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Banach algebra structure in Fourier spaces and generalization of harmonic analysis on locally compact groups

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Abstract.

Let (M, G, α) be a continuous W^*-dynamical system. Then, the predual $(G \otimes_\alpha M)_*$ of the crossed product $G \otimes_\alpha M$ of M by G can be turned into a Banach algebra and some of the notions and theorems in harmonic analysis on locally compact groups are extended to the corresponding ones in the crossed products. Among others, one can get a criterion for T in $G \otimes_\alpha M$ to fall in M: $T=\pi_\alpha(x)$ for some $x \neq 0$ in M if and only if the support of T reduces to the unit e in G.

1. Introduction.

Generalizing the so-called Pontryagin's duality theorem for locally compact abelian groups, K. Saito [6] proved that, for a general locally compact group G, the predual $\mathfrak{M}(G)_*$ of $\mathfrak{M}(G)$, the von Neumann algebra generated by the left regular representation of G, becomes an involutive commutative Banach algebra by suitably introducing the multiplication in it and the spectrum space $\widehat{\mathfrak{M}(G)_*}$ of $\mathfrak{M}(G)_*$ is homeomorphic to the original group G.

P. Eymard [3], on the other hand, regarded $\mathfrak{M}(G)_*$ as a regular function algebra $A(G)$ on G. He called it the Fourier algebra of G and he showed, by using techniques in function algebras and in von Neumann algebras, that some of the notions and results of harmonic analysis on locally compact abelian groups can be extended to the non-abelian case.

Recently, in order to have an explicit form of the element of a continuous W^*-crossed product $G \otimes_\alpha M$, H. Takai [8] introduced the notion of Fourier spaces and, under the condition that (M, G, α) be a G-finite separable continuous W^*-dynamical system, he showed that Gelfand-Raikov's and Godement's theorems in harmonic analysis on locally compact groups can be generalized in (M, G, α). And he added a remark that the predual $(G \otimes_\alpha M)_*$ can be regarded as the space of continuous functions of G into the predual M_* and this, in the case where M is the complex number field, coincides with the Fourier algebra $A(G)$ of G in Eymard's sense.

In the present paper, we define Fourier spaces in the most general cases. They are Banach algebras as they were in the scalar case, and some of the notions and results in harmonic analysis can be naturally extended. In particular, a generalization of Beurling's theorem, which is a criterion for an operator in $G\otimes_{\alpha}M$ to fall in M, will be proved. Similar results are also found in [4], [5]. But one will see that we get the theorem as a natural extension of the harmonic analysis to crossed products and the way we have followed is different from theirs.

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2. Notations and preliminaries.

In this section, we give some remarks on notations and recall some basic properties of Fourier algebras of P. Eymard which will be frequently used in the sequel.

Let G be a locally compact group. Concerning the Haar integrals on G, we follow the notations in [3]: the only exception is that we denote $\mathfrak{M}(G)$ the von Neumann algebra generated by the left regular representation ρ of G on $L^2(G)$.

Let (M, G, α) be a continuous W^*-dynamical system (abbreviated as W^*-d. s.). This means that G is a locally compact group, M is a von Neumann algebra and α is a homomorphism of G into the group $\mathrm{Aut}(M)$ of all *-automorphisms of M such that the function $G\ni g\mapsto\varphi\circ\alpha_g(x)$ is continuous for all $\varphi\in M_*$ (the predual of M) and $x\in M$. We assume that M acts on a Hilbert space $\mathfrak{H}$. Then, the continuous W^*-crossed product $G\otimes_{\alpha}M$ of M by G is a von Neumann algebra acting on the Hilbert space $L^2(G;\mathfrak{H})$, which is generated by the operators of the following types;

i) $(\pi_{\alpha}(x)\xi)(h)=\alpha_h^{-1}(x)\xi(h),\quad \xi\in L^2(G;\mathfrak{H}),\quad x\in M$

ii) $(\lambda(g)\xi)(h)=\xi(g^{-1}h)$.

Then, π_{α} defines a faithful normal *-representation of M into $G\otimes_{\alpha}M$ and λ is a continuous unitary representation of G on $L^2(G;\mathfrak{H})$. Moreover, (π_{α}, λ) is a covariant representation of M, in the sense that

$$\lambda(g)\pi_{\alpha}(x)\lambda(g)^{-1}=\pi_{\alpha}\circ\alpha_g(x)\qquad\text{for any}\quad x\in M,\quad g\in G,$$

(see [9]). We denote by $M(G)$ the von Neumann subalgebra of $G\otimes_{\alpha}M$ generated by $\lambda(g)$ for all $g\in G$.

Let Ω be a locally compact space and X a Banach space. Then, we denote by $C^b(\Omega; X)$ (resp. $C_0(\Omega; X)$, $K(\Omega; X)$) the space of all continuous and bounded (resp. vanishing at infinity, with compact support) functions f of Ω into X with the norm $\|f\|_\infty = \sup_{s\in\Omega}|f(s)|$. When X is the complex number field, we simply denote it by $C^b(\Omega)$ (resp. $C_0(\Omega)$, $K(\Omega)$).

Finally, we collect here some of the basic properties of Fourier algebra $A(G)$ of G which will be freely used in the sequel:

(i) $A(G)$ is the set of all complex valued functions on G of the form $f*\tilde{g}$ ($f, g\in L^2(G)$), where $\tilde{g}(a)=\overline{g(a^{-1})}$.

(ii) $A(G)$ is a closed ideal of $B(G)$, where $B(G)$ is the commutative Banach algebra of finite linear combinations u of continuous positive definite functions on G with the norm

$$\|u\| = \sup_{\|\pi(f)\|\leq 1,\, f\in K(G)} \left|\int f(g)\, u(g)\, dg\right|,$$

where π varies over all continuous unitary representations of G, the multiplication in $B(G)$ being the pointwise multiplication. When $u\in A(G)$,

$$\|u\| = \sup_{\|\rho(f)\|\leq 1,\, f\in K(G)} \left|\int f(g)\, u(g)\, dg\right|.$$

(iii) For every compact set K and open set U of G such that $K\subset U$, there exists $u\in A(G)$ with the properties

1) $0\leqq u(s)\leqq 1$ for all $s\in G$;

2) $u=1$ on K

3) $\mathrm{supp}(u)\subset U$.

(iv) $K(G)\cap A(G)$ is dense in $A(G)$.

For details, see [3]. For von Neumann algebras and C^*-algebras, we refer to [1] and [2].

3. The Banach algebra structure in Fourier spaces.

In this section, we shall define the Fourier space $F_\alpha(G; M_*)$ associated with a continuous W^*-d.s. (M, G, α) and the Banach algebra structure in $F_\alpha(G; M_*)$ so that, when M is the complex number field, $F_\alpha(G; M_*)$ is reduced to the Fourier algebra $A(G)$. In general, $F_\alpha(G; M_*)$ is not commutative. We study then the relation between $F_{\alpha}(G;M_{*})$ and $A(G)$, and prove that G is homeomorphic to the space $\widehat{F_{\alpha}(G;M_{*})}$ of all non-zero complex homomorphisms of $F_{\alpha}(G;M_{*})$, when $\widehat{F_{\alpha}(G;M_{*})}$ is equipped with the topology inherited from the weak*-topology of $F_{\alpha}(G;M_{*})^{*}$. We have also a generalization of Wiener's tauberian theorem concerning a dense left ideal in $F_{\alpha}(G;M_{*})$.

Let (M, G, α) be a continuous W^{*}-d. s. and M_{α} be the set of all elements $x\in M$ such that $G\ni g\mapsto\alpha_{g}(x)\in M$ is continuous by norm. Then, M_{α} is a G-invariant σ-weakly dense C^{*}-subalgebra of M with identities in common. Define in $K(G;M_{\alpha})$ the involutive normed algebra structure by

$$x*y(g)=\int x(h)\alpha_{h}(y(h^{-1}g))dh$$

$$x^{*}(g)=\Delta(g)^{-1}\alpha_{g}(x(g^{-1})^{*})$$

$$\|x\|_{1}=\int\|x(g)\|dg \qquad x,\ y\in K(G;M_{*}).$$

For each $x\in K(G;M_{\alpha})$, we associate an operator $\pi_{\alpha}(x)$ of $G\otimes_{\alpha}M$ by

$$\hat{\pi}_{\alpha}(x)=\int\pi_{\alpha}(x(g))\lambda(g)dg,$$

the integral being taken in the σ-weak topology of $G\otimes_{\alpha}M$. Then, it is not hard to see that $\hat{\pi}_{\alpha}$ is a faithful *-representation of $K(G;M_{\alpha})$ into $G\otimes_{\alpha}M$ and the image of $K(G;M_{\alpha})$ by $\hat{\pi}_{\alpha}$ is a σ-weakly dense *-subalgebra of $G\otimes_{\alpha}M$. For each continuous function u on G with values in the Banach space M_{*}, define

$$\|u\|=\sup_{\|\hat{\pi}_{\alpha}(x)\|\leqq 1\ x\in K(G;M_{\alpha})}\left|\int u(g)(x(g))dg\right|. \tag{3.1}$$

Let $B_{\alpha}(G;M_{*})$ be the set of all continuous functions u on G into M_{*} such that $\|u\|<+\infty$. Then, (3.1) defines a norm on $B_{\alpha}(G;M_{*})$ and $B_{\alpha}(G;M_{*})$ is a Banach space under this norm. Moreover,

$$\|u\|_{\infty}\leqq\|u\|,$$

so that, in particular, $B_{\alpha}(G;M_{*})$ is contained in $C^{b}(G;M_{*})$. This fact is proved as follows; for each $g\in G$, let $\mathfrak{B}(g)$ be a filter basis of neighborhoods of g in G and, for each V in $\mathfrak{B}(g)$, let ϕ_{V} be a non-negative function in $K(G)$ with $\operatorname{supp}(\phi_{V})\subset V$ and the integral equal to one. For $x\in M_{\alpha}$, put

$$x(\psi_V)(s)=\psi_V(s)\,x.$$

Clearly, $x(\psi_V)\in K(G;M_\alpha)$ and

$$\|x(\psi_V)\|_1=\|x\|.$$

Since $\|\hat{\pi}_\alpha(X)\|\leqq\|X\|_1$ for all $X\in K(G;M_\alpha)$,

$$\|u\|\geqq\left|\int u(s)(x(\psi_V)(s))\,ds\right|=\left|\int\psi_V(s)\,u(s)(x)\,ds\right|$$

if $x\in M_\alpha$ with $\|x\|\leqq 1$. The function $G\ni s\mapsto u(s)(x)$ being continuous, we have

$$\lim_V\int\psi_V(s)\,u(s)(x)\,ds=u(g)(x),$$

so that

$$\|u\|\geqq|u(g)(x)|$$

for all $g\in G$ and $x\in M_\alpha$ with $\|x\|\leqq 1$. Now the σ-weak continuity of $M\ni x\mapsto u(g)(x)$ and the σ-weak density of M_α in M implies $\|u\|\geqq\|u\|_\infty$. The rest will be almost clear and we omit the proof.

On the other hand, each element u of the predual $(G\otimes_\alpha M)_*$ of $G\otimes_\alpha M$ may be regarded as an element of $B_\alpha(G;M_*)$ by defining

$$u(g)(x)=\langle\pi_\alpha(x)\lambda(g),\,u\rangle\quad(\equiv u(\pi_\alpha(x)\lambda(g))).$$

It is obvious by definition that the above map from $(G\otimes_\alpha M)_*$ into $B_\alpha(G;M_*)$ is linear and isometric, and so there will be no confusion by using the same notation whichever u belongs to. The following definition is due to H. Takai [8].

(3.2) DEFINITION. The space $(G\otimes_\alpha M)_*$ considered as the closed subspace of $B_\alpha(G;M_*)$ by the above correspondence will be called the Fourier space associated with (M, G, α) and will be denoted by $F_\alpha(G;M_*)$.

(3.3) REMARK 1°. When M reduces to scalars, $B_\alpha(G;M_*)=B_\rho(G)$ (see [3]).

2°. The most general form of Proposition 3.1 in [8] can be obtained in the C^*-d. s. (M_α, G, α) too, but we will not discuss it here.

Here we get the following lemma in the most general setting which plays a fundamental role in the sequel;

(3.4) LEMMA. *The elements of* $F_\alpha(G;M_*)$ *with compact support form a dense subspace of* $F_\alpha(G;M_*)$. *Hence, in particular,* $F_\alpha(G;M_*)$ *is contained in* $C_0(G;M_*)$.

In fact, for $u=\omega_{\zeta,\eta}\in F_\alpha(G;M_*)$, where $\eta=e_1\otimes\psi_1$, $\zeta=e_2\otimes\psi_2\in\mathfrak{H}\otimes L^2(G)$ $=L^2(G;\mathfrak{H})$ and $\psi_1,\psi_2\in K(G)$,

$$\begin{aligned}|u(g)(x)|&=|(\pi_\alpha(x)\lambda(g)\zeta|\eta)|\\&=\left|\int(\alpha_h^{-1}(x)\zeta(g^{-1}h)|\eta(h))\,dh\right|\\&=\left|\int(\alpha_h^{-1}(x)e_2|e_1)\psi_2(g^{-1}h)\overline{\psi_1(h)}\,dh\right|\\&\leqq\|x\|\ \|e_1\|\ \|e_2\|\ |\psi_1|*|\widetilde{\psi_2}|(g),\end{aligned}$$

so that we have

$$\|u(g)\|\leqq\|e_1\|\ \|e_2\|\ |\psi_1|*|\widetilde{\psi_2}|(g).$$

Since $|\psi_1|*|\widetilde{\psi_2}|$ has a compact support, so does u. Next, let E be the dense subspace of $L^2(G;\mathfrak{H})$ generated algebraically by elements of the form $e\otimes\psi$, where $e\in\mathfrak{H}$, $\psi\in K(G)$. Then, for each $\zeta,\eta\in L^2(G;\mathfrak{H})$, there exist sequences ζ_n, η_n in E such that $\zeta=\lim_{n\to\infty}\zeta_n$, $\eta=\lim_{n\to\infty}\eta_n$. Since

$$\begin{aligned}&|\langle T,\omega_{\zeta,\eta}\rangle-\langle T,\omega_{\zeta_n,\eta_n}\rangle|\\&\quad=|(T\zeta|\eta)-(T\zeta_n|\eta_n)|\\&\quad\leqq\|T\|(\|\zeta-\zeta_n\|\ \|\eta\|+\|\eta-\eta_n\|\ \|\zeta_n\|),\end{aligned}$$

we obtain

$$\|\omega_{\zeta,\eta}-\omega_{\zeta_n,\eta_n}\|\leqq\|\zeta-\zeta_n\|\ \|\eta\|+\|\eta-\eta_n\|\ \|\zeta_n\|\longrightarrow 0,$$

when $n\to\infty$. By the preceding argument, ω_{ζ_n,η_n} belongs to $F_\alpha(G;M_*)\cap K(G;M_*)$, so that $\omega_{\zeta,\eta}$ belongs to the closure of $F_\alpha(G;M_*)\cap K(G;M_*)$. Every element of $F_\alpha(G;M_*)$ being the limit of finite linear combinations of such elements, $F_\alpha(G;M_*)$ is contained in the closure of $F_\alpha(G;M_*)\cap K(G;M_*)$. Since $\|u\|_\infty\leqq\|u\|$, $F_\alpha(G;M_*)$ is contained in $C_0(G;M_*)$.

Now, define a unitary operator W of $L^2(G\times G;\mathfrak{H}\otimes\mathfrak{H})$ onto itself by

$$(W\xi)(s,t)=\xi(s,st),$$

where $\xi\in L^2(G\times G;\mathfrak{H}\otimes\mathfrak{H})$. Then, it is not hard to see that

$$W^{-1}(\pi_\alpha(x)\lambda(g)\otimes 1)W=\pi_\alpha(x)\lambda(g)\otimes\lambda(g)$$

for every $x\in M$ and $g\in G$. Put

$$\delta(T)=W^{-1}(T\otimes 1)W\qquad T\in G\otimes_\alpha M.$$

Then, δ is a faithful normal *-representation of $G\otimes_\alpha M$ into $(G\,{}_\alpha\!\otimes M)\otimes(G\otimes_\alpha M)$ such that

$$\delta(\pi_\alpha(x)\lambda(g))=\pi_\alpha(x)\lambda(g)\otimes\lambda(g),$$

therefore

$$\delta(G\otimes_\alpha M)\subset(G\otimes_\alpha M)\otimes M(G).$$

With the above preparations, we get the following

(3.5) THEOREM. *For $u, v\in F_\alpha(G;M_*)$, define the product by*

$$u*v(g)(x)=u(g)(x)v(g)(1)\qquad g\in G,\quad x\in M.$$

Then, $u*v$ falls in $F_\alpha(G;M_*)$ and makes $F_\alpha(G;M_*)$ into an associative Banach algebra. Clearly, when M reduces to the complex number field, this product coincides with the usual pointwise product of scalar functions in $A(G)$. (For a clear meaning of this theorem, see [7], p. 1429, 1. 6-7.)

PROOF. For each $u, v\in F_\alpha(G;M_*)$, the functional given by the map $G\otimes_\alpha M\ni T\mapsto\langle\delta(T), u\otimes v\rangle$ is in $(G\otimes_\alpha M)_*$, where $u\otimes v\in(G\otimes_\alpha M)_*\hat{\otimes}_{\alpha_0*}(G\otimes_\alpha M)_*$ in the sense of Turumaru [10]. Therefore there exists a unique $w\in F_\alpha(G;M_*)$ such that

$$\langle T, w\rangle=\langle\delta(T), u\otimes v\rangle$$

for all $T\in G\otimes_\alpha M$ and we have

$$\|w\|\leqq\|u\|\,\|v\|. \tag{3.6}$$

Putting $T=\pi_\alpha(x)\lambda(g)$,

$$\begin{aligned}\langle\pi_\alpha(x)\lambda(g), w\rangle&=\langle\pi_\alpha(x)\lambda(g)\otimes\lambda(g), u\otimes v\rangle\\&=\langle\pi_\alpha(x)\lambda(g), u\rangle\,\langle\lambda(g), v\rangle,\end{aligned}$$

so that we have

$$w(g)(x)=u(g)(x)v(g)(1)=u*v(g)(x).$$

By this expression, it is easily seen that $F_\alpha(G;M_*)$ is an associative algebra and also a Banach algebra by (3.6), which completes the proof.

We now arrived at the starting point of the study of the algebra structure of $F_\alpha(G;M_*)$, but before proceeding further, we notice the left and right module structure of $G\otimes_\alpha M$ over $F_\alpha(G;M_*)$ which will play an essential role in §4.

(3.7) DEFINITION. For $T \in G \otimes_\alpha M$ and $u \in F_\alpha(G; M_*)$, the mapping

$$F_\alpha(G; M_*) \ni v \longmapsto \langle T, u*v \rangle$$

defines a bounded linear functional on $F_\alpha(G; M_*)$ and hence, by the duality between $F_\alpha(G; M_*)$ and $G \otimes_\alpha M$, there exists a unique element of $G \otimes_\alpha M$, denoted by Tu, such that

$$\langle Tu, v \rangle = \langle T, u*v \rangle$$

for all $v \in F_\alpha(G; M_*)$. Analogously, we define the product vT of T by v on the left by $\langle vT, u \rangle = \langle T, u*v \rangle$.

(3.8) REMARK. If M is the complex number field, the above definition coincides with that of [3] (Chap. 4, Definition 4.1), and in that case $uT = Tu$. But, in general, this is not the case.

(3.9) By the operation $G \otimes_\alpha M \times F_\alpha(G; M_*) \ni (T, u) \mapsto Tu \in G \otimes_\alpha M$, $G \otimes_\alpha M$ becomes a right module over $F_\alpha(G; M_*)$ and

$$\|Tu\| \leqq \|T\| \; \|u\|.$$

The left module structure of $G \otimes_\alpha M$ is to be similarly understood and

$$\|uT\| \leqq \|u\| \; \|T\|.$$

Moreover,

$$\begin{aligned} &[\pi_\alpha(x)\lambda(g)]u = u(g)(x)\lambda(g) \\ &u[\pi_\alpha(x)\lambda(g)] = u(g)(1)\pi_\alpha(x)\lambda(g) \end{aligned} \tag{3.10}$$

so that

(3.11) $\qquad Tu \in M(G) \quad$ for all $\; T \in G \otimes_\alpha M \;$ and $\; u \in F_\alpha(G; M_*)$.

(3.12) LEMMA. *Put* $\tau(u)(g) = u(g)(1)$ $(g \in G, u \in F_\alpha(G; M_*))$. *Then,* τ *defines a continuous homomorphism of* $F_\alpha(G; M_*)$ *onto* $A(G)$ *with norm* $\|\tau\| \leqq 1$. *Moreover* τ *maps the cone* $F_\alpha(G; M_*)_+$ *of all* α*-positive definite functions (see* [8], p. 856) *of* $F_\alpha(G; M_*)$ *onto the cone* $A(G)_+$ *of all positive definite functions of* $A(G)$ *isometrically.*

REMARK. In the following, τ will always denote this homomorphism.

PROOF. First we remark the cone of all α-positive definite (resp. positive definite) functions in $F_\alpha(G; M_*)$ (resp. $A(G)$) can be identified with the cone of all positive normal linear functionals of $G \otimes_\alpha M$ (resp. $\mathfrak{M}(G)$). Since, for each $u \in F_\alpha(G; M_*)$, the functional

$$\mathfrak{M}(G) \ni t \longmapsto \langle 1 \otimes t, u \rangle$$

is in $\mathfrak{M}(G)_*$, there exists a unique $\bar{u}\in A(G)$ such that

$$\langle t, \bar{u}\rangle=\langle 1\otimes t, u\rangle \qquad \text{for all} \quad t\in\mathfrak{M}(G).$$

By making $t=\rho(g)$ $(g\in G)$, we have

$$\tau(u)(g)=u(g)(1)=\bar{u}(g).$$

Conversely, if $\omega\in A(G)_+$, there exists an $f\in L^2(G)$ such that

$$\omega(g)=\bar{f}*\check{f}(g)=(\rho(g)f\,|\,f) \qquad \text{for all} \quad g\in G, \quad \text{where} \quad \check{f}(g)=f(g^{-1}).$$

Hence, if we take $e\in\mathfrak{H}$ with $\|e\|=1$ and put $u=\omega_{\eta,\eta}$ $(\eta=e\otimes f,\ e\otimes f\in L^2(G\,;\mathfrak{H}))$, u belongs to $F_\alpha(G\,;M_*)_+$ and

$$\tau(u)=\omega.$$

Since every element of $F_\alpha(G\,;M_*)$ (resp. $A(G)$) is a finite linear combination of elements in $F_\alpha(G\,;M_*)_+$ (resp. $A(G)_+$), we may conclude that τ is a map of $F_\alpha(G\,;M_*)$ onto $A(G)$. The rest will be almost clear and this completes the proof.

(3.13) DEFINITION. If A is a complex algebra, we denote by $\hat{A}$ the set of all non-zero complex homomorphisms with $\sigma(A_a^*, A)$-topology, where A_a^* is the algebraic dual of A. Note that $\hat{A}$ is contained in the topological dual of A if A is a Banach algebra.

The following theorem may be regarded as a generalization of [**3**], Theorem 3.34 (see also [**6**] p. 358, Theorem). We prove it by a reduction to the scalar case so that the relation between $F_\alpha(G\,;M_*)$ and $A(G)$ will get clear.

(3.14) THEOREM. *G is homeomorphic to $\widehat{F_\alpha(G\,;M_*)}$; the correspondence $G\ni g\mapsto \Phi_g\in\widehat{F_\alpha(G\,;M_*)}$ is given by*

$$\Phi_g(u)=u(g)(1) \qquad \textit{for all} \quad u\in F_\alpha(G\,;M_*).$$

PROOF. Let J be the kernel of τ. Then, J is a closed two-sided ideal of $F_\alpha(G\,;M_*)$ (Lemma 3.12), and $F_\alpha(G\,;M_*)/J$ is algebraically isomorphic to $A(G)$, so that it is a commutative Banach algebra. Hence $\widehat{F_\alpha(G\,;M_*)/J}$ is homeomorphic to $\widehat{A(G)}$ in the canonical way. Since $\widehat{A(G)}$ is homeomorphic to G ([**3**], Theorem 3.34), the only thing to be proved is that every $\Phi\in\widehat{F_\alpha(G\,;M_*)}$ vanishes on J. By the duality between $F_\alpha(G\,;M_*)$ and $G\otimes_\alpha M$, we may regard Φ as an element of $G\otimes_\alpha M$. Then, since Φ is a complex homomorphism on $F_\alpha(G\,;M_*)$, we have

$$\langle \Phi, u*v \rangle = \langle \Phi, u \rangle \langle \Phi, v \rangle$$

for all $u \in F_\alpha(G; M_*)$ and $v \in F_\alpha(G; M_*)$, which implies

$$\Phi u = \langle \Phi, u \rangle \Phi.$$

Since $\Phi \neq 0$, there exists $u_0 \in F_\alpha(G; M_*)$ such that $\langle \Phi, u_0 \rangle \neq 0$. Hence we have, by 3.11,

$$\Phi = \langle \Phi, u_0 \rangle^{-1} \Phi u_0 \in M(G).$$

Noticing that $J = M(G)^0$, where $M(G)^0$ is the polar of $M(G)$ in $F_\alpha(G; M_*)$ we have

$$\langle \Phi, u \rangle = 0 \qquad \text{for all} \quad u \in J,$$

which completes the proof.

(3.15) COROLLARY. *$F_\alpha(G; M_*)/J$ is a semisimple commutative Banach algebra which is algebraically isomorphic to $A(G)$. In particular, $\widehat{F_\alpha(G; M_*)/J}$ is homeomorphic to $\widehat{A(G)}$.*

(3.16) COROLLARY. *Let (M_1, G_1, α_1) and (M_2, G_2, α_2) be two continuous W^*-d. ss. If $F_{\alpha_1}(G_1; M_{1*})$ and $F_{\alpha_2}(G_2; M_{2*})$ are algebraically isomorphic, G_1 and G_2 are homeomorphic.*

The following theorem may be seen as one version of Wiener's tauberian theorem, but it is not a direct consequence of Theorem 3.14 (cf. [**3**]).

(3.17) THEOREM. *Let I be a left ideal of $F_\alpha(G; M_*)$. If, for every $g \in G$, there exists $u \in I$ such that $u(g)(1) \neq 0$, then, I is dense in $F_\alpha(G; M_*)$.*

PROOF. Since $\tau(I)$ is an ideal of $A(G)$ satisfying the same condition, $\tau(I)$ contains $A(G) \cap K(G)$ (see the proof of [**3**] Corollary 3.38). Hence, the lemma 3.2 of [**3**] implies that, for every compact set K of G, there exists $u \in I$ such that $u(g)(1) = 1$ for all $g \in K$. Therefore, for every $v \in F_\alpha(G; M_*) \cap K(G; M_*)$, there exists $u \in I$ such that

$$v*u = v.$$

Since I is a left ideal, I must contain $F_\alpha(G; M_*) \cap K(G; M_*)$. As $F_\alpha(G; M_*) \cap K(G; M_*)$ is dense in $F_\alpha(G; M_*)$ (3.4 Lemma) we get the theorem.

4. Generalized Beurling's theorem.

In this section, we generalize the Beurling's theorem in harmonic analysis to the case of crossed products.

In [**5**], Section 5, Nakagami defined the "spectrum" of an operator T in $G \otimes_\alpha M$ which corresponds to our notion of support given below. He also obtained a generalization of Beurling's theorem in terms of his notion of spectrum by making use of the theory of vector valued weights. Our method here is a reduction to the scalar case which will make the whole argument more transparent.

We begin by the following proposition.

(4.1) PROPOSITION. (i) *Let* g (*resp.* T) *be an element of* G (*resp.* $G \otimes_{\alpha} M$). *Then, the following conditions are equivalent:*

1°. $\lambda(g)$ *belongs to the* σ*-weak closure of* $TF_{\alpha}(G; M_*)$;

2°. $vT=0$ *implies* $\tau(v)(g)=0$;

3°. *for every neighborhood* V *of* g, *there exists* $u \in F_{\alpha}(G; M_*)$ *with* $\mathrm{supp}(u) \subset V$ *such that* $\langle T, u\rangle \neq 0$.

(ii) *Let* g (*resp.* T) *be an element of* G (*resp.* $G \otimes_{\alpha} M$). *Then, the following conditions are equivalent:*

1°. $\pi(x)\lambda(g)$ *belongs to the* σ*-weak closure of* $F_{\alpha}(G; M_*)T$;

2°. $Tv=0$ *implies* $v(g)(x)=0$.

Moreover, if $x \neq 0$, *then* 2° *implies* (i)-3°.

PROOF. (i) Let I_T be the set of all $u \in F_{\alpha}(G; M_*)$ such that $uT=0$ and E_T the subspace of $G \otimes_{\alpha} M$ consisting of all elements of the form $Tu (u \in F_{\alpha}(G; M_*))$. Note that I_T forms a closed left ideal of $F_{\alpha}(G; M_*)$. And the polar E_T^0 of E_T in $F_{\alpha}(G; M_*)$ is equal to I_T. Therefore $\lambda(g)$ belongs to the σ-weak closure of E_T if and only if $\langle\lambda(g), u\rangle=0$ for all $u \in I_T$, which proves $1° \Leftrightarrow 2°$.

Next we prove $2° \Rightarrow 3°$; in fact, if 3° is not the case, there exists a neighborhood V of g such that $\langle T, u\rangle=0$ for all $u \in F_{\alpha}(G; M_*)$ with $\mathrm{supp}(u) \subset V$. On the other hand, there exists $u_0 \in F_{\alpha}(G; M_*)$ such that $\mathrm{supp}(\tau(u_0)) \subset V$ and $\tau(u_0)(g) \neq 0$ (Lemma 3.12). Hence, $\mathrm{supp}(u*u_0) \subset V$ for all $u \in F_{\alpha}(G; M_*)$ and, by assumption, we have

$$\langle u_0 T, u\rangle=\langle T, u*u_0\rangle=0 \qquad \text{for all} \quad u \in F_{\alpha}(G; M_*).$$

Hence, $u_0 T=0$ which is a contradiction and this proves $2° \Rightarrow 3°$.

Finally we prove $3° \Rightarrow 2°$; for this purpose, it suffices to show $\tau(u)(g) \neq 0$ implies $uT \neq 0$. We may assume $|\tau(u)(g)|>1$. Then, there exists a compact neighborhood V of g such that $|\tau(u)(s)| \geqq 1$ for all $s \in V$. Take then a $w \in F_{\alpha}(G; M_*)$ such that

$$\tau(w)(s)=\tau(u)(s)^{-1} \qquad \text{for all} \quad s \in V,$$

which is possible by $\tau(F_{\alpha}(G; M_*))=A(G)$. By hypothesis, there exists $h \in F_{\alpha}(G; M_*)$ with $\mathrm{supp}(h) \subset V$ and $\langle T, h\rangle \neq 0$. Put $v=h*w$. Then, it is not hard to see

$$v*u=h,$$

which implies

$$\langle uT, v\rangle=\langle T, v*u\rangle=\langle T, h\rangle\neq 0.$$

Therefore $uT\neq 0$.

Now, the proof of (i) is complete.

(ii) can be proved in a similar way.

(4.2) DEFINITION. Let $T\in G\otimes_{\alpha}M$. Then the support of T is defined as the set of all $g\in G$ satisfying the equivalent conditions of Proposition 4.1-(i) and it will be denoted by $\mathrm{supp}(T)$.

This is a direct generalization of that given in [3] (Definition 4.5).

In the following proposition, we show some basic properties of the support of operators, some of which will be used in the proof of the next theorem.

(4.3) PROPOSITION. (i) *Let* $T\in G\otimes_{\alpha}M$. *Then,*

1°. $\mathrm{supp}(T)$ *is a closed subset of* G;

2°. $T=0$ *if and only if* $\mathrm{supp}(T)=\emptyset$ ($\emptyset$=*empty set*).

(ii) *Let* $u\in F_{\alpha}(G\,;M_{*})$. *Then,*

1°. $\mathrm{supp}(Tu)\subset\mathrm{supp}(u)\cap\mathrm{supp}(T)$.

2°. $\mathrm{supp}(uT)\subset\mathrm{supp}(\tau(u))\cap\mathrm{supp}(T)$.

(iii) $\mathrm{supp}(T)$ *is the smallest closed set* F *of* G *satisfying the following condition:*

$$u\in F_{\alpha}(G\,;M_{*})\cap K(G\,;M_{*}) \quad \textit{and} \quad \mathrm{supp}(u)\cap F=\emptyset \quad \textit{imply} \quad \langle T, u\rangle=0.$$

(iv) $\mathrm{supp}(T)$ *is the smallest closed set* F *of* G *satisfying the following condition:*

for every closed neighborhood Ω *of* F *with a relatively compact complement* Ω' *in* G, T *is the* σ*-weak limit of operators of the form;*

$$\pi_{\alpha}(x_1)\lambda(g_1)+\cdots+\pi_{\alpha}(x_n)\lambda(g_n),$$

$$\textit{where} \quad x_i\in M \quad \textit{and} \quad g_i\in\Omega \ (i=1, 2, \cdots, n).$$

(v) *Let* Σ *be a closed subset of* G *and* T_{λ} $(\lambda\in\Lambda)$, T *be in* $G\otimes_{\alpha}M$. *If* $\mathrm{supp}(T_{\lambda})\subset\Sigma$ *for all* $\lambda\in\Lambda$ *and* T *is the* σ*-weak limit of* T_{λ}, $\mathrm{supp}(T)\subset\Sigma$.

(vi) 1°. $\mathrm{supp}(\pi_{\alpha}(x)T)\subset\mathrm{supp}(T)$ *for every* $x\in M$. *Hence, if* x *is invertible in* M, $\mathrm{supp}(\pi_{\alpha}(x)T)=\mathrm{supp}(T)$.

2°. $\mathrm{supp}(T^{*})=\mathrm{supp}(T)^{-1}$.

3°. $\mathrm{supp}(T_1+T_2)\subset\mathrm{supp}(T_1)\cup\mathrm{supp}(T_2)$, *and if* $\mathrm{supp}(T_1)\cap\mathrm{supp}(T_2)=\emptyset$, *the equality holds.*

4°. *Suppose* $\mathrm{supp}(T_1)$ *or* $\mathrm{supp}(T_2)$ *compact. Then,*

$$\mathrm{supp}(T_1 T_2) \subset \mathrm{supp}(T_1)\,\mathrm{supp}(T_2).$$

PROOF. This proposition is essentially the same as Proposition 4.8 of [3], so we only give the outline of the proof. Because of frequent use, we quote Proposition 4.8 of [3] as (E).

(i) 1° is obvious by Proposition 4.1-(i)-2° and $T=0$ implies $I_T=F_\alpha(G; M_*)$. Since $\tau(F_\alpha(G; M_*))=A(G)$, $\mathrm{supp}(T)=\emptyset$ if $T=0$ (Proposition 4.1-(i)-2°). Conversely, if $T\neq 0$, then $I_T\neq F_\alpha(G; M_*)$. This can be easily derived by an argument similar to that in (E). Since I_T is a closed left ideal of $F_\alpha(G; M_*)$, the validity of the tauberian theorem and Proposition 4.1-(i)-2° imply $\mathrm{supp}(T)\neq\emptyset$, which proves 2°.

(ii) can be easily derived from (i)-1° and Proposition 4.1-(i)-1°, 3° by an argument similar to that in (E).

(iii) is also deduced from (E) by noticing that for every $u\in F_\alpha(G; M_*)\cap K(G; M_*)$, there exists $\omega\in F_\alpha(G; M_*)$ such that $u*\omega=u$.

(iv) First we notice the following point; let I_Ω be the set of all $u\in F_\alpha(G; M_*)$ such that $u(g)(x)=0$ for all $g\in\Omega$ and $x\in M$. Then, I_Ω is the polar of $\boldsymbol{C}\langle\pi_\alpha(M)\lambda(\Omega)\rangle$ in $F_\alpha(G; M_*)$, where $\boldsymbol{C}\langle\pi_\alpha(M)\lambda(\Omega)\rangle$ is the subspace $G\otimes_\alpha M$ generated by the set $\pi_\alpha(M)\lambda(\Omega)=\{\pi_\alpha(x)\lambda(g); g\in\Omega, x\in M\}$. Hence, T vanishes on I_Ω if and only if T belongs to the σ-weak closure of $\boldsymbol{C}\langle\pi_\alpha(M)\lambda(\Omega)\rangle$. Now it is not hard to see that the closed sets of G satisfying the condition in (iv) are equal to those satisfying the condition in (iii) (cf. (E)) and this completes the proof.

(v) is a consequence of (iii) (cf. (E)).

(vi) 1° is a consequence of (iv). The use of (iv) proves 2° and 3° by noticing that the *-operation is σ-weakly continuous and that (π_α, λ) is a covariant representation of M (cf. (E)). Finally we prove 4°. First we remark that $\mathrm{supp}(\pi_\alpha(x)\lambda(g))=\{g\}$ if $x\neq 0$: this is an easy consequence of $v[\pi_\alpha(x)\lambda(g)]=v(g)(1)\pi_\alpha(x)\lambda(g)$ and Proposition 4.1-(i)-2°. Then, by using (iv) and noticing that (π_α, λ) is a covariant representation of M, we get

$$\mathrm{supp}(\pi_\alpha(x)\lambda(g)T)\subset\mathrm{supp}(\pi_\alpha(x)\lambda(g))\,\mathrm{supp}(T).$$

Now the validity of (iv), (v), (vi)-2° and (vi)-3° completes the proof (cf. (E)).

At this stage, we can prove the following generalized version of Beurling's theorem (cf. [3] Theorem 4.9) which gives a criterion that an operator $T\in G\otimes_\alpha M$ falls in M.

(4. 4) THEOREM. *Let T be an operator in $G \otimes_{\alpha} M$. Then, the support of T reduces to one point g in G if and only if T is of the form $T=\pi_{\alpha}(x)\lambda(g)$ for some $x \neq 0$ in M.*

PROOF. First we prove the following; if $S=1 \otimes S_1 \in 1 \otimes \mathfrak{M}(G)=M(G)$, then

$$\mathrm{supp}(S)=\mathrm{supp}(S_1).$$

The inclusion $\mathrm{supp}(S) \subset \mathrm{supp}(S_1)$ can be easily derived from Proposition 4. 3-(iv). Hence, we prove the opposite inclusion. If $u \in F_{\alpha}(G; M_*)$, $\langle \lambda(g), u \rangle = \tau(u)(g) = \langle \rho(g), \tau(u) \rangle$, so that

$$\langle S, u \rangle = \langle S_1, \tau(u) \rangle.$$

Put $F=\mathrm{supp}(S)$. Take $\omega \in A(G) \cap K(G)$ such that $\mathrm{supp}(\omega) \cap F = \emptyset$. Then, there exists u in $F_{\alpha}(G; M_*)$ such that $\tau(u)=\omega$. Since $\mathrm{supp}(\omega)$ is compact and F is closed, there exists v in $F_{\alpha}(G; M_*)$ such that $\tau(v)(s)=1$ if $s \in \mathrm{supp}(\omega)$ and $\mathrm{supp}(\tau(v))$ is compact and disjoint from F. Since $\mathrm{supp}(u*v) \subset \mathrm{supp}(\tau(v))$, $u*v$ belongs to $F_{\alpha}(G; M_*) \cap K(G; M_*)$ and $\mathrm{supp}(u*v) \cap F = \emptyset$. Hence, by Proposition 4. 3-(iii), $\langle S, u*v \rangle = 0$. Since

$$\langle S, u*v \rangle = \langle S_1, \tau(u*v) \rangle = \langle S_1, \tau(u)\tau(v) \rangle = \langle S_1, \omega\tau(v) \rangle = \langle S_1, \omega \rangle,$$

we obtain $\langle S_1, \omega \rangle = 0$. Now, Proposition 4. 8 of [3] implies

$$\mathrm{supp}(S_1) \subset F = \mathrm{supp}(S),$$

which proves $\mathrm{supp}(S)=\mathrm{supp}(S_1)$.

The "if" part is already shown in the proof of Proposition 4. 3. So we are to prove the "only if" part. By Proposition 4. 3-(vi)-4°, we may assume $\mathrm{supp}(T)=\{e\}$. Then, by Proposition 4. 3-(ii)-1°,

$$\mathrm{supp}(Tu)=\{e\} \quad \text{or} \quad \emptyset$$

for all $u \in F_{\alpha}(G; M_*)$. Since Tu belongs to $M(G)$ (3. 11), the argument in the above and Theorem 4. 9 of [3] imply

$$Tu=\nu(u)1,$$

where $\nu(u)$ is a complex number. It is clear that $F_{\alpha}(G; M_*) \ni u \mapsto \nu(u)$ is a bounded linear functional. Let u be an element of $F_{\alpha}(G; M_*) \cap K(G; M_*)$. Then, there exists $v \in F_{\alpha}(G; M_*)$ such that $\tau(v)(s)=1$ if $s \in \mathrm{supp}(u) \cup \{e\}$. Thus we get

$$\langle T, u \rangle = \langle T, u*v \rangle = \langle Tu, v \rangle = \nu(u)\langle 1, v \rangle = \nu(u)\tau(v)(e) = \nu(u).$$

The above equality holds for every $u \in F_{\alpha}(G; M_*)$ since $F_{\alpha}(G; M_*) \cap K(G; M_*)$ is dense in $F_{\alpha}(G; M_*)$ and ν is continuous. Hence, we get

$$\langle T, u*v\rangle=\langle T, u\rangle\langle 1, v\rangle,$$

for all $u, v \in F_\alpha(G; M_*)$, which implies

$$\langle\delta(T), u\otimes v\rangle=\langle T\otimes 1, u\otimes v\rangle.$$

Hence, we get $\delta(T)=T\otimes 1$. Therefore $T=\pi_\alpha(x)$ for some $x\in M$ by a slight modification of [4] Proposition 2.3 (cf. [5]) and this completes the proof.

REMARK. The author is informed the use of [4] Proposition 2.3 by Y. Katayama.

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