J. Math. Soc. Japan
Vol. 31, No. 3, 1979

# Algebraic differential equations of Clairaut type from the differential-algebraic standpoint

By Keiji NISHIOKA

(Received Jan. 17, 1978)
(Revised Aug. 19, 1978)

## §0. Introduction.

Let $k$ be an algebraically closed ordinary differential field of characteristic zero, and $K$ be a one-dimensional algebraic function field over $k$. We assume that $K$ is a differential extension of $k$.

The following definition is due to M. Matsuda [3]: $K$ is said to be *free from parametric singularities* if $\nu_P(t')\geqq 0$ for each prime divisor $P$ of $K$, where $\nu_P$ is the normalized valuation belonging to $P$ and $t$ is a prime element in $P$.

Let $F$ be an algebraically irreducible element of $k\{y\}$ of the first order, and $K(F; k)$ be the associated differential algebraic function field with $F$ over $k$. Then, M. Matsuda [3] obtained the following theorem: Suppose that $K(F; k)$ is free from parametric singularities. Then, it is of Riccati type over $k$ if its genus $g$ is zero; it is a differential elliptic function field over $k$ if $g$ is one.

We say that $K$ is of *Clairaut type* over $k$ if the following two conditions are satisfied:

(i) $k$ contains an element $x$ such that $x'=1$;

(ii) There exists an element $y$ of $K$ such that $K=k(y, y')$ with $G(y-xy', y')=0$, where $G$ is an irreducible polynomial over the field of constants of $k$.

Under the assumption (i), $K$ is of Clairaut type if and only if $K=k(K_0)$, where $K_0$ is the field of constants of $K$ (cf. § 1).

M. Matsuda [4, pp. 5-6] expected that the following statement is true: Suppose that $K(F; k)$ is free from parametric singularities. Then, there exists a differential extension field $k^*$ of $k$ which satisfies the following two conditions:

(iii) The field of constants of $k^*$ is the same as that of $k$;

(iv) $K(F; k^*)$ is of Clairaut type over $k^*$.

This statement is true in the case where $g=0, 1$. It is derived from known results (cf. § 5). We shall prove that it is true in the case where $g>1$ and that $k$ itself can be taken as $k^*$ in this case if $k$ satisfies the condition (i).

THEOREM. *Suppose that $K$ is free from parametric singularities and the genus $g$ is greater than one. Then, $K=k(K_0)$.*

In the special case where $K$ is a hyperelliptic function field over $k$, this theorem is known (Painlevé [6, p. 68], Schlesinger [8, p. 118]).

By Briot-Bouquet's theorem, $k$ contains a nonconstant element if $K(F; k)$ is free from parametric singularities and if its genus is greater than one (Matsuda [3]).

In § 2 we shall state some known results on Weierstrass points (cf. Hurwitz [1], Iwasawa [2]).

The author wishes to express his sincere gratitude to Dr. M. Matsuda for his kind advices. He simplified the author's original proof of Theorem by introducing Lemma in § 3.

## § 1. Clairaut type.

PROPOSITION 1. *Under the assumption* (i), *$K$ is of Clairaut type if and only if $K=k(K_0)$.*

PROOF. Suppose that the condition (ii) is satisfied by $K$. Set $a=y-xy'$ and $b=y'$. Then, the differentiation of $G(a, b)=0$ gives us

$$y''(G_b-G_a)=0.$$

Here, the term in the parenthesis does not vanish, since its degree in $y'$ is less than that of $G$. Hence, $y''=0$. We have $K=k(a, b)$ and $a'=b'=0$. Conversely suppose that $K=k(K_0)$. We show that $K_0$ is a one-dimensional algebraic function field over $k_0$ the field of constants of $k$. Since $K_0\neq k_0$, there exists a transcendental constant $a$ of $K$ over $k_0$. It is transcendental over $k$. Any constant $c$ of $K$ is algebraic over $k_0(a)$, because it is algebraic over $k(a)$. We have

$$[k_0(a, c): k_0(a)]\leqq[K: k(a)].$$

Hence, $[K_0: k_0(a)]\leqq[K: k(a)]$. We have $K_0=k_0(a, b)$ for some element $b$ of $K_0$, and $K=k(K_0)=k(a, b)$. Let us set $y=a+bx$. Then, $y'=b$ and $K=k(y, y')$.

## § 2. Weierstrass points.

Let $k$ be an algebraically closed field of characteristic zero and $K$ be a one-dimensional algebraic function field over $k$ of genus $g$. For a divisor $A$ of $K$ we shall define a vector space over $k$:

$$L(A)=\{x\in K;\ (x)A \text{ is an integral divisor}\}\cup\{0\};$$

here $(x)$ is the principal divisor of $x$ in $K$ different from zero. Its dimension is finite and denoted by $l(A)$. Let $P$ be a prime divisor of $K$. Then, there exist $g$ positive integers $n_1, \cdots, n_g$ such that $L(P^n) \neq L(P^{n-1})$ for any positive integer $n$ different from $n_1, \cdots, n_g$. The prime divisor $P$ is called a Weierstrass point of $K$ if the set $\{n_1, \cdots, n_g\}$ is not $\{1, 2, \cdots, g\}$. If $g$ is greater than one, the set of Weierstrass points of $K$ is finite and its cardinarity $N$ satisfies

$$N \geqq 2g+2.$$

The equality holds if and only if $K$ is a hyperelliptic function field over $k$.

### § 3. Hurwitz's formula.

Let $M$ be a subfield of $K$ containing $k$ properly. Then, it is a one-dimensional algebraic function field over $k$. We have Hurwitz's formula:

$$2g = \sum_{P'} (e_{P'}-1) + 2eg_0 - 2(e-1);$$

here $e=[K: M]$; $g_0$ denotes the genus of $M$; $P'$ runs over prime divisor of $M$; $e_{P'}$ denotes the ramification exponent of $P'$ with respect to $M$.

LEMMA. *Suppose that $P_i$ and $u_i$ $(1 \leqq i \leqq r)$ are prime divisors and elements of $K$ respectively such that for each $i$ we have $\nu_{P_i}(u_i)<0$ and $\nu_Q(u_i) \geqq 0$ if $Q$ is a prime divisor different from $P_i$. Take $k(u_1, \cdots, u_r)$ as $M$. Assume that $g>0$ and $r \geqq 2g+2$. Then, either $K=M$ or $e=2$ and $g_0=0$.*

PROOF. Let $Q_i$ be the restriction of $P_i$ to $M$. Then, $\nu_{Q_i}(u_i)<0$, where $\nu_{Q_i}$ is the normalized valuation belonging to $Q_i$. For any prime divisor $P$ of $K$ different from $P_i$, $P$ is not an extension of $Q_i$ because of $\nu_P(u_i) \geqq 0$. Hence $Q_i = P_i^e$. We have

$$\sum_{P'} (e_{P'}-1) \geqq r(e-1).$$

By Hurwitz's formula

$$g \geqq (e-1)g + eg_0,$$

since $r \geqq 2g+2$. Therefore, either $e=1$ or $e=2$ and $g_0=0$, because $g>0$.

### § 4. Proof of Theorem.

Let $\{P_1, \cdots, P_r\}$ be the set of all Weierstrass points of $K$. Then,

$$r \geqq 2g+2.$$

Set $P=P_1$. There exists a positive integer $n$ such that

$$l(P^{n-1}) \leqq l(P^n) = l(P^{n+1}).$$

Take an element $u$ of $L(P^n)$ which does not belong to $L(P^{n-1})$. Let $Q$ be any prime divisor of $K$ different from $P$. By our assumption that $K$ is free from parametric singularities, we have $\nu_Q(t'_Q)\geqq 0$, where $t_Q$ is a prime element in $Q$. Hence, $\nu_Q(u')\geqq 0$, because $\nu_Q(u)\geqq 0$. We shall prove that $\nu_P(t'_P)>0$. To the contrary, let us suppose that $\nu_P(t'_P)=0$. Then $u'$ is contained in $L(P^{n+1})$ but not in $L(P^n)$. This contradicts our assumption on $n$. Hence, $\nu_P(t'_P)>0$. There is a positive integer $m$ such that $m\leqq g$ and $l(P^m)=2$. Let $\{1, u\}$ be a basis of $L(P^m)$. Then,

$$u'=a+bu$$

with $a, b$ in $k$. For any $P_i$ different from $P$, there is an element $\varepsilon_i$ of $k$ such that

$$\nu_{P_i}(u-\varepsilon_i)>0\,.$$

We have

$$\varepsilon'_i=a+b\varepsilon_i\,,$$

since $\nu_{P_i}(t'_{P_i})>0$. Set $v=u-\varepsilon_2$. Then, $(v)=EP^{-m}$, where $E$ is an integral divisor of degree $m$. By inequalities

$$m\leqq g<2g+1\leqq r-1$$

there is an index $i$ such that $\nu_{P_i}(v)=0$. Take such an index $i$. We have $\varepsilon_i\neq\varepsilon_2$. Set

$$w=(u-\varepsilon_2)/(\varepsilon_i-\varepsilon_2)\,.$$

Then, it is an element of $L(P^m)$ being a transcendental constant over $k$. Thus, for every Weierstrass point $P_i$, there exists a transcendental constant $w_i$ over $k$ such that for each $i$ we have $\nu_{P_i}(w_i)<0$ and $\nu_Q(w_i)\geqq 0$ if $Q$ is a prime divisor different from $P_i$. Let $M$ denote $k(w_1, \cdots, w_r)$. By Lemma, either $K=M$ or $e=2$ and $g_0=0$. Suppose that we are in the latter case. Then, $r=2g+2$. Let $k_0$ and $M_0$ denote the fields of constants of $k$ and $M$ respectively. Then, $M=k(M_0)$ and $M_0$ is $k_0(w_1, \cdots, w_r)$ being a one-dimensional algebraic function field over $k_0$. The genus of $M_0$ is zero, since it is not greater than $g_0$ (cf. Rosenlicht [7, Lemma 3]). Hence, there exists a transcendental constant $\gamma$ over $k$ such that $M_0=k_0(\gamma)$. We have $M=k(\gamma)$. There exists an element $y$ of $K$ such that $K=M(y)$ and

$$y^2=\prod(\gamma-\alpha_i)\qquad(1\leqq i\leqq s)\,,$$

where $s=2g+1$ or $2g+2$; $\alpha_i$ is in $k$ and $\alpha_i\neq\alpha_j$ $(i\neq j)$. Changing indices $i$ if necessary, we assume that

$$\nu_{P_i}(\gamma-\alpha_i)>0\qquad(1\leqq i\leqq s)\,.$$

We have $\alpha'_i=0$, since $\gamma'=0$ and $\nu_{P_i}(t'_i)>0$; here $t_i$ is a prime element in $P_i$. This proves $y'=0$.

## § 5. Case $g=0$, 1.

Let $\Omega$ be a universal extension of $k$.

PROPOSITION 2. *Suppose that $K$ is free from parametric singularities, and that $g$ is either 0 or 1. Then, there exists a differential subfield $k^*$ of $\Omega$ finitely generated over $k$ which satisfies* (iii) *and the following two conditions:*

(v) *$K$ and $k^*$ are linearly disjoint over $k$;*

(vi) *$k^*(K)$ is generated by its constants over $k^*$.*

PROOF. Case $g=0$. We have $K=k(t)$ with

$$t'=a+bt+ct^2\,;\ a,\ b,\ c\in k \tag{1}$$

for some element $t$ of $K$ (cf. [3, Theorem F]). Let us set $k_1^*=k$ and define $k_r^*$ inductively as follows: If the set of all solutions of (1) in $k_r^*$ is infinite, then we set $k_{r+1}^*=k_r^*$. In the contrary case let us take a generic point $u$ of the general solution of (1) over $k_r^*(t)$ in $\Omega$. If the field of constants of $k_r^*(u)$ is $k_0$, then we set $k_{r+1}^*=k_r^*(u)$. In the contrary case, there exist in $\Omega$ infinitely many solutions of (1) which are algebraic over $k_r^*$: For we consider the rational function field $\Sigma(u)$ over $\Sigma$, where $\Sigma$ is the algebraic closure of $k_r^*$: Let $\gamma$ be a transcendental constant of $\Sigma(u)$ over $\Sigma$, and let $P$ be a prime divisor of $\Sigma(u)$ such that $\nu_P(\gamma-\gamma_1)>0$ and $\nu_P(u-\xi)>0$ for some constant $\gamma_1$ of $\Sigma$ and some element $\xi$ of $\Sigma$: We have $\nu_P(u'-\xi')>0$, and $\xi$ is a solution of (1): Since such prime divisors exist infinitely, it follows that $\Sigma$ contains infinitely many solutions of (1). We take a solution $v$ of (1) in $\Sigma$ which is different from any solution of (1) in $k_r^*$, and set $k_{r+1}^*=k_r^*(v)$. Thus, $k_r^*$ is defined inductively. Let us set $k^*=k_4^*$. Then, $k^*$ satisfies (iii), (v), and it contains three solutions $t_1$, $t_2$, $t_3$ of (1) different from each other. The cross-ratio

$$\{(t-t_1)(t_3-t_2)\}\,/\,\{(t-t_2)(t_3-t_1)\}$$

is a transcendental constant $c^*$ over $k^*$. We have $k^*(t)=k^*(c^*)$.

Case $g=1$. Since $g=1$, we have $K=k(u,\ v)$ with

$$v^2=R(u)=u(u^2-1)(u-\delta)\,;\ \delta\in k\,;\ \delta^2\neq 0,\ 1$$

for some elements $u$ and $v$ of $K$. By our assumption $K$ is free from parametric singularities, if $u'=0$, then $\delta'=0$ and $v'=0$ (cf. [3, p. 452]). In this case we can set $k^*=k$. Suppose that $u'\neq 0$. Then, $\delta'=0$, $K=k(u,\ u')$ and

$$(u')^2=\lambda R(u)\,;\ \lambda\in k\,;\ \lambda\neq 0 \tag{2}$$

(cf. [3, p. 451]). If $K$ contains a transcendental constant over $k$, then there exists in $k$ a nonsingular solution of (2) (cf. [3, p. 453]). In this case we set $k^*=k$. In the contrary case let us take a generic point $\xi$ of the general solution over $K$ in $\Omega$: We set $k^*=k(\xi,\ \xi')$: The field of constants of $k^*$ is the same as that of $k(u, u')$, that is $k_0$. In any case $k^*$ satisfies (iii), (v), and it contains a nonsingular solution $\xi$ of (2). Let us define a new differentiation signed by the dot in $k^*(u, u')$ by

$$\dot{x}=\mu x', \quad \mu^2=\lambda^{-1}(2/\delta),$$

and set

$$w=2\xi/(1+\xi), \quad z=2u/(1+u).$$

Then, $k^*(u, u')=k^*(z, \dot{z})$, $w\in k^*$, and $w$, $z$ satisfy

$$(\dot{y})^2/4=S(y)=y(1-y)(1-\kappa^2 y), \quad \ddot{y}=2S_y, \quad \kappa^2=(1+\delta)/(2\delta).$$

We define two elements $a$ and $b$ of $k^*(z, \dot{z})$ by

$$a=\{z(1-w)(1-\kappa^2 w)-\dot{z}\dot{w}/2+w(1-z)(1-\kappa^2 z)\}/(1-\kappa^2 zw)^2,$$

$$2b=\{C(w, z)\dot{z}-C(z, w)\dot{w}\}/(1-\kappa^2 zw)^3,$$

where

$$C(w, z)=\kappa^2 z\{w(1-z)(1-\kappa^2 z)+z(1-w)(1-\kappa^2 w)\} -(1-\kappa^2 zw)^2+2S(z)/z.$$

Then, $\dot{a}=\dot{b}=0$, $b^2=S(a)$ and

$$z=\{a(1-w)(1-\kappa^2 w)+b\dot{w}+w(1-a)(1-\kappa^2 a)\}/(1-\kappa^2 aw)^2,$$

$$\dot{z}=\{C(a, w)\dot{w}+2C(w, a)b\}/(1-\kappa^2 aw)^3$$

(cf. [3, pp. 452-453], [4]). We have

$$k^*(u, u')=k^*(z, \dot{z})=k^*(a, b).$$

## References

[1] A. Hurwitz, Ueber algebraische Gebilde mit eindeutigen Transformationen in sich, Math. Ann., **41** (1892), 401-442.

[2] K. Iwasawa, Theory of algebraic function fields (in Japanese), Iwanami Shoten, Tokyo, 1952.

[3] M. Matsuda, Algebraic differential equations of the first order free from parametric singularities from the differential-algebraic standpoint, J. Math. Soc. Japan, **30** (1978), 447-455.

[4] M. Matsuda, Lectures on algebraic differential equations of the first order, preprint.

[5] K. Nishioka, Transcendental constants over the coefficient fields in differential elliptic function fields, Pacific J. Math., **74** (1978), 191-197.

[6] P. Painlevé, Leçons sur la théorie analytique des équations différentielles, professées à Stockholm, Paris, 1897.

[7] M. Rosenlicht, The nonminimality of the differential closure, Pacific J.Math., **52** (1974), 529-537.

[8] L. Schlesinger, Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf functionentheorisher Grundlage, Walter de Gruyter, 1922.

Keiji NISHIOKA
Department of Mathematics
Osaka University
Toyonaka, Osaka
560 Japan