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Homogeneous Riemannian manifolds with a fixed isotropy representation

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1. Introduction.

In this paper we give a classification of simply-connected homogeneous Riemannian manifolds $M=G/H$ where H is isomorphic to a product of rotation groups and the linear isotropy representation of H is a direct sum of standard representations with a trivial representation. This situation arises naturally in the study of homogeneous Riemannian manifolds which admit a large group of isometries. In fact if $M=I_0(M)/H$, where

$$\dim I(M)>\frac{n^2}{4}+n\,,\quad n=\dim M\geqq 11 \tag{1.1}$$

then it follows that $H\cong SO(k)\times K$, with $k>n/2$, $K\subseteqq SO(n-k)$ and the linear isotropy representation of H splits [**5**, Theorem 1.18].

Our results are quite simple to state if each of the rotation groups has order at least 3. In that case M is isometric to a product of a certain number of simply-connected manifolds of constant curvature together with a simply-connected Lie group with a left-invariant metric. (Theorem B). If H is isomorphic to a single rotation group, this appears to be consistent with some local results obtained by Kurita [8] a number of years ago. If some of the rotation groups in the decomposition of H have order 2, then the description of the corresponding manifolds becomes more complicated. This is done in Section 4, where, in particular, we obtain a generalization of Cartan's classification [3] of 3-dimensional manifolds which admit a transitive group of motions of dimension 4.

In Section 5, we apply the above results to give an explicit description of those manifolds satisfying (1.1) and $n-3\leqq k\leqq n$. This turns up some inaccuracies and extends some results in [7], while at the same time exhibiting the differences with the compact case studied by Lukesh. In [**9**] it is shown that if M is compact and satisfies (1.1), then it must split isometrically with one factor being a standard sphere S^k, $k>n/2$. As we shall see in Section 5, there

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are uncountably many homogeneous metrics of strictly negative curvature on $\boldsymbol{R}^n$ with non-isomorphic isometry groups all of dimension $\frac{1}{2}(n^2-3n+6)$.

2. Algebraic preliminaries.

$M=G/H$ will denote a connected homogeneous n-dimensional Riemannian manifold. Throughout this paper we shall assume that G is connected and that the transitive action of G on M is effective. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. Since H is compact we can choose a complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{m}]\subseteqq\mathfrak{m}$. Moreover we can naturally identify $\mathfrak{m}$ with the tangent space of M at the base point $z_0=\{H\}\in M$, and hence $\mathfrak{m}$ carries an inner product $\langle , \rangle$ induced by the Riemannian structure of M. If we let

$$\tilde{\mathfrak{m}}=\{X\in\mathfrak{m}:\ [\mathfrak{h}, X]=0\},$$

then $\mathfrak{m}$ splits as

$$\mathfrak{m}=\mathfrak{m}_1+\tilde{\mathfrak{m}} \tag{2.1}$$

where $\mathfrak{m}_1$ is the orthogonal complement of $\tilde{\mathfrak{m}}$ and therefore is ad($\mathfrak{h}$)-invariant.

(2.2) PROPOSITION. *Assume $H\cong SO(q)$, $q\geqq 3$, and that the linear isotropy action of H on $\mathfrak{m}_1$ is standard. Then*

$$[\mathfrak{m}_1, \mathfrak{m}_1]\subseteqq\mathfrak{h} \tag{2.3}$$

$$[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1 . \tag{2.4}$$

PROOF. We begin by showing that $[\mathfrak{m}_1, \mathfrak{m}_1]_{\tilde{\mathfrak{m}}}=0$, where $[\mathfrak{m}_1, \mathfrak{m}_1]_{\tilde{\mathfrak{m}}}$ denotes the projection of $[\mathfrak{m}_1, \mathfrak{m}_1]$ on $\tilde{\mathfrak{m}}$, relative to the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\tilde{\mathfrak{m}}$. Given E_1, E_2, orthonormal vectors in $\mathfrak{m}_1$, let $A\in H$ be so that $A(E_1)=-E_1$, $A(E_2)=E_2$, where $A(E_i)$ denotes $\mathrm{Ad}(A)E_i$. Then $[E_1, E_2]_{\tilde{\mathfrak{m}}}=A([E_1, E_2]_{\tilde{\mathfrak{m}}})=[AE_1, AE_2]_{\tilde{\mathfrak{m}}}=-[E_1, E_2]_{\tilde{\mathfrak{m}}}$ which implies $[E_1, E_2]_{\tilde{\mathfrak{m}}}=0$ and hence $[\mathfrak{m}_1, \mathfrak{m}_1]_{\tilde{\mathfrak{m}}}=0$.

In order to show that $[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_1}=0$, we choose an orthonormal basis $E_1, \cdots, E_q$ of $\mathfrak{m}_1$ and let

$$[E_i, E_j]_{\mathfrak{m}_1}=\sum_{k=1}^{q} C_{ij}^k E_k .$$

It is then enough to show that $C_{ij}^k=0$ for each i, j, k. If $q\geqq 4$, let $l\neq i, j, k$ and $A\in H$ the element defined by

$$A(E_i)=-E_i,\quad A(E_l)=-E_l \quad \text{and}$$
$$A(E_s)=E_s \quad \text{for} \quad s\neq i, l .$$

We then have

$$\sum_{s=1}^{q} C_{ij}^s A(E_s)=A[E_i, E_j]_{\mathfrak{m}_1}=-[E_i, E_j]_{\mathfrak{m}_1}=-\sum_{s=1}^{q} C_{ij}^s E_s$$

and comparing terms for $s=k$, this gives $C_{ij}^k=0$.

Suppose now that $q=3$. Let E_i, $1\leqq i\leqq 3$, be as before, and let $B\in H$ be defined by

$$B(E_1)=-E_1, \quad B(E_2)=-E_2, \quad B(E_3)=E_3.$$

Since $B([E_i, E_j]_{\mathfrak{m}_1})=[B(E_i), B(E_j)]_{\mathfrak{m}_1}$, it is straightforward to check that

$$[E_i, E_j]_{\mathfrak{m}_1}=\gamma_k E_k, \qquad i\neq j\neq k\neq i, \quad \gamma_k\in\boldsymbol{R}.$$

On the other hand for an arbitrary $A\in H$, let

$$A(E_i)=\sum_{j=1}^{3} a_{ji}E_j.$$

We then have

$$\gamma_3 A(E_3)=A([E_1, E_2]_{\mathfrak{m}_1})=[\sum_{j=1}^{3} a_{j1}E_j, \ \sum_{k=1}^{3} a_{k2}E_k]_{\mathfrak{m}},$$

and comparing the coefficients of E_3 we obtain

$$\gamma_3 a_{33}=\gamma_3(a_{11}a_{22}-a_{21}a_{12})$$

for every $A\in H$. It then follows that $\gamma_3=0$, and consequently $[E_1, E_2]_{\mathfrak{m}_1}=0$. This proves (2.3)

It remains to show $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1$. Set

$$\bar{\mathfrak{m}}_1=\{E\in\mathfrak{m}_1 : [\tilde{\mathfrak{m}}, E]_{\tilde{\mathfrak{m}}}=0\}.$$

Then $\bar{\mathfrak{m}}_1$ is an Ad(H)-invariant subspace of $\mathfrak{m}_1$ and thus either $\bar{\mathfrak{m}}_1=\mathfrak{m}_1$ or $\bar{\mathfrak{m}}_1=\{0\}$. On the other hand it is easy to check that if $E\in\mathfrak{m}_1$, $A\in H$ are such that $A(E)\neq E$, then $E-A(E)\in\bar{\mathfrak{m}}_1$. Hence $\bar{\mathfrak{m}}_1=\mathfrak{m}_1$ and therefore $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]_{\tilde{\mathfrak{m}}}=0$.

For each $X\in\tilde{\mathfrak{m}}$, let $\bar{\mathfrak{m}}_1(X)$ be the Ad(H)-invariant subspace of $\mathfrak{m}_1$ defined by

$$\bar{\mathfrak{m}}_1(X)=\{E\in\mathfrak{m}_1 : [X, E]_{\mathfrak{h}}=0\}.$$

If $\bar{\mathfrak{m}}_1(X)=\mathfrak{m}_1$ for all $X\in\tilde{\mathfrak{m}}$, then $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]_{\mathfrak{h}}=0$ and (2.4) follows. Assume $\bar{\mathfrak{m}}_1(X)=0$ for some $X\in\tilde{\mathfrak{m}}$. This is clearly impossible if $q\geqq 4$ (or $q=2$) since $[X, \mathfrak{m}_1]_{\mathfrak{h}}$ would be a q-dimensional ideal of $\mathfrak{h}\cong so(q)$. The case $q=3$ again requires a separate proof. Let E_i, $1\leqq i\leqq 3$, be as before and let A_{ij}, $1\leqq i\neq j\leqq 3$, be elements of $\mathfrak{h}\cong so(3)$ defined by

$$\text{(2.5)} \qquad \begin{aligned} &[A_{ij}, E_i]=-E_j, \quad [A_{ij}, E_j]=E_i \\ &[A_{ij}, E_k]=0, \qquad k\neq i, j. \end{aligned}$$

Clearly $\{A_{ij}, 1\leqq i<j\leqq 3\}$ is a basis of $\mathfrak{h}$ and $A_{ij}=-A_{ji}$. One can readily check that

$$\text{(2.6)} \qquad [A_{ij}, A_{jk}]=A_{ik}.$$

Let

$$[X, E_i]_{\mathfrak{h}}=\alpha_i A_{12}+\beta_i A_{13}+\gamma_i A_{23}\,.$$

Since

$$[A_{23}, [X, E_1]_{\mathfrak{h}}]=[A_{23}, [X, E_1]]_{\mathfrak{h}}$$

$$=[[A_{23}, X], E_1]_{\mathfrak{h}}+[X, [A_{23}, E_1]]_{\mathfrak{h}}=0\,,$$

using (2.5) and (2.6) we obtain

$$0=[A_{23}, \alpha_1 A_{12}+\beta_1 A_{13}+\gamma_1 A_{23}]=-\alpha_1 A_{13}+\beta_1 A_{12}$$

which implies $\alpha_1=\beta_1=0$. Similarly one can show that $\alpha_2=\gamma_2=\beta_3=\gamma_3=0$ and $\alpha_3=-\beta_2=\gamma_1=\lambda$.

On the other hand, since by (2.3), $[\mathfrak{m}_1, \mathfrak{m}_1]\subseteqq\mathfrak{h}$ we get

$$0=[X, [E_1, E_3]]_{\mathfrak{m}_1}=[[X, E_1], E_3]_{\mathfrak{m}_1}+[E_1, [X, E_3]]_{\mathfrak{m}_1}$$

$$=[[X, E_1]_{\mathfrak{h}}, E_3]+[E_1, [X, E_3]_{\mathfrak{h}}]$$

$$=[\lambda A_{23}, E_3]+[E_1, \lambda A_{12}]=\lambda E_2+\lambda E_2=2\lambda E_2\,.$$

Hence $\lambda=0$ and $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1$. This completes the proof of Proposition (2.2).

For each $X\in\tilde{\mathfrak{m}}$, the linear transformation

$$\mathrm{ad}(X): \mathfrak{m}_1\rightarrow\mathfrak{m}_1$$

commutes with the standard action of $H\cong SO(q)$ on $\mathfrak{m}_1$. Hence there exists a linear functional $\alpha\in\tilde{\mathfrak{m}}^*$, such that

(2.7) $$[X, E]=\alpha(X)E\,,\qquad X\in\tilde{\mathfrak{m}}\,,\quad E\in\mathfrak{m}_1\,.$$

We set

(2.8) $$\tilde{\mathfrak{m}}'=\ker\alpha=\{X\in\tilde{\mathfrak{m}}: [X, \mathfrak{m}_1]=0\}$$

Then, either $\tilde{\mathfrak{m}}=\tilde{\mathfrak{m}}'$ or $\dim\tilde{\mathfrak{m}}'=\dim\tilde{\mathfrak{m}}-1$.

(2.9) PROPOSITION. *$\tilde{\mathfrak{m}}$ is a subalgebra of $\mathfrak{g}$ and $\tilde{\mathfrak{m}}'$ is an ideal in $\mathfrak{g}$.*

PROOF. First of all we notice that $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]_{\mathfrak{m}_1}=0$ since $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]_{\mathfrak{m}_1}$ would be a subspace of $\mathfrak{m}_1$ where $\mathfrak{h}$ acts trivially. Similarly we have

$$[\mathfrak{h}, [\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]_{\mathfrak{h}}]=0$$

which implies $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]_{\mathfrak{h}}=0$ since $\mathfrak{h}\cong so(q)$ contains no non-trivial abelian ideals for $q\geqq 3$. This proves the first statement in (2.9).

Let $X_1, X_2\in\tilde{\mathfrak{m}}$, $E\in\mathfrak{m}_1$. Then

$$[[X_1, X_2], E]=(\alpha(X_1)\alpha(X_2)-\alpha(X_2)\alpha(X_1))E=0\,.$$

This implies $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]\subseteqq\tilde{\mathfrak{m}}'$ and since $[\mathfrak{h}+\mathfrak{m}_1, \tilde{\mathfrak{m}}']=0$, the proposition follows.

(2.10) COROLLARY. *If $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$ then the decomposition*

$$\mathfrak{g}=(\mathfrak{h}+\mathfrak{m}_1)\oplus\tilde{\mathfrak{m}}$$

is a direct sum of ideals.

(2.11) LEMMA. *If* $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$, *then* $\mathfrak{m}_1$ *is an abelian ideal in* $\mathfrak{g}$.

PROOF. Using (2.2) it is enough to show that $[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{h}}=0$. Let $E_1, E_2\in\mathfrak{m}_1$ and choose $X\in\tilde{\mathfrak{m}}$ such that $\alpha(X)=1$. Then

$$\begin{aligned}0&=[X, [E_1, E_2]_{\mathfrak{h}}]=[X, [E_1, E_2]]_{\mathfrak{h}}\\&=[\alpha(X)E_1, E_2]+[E_1, \alpha(X)E_2]=2[E_1, E_2]\end{aligned}$$

and the result follows.

(2.12) PROPOSITION. *If* $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$, *then* $\mathfrak{m}=\mathfrak{m}_1+\tilde{\mathfrak{m}}$ *is an ideal in* $\mathfrak{g}$.

PROOF. This is a consequence of Proposition (2.2) and the above lemma.

3. Global results.

Throughout this section M will denote a connected and simply-connected homogeneous Riemannian manifold $M=G/H$, with G a connected subgroup of $I(M)$ acting effectively on M. We assume further that

$$H\cong H_1\times\cdots\times H_k$$

and the linear isotropy representation of H splits. The Lie algebra $\mathfrak{g}$ of G has therefore a decomposition

$$\mathfrak{g}=\mathfrak{h}_1+\cdots+\mathfrak{h}_k+\mathfrak{m}_1+\cdots+\mathfrak{m}_k+\tilde{\mathfrak{m}} \tag{3.1}$$

where $\mathfrak{h}_i$ leaves $\mathfrak{m}_i$ invariant and acts trivially on $\tilde{\mathfrak{m}}$ and $\mathfrak{m}_j$, $j\neq i$. It is clear that the decomposition $\mathfrak{m}=\mathfrak{m}_1+\cdots+\mathfrak{m}_k+\tilde{\mathfrak{m}}$ is orthogonal relative to the inner product induced in $\mathfrak{m}$ by the Riemannian structure of M.

(3.2) LEMMA. *Let* $M=G/(H_1\times H_2)$ *be as above,* $\mathfrak{g}=\mathfrak{h}_1+\mathfrak{h}_2+\mathfrak{m}_1+\mathfrak{m}_2+\tilde{\mathfrak{m}}$ *as in* (3.1). *If* $\mathfrak{s}=\mathfrak{h}_2+\mathfrak{m}_1+\mathfrak{m}_2+\tilde{\mathfrak{m}}$ *is an ideal of* $\mathfrak{g}$ *and* S *denotes the corresponding analytic subgroup of* G, *then* $M=S/H_2$.

PROOF. The subgroup S acts as an effective group of isometries of M. Since $\mathfrak{s}\cap(\mathfrak{h}_1+\mathfrak{h}_2)=\mathfrak{h}_2$, and S is normal in G, we have for any $z\in M$

$$\begin{aligned}\dim S_z&=\dim(S\cap G_{z_0})=\dim(S\cap(H_1\times H_2))\\&=\dim\mathfrak{h}_2\,,\end{aligned}$$

where $z_0=\{H\}$ is the base point in M. Hence for any $z\in M$, the orbit $S(z)$ has the same dimension as M and is therefore open in M, and consequently every orbit is also closed. Since M is assumed to be connected this proves (3.2).

In particular if $H_2=\{e\}$, $\mathfrak{s}=\mathfrak{m}$ and we obtain the following standard result.

(3.3) LEMMA. *Let* $M=G/H$ *be as above and assume that the Lie algebra* $\mathfrak{g}$ *of*

G admits a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an ideal of $\mathfrak{g}$. Then M is isometric to a Lie group with a left-invariant metric.

We shall also need the following:

(3.4) LEMMA. *Let $M=G/(H_1\times H_2)$ be a connected simply-connected, homogeneous Riemannian manifold. Assume further that the decomposition* (3.1)

$$\mathfrak{g}=(\mathfrak{h}_1+\mathfrak{m}_1)\oplus(\mathfrak{h}_2+\mathfrak{m}_2+\tilde{\mathfrak{m}})=\mathfrak{g}_1\oplus\mathfrak{g}_2$$

is a direct sum of ideals. Then there exists closed connected normal subgroups G_i, $i=1, 2$ of G with Lie algebras $\mathfrak{g}_i$, $i=1, 2$, such that M is isometric to the product $M_1\times M_2$ where M_i is the simply-connected homogeneous Riemannian manifold G_i/H_i.

PROOF. Let $\hat{G}$ denote the universal covering group of G, and $\hat{G}_i$, $i=1, 2$, the analytical subgroups of $\hat{G}$ with Lie algebras $\mathfrak{g}_i$, $i=1, 2$, respectively. Then $\hat{G}_i$ is a simply-connected closed normal subgroup of $\hat{G}$. Since $\mathfrak{h}_i\subseteqq\mathfrak{g}_i$, let $\hat{H}_i\subseteqq\hat{G}_i$ be the corresponding connected subgroup. Now $\hat{G}$ acts as a transitive group of isometries (although possibly not effectively) on M. So

$$\hat{M}=\hat{G}/G_{z_0}$$

where $\hat{G}_{z_0}$ is the isotropy subgroup at the base point $z_0\in M$. It is clear however that

$$\hat{G}_{z_0}=\hat{H}_1\times\hat{H}_2$$

since they are connected subgroups with the same Lie algebra. Therefore M splits diffeomorphically as

$$M=\hat{G}_1/\hat{H}_1\times\hat{G}_2/\hat{H}_2\,. \tag{3.5}$$

Let $\pi:\hat{G}\to G$ be the natural projection, $G_i=\pi(\hat{G}_i)$, H_i as before. The subgroup $N=\operatorname{Ker}\pi$ is normal in $\hat{G}$ and is contained in $\hat{G}_{z_0}$. Moreover since N acts trivially on M, $N_i=\hat{G}_i\cap N$ acts trivially on $M_i=\hat{G}_i/\hat{H}_i$, hence

$$M_i=\hat{G}_i/\hat{H}_i=\frac{\hat{G}_i/(\hat{G}_i\cap N)}{\hat{H}_i/(\hat{H}_i\cap N)}=G_i/H_i^*\,.$$

But H_i and H_i^* are both connected and have the same Lie algebra $\mathfrak{h}_i$. Therefore $H_i^*=H_i$ and $M_i=G_i/H_i$.

It remains to show that (3.5) is an isometric splitting, or equivalently, that for any $z=(z_1, z_2)\in M$, the subspaces $T_{z_1}(M_1)$ and $T_{z_2}(M_2)$ are orthogonal with respect to the Riemannian inner product in $T_z(M)$. But this is clear at the base point $z_0=(z_1^0, z_2^0)$ since $T_{z_1^0}(M_1)\cong\mathfrak{m}_1$ and $T_{z_2^0}(M_2)\cong\mathfrak{m}_2+\tilde{\mathfrak{m}}$, and at any other point by homogeneity.

In what follows, ρ_q and θ_k will denote the standard and trivial representations of $SO(q)$ on $\boldsymbol{R}^q$ and $\boldsymbol{R}^k$ respectively.

THEOREM A. *Suppose $M=G/H$ is a connected simply-connected n-dimensional homogeneous Riemannian manifold. If H is isomorphic to $SO(q)$, $3\leqq q\leqq n$, and the linear isotropy representation of H is $\rho_q\oplus\theta_{n-q}$, then either*

(1) *M is isometric to $M_1^{(q)}\times M_2^{(n-q)}$ where M_1 is a q-dimensional simply-connected space of constant curvature and M_2 is isometric to an $(n-q)$-dimensional simply-connected Lie group with a left-invariant metric. Furthermore $G\cong I_0(M_1)\times M_2$, where $I_0(M_1)$ is the identity connected component of the full group of isometries of M_1, or*

(2) *M is isometric to a Lie group with a left-invariant metric and G is isomorphic to a semi-direct product of $SO(q)$ with M.*

PROOF. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\tilde{\mathfrak{m}}$ be as in (2.1). If $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$, then by Corollary (2.10)

$$\mathfrak{g}=(\mathfrak{h}+\mathfrak{m}_1)\oplus\tilde{\mathfrak{m}}$$

is a direct sum of ideals. Applying lemma (3.4) we can conclude that M splits isometrically

$$M=G_1/SO(q)\times G_2\,.$$

But G_1 acts effectively on the q-dimensional manifold M_1 and $\dim G_1=\frac{1}{2}q(q+1)$, hence M_1 has constant curvature. We thus obtain (1).

If $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$, then by (2.12) $\mathfrak{m}$ is an ideal in $\mathfrak{g}$. Lemma (3.3) now applies to give (2).

(3.6) REMARK. In (2) of Theorem A, observe that by (2.11) $\mathfrak{m}_1$ is an abelian ideal of $\mathfrak{g}$, and hence the Lie group M contains a simply-connected closed normal abelian subgroup of dimension q. It then follows that if M is compact we must have $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$ and thus case (1) in Theorem A. This is the case studied by Lukesh [**9**].

Also, since by (2.9) $\tilde{\mathfrak{m}}'$ is an ideal in $\mathfrak{g}$, the corresponding analytic subgroup K of G is normal. Moreover K coincides with the identity connected component of the centralizer of $\mathfrak{h}+\mathfrak{m}_1$ in G, and is therefore closed. One can easily check that $K\cap H=\{e\}$, from which it follows, since K is normal, that K acts freely on M. Moreover the orbit space M/K with its induced metric can be seen to be a space of constant negative curvature [**6**, Theorem 3.3], hence diffeomorphic to Euclidean space. Since M is a principal fiber bundle over M/K, M is diffeomorphic to the product of K with a Euclidean space.

THEOREM B. *Suppose $M=G/H$, is a connected simply-connected n-dimensional homogeneous Riemannian manifold. If H is isomorphic to a product $SO(q_1)\times SO(q_2)\times\cdots\times SO(q_k)$ where*

$$q_i\geqq 3 \quad \textit{for all } i \textit{ and } \quad \sum_{i=1}^{k} q_i\leqq n$$

and if the linear isotropy representation of H splits as

$$\rho_{q_1}\oplus\rho_{q_2}\oplus\cdots\oplus\rho_{q_k}\oplus\theta_{n-\Sigma q_i},$$

then there exists some subset $q_{i_1}, \cdots, q_{i_l}$ *of the* q_i*'s such that* M *is isometric to*

$$M_1\times M_2\times\cdots\times M_l\times M_{l+1}$$

where M_j, $1\leqq j\leqq l$, *is a* q_{i_j}*-dimensional simply-connected manifold of constant curvature and* M_{l+1} *is an* $(n-\sum_{j=1}^{l} q_{i_j})$*-dimensional simply-connected Lie group with a left-invariant metric.*

PROOF. We decompose $\mathfrak{g}$ according to (3.1) as

$$\mathfrak{g}=\mathfrak{h}_1+\cdots+\mathfrak{h}_k+\mathfrak{m}_1+\cdots+\mathfrak{m}_k+\tilde{\mathfrak{m}}.$$

Let

$$\mathfrak{s}=\mathfrak{h}_2+\cdots+\mathfrak{h}_k+\mathfrak{m}_1+\cdots+\mathfrak{m}_k+\tilde{\mathfrak{m}}$$

$$\mathfrak{s}_1=\mathfrak{m}_1$$

$$\tilde{\mathfrak{s}}=\mathfrak{h}_2+\cdots+\mathfrak{h}_k+\mathfrak{m}_2+\cdots+\mathfrak{m}_k+\tilde{\mathfrak{m}}$$

and observe that $[\mathfrak{h}_1, \tilde{\mathfrak{s}}]=0$. Then

$$\mathfrak{g}=\mathfrak{h}_1+\mathfrak{s}_1+\tilde{\mathfrak{s}}$$

and we set $\tilde{\mathfrak{s}}'=\{X\in\tilde{\mathfrak{s}}: [X, \mathfrak{s}_1]=0\}$. If $\tilde{\mathfrak{s}}'=\tilde{\mathfrak{s}}$, then $\mathfrak{g}=(\mathfrak{h}_1+\mathfrak{s}_1)\oplus\tilde{\mathfrak{s}}$ is by (2.10) a direct sum of ideals. It follows then from Lemma (3.4) that M is isometric to $M_1\times M^*$ where M_1 is a q_1-dimensional simply-connected manifold of constant curvature and $M^*=G_2/(H_2\times\cdots\times H_k)$, where G_2 is the analytic subgroup of G with Lie algebra $\tilde{\mathfrak{s}}$. We proceed inductively on M^* with respect to k.

If $\tilde{\mathfrak{s}}'\neq\tilde{\mathfrak{s}}$, then $\mathfrak{s}=\mathfrak{s}_1+\tilde{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$. Lemma (3.2) now implies that $M=S/(H_2\times\cdots\times H_k)$, where S is the analytic subgroup corresponding to $\mathfrak{s}$. Again an inductive process completes the proof.

(3.7) REMARK. Using the results of Section 2, it is possible to give rather explicit descriptions of the Lie algebra $\mathfrak{g}$ and the group G. Although in the general case this is not particularly enlightening, we will do it in Section 5 for some special cases.

We end this section with the following result, a local version of which is due to Wakakuwa [**11**, Theorem 2].

THEOREM C. (Wakakuwa). *Suppose* $M=G/H$ *is a connected simply-connected* n*-dimensional homogeneous Riemannian manifold. Assume that* $H\cong H_1\times\cdots\times H_k$ *and the linear isotropy representation of* H *is faithful and splits. Then* M *is isometric to a product*

$$M\cong M_1\times\cdots\times M_k,$$

where $M_i=G_i/H_i$ *for* G_i *some connected normal subgroup of* G.

PROOF. We sketch a proof using the techniques of Section 2. Infinitesimally we have

$$\mathfrak{g}=\mathfrak{h}_1+\cdots+\mathfrak{h}_k+\mathfrak{m}_1+\cdots+\mathfrak{m}_k\,.$$

It can be shown that $\mathfrak{g}_i=\mathfrak{h}_i+\mathfrak{m}_i$ is an ideal of $\mathfrak{g}$ for $1\leqq i\leqq k$, and therefore $\mathfrak{g}$ splits as $\mathfrak{g}=\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_k$. The result now follows inductively from Lemma (3.4).

4. Rotation groups of order 2.

In this section we will study Riemannian homogeneous spaces of the form G/H, where $H\cong SO(2)$ and the linear isotropy representation of H is equivalent to $\rho_2\oplus\theta_{n-2}$. In particular, in the case $\dim M=3$ we will recover Cartan's classification [3] of 3-dimensional manifolds admitting a transitive group of isometries of dimension 4. The general case where H is isomorphic to a product of rotation groups, some of which are of order 2, can be treated along the same lines as Theorem B of the preceding section. In Section 5 we shall study one such case in detail.

Keeping the notation of Section 2 we can write

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\tilde{\mathfrak{m}} \tag{4.1}$$

where $[\mathfrak{h}, \tilde{\mathfrak{m}}]=0$, $\dim\mathfrak{m}_1=2$, $\mathfrak{h}\cong so(2)$ acts on $\mathfrak{m}_1$ in the natural way and $\mathfrak{m}_1$ is orthogonal to $\tilde{\mathfrak{m}}$ relative to the natural inner product in $\mathfrak{m}=\mathfrak{m}_1+\tilde{\mathfrak{m}}$. It is easy to check that (2.4) is still valid in this case, that is

$$[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1\,. \tag{4.2}$$

Let now E_1, E_2 be an orthonormal basis of $\mathfrak{m}_1$ and H_0 the element of $\mathfrak{h}$ defined by

$$[H_0, E_1]=-E_2\,,\quad [H_0, E_2]=E_1\,. \tag{4.3}$$

Given $X\in\tilde{\mathfrak{m}}$ we can write

$$[X, E_i]=\sum_{j=1}^{2} a_{ij}E_j\,,\qquad i=1, 2\,.$$

Since $[H_0, [X, E_1]]=-[X, E_2]$ we deduce that $a_{11}=a_{22}$ and $a_{12}=-a_{21}$. Hence there exist linear functionals α, $\beta\in\tilde{\mathfrak{m}}^*$ such that

$$[X, E_1]=\alpha(X)E_1-\beta(X)E_2$$

$$[X, E_2]=\beta(X)E_1+\alpha(X)E_2\,.$$

Let $X_1, \cdots, X_n$ be an orthonormal basis of $\tilde{\mathfrak{m}}$ such that $\beta(X_i)=0$ for $1\leqq i\leqq n-1$ and let $\beta(X_n)=b$. Then replacing $\tilde{\mathfrak{m}}$ by the subspace spanned by $X_1, \cdots, X_{n-1}$, X_n-bH_0, and making the corresponding change in $\mathfrak{m}$, we can assume that

$$[X, E]=\alpha(X)E\,,\qquad X\in\tilde{\mathfrak{m}}\,,\quad E\in\mathfrak{m}_1\,. \tag{4.4}$$

(4.5) LEMMA. $\tilde{\mathfrak{m}}$ *is a subalgebra of* $\mathfrak{g}$ *and* $\tilde{\mathfrak{m}}'=\operatorname{Ker}\alpha\subseteqq\tilde{\mathfrak{m}}$ *is an ideal in* $\mathfrak{g}$.

PROOF. As in (2.9) we have $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]_{\mathfrak{m}_1}=0$. On the other hand, let $X_i\in\tilde{\mathfrak{m}}$, $i=1, 2$, and set

$$[X_1, X_2]_{\mathfrak{h}}=aH_0 .$$

Then

$$0=[[X_1, X_2], E_1]=a[H_0, E_1]+[[X_1, X_2]_{\tilde{\mathfrak{m}}}, E_1]$$

$$=-aE_2+\alpha([X_1, X_2]_{\tilde{\mathfrak{m}}})E_1 .$$

Hence $a=0$ and thus $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]\subseteqq\tilde{\mathfrak{m}}$. The second statement follows as in (2.9).

As in the proof of Proposition (2.2) one can show that

$$[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{m}_1}=0 . \tag{4.6}$$

However, it is not true in general that $[\mathfrak{m}_1, \mathfrak{m}_1]_{\tilde{\mathfrak{m}}}=0$. This is what distinguishes this case from the one discussed in Section 2.

(4.7) THEOREM. *Let* $M=G/H$ *be a connected simply-connected n-dimensional homogeneous Riemannian manifold. Assume* $H\cong SO(2)$ *and the linear isotropy representation of* H *is equivalent to* $\rho_2\oplus\theta_{n-2}$. *Then* M *is one of the following:*

(1) M *is isometric to a product* $M=M_1^{(2)}\times M_2^{(n-2)}$ *where* M_1 *is a* 2*-dimensional simply-connected space of constant curvature and* M_2 *is a simply-connected Lie group with a left-invariant metric. Moreover* $G\cong I_0(M_1)\times M_2$.

(2) M *is isometric to a simply-connected Lie group with a left-invariant metric and* G *is isomorphic to a semi-direct product of* $SO(2)$ *with* M.

(3) M *is a principal fiber bundle, with abelian structural group, over the product of a* 2*-dimensional space with non-zero constant curvature and a simply-connected Lie group with a left-invariant metric.*

PROOF. We begin by considering the case $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$, that is $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]=0$. Let $\mathfrak{z}(\mathfrak{g})$, $\mathfrak{z}(\tilde{\mathfrak{m}})$ denote the centers of $\mathfrak{g}$ and $\tilde{\mathfrak{m}}$ respectively. Then

$$\mathfrak{z}(\mathfrak{g})=\mathfrak{z}(\tilde{\mathfrak{m}}) . \tag{4.8}$$

In fact, let $Z\in\mathfrak{z}(\mathfrak{g})$. If we decompose Z according to (4.1) as $Z=Z_{\mathfrak{h}}+Z_{\mathfrak{m}_1}+Z_{\tilde{\mathfrak{m}}}$, then since $Z_{\mathfrak{h}}$ acts trivially on $\mathfrak{m}_1$ we must have $Z_{\mathfrak{h}}=0$. Similarly $Z_{\mathfrak{m}_1}$ defines a subspace of $\mathfrak{m}_1$ where $\mathfrak{h}$ acts trivially, hence $Z_{\mathfrak{m}_1}=0$ and $\mathfrak{z}(\mathfrak{g})\subseteqq\tilde{\mathfrak{m}}$. Since clearly $\mathfrak{z}(\tilde{\mathfrak{m}})\subseteqq\mathfrak{z}(\mathfrak{g})$ we obtain (4.8).

Let E_i, $i=1, 2$, be an orthonormal basis of $\mathfrak{m}_1$ and set

$$[E_1, E_2]=\lambda H_0+\mu Z , \qquad \mu\geqq 0 ,$$

where $Z\in\tilde{\mathfrak{m}}$ is a unit vector. Notice that for $\mu\neq0$, $Z\in\mathfrak{z}(\tilde{\mathfrak{m}})=\mathfrak{z}(\mathfrak{g})$. If $\mu=0$, then the decomposition

$$\mathfrak{g}=(\mathfrak{h}\oplus\mathfrak{m}_1)\oplus\tilde{\mathfrak{m}}=\mathfrak{g}_\lambda\oplus\tilde{\mathfrak{m}} ,$$

where $\mathfrak{g}_\lambda$ is the 3-dimensional Lie algebra $\{H_0, E_1, E_2 : [H_0, E_1]=-E_2, [H_0, E_2]$

$=E_1$, $[E_1, E_2]=\lambda H_0\}$, is a direct sum of ideals and, consequently, it follows from Lemma (3.4) that M splits isometrically as $M_1^{(2)}\times M_2^{(n-2)}$. Moreover M_1 is a 2-dimensional space of constant curvature, positive if $\lambda<0$, negative if $\lambda>0$ and zero if $\lambda=0$. This gives (1) in (4.7).

If $\mu\neq 0$, $\lambda=0$; then $\mathfrak{m}$ is an ideal in $\mathfrak{g}$ and applying Lemma (3.3) we obtain case (2).

Assume now that $\lambda\neq 0$, $\mu\neq 0$. Let $\mathfrak{g}_1=\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Then $\mathfrak{g}_1$ is a split Lie algebra:

(4.9) $$\mathfrak{g}_1\cong\mathfrak{g}_\lambda\oplus\mathfrak{L}\,.$$

If C denotes the analytic subgroup of G whose Lie algebra is $\mathfrak{z}(\mathfrak{g})$, then C is closed and normal. Moreover C acts freely on M since $C\cap H=\{e\}$. Therefore

$$C\to M\to M/C$$

is a principal fiber bundle. The group G/C acts as an effective group of isometries on M/C, and the isotropy subgroup at any point is isomorphic to $SO(2)$. It follows then from (4.9) that M/C is as in (1) of (4.7). We thus obtain case (3).

Finally, suppose $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$. Let $X\in\tilde{\mathfrak{m}}$ be a unit vector such that $\alpha(X)=a>0$. As in (2.11) we have $[\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{h}}=0$ and in fact $[\mathfrak{m}_1, \mathfrak{m}_1]\subseteq\tilde{\mathfrak{m}}'$. Therefore $\mathfrak{m}$ is an ideal in $\mathfrak{g}$ and using (3.3) we obtain case (2) again. Notice that $\tilde{\mathfrak{m}}'$ is an ideal in $\mathfrak{m}$ and $\mathfrak{m}/\tilde{\mathfrak{m}}'$ is isomorphic to the 3-dimensional Lie algebra

(4.10) $$\{X, E_1, E_2 : [X, E_i]=aE_i,\ [E_1, E_2]=0\}.$$

(4.11) EXAMPLE. Suppose $\dim M=3$, hence $\dim G=4$ and we are in the situation studied by E. Cartan in [**3**]. Case (1) Theorem (4.7) gives, of course, an isometric product of a 2-dimensional space of constant curvature and a line. In case (2) we have two possibilities depending upon whether $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$ or $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$. In the first situation M is isometric to the Heisenberg group (strictly upper triangular 3×3-matrices), endowed with a left-invariant metric, while in the second M is a solvable Lie group whose Lie algebra is described by (4.10). Moreover, with respect to any left-invariant metric M will have strictly negative curvature [**4**].

The most interesting case is that described in (3) of (4.7). If $\lambda>0$, then M is a principal fiber bundle over a space of constant negative curvature and hence it is diffeomorphic to Euclidean 3-space. If $\lambda<0$, however, one can check that for $\mu^2=-\lambda$, M has constant positive curvature and is, therefore, diffeomorphic to S^3. Since a change in the metric in the direction of Z allows us to change the value of μ arbitrarily we see that if $\lambda<0$ then M is diffeomorphic to S^3 although not, in general, isometric.

(4.12) REMARK. In the case $\dim M=4$, the results in this section complete the classification in [7], where the cases $\tilde{\mathfrak{m}}'=\tilde{\mathfrak{m}}$ and $\mu=0$ are treated.

5. Some special cases.

In this section we apply the preceding results to give a classification of the n-dimensional connected, simply-connected homogeneous Riemannian manifolds $M=G/H$ where $H\cong SO(k)\times K$, $n-3\leqq k\leqq n$ and such that the linear isotropy representation of H is standard. As is well-known, if $k=n$ then M is a space of constant curvature; while if $k=n-1$ then M is either an n-dimensional space of constant negative curvature or a product of an $(n-1)$-dimensional space of constant curvature and a line [6].

If $k=n-2$ and $K=SO(2)$ it follows from Theorem C that M is isometrically equivalent to a product $M=M_1^{(n-2)}\times M_2^{(2)}$ of simply-connected spaces of constant curvature. The case $K=\{e\}$ has been studied by Kobayashi and Nagano in [7]; however their results turn out to be valid only under the additional assumption that M be *naturally reductive*. When this restriction is removed one obtains a one-parameter family of new examples. The case $H=SO(2)$ has been studied in Section 4; for $n-2\geqq 3$ we have

(5.1) THEOREM. *Let $M=G/H$ be a simply-connected n-dimensional homogeneous Riemannian manifold and assume that $H\cong SO(n-2)$, $n-2\geqq 3$ and the linear isotropy representation of H is the standard one. Then M is one of the following:*

(1) *M is isometric to a product $M_1^{(n-2)}\times M_2^{(2)}$ where M_1 is a simply-connected $(n-2)$-dimensional space of constant curvature and M_2 is a simply-connected Lie group with a left-invariant metric. Moreover $G\cong I_0(M_1)\times M_2$.*

(2) *$M\cong M_1^{(n-1)}\times \boldsymbol{R}$, where M_1 is a space of constant negative curvature and $G\cong G_1\times \boldsymbol{R}$, where the Lie algebra $\mathfrak{g}_1$ of G_1 is the one described by Kobayashi in* [**6**, *Theorem* 3.3].

(3) *M is isometric to a solvable Lie group $M(\lambda)$, $\lambda\neq 0$, with a left-invariant metric. For $\lambda>0$, $M(\lambda)$ has strictly negative curvature, constant for $\lambda=1$. Moreover, $M(\lambda)$ is a principal fiber bundle, over an $(n-1)$-dimensional space of constant negative curvature.*

PROOF. Let $\mathfrak{g}$, $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Let $\mathfrak{m}$ be an ad($\mathfrak{h}$)-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, and $\mathfrak{m}=\mathfrak{m}_1+\tilde{\mathfrak{m}}$ as in (2.1). By (2.4) we have $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1$. If $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]=0$ then Theorem A implies case (1).

Assume then that $[\tilde{\mathfrak{m}}, \mathfrak{m}_1]\neq 0$, and let $\alpha\in\tilde{\mathfrak{m}}^*$ be as in (2.7). Choose a unit vector $X\in\tilde{\mathfrak{m}}'=\mathrm{Ker}(\alpha)$ and let $Y\in\tilde{\mathfrak{m}}$ be such that $\langle X, Y\rangle=0$ and $\alpha(Y)=1$. Up to scalar multiplication of the inner product $\langle\,,\rangle$, we can assume that $\|Y\|=1$. Notice that these choices determine a particular metric in each homothety class. By Proposition (2.9) we have that

$$[Y, X]=\lambda X, \qquad \lambda\in \boldsymbol{R}.$$

If $\lambda=0$, then the decomposition

$$\mathfrak{g}=(\mathfrak{h}+\mathfrak{m}_1+\boldsymbol{R}Y)\oplus \boldsymbol{R}X$$

is a direct sum of ideals and applying Lemma (3.4) we obtain case (2).

If $\lambda\neq 0$, then $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}$ and hence by Lemma (3.3) M is isometric to a Lie group $M(\lambda)$ with a left-invariant metric. The derived algebra $[\mathfrak{m}, \mathfrak{m}]=\mathfrak{m}_1+\boldsymbol{R}X$ is an abelian ideal of codimension 1. Therefore $\mathfrak{m}$ is solvable and, moreover, we can apply Theorem 1 in [4] to conclude that if $\lambda>0$ $M(\lambda)$ has negative sectional curvature. The last statement in (3) follows from Remark (3.6).

(5.2) PROPOSITION. *The sectional curvatures of $M(\lambda)$ satisfy:*

(i) *For* $\lambda>0$, $\min(-1, -\lambda^2)\leqq K\leqq \max(-1, -\lambda^2)$

(ii) *For* $\lambda<0$, $\min(-1, -\lambda^2)\leqq K\leqq -\lambda$.

PROOF. The Lie algebra $\mathfrak{m}(\lambda)$ decomposes as

$$\mathfrak{m}(\lambda)=\mathfrak{m}_1+\boldsymbol{R}X+\boldsymbol{R}Y$$

with $\alpha(Y)=1$ and $[Y, X]=\lambda X$. It is enough to consider 2-dimensional subspaces of $\mathfrak{m}(\lambda)$ of the form

$$\mathfrak{p}=\mathrm{span}_{\boldsymbol{R}}\{aY+bZ_1, Z_2\}$$

where $Z_i\in\mathfrak{m}_1+\boldsymbol{R}X$, $i=1, 2$, are orthonormal and $a^2+b^2=1$. We then have [4]

$$K(\mathfrak{p})=-a^2\langle T^2Z_2, Z_2\rangle+b^2(\langle TZ_1, Z_2\rangle^2-\langle TZ_1, Z_1\rangle\langle TZ_2, Z_2\rangle) \tag{5.3}$$

where $T=\mathrm{ad}(Y)$: $\mathfrak{m}_1+\boldsymbol{R}X\to\mathfrak{m}_1+\boldsymbol{R}X$. If we now write

$$Z_i=Z_i'+z_iX, \qquad Z_i'\in\mathfrak{m}_1, \quad z_i\in\boldsymbol{R},$$

then $TZ_i=Z_i+(\lambda-1)z_iX$ and (5.3) becomes

$$\begin{aligned}K(\mathfrak{p})=&-a^2\langle Z_2+(\lambda^2-1)z_2X, Z_2\rangle^2+b^2(\langle Z_1+(\lambda-1)z_1X, Z_2\rangle^2\\ &-\langle Z_1+(\lambda-1)z_1X, Z_1\rangle\langle Z_2+(\lambda-1)z_2X, Z_2\rangle)\\ =&-1-[(a^2(\lambda^2-1)+b^2(\lambda-1))z_2^2+b^2(\lambda-1)z_1^2]\,.\end{aligned}$$

It is then clear that for $\lambda=1$, $K\equiv -1$. If $\lambda>1$ we have $K(\mathfrak{p})\leqq -1$; on the other hand the expression between brackets attains its maximum for $z_1=0$, $z_2=1$, $a=1$, $b=0$ and thus $-\lambda^2\leqq K(\mathfrak{p})\leqq -1$. Similarly, if $0<\lambda<1$, $-1\leqq K(\mathfrak{p})$ and the maximum of $K(\mathfrak{p})$ is attained at the same point giving $-1\leqq K(\mathfrak{p})\leqq -\lambda^2$. This proves (i); an analogous argument shows (ii).

(5.4) COROLLARY. *If $\lambda_1\neq\lambda_2$ then $M(\lambda_1)$ is not homothetic to $M(\lambda_2)$.*

The spaces $M(\lambda)$, $\lambda>0$, constitute therefore a "one-parameter" family of solvable Lie groups admitting a left-invariant metric of strictly negative curvature. These spaces have been studied by Heintze [4] and by Azencott and Wilson [1], [2], who have given an infinitesimal characterization of the full isometry group of such a solvmanifold. In our case we have

(5.5) THEOREM. *Let* $M(\lambda)=G/H$, $0<\lambda\neq 1$, *be as in* (3) *of Theorem* (5.1). *Then* $G\cong I_0(M(\lambda))$.

PROOF. If $\dim I_0(M(\lambda))>\dim G$, then the isotropy subgroup of $I_0(M(\lambda))$ at the origin $o=\{H\}\in M(\lambda)$ must be isomorphic to one of the following: $SO(n)$, $SO(n-1)$ or $SO(n-2)\times SO(2)$. Since for $\lambda\neq 1$, $M(\lambda)$ is not a space of constant curvature it is clear that the $SO(n)$-case cannot occur. In either of the remaining two cases $M(\lambda)$ would have a Euclidean factor which is impossible since $M(\lambda)$ has strictly negative curvature.

We shall next consider the case $H=SO(n-3)\times K$. If $K\cong SO(3)$ then Theorem C implies that $M\cong M_1^{(n-3)}\times M_2^{(3)}$, where M_i is a simply-connected space of constant curvature and $G\cong I_0(M_1)\times I_0(M_2)$. If $K=\{e\}$ and $n-3\geqq 3$, then we may apply Theorem A to conclude that either M is isometric to a product $M\cong M_1^{(n-3)}\times M_2^{(3)}$, where M_1 is an $(n-3)$-dimensional space of constant curvature and M_2 is a 3-dimensional simply-connected Lie group with a left-invariant metric, (For a classification of these Lie groups together with their curvature properties, relative to a left-invariant metric, we refer to Milnor [10]), or M is itself isometric to a Lie group with a left-invariant metric. We recall how this latter case arises: Let

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m}_1+\tilde{\mathfrak{m}}$$

be as in (2.1), and assume $\tilde{\mathfrak{m}}'\neq\tilde{\mathfrak{m}}$. By (2.11) we then have that $\mathfrak{m}_1$ is an abelian ideal in $\mathfrak{g}$. As before let $Y\in\tilde{\mathfrak{m}}$, be a unit vector, orthogonal to $\tilde{\mathfrak{m}}'$ and such that $\alpha(Y)=1$. The study of the Lie algebra $\mathfrak{m}$ (and thus of $\mathfrak{g}$) now reduces to the study of the 3-dimensional sub-algebra $\tilde{\mathfrak{m}}$. We consider the following cases:

(i) $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]=0$. In particular the decomposition

$$\mathfrak{g}=(\mathfrak{h}+\mathfrak{m}_1+\boldsymbol{R}Y)\oplus\tilde{\mathfrak{m}}'$$

is a direct sum of ideals and therefore M is isometric to a product $M\cong M_1^{(n-2)}\times\boldsymbol{R}^2$ where M_1 is an $(n-2)$-dimensional space of constant negative curvature.

(ii) $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]=\tilde{\mathfrak{m}}'$, i.e. $\dim[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]=2$. We first prove

(5.6) LEMMA. $\tilde{\mathfrak{m}}'$ *is an abelian ideal.*

PROOF. Let X_1, X_2 be a basis of $\tilde{\mathfrak{m}}'$ such that

$$[X_1, X_2]=\lambda X_2\,.$$

Let $[Y, X_i]=\sum_{j=1}^{2} a_{ij}X_j$, $i=1, 2$. Then

$$\lambda[Y, X_2]=[Y, [X_1, X_2]]=[[Y, X_1], X_2]+[X_1, [Y, X_2]]$$

and we have

$$\lambda \sum_{j=1}^{2} a_{2j}X_j=\lambda a_{11}X_2+\lambda a_{22}X_2$$

which implies that if $\lambda\neq 0$, $a_{11}=a_{21}=0$. But this would mean $[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]\subseteqq \boldsymbol{R}X_2$, contradicting assumption (ii). Hence $\lambda=0$ and (5.6) is proved.

It is clear now that in this case the derived algebra $[\mathfrak{m}, \mathfrak{m}]=\mathfrak{m}_1+\tilde{\mathfrak{m}}'$ is an abelian ideal of codimension 1. We can therefore apply Theorem 1 in [**4**] to conclude

(5.7) PROPOSITION. *Let D (respectively S) denote the symmetric (respectively skew-symmetric) part of the linear transformation*

$$\mathrm{ad}(Y): \tilde{\mathfrak{m}}'\to\tilde{\mathfrak{m}}'.$$

Then M admits a left-invariant metric with strictly negative curvature if and only if

(a) *D is positive definite*

(b) *$D^2-DS-SD$ is positive definite.*

If in addition we assume that $\tilde{\mathfrak{m}}$ is unimodular (i.e. $\mathrm{tr}(\mathrm{ad}\, Y)=0$) then a straightforward argument shows that there exists a basis X_1, X_2 of $\tilde{\mathfrak{m}}'$ such that $[Y, X_i]=\mu X_j$, $i\neq j$, $\mu\neq 0$. Therefore $\tilde{\mathfrak{m}}\cong E(1, 1)$, the Lie algebra of the group of rigid motions of Minkowski 2-space [**10**]. Moreover it follows from (5.7) that M does not admit a left-invariant metric with strictly negative curvature.

On the other hand if $\tilde{\mathfrak{m}}$ is not unimodular then $\mathrm{tr}(\mathrm{ad}\, Y)$ and $\det(\mathrm{ad}\, Y)$ are a complete set of isomorphism invariants for the Lie algebra $\mathfrak{m}$ [**10**]. In this case the ideal $\tilde{\mathfrak{m}}'$ may be characterized as the unimodular kernel of $\tilde{\mathfrak{m}}$.

(iii) $\dim[\tilde{\mathfrak{m}}, \tilde{\mathfrak{m}}]=1$. In this case $[\mathfrak{m}, \mathfrak{m}]$ is an abelian ideal of codimension 2 and [**4**, Proposition 2] implies that M does not admit a left-invariant metric with strictly negative curvature. Moreover, $\tilde{\mathfrak{m}}$ is not unimodular and the trace of $\mathrm{ad}(Y)$ acting on the unimodular kernel of $\tilde{\mathfrak{m}}$ is a complete isomorphism invariant for $\mathfrak{m}$.

Now we consider the case $M=G/H$, $H\cong SO(n-3)\times SO(2)$, $n-3\geqq 3$. In this case $\dim\tilde{\mathfrak{m}}=1$ and $\mathfrak{g}$ decomposes according to (3.1) as

$$\mathfrak{g}=\mathfrak{h}_1+\mathfrak{h}_2+\mathfrak{m}_1+\mathfrak{m}_2+\tilde{\mathfrak{m}},$$

$$\mathfrak{h}_1\cong so(n-3), \quad \mathfrak{h}_2\cong so(2).$$

We set $\tilde{\mathfrak{m}}_1=\mathfrak{h}_2+\mathfrak{m}_2+\tilde{\mathfrak{m}}$. We have

$$[\mathfrak{h}_1, \tilde{\mathfrak{m}}_1]=0, \quad [\tilde{\mathfrak{m}}_1, \mathfrak{m}_1]\subseteqq\mathfrak{m}_1.$$

Let $\tilde{\mathfrak{m}}_1'=\{X\in\tilde{\mathfrak{m}}_1 : [X, \mathfrak{m}_1]=0\}$. If $\tilde{\mathfrak{m}}_1=\tilde{\mathfrak{m}}_1'$ then $\mathfrak{g}$ is a split Lie algebra

$\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ where

$$\mathfrak{g}_1=\mathfrak{h}_1+\mathfrak{m}_1$$

$$\mathfrak{g}_2=\mathfrak{h}_2+\mathfrak{m}_2+\tilde{\mathfrak{m}}\,.$$

It then follows from (3.4) that
(5.8) *M is isometric to a product*

$$M\cong M_1^{(n-3)}\times M_2^{(3)}$$

where M_1 is a simply-connected space of constant curvature and M_2 is a 3-dimensional simply-connected manifold admitting a 4-dimensional transitive group of isometries. These spaces have been classified in (4.11).

Assume now $\tilde{\mathfrak{m}}_1'\neq\tilde{\mathfrak{m}}_1$. Then $\dim\tilde{\mathfrak{m}}_1'=3$. Moreover, $\mathfrak{h}_2\subseteqq\tilde{\mathfrak{m}}_1'$ and since $\tilde{\mathfrak{m}}_1'$ is an ideal, $\mathfrak{m}_2=[\mathfrak{h}_2, \mathfrak{m}_2]\subseteqq\tilde{\mathfrak{m}}_1'$. Hence

$$\tilde{\mathfrak{m}}_1'=\mathfrak{h}_2+\mathfrak{m}_2\,.$$

Let Y be a unit vector in $\tilde{\mathfrak{m}}$ such that $\mathrm{ad}\,Y:\mathfrak{m}_1\to\mathfrak{m}_1$ is the identity map (again this is possible up to homothety). Furthermore, as in Section 4 we can also assume (changing $\mathfrak{m}_2$ if necessary) that there exists $\lambda\in\boldsymbol{R}$ such that

$$[Y, X]=\lambda X\,,\qquad X\in\mathfrak{m}_2\,.$$

We have also shown in Section 4 that $[\mathfrak{m}_2, \mathfrak{m}_2]\subseteqq\tilde{\mathfrak{m}}$. Therefore if E_1, E_2 is a basis of $\mathfrak{m}_2$ we have

$$[E_1, E_2]=\alpha Y\,,\qquad \alpha\in\boldsymbol{R}\,.$$

But if $0\neq Z\in\mathfrak{m}_1$, we get

$$0=[[E_1, E_2], Z]=\alpha[Y, Z]=\alpha Z\,.$$

Consequently $[\mathfrak{m}_2, \mathfrak{m}_2]=0$, the subalgebra $\mathfrak{m}$ is solvable and the derived algebra $[\mathfrak{m}, \mathfrak{m}]=\mathfrak{m}_1+\mathfrak{m}_2$ is an abelian ideal of codimension 1.

If $\lambda=0$ then $\mathfrak{g}$ splits as $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$, with

$$\mathfrak{g}_1=\mathfrak{h}_1+\mathfrak{m}_1+\boldsymbol{R}Y\,,\quad \mathfrak{g}_2=\mathfrak{h}_2+\mathfrak{m}_2$$

and therefore
(5.9) *M is isometric to a product $M_1^{(n-2)}\times\boldsymbol{R}^2$, where M_1 is a space of constant negative curvature. Moreover $G\cong G_1\times E(2)$, where G_1 is the group described in* [**6**, *Theorem* 3.3] *and* $E(2)$ *is the group of motions of Euclidean* 2*-space.*

For $\lambda\neq 0$, M is isometric to a solvable Lie group $M(\lambda)$ with a left-invariant metric and it follows from [**4**, Theorem 1] that
(5.10) *If $\lambda>0$, $M(\lambda)$ has strictly negative curvature; for $\lambda<0$, $M(\lambda)$ has both positive and negative sectional curvatures.*

It is straightforward to check that (5.2), (5.4) and (5.5) carry over to this case without modifications.

Finally, we consider the case of a 5-dimensional homogeneous Riemannian manifold $M=G/H$ where $H\cong SO(2)\times SO(2)$ and the linear isotropy representation of H is equivalent to $\rho_2\oplus\rho_2\oplus\theta_1$. As in (3.1) we can write

$$\mathfrak{g}=\mathfrak{h}_1+\mathfrak{h}_2+\mathfrak{m}_1+\mathfrak{m}_2+\tilde{\mathfrak{m}}\,.$$

Let $\tilde{\mathfrak{m}}_1=\mathfrak{h}_2+\mathfrak{m}_2+\tilde{\mathfrak{m}}$, $\tilde{\mathfrak{m}}_2=\mathfrak{h}_1+\mathfrak{m}_1+\tilde{\mathfrak{m}}$; then $[\mathfrak{h}_i, \tilde{\mathfrak{m}}_i]=0$, $[\tilde{\mathfrak{m}}_i, \mathfrak{m}_i]\subseteqq\mathfrak{m}_i$ and $[\tilde{\mathfrak{m}}_i, \tilde{\mathfrak{m}}_i]\subseteqq\tilde{\mathfrak{m}}_i'=\{X\in\tilde{\mathfrak{m}}_i : [X, \mathfrak{m}_i]=0\}$.

If $\tilde{\mathfrak{m}}_1\neq\tilde{\mathfrak{m}}_1'$, then $\mathfrak{g}_2=\mathfrak{h}_2+\mathfrak{m}_1+\mathfrak{m}_2+\tilde{\mathfrak{m}}$ is a subalgebra of $\mathfrak{g}$ and the corresponding analytic subgroup $G_2\subseteqq G$ acts transitively on M with isotropy $H_2\cong SO(2)$. Hence this case reduces to the one studied in Section 4.

We can therefore assume that $\tilde{\mathfrak{m}}_i=\tilde{\mathfrak{m}}_i'$ for $i=1, 2$. In particular we have

$$[\mathfrak{m}_1, \mathfrak{m}_2]=[\mathfrak{m}_1, \tilde{\mathfrak{m}}]=[\mathfrak{m}_2, \tilde{\mathfrak{m}}]=0\,,$$

$$[\mathfrak{m}_i, \mathfrak{m}_i]\subseteqq\mathfrak{h}_i+\tilde{\mathfrak{m}}\,.$$

Thus $\tilde{\mathfrak{m}}$ is an ideal in $\mathfrak{g}$ and the quotient $\mathfrak{g}/\tilde{\mathfrak{m}}$ is a split Lie algebra

$$\mathfrak{g}/\tilde{\mathfrak{m}}\cong\mathfrak{g}_1\oplus\mathfrak{g}_2\,,\quad \mathfrak{g}_i=\mathfrak{h}_i+\mathfrak{m}_i+\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}\,.$$

Moreover, the one-parameter group $\exp\tilde{\mathfrak{m}}$ acts freely on M and hence M is a principal fiber bundle over $M/\exp\tilde{\mathfrak{m}}$. It is easy to check that, relative to the induced metric, the space splits isometrically as $M_1\times M_2$ where M_i, $i=1, 2$ is a simply-connected 2-dimensional space of constant curvature.

References

[1] Robert Azencott and Edward N. Wilson, Homogeneous manifolds with negative curvature, I, Trans, Amer. Math. Soc., **215** (1976), 323-362.

[2] Robert Azencott and Edward N. Wilson, Homogeneous manifolds with negative curvature, II, Mem, Amer. Math. Soc., **8**, no. 178 (1976).

[3] E. Cartan, Leçons sur la Géométrie des Espaces de Riemann, Gauthier-Villars, Paris, 1963.

[4] Ernst Heintze, On homogeneous manifolds of negative curvature, Math. Ann., **211** (1974), 23-34.

[5] Wu-chung Hsiang and Wu-yi Hsiang, Differentiable actions of compact connected classical groups I, Amer. J. Math., **89** (1967), 705-786.

[6] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, 1972.

[7] S. Kobayashi and T. Nagano, Riemannian manifolds with abundant isometries, Differential Geometry in Honor of K. Yano, Kinokuniya, Tokyo, 1972, 195-220.

[8] Minoru Kurita, On the isometry of a homogeneous Riemann Space, Tensor, **3** (1954), 91-100.

[9] Gordon W. Lukesh, Compact homogeneous Riemannian manifolds, Geometriae Dedicata, **7** (1978), 131-137.

[10] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Math., **21** (1976), 293-329.

[11] H. Wakakuwa, On n-dimensional Riemannian spaces admitting some groups of motions of order less than $\frac{1}{2}n(n-1)$, Tôhoku Math. J., 6 (1954), 121-134.

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