamma(t)$, we have

$$\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},0)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},0)}{\alpha_\varepsilon(\tilde{x},0)}|u_{0\varepsilon}(x)|^2d\tilde{x}\leqq C\|u_0\|^2_1 \tag{4.1}$$

where the constant C is independent of ε.

From the inequality (3.6) and the remark above, we have

$$\|u_{\varepsilon t}(t)\|^2+\|u_\varepsilon(t)\|_1^2+\int_{\substack{\partial\Omega\\ \alpha_\varepsilon(\tilde{x},t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},t)}{\alpha_\varepsilon(\tilde{x},t)}|u_\varepsilon(\tilde{x},t)|^2d\tilde{x}$$
$$\leqq C\{\|u_1\|^2+\|u_0\|_1^2+\int_0^T\|f(s)\|^2ds\}, \tag{4.2}$$

where C is independent of ε.

So taking a subsequence suitably from $\{u_\varepsilon(x,t)\}_{\varepsilon>0}$, if necessary, we may suppose that

(a) there exists a function $u(x,t)$ which is a limit of $u_\varepsilon(x,t)$ weak starly in $L^\infty(0,T;H^1(\Omega))$ and weakly in $H^1(Q)$, and

(b) $\{u_{\varepsilon t}(x,t)\}$ converges to $v(x,t)$ in weak starly in $L^\infty(0,T;L^2(\Omega))$. But necessarily $v=u_t$.

Multiplying a test function $\eta(x,t)$ to (3.1), we have by integration by parts,

$$-\int_0^T(u_{\varepsilon t},\eta_t)dt+\int_0^T\sum_{k=1}^n(u_{\varepsilon x_k},\eta_{x_k})dt$$
$$=(u_{1\varepsilon},\eta(0))+\int_0^T(f_\varepsilon,\eta)dt-\iint_{\substack{\Sigma\\ \alpha_\varepsilon(\tilde{x},t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},t)}{\alpha_\varepsilon(\tilde{x},t)}u_\varepsilon(\tilde{x},t)\overline{\eta(\tilde{x},t)}d\tilde{x}dt \tag{4.3}$$

where $\{u_\varepsilon\}$ is itself assumed to be a subsequence.

As $\eta(x,t)\in V_0(t)$ a. e. t, we calculate the boundary integral

$$\left|\iint_{\substack{\Sigma\\ \alpha_\varepsilon(x,t)\neq 0}}\frac{1-\alpha_\varepsilon(\tilde{x},t)}{\alpha_\varepsilon(\tilde{x},t)}u_\varepsilon(\tilde{x},t)\overline{\eta(\tilde{x},t)}d\tilde{x}dt\right|$$
$$\leqq\sqrt{\iint_\Sigma|u_\varepsilon(\tilde{x},t)|^2d\tilde{x}dt}\cdot\sqrt{\iint_\Sigma\chi_\varepsilon(\tilde{x},t)|\eta(\tilde{x},t)|^2d\tilde{x}dt} \tag{4.4}$$

where $\chi_\varepsilon(\tilde{x},t)=1$ on $\partial_N\Omega(t)-\{(\tilde{x},t)\in\Sigma;\ \alpha_\varepsilon(\tilde{x},t)=0\}$, and $=0$ otherwise. Applying Lebesgue's convergence theorem, we see that the boundary integral converges to 0 when ε tends to 0.

So, making ε tend to 0 in (4.3), we have, for any test function η,

$$-\int_0^T(u_t,\eta_t)dt+\int_0^T\sum_{k=1}^n(u_{x_k},\eta_{x_k})dt=(u_1,\eta(0))+\int_0^T(f,\eta)dt. \tag{4.5}$$

LEMMA 4.1. *Let $\chi(\tilde{x},t)$ be the characteristic function of $\partial_0\Omega(t)$, and let u belong to $H^1(\Omega)$. Then, u belongs to $V_0(t)$ if and only if*

$$\int_{\partial\Omega}\chi(\tilde{x},t)u(\tilde{x})\phi(\tilde{x})d\tilde{x}=0 \qquad \text{for any}\quad \phi(\tilde{x})\in C^\infty(\partial\Omega). \tag{4.6}$$

This lemma is easily proved by applying the well-known technique of elliptic boundary value problems. See Lemma 3.2 of [**6**].

For any $\phi(\tilde{x})\in C^{\infty}(\partial\Omega)$, we have, by Lemma 4.1,

$$\int_{\partial\Omega}\chi(\tilde{x},\ t)u(\tilde{x},\ t)\phi(\tilde{x})d\tilde{x}=\lim_{\varepsilon\to 0}\int_{\partial\Omega}\chi(\tilde{x},\ t)u_{\varepsilon}(\tilde{x},\ t)\phi(\tilde{x})d\tilde{x}=0\,. \tag{4.7}$$

This stands for $u(\cdot,\ t)\in V_0(t)$ a. e. t.

Then, following the same argument as was done in [**6**], we have the desired result. See also Lions-Magenes [**9**].

Moreover, from the inequality (4.2), the inequality (1.8) follows.

§5. Proof of Theorem B.

Let u be a weak solution satisfying $u_0=0$ and

$$\int_0^T\{-(u_t,\ \eta_t)+\sum_{j=1}^{n}(u_{x_j},\ \eta_{x_j})\}\,dt=0 \qquad \text{for any test function} \quad \eta. \tag{5.1}$$

We claim that $u=0$.

To prove this, we consider the following 'adjoint' problem.

$$\begin{cases} \square\,\eta=u & \text{in} \quad Q\,, \\ \eta(T)=\eta_t(T)=0\,, & \end{cases} \tag{5.2}$$

$$Y(\tilde{x},\ t)\frac{\partial\eta}{\partial\nu}+(1-Y(\tilde{x},\ t))\eta=0 \qquad \text{on} \quad \Sigma\,. \tag{5.3}$$

From the argument in the preceding sections, reversing the time direction, we have

LEMMA 5.1. *As Γ is time-like, there exists a function $\tilde{\eta}$ such that*

$$\tilde{\eta}\in L^{\infty}(0,\ T\,;\ H^1(\Omega))\,, \qquad \tilde{\eta}_t\in L^{\infty}(0,\ T\,;\ L^2(\Omega)) \tag{5.4}$$

$$\textit{and} \quad \tilde{\eta}(\cdot,\ t)\in V_0(t) \quad \textit{for a. e. } t\,,$$

$$\int_0^T\{-(v_t,\ \tilde{\eta}_t)+\sum_{j=1}^{n}(v_{x_j},\ \tilde{\eta}_{x_j})\}\,dt=\int_0^T(v,\ u)dt \tag{5.5}$$

holds for any $v\in L^1(0,\ T\,;\ H^1(\Omega))$, $v_t\in L^1(0,\ T\,;\ H^2(\Omega))$, $v(\cdot,\ t)\in V_0(t)$ for a. e. t, and

$$v(0)=v_t(0)=0\,. \tag{5.6}$$

From the definition of weak solution, as the function η in (5.1), we take the solution $\tilde{\eta}$ in Lemma 5.1. Moreover, as the function v in (5.5), we may take u of (5.1). These mean that $\int_0^T \|u(t)\|^2 dt = 0$. This is what we want to prove.

Q. E. D.

COROLLARY OF THEOREM B. *As the weak solution u of* (1.1) *with* (1.2) *is unique, that sequence $\{u_\varepsilon\}_{\varepsilon>0}$ itself converges to u where u_ε is the solution of the auxiliary problem* (3.1) *with* (3.2).

Finally, Theorem C is easily proved by remarking that the approximating solution $u_\varepsilon(x, t)$ has the dependence domain independent of ε.

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