

## On the bifurcation of the multiplicity and topology of the Newton boundary

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### 0. Introduction.

In [3], A.G. Kouchnirenko presented a beautiful formula about the multiplicity of an isolated singularity of a hypersurfaces  $V=f^{-1}(0)$  in  $\mathbf{C}^n$  where  $f(z)$  is assumed to have a non-degenerate Newton principal part. It states that the multiplicity of  $V$  at the origin (=the Milnor number) is equal to the Newton number of  $\Gamma_-(f)$  (and thus is independent of a particular choice of the coefficients of  $f(z)$ ).

The purpose of this paper is to give a geometrical proof of Kouchnirenko's Theorem from the viewpoint of the bifurcation of the multiplicity. First we prove that *the Milnor fibration of  $f(z)$  is determined by the Newton boundary  $\Gamma_-(f)$  if the Newton principal part of  $f$  is non-degenerate* (Theorem 2.1).

For the calculation of the multiplicity, we consider the bifurcating equation :

$$z_1 \frac{\partial f}{\partial z_1} - t\gamma_1 = \cdots = z_n \frac{\partial f}{\partial z_n} - t\gamma_n = 0.$$

If  $\gamma=(\gamma_1, \dots, \gamma_n)$  is generic and  $t$  is sufficiently small, the bifurcating solutions of the above equation are all simple and one finds exactly  $n!$  volume  $\Gamma_-(f)$  solutions ( $t \neq 0$ ) (Theorem 4.2).

### 1. Milnor fibration.

Let  $f(z_1, z_2, \dots, z_n)$  be an analytic function in an open neighbourhood  $U$  of  $\mathbf{C}^n$  ( $f(0)=0$ ) and assume that  $f(z)$  has an isolated critical point at the origin. We can take a positive number  $\epsilon$  so that the sphere  $S(r)=\{z \in \mathbf{C}^n; \|z\|^2=|z_1|^2 + \cdots + |z_n|^2=r^2\}$  cuts the hypersurface  $V_0=f^{-1}(0)$  transversely for any  $0 < r \leq \epsilon$ . (Therefore  $V_0 \cap S(r)$  is a smooth manifold.) Fixing such an  $\epsilon$ , we can take  $\delta > 0$  such that  $V_\eta=f^{-1}(\eta)$  is non-singular in  $D(\epsilon)$  and is transverse to  $S(\epsilon)$  for  $0 < |\eta| \leq \delta$  where  $D(\epsilon)=\{z \in \mathbf{C}^n; \|z\| \leq \epsilon\}$ . Then we have a so-called Milnor

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fibration

$$f: X \longrightarrow S$$

where  $S = \{\eta \in \mathbf{C}; 0 < |\eta| \leq \delta\}$  and  $X = f^{-1}(S) \cap D_\varepsilon$ . This fibration does not depend on the choice of  $\varepsilon$  and  $\delta$  up to a fibre preserving diffeomorphism and we call  $\varepsilon$  a stable radius of the Milnor fibration for  $f(z)$ . The fibre is  $(n-2)$ -connected and its  $(n-1)$ -th Betti number is called the Milnor number of  $f(z)$  and we denote this number by  $\mu(f, 0)$  (Milnor [5]).

## 2. Newton boundary.

Now we recall basic notions about the Newton boundary (see [3] for detail.) and state the first result. Let  $f(z)$  be the analytic function defined by a convergent power series  $\sum a_\nu z^\nu$  where  $\nu = (\nu_1, \dots, \nu_n)$  goes through multi-integers  $(\mathbf{N})^n$  and  $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \dots z_n^{\nu_n}$  as usual. A polyhedron  $\Gamma_+(f) \subset \mathbf{R}^n$  is defined by {the convex closure of  $\bigcup_\nu (\nu + (\mathbf{R}_+)^n)$  where the union is taken for all  $\nu$  such that  $a_\nu \neq 0$ }. ( $\mathbf{R}_+ = \{x \in \mathbf{R}; x \geq 0\}$ ). The *Newton boundary*  $\Gamma(f)$  is by definition the union of the compact boundaries of  $\Gamma_+(f)$ . The polynomial  $\sum_{\nu \in \Gamma_+(f)} a_\nu z^\nu$  is called the *Newton principal part* of  $f(z)$ . For a face  $\Delta$  of  $\Gamma(f)$ , we put  $f_\Delta(z) = \sum_{\nu \in \Delta} a_\nu z^\nu$ . We say that  $f$  is *nondegenerate on  $\Delta$*  if the equation  $\frac{\partial f_\Delta}{\partial z_1} = \frac{\partial f_\Delta}{\partial z_2} = \dots = \frac{\partial f_\Delta}{\partial z_n} = 0$  has no solution in  $(\mathbf{C}^*)^n$ . When  $f$  is non-degenerate on every face  $\Delta$  of  $\Gamma(f)$ , we say that  $f$  has a *non-degenerate Newton principal part*.  $f(z)$  is called to be *convenient* if the intersection of  $\Gamma(f)$  with each coordinate axe is non-empty.  $\Gamma_-(f)$  is the compact polyhedron which is the cone over  $\Gamma(f)$  with the origin as a vertex. When  $f$  is convenient, the *Newton number*  $\nu(\Gamma_-(f))$  is by definition  $n! V_n - (n-1)! V_{n-1} + \dots + (-1)^{n-1}! V_1 + (-1)^n$  where  $V_n$  is the  $n$ -dimensional volume of  $\Gamma_-(f)$  and  $V_k$  is the sum of  $k$ -dimensional volumes of  $\Gamma_-(f) \cap \mathbf{R}^I$  where  $I = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$  and  $\mathbf{R}^I = \{x = (x_1, \dots, x_n); x_i = 0 \text{ if } i \notin I\}$ .

Our first result is:

**THEOREM 2.1.** *Suppose that  $f(z)$  has an isolated critical point at the origin and that  $f$  has a non-degenerate Newton principal part. Then the Milnor fibration at the origin is determined by the Newton boundary  $\Gamma(f)$ .*

**COROLLARY 2.2.** *The topology of singularity and  $\mu(f, 0)$  are independent of the particular choice of  $f$  for a fixed  $\Gamma(f)$ .*

**REMARK 2.3.** The latter statement in Corollary 2.2 was proved by Kouchnirenko [3]. As for the topology of the singularity, the case  $n=3$  is the only new result. ([3], [4]).

First we prepare a few lemmas.

LEMMA 2.4. Let  $f_t(z)$  ( $0 \leq t \leq 1$ ) be a smooth family of analytic functions such that  $f_t(0)=0$  and  $f_t(z)$  has an isolated critical point at the origin for each  $t$ ,  $0 \leq t \leq 1$ . Assume that  $f_t$  ( $0 \leq t \leq 1$ ) has a uniform stable radius  $\epsilon$  for Milnor fibration i.e. for any  $0 < r \leq \epsilon$  and  $0 \leq t \leq 1$ ,  $S(r)$  cuts  $f_t^{-1}(0)$  transversely. Then the Milnor fibrations of  $f_0$  and  $f_1$  are equivalent.

The proof is easy. (See for example, Oka [7].)

The following lemma is slightly stronger than Proposition 3.6 of [3].

LEMMA 2.5. Let  $f(z)$  be an analytic function which has an isolated critical point at the origin. Consider the family  $F(z, t)=f(z)+tz^\nu$  ( $0 \leq t \leq 1$ ) where  $\nu=(\nu_1, \dots, \nu_n)$  is a fixed multiinteger. If  $|\nu|=\nu_1+\nu_2+\dots+\nu_n \geq \mu(f, 0)+2$ ,  $F(z, t)$  ( $0 \leq t \leq 1$ ) has a uniform stable radius for Milnor fibration.

PROOF. By Lemma 3.5 and Proposition 3.6 of [3],  $\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}$  generate an ideal  $J$  of  $\mathbf{C}[[z_1, \dots, z_n]]$  (=ring of the formal power series in  $z_1, \dots, z_n$ ) which is independent of  $t$  and  $\mathfrak{M}^{\mu(f, 0)}$  is contained in  $J$  where  $\mathfrak{M}$  is the maximal ideal. Let  $W = \{(z, t) \in \mathbf{C}^n \times \mathbf{R}; F(z, t)=0 \text{ and } \bar{z} \text{ and } (\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_n}) \text{ are linearly dependent over } \mathbf{C}\}$ . Suppose that  $F(z, t)$  is not uniform. Then for some  $0 \leq t_0 \leq 1$ ,  $(0, t_0) \in$  the closure of  $W \cap \{z \neq 0\}$ . By the Curve Selection lemma (Milnor [5]) we can find a real analytic curve  $p(s)=(z(s), t(s))$ ,  $0 \leq s \leq \epsilon$ , in  $W$  such that  $z(s) \neq 0$  for  $s > 0$  and  $t(0)=t_0$ . We consider the Taylor expansion:

$$(2.5.1) \quad \begin{pmatrix} z(s) \\ t(s) \end{pmatrix} = \begin{pmatrix} z_1(s) \\ \vdots \\ z_n(s) \end{pmatrix} = \begin{pmatrix} \alpha_1 s^{a_1} + \text{higher} \\ \vdots \\ \alpha_n s^{a_n} + \text{higher} \\ t_0 + \text{higher} \end{pmatrix}.$$

We also have  $\lambda(s) \in \mathbf{C}$  expanded in a Laurent series

$$(2.5.2) \quad \lambda(s) = \lambda_0 s^b + \text{higher}$$

so that

$$(2.5.3) \quad F(z(s), t(s)) = 0 \quad \text{and}$$

$$(2.5.4) \quad \frac{\partial F}{\partial z_j}(z(s), t(s)) = \lambda(s) \bar{z}_j(s) \quad (j=1, \dots, n).$$

$\{a_j\}$  ( $j=1, \dots, n$ ) are positive integers (possibly  $\infty$ ).  $a_j = \infty$  implies by definition  $z_j(s) \equiv 0$ . Otherwise  $\alpha_j$  is assumed to be a non-zero complex number. Taking the differential of (2.5.3) in  $s$  we get

$$(2.5.5) \quad \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} + z(s)^\nu \cdot \frac{dt(s)}{ds} = 0.$$

Let  $a = \text{minimum}\{a_1, a_2, \dots, a_n\}$ . Using (2.5.4), we can write

$$\frac{\partial F}{\partial z_j}(z(s), t(s)) \cdot \frac{dz_j}{ds}(s) = \lambda_0 a_j |\alpha_j|^2 s^{2a_j+b-1} + \dots$$

That  $a_j |\alpha_j|^2$  ( $j=1, \dots, n$ ) are positive implies

$$\lim_{s \rightarrow 0} \left( \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j}{ds}(s) \right) / s^{2a+b-1}$$

is a non-zero complex number. Assertion:  $a+b \leq \mu(f, 0) \cdot a$ . Assuming this, we get  $\lim_{s \rightarrow 0} z^\nu(s) \frac{dt}{ds} / s^{2a+b-1} = 0$  which is a contradiction to the equation (2.5.5).

Now we prove the assertion. We may assume  $a = a_1$ . The inclusion  $m^{\mu(f, 0)} \subset J$  gives us an equation

$$(2.5.6) \quad \begin{aligned} z_1^{\mu(f)} &= \sum_{k=1}^n g_k(z) \frac{\partial f}{\partial z_k} \\ &= \sum_{k=1}^n G_k(z, t) \frac{\partial F}{\partial z_k} \end{aligned}$$

where  $g_k(z)$  and  $G_k(z, t)$  are analytic functions. Assuming  $a+b > \mu(f, 0) \cdot a$ , we obtain an absurd equality:

$$\begin{aligned} 0 \neq \alpha_1^{\mu(f, 0)} &= \lim_{s \rightarrow 0} z_1(s)^{\mu(f, 0)} / s^{\mu(f, 0) \cdot a} \\ &= \lim_{s \rightarrow 0} \sum_{k=1}^n G_k(z(s), t(s)) \frac{\partial F}{\partial z_k}(z(s), t(s)) / s^{\mu(f, 0) \cdot a} \\ &= 0. \end{aligned}$$

PROOF OF THEOREM 2.1. Suppose that we have two analytic functions  $f(z)$  and  $g(z)$  which have non-degenerate Newton principal parts on the same boundary  $\Gamma (= \Gamma(f) = \Gamma(g))$ . By Lemma 2.5 we may assume that  $f$  and  $g$  are convenient. Because the non-degenerate condition of the Newton principal part is an open condition (see the appendix and [3]), we can take a piecewise analytic family  $F(z, t)$  such that  $F(z, 0) = f(z)$  and  $F(z, 1) = g(z)$  and  $F(z, t)$  as a function of  $z$  has a non-degenerate Newton principal part on  $\Gamma$  for each  $t$ . Thus the proof reduces to the following lemma by virtue of Lemma 2.4.

LEMMA 2.6. *Let  $F(z, t)$  ( $0 \leq t \leq 1$ ) be an analytic family such that for each fixed  $t$  ( $0 \leq t \leq 1$ )  $F(z, t)$  is convenient and has a non-degenerate Newton principal part on the same boundary  $\Gamma$ . Then  $F(z, t)$  ( $0 \leq t \leq 1$ ) has a uniform stable radius for the Milnor fibrations.*

PROOF. Assume the contrary. Then using the Curve Selection lemma we can find a real analytic curve:

$$(2.6.1) \quad \begin{pmatrix} z(s) \\ t(s) \end{pmatrix} = \begin{pmatrix} z_1(s) \\ \vdots \\ z_n(s) \\ t(s) \end{pmatrix} = \begin{pmatrix} \alpha_1 s^{a_1} + \text{higher} \\ \vdots \\ \alpha_n s^{a_n} + \text{higher} \\ t_0 + \text{higher} \end{pmatrix}$$

and a Laurent series

$$\lambda(s) = \lambda_0 s^b + \text{higher} \quad (\lambda_0 = 0 \Leftrightarrow \lambda(s) \equiv 0)$$

satisfying the following equations.

$$(2.6.2) \quad F(z(s), t(s)) = 0 \quad \text{and}$$

$$(2.6.3) \quad \frac{\partial F}{\partial z_j}(z(s), t(s)) = \lambda(s) \bar{z}_j(s) \quad (j=1, 2, \dots, n).$$

From (2.6.2) we get

$$(2.6.4) \quad \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} + \frac{\partial F}{\partial t}(z(s), t(s)) \frac{dt(s)}{ds} = 0.$$

We may assume that  $z_1(s) \neq 0, \dots, z_k(s) \neq 0$  and  $z_{k+1}(s) \equiv \dots \equiv z_n(s) = 0$ . Let  $\mathbf{R}^k$  be the subspace of  $\mathbf{R}^n$  defined by  $x_j = 0$  for  $j > k$ . Consider the continuous function  $l_a(x)$  on  $\Gamma \cap \mathbf{R}^k$  defined by  $l_a(x) = (x, a) = \sum_{j=1}^k a_j x_j$ . By the convenience assumption on  $f$ ,  $\Gamma \cap \mathbf{R}^k \neq \emptyset$ . So let  $\Delta(a)$  be the unique face of  $\Gamma \cap \mathbf{R}^k$  where  $l_a(x)$  takes the minimal value, say  $d$ . See Figure A.

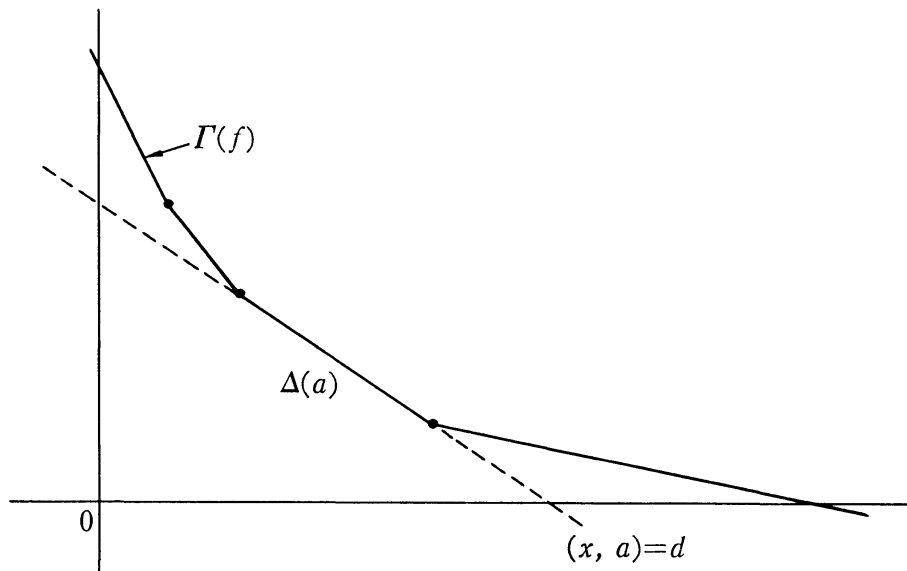


Figure A

Then  $\frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} = a_j \alpha_j \frac{\partial F_{\Delta(a)}}{\partial z_j}(\alpha, t_0) t^{a-1} + \text{higher terms}$  where  $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots, 0)$ . Here  $F_{\Delta(a)}(z, t)$  is defined as before considering it as a function of  $z$ . Note that  $F_{\Delta(a)}(z, t)$  is a function of  $z_1, \dots, z_k$  and  $t$ . Thus the non-degeneracy assumption for  $F(z, t_0)$  implies that  $\frac{\partial F_{\Delta(a)}}{\partial z_j}(\alpha, t_0) \neq 0$  for some  $1 \leq j \leq k$ , say  $j_0$ . On the other hand, by (2.6.3) we have

$$\frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} = \lambda_0 |\alpha_j|^2 a_j s^{2a_j+b-1} + \text{higher.}$$

Thus  $\lambda_0 \neq 0$  and  $2a_j + b - 1 \geq d - 1$ . Moreover  $\lim_{s \rightarrow 0} \text{argument} \left( \frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} \right) = \text{argument } \lambda_0$  for  $1 \leq j \leq k$ .

Therefore

$$\lim_{s \rightarrow 0} \sum_{j=1}^n \frac{\partial F}{\partial z_j}(z(s), t(s)) \frac{dz_j(s)}{ds} / s^{d-1}$$

is a non-zero complex number, while

$$\lim_{s \rightarrow 0} \frac{\partial F}{\partial t}(z(s), t(s)) / s^{d-1} = 0$$

because of the assumption on the Newton boundary. However this trivially contradicts the equation (2.6.4), completing the proof of Lemma 2.6.

REMARK. 2.7. In the above proof it is proved that  $f(z)$  has an isolated critical point at the origin if  $f(z)$  is convenient and its Newton principal part is non-degenerate.

### 3. Bifurcation of multiplicity.

Consider an analytic mapping

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) : U \longrightarrow \mathbb{C}^n$$

where  $U$  is an open neighbourhood of the origin in  $\mathbb{C}^n$ . Assume that 0 is isolated in  $\varphi^{-1}(0)$  and let  $D$  be a small disk  $\{z \in \mathbb{C}^n; \|z\| \leq \varepsilon\}$  such that  $D \cap \varphi^{-1}(0) = \{0\}$ . Then the multiplicity of  $\varphi_1(z) = \dots = \varphi_n(z) = 0$  at the origin is defined by the mapping degree of the normalized map

$$\tilde{\varphi} = \varphi / \|\varphi\| : \partial D \longrightarrow S^{2n-1}.$$

We denote this number by  $\mu(\varphi, 0)$ . More generally if  $\partial D \cap \varphi^{-1}(0) = \emptyset$  and  $D$  contains several points of  $\varphi^{-1}(0)$ , we denote the above mapping degree by  $\mu(\varphi, D)$ . This is equal to  $\sum_P \mu(\varphi, P)$  where  $P \in \varphi^{-1}(0) \cap D$ .

In the case that  $\varphi_i(z) = -\frac{\partial f}{\partial z_i}$  ( $i=1, \dots, n$ ) for an analytic function  $f(z)$ ,  $\mu(\varphi, 0)$  coincides with  $\mu(f, 0)$ . ([5]).

Now consider an analytic family  $\{\varphi_t\}$  ( $0 \leq t \leq 1$ ) of analytic mapping,  $\varphi_t : U \rightarrow \mathbb{C}^n$  such that  $\varphi_0 = \varphi$  and  $\varphi^{-1}(0) \cap D = \{0\}$  and  $\varphi_t(z) \neq 0$  for  $z \in \partial D$  and  $0 \leq t \leq 1$  and that  $\varphi_t^{-1}(0)$  are isolated in  $D$  for  $0 \leq t \leq 1$ . Let  $\varphi_t^{-1}(0) \cap D = \{\alpha_1(t), \dots, \alpha_k(t)\}$ . By taking a suitable positive integer  $m_j$  and putting  $t = s^{m_j}$ ,  $\alpha_j(t)$  ( $j=1, \dots, k$ ) become analytic functions in  $s$ , for  $t$  small enough (Curve Selection Lemma).

DEFINITION. We call  $\lim_{t \rightarrow 0} \sum_{\alpha_j(t) \neq 0} \mu(\varphi_t, \alpha_j(t))$  the bifurcated multiplicity of  $\{\varphi_t=0\}$  at the origin and we denote this number by  $\text{Bf.}\mu(\{\varphi_t\}, 0)$ . Note that  $\text{Bf.}\mu(\{\varphi_t\}, 0)$  is equal to  $\mu(\varphi_0, 0) - \lim_{t \rightarrow 0} \mu(\varphi_t, 0)$  (Milnor [5]). In the case that  $\varphi_t = \left(\frac{\partial f_t}{\partial z_1}, \dots, \frac{\partial f_t}{\partial z_n}\right)$  where  $\{f_t\}_{0 \leq t \leq 1}$  is an analytic family of analytic functions having isolated critical points in  $D$ ,  $\text{Bf.}\mu(\{\varphi_t\}, 0)$  is the number of the bifurcated vanishing cycles. Vanishing cycles at non-zero  $\alpha_j(t)$  for  $f_t$ , together with the vanishing cycles of  $f_t$  at 0, form the basis of the vanishing cycles for  $f_0$  at the origin.

4. A nice bifurcation.

Let  $f(z)$  be an analytic function of  $n$  variables whose Newton principal part is non-degenerate and convenient. We want to calculate geometrically the multiplicity of the next equation at the origin:

$$A: \quad z_1 \frac{\partial f}{\partial z_1} = \dots = z_n \frac{\partial f}{\partial z_n} = 0.$$

A vector  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$  is called *regular* for  $A$  if for any finite vertices  $\nu^1, \nu^2, \dots, \nu^k$  of  $\Gamma(f)$  such that  $\text{rank}(\nu^1, \dots, \nu^k) < n$ ,  $\gamma$  is not a linear combination of  $\nu^j$  with complex coefficients. In particular,  $\gamma_j \neq 0$  ( $j=1, \dots, n$ ) by the convenience assumption of  $f$ . Hereafter we fix a regular  $\gamma$ .

Instead of  $A$ , we consider the equation

$$A(\gamma, t): \quad z_j \frac{\partial f}{\partial z_j} - t\gamma_j = 0 \quad (j=1, \dots, n).$$

Let  $\mathcal{S}$  be the set of faces of dimension  $n-1$  of  $\Gamma(f)$  and let  $\mathcal{A}(0)$  be the cone over  $\mathcal{A}$  with the origin. Then  $\Gamma_-(f) = \bigcup_{\mathcal{A} \in \mathcal{S}} \mathcal{A}(0)$  and they make a natural polyhedral decomposition of  $\Gamma_-(f)$ . Let  $L(\mathcal{A})$  be the hyperplane in  $\mathbb{R}^n$  containing  $\mathcal{A} \in \mathcal{S}$  and let

$$(4.1) \quad m(\mathcal{A})_1 x_1 + m(\mathcal{A})_2 x_2 + \dots + m(\mathcal{A})_n x_n = d(\mathcal{A})$$

be the defining equation of  $L(\mathcal{A})$  such that  $m(\mathcal{A})_j$  ( $j=1, \dots, n$ ) are positive integers and have 1 as their greatest common divisor. Let  $\{C_\nu\}$  be the coefficients of  $z^\nu$  for  $\nu \in \Gamma(f)$  in the Taylor expansion of  $f(z)$ .

Now we can interpret Kouchnirenko's formula ([3]) as follows:

THEOREM 4.2. *If the coefficients  $\{c_\nu\}$  are generic and if  $\gamma$  is regular,  $\mu\left(z_1 \frac{\partial f}{\partial z_1} = \dots = z_n \frac{\partial f}{\partial z_n}, 0\right) = \text{Bf.}\mu(A(\gamma, t), 0) = n!$  volume  $\Gamma_-(f)$ . More precisely for each  $\mathcal{A} \in \mathcal{S}$ , there exist  $n!$  volume  $\mathcal{A}(0)$  simple solutions of  $A(\gamma, t)$  ( $t$ : small) which can be parametrized real analytically as:*

$$z(s) = (\alpha_1 s^{a(\Delta)_1} + \text{higher}, \dots, \alpha_n s^{a(\Delta)_n} + \text{higher})$$

and

$$t(s) = s^{b(\Delta)}$$

where the integral vector  $a(\Delta) = (a(\Delta)_1, \dots, a(\Delta)_n)$  is equal to  $q(\Delta)m(\Delta) = (q(\Delta)m(\Delta)_1, \dots, q(\Delta)m(\Delta)_n)$  for a positive integer  $q(\Delta)$  and  $b(\Delta) = q(\Delta)d(\Delta)$ .

For a subset  $I \subset \{1, 2, \dots, n\}$ , let  $C^I$  be the subspace  $\{z = (z_1, \dots, z_n) \in C^n; z_j = 0 \text{ for } j \in I\}$  and let  $f_I$  be the restriction of  $f$  to  $C^I$ . Then the additivity of the intersection multiplicity and Theorem 4.2 implies:

COROLLARY 4.2.1 (Kouchnirenko [3]). *Various Milnor numbers are related by  $\sum_{I \subset \{1, 2, \dots, n\}} \mu(f_I, 0) = n! \Gamma_-(f)$  where  $\mu(f_B, 0) = 1$  by definition.*

Thus by the induction on  $n$ , we get

COROLLARY 4.2.2 (Kouchnirenko [3]).  $\mu(f, 0) = \nu(\Gamma_-(f))$ . (For the definition of the Newton number  $\nu(\Gamma_-(f))$ , see § 2.)

REMARK 4.2.3. For a non-regular  $\gamma$ , the bifurcating solutions of  $A(\gamma, t)$  are not simple in general and the statement about the parametrization of Theorem 4.2 is not true.

The proof is based on several lemmas which are of independent interest. Take a solution  $z^0$  of  $A(\gamma, t^0)$  for  $t^0$  sufficiently small. Using the Curve Selection lemma ([2], [5]), we can find a real analytic curve  $(z(s), t(s)), 0 \leq s \leq \epsilon$  ( $\epsilon$ : small), such that  $(z(s_0), t(s_0)) = (z^0, t^0)$  for some  $s_0 > 0$ ,  $(z(0), t(0)) = (0, 0)$  and

$$(4.2.4) \quad z_j(s) \frac{\partial f}{\partial z_j}(z(s)) = t(s) \gamma_j \quad (j=1, 2, \dots, n).$$

Changing the parametrization if necessary, we may assume that  $t(s) = s^b$  ( $b$ : positive integer) and consider the Taylor expansion of  $z(s)$ :

$$(4.2.5) \quad z_j(s) = \alpha_j s^{a_j} + \text{higher terms}, \quad j=1, 2, \dots, n.$$

In this expression,  $\alpha_j \neq 0$  if  $z_j(s) \neq 0$ . Let  $I = \{1 \leq i \leq n; \alpha_i \neq 0\}$ . Consider the linear function  $l_a(x) = \sum_{i \in I} a_i x_i$  on  $\Gamma(f_I) = \Gamma(f) \cap R^I$  and let  $\Delta$  be the face of  $\Gamma(f_I)$  where  $l_a(x)$  takes the minimal value, say  $d$ .

ASSERTION.  $I = \{1, 2, \dots, n\}$  and  $\dim \Delta = n - 1$  (i. e.  $\Delta \in \mathcal{S}$ ).

PROOF. We consider the equation of the lowest terms in (4.2.4). The case  $d > b$  is trivially impossible. If  $d < b$ , we get  $\alpha_j \frac{\partial f_\Delta}{\partial z_j}(\alpha) = 0$  for  $j \in I$ . This has no solution in  $(C^*)^I$  by the non-degeneracy assumption of  $f$ . Assume  $d = b$ . Let  $\nu^1, \nu^2, \dots, \nu^m$  be the integral points of  $\Delta$  and let  $c_j = c_{\nu^j}$  ( $j=1, 2, \dots, m$ ). Then we get

$$\gamma = \sum_{j=1}^m c_j \alpha^{\nu^j} \cdot \nu^j.$$



By the regularity for  $\gamma$ , this implies  $\dim \Delta = n - 1$ , completing the proof of the assertion.

Thus the equation of the lowest terms is :

$$(4.2.6; \Delta) : \quad \alpha_j \frac{\partial f_\Delta}{\partial z_j}(\alpha) = \gamma_j \quad (j=1, 2, \dots, n)$$

for  $\Delta \in \mathcal{S}$ .

LEMMA 4.3. *We assume that the coefficients of  $f_\Delta(z)$  are generic. For a regular  $\gamma$  and  $\Delta \in \mathcal{S}$ , (4.2.6;  $\Delta$ ) has  $n!$  volume  $\Delta(0)$  solutions. They are all simple and are contained in  $(\mathbb{C}^*)^n$ .*

This lemma is one of the main steps and the proof is done in the next section.

Assuming this for a while, we want to show the bijective correspondence between the “formal” solutions of (4.2.6;  $\Delta$ ) ( $\Delta \in \mathcal{S}$ ) and the bifurcating solutions of  $A(\gamma, t)$  “along”  $\Delta$ . Let  $\alpha^1(\Delta), \alpha^2(\Delta), \dots, \alpha^{k(\Delta)}(\Delta)$  be the solutions of (4.2.6;  $\Delta$ ) ( $k(\Delta) = n!$  volume  $\Delta(0)$ ) and for a sufficiently small  $\varepsilon > 0$  (fixed) let  $D_j(\Delta) = \{\alpha \in \mathbb{C}^n; \|\alpha - \alpha^j(\Delta)\| \leq \varepsilon\}$ . Then  $D_j(\Delta) \subset (\mathbb{C}^*)^n$  and  $D_j(\Delta) \cap D_k(\Delta) = \emptyset$  if  $j \neq k$ . Let  $D_j(\Delta, t) = \{z = (\alpha_1 s^{m(\Delta)_1}, \dots, \alpha_n s^{m(\Delta)_n}); \alpha \in D_j(\Delta)\}$  and  $t = s^{d(\Delta)}$ .  $D_j(\Delta, t)$  ( $j=1, \dots, n!$  volume  $\Delta(0)$ ;  $\Delta \in \mathcal{S}$ ) are naturally diffeomorphic to the standard  $2n$ -dimensional disk ( $t \neq 0$ ) and are considered to be contained in  $\mathbb{C}^n \times t$ . Note that  $D_j(\Delta, t) \cap D_k(\Delta', t) = \emptyset$  for  $\Delta \neq \Delta'$  and  $t$  sufficiently small.

LEMMA 4.4. (i) *For a sufficiently small  $t^0 > 0$ , any solution  $z^0$  for  $A(\gamma, t^0)$  is contained in the interior of some (and unique)  $D_j(\Delta, t)$ .* (ii) *Each  $D_j(\Delta, t)$  for  $t$  small contains exactly one solution  $z^j(\Delta, t)$  of  $A(\gamma, t)$  and it is simple.  $z^j(\Delta, t)$  can be parametrized as*

$$z_k^j(\Delta, t) = \alpha_k^j(\Delta) s^{q(\Delta) m(\Delta)_k} + \text{higher} \quad (k=1, 2, \dots, n)$$

and

$$t = s^{q(\Delta) d(\Delta)}$$

for some positive integer  $q(\Delta)$ . ( $\alpha^j(\Delta) = (\alpha_1^j(\Delta), \dots, \alpha_n^j(\Delta))$ ).

PROOF. (i) is already proved in the above argument. Therefore we need only prove that each  $D_j(\Delta, t)$  contains a simple root. Let

$$\varphi(z, t) = \left( z_1 \frac{\partial f}{\partial z_1} - t\gamma_1, \dots, z_n \frac{\partial f}{\partial z_n} - t\gamma_n \right)$$

and

$$\varphi_\Delta(z, t) = \left( z_1 \frac{\partial f_\Delta}{\partial z_1} - t\gamma_1, \dots, z_n \frac{\partial f_\Delta}{\partial z_n} - t\gamma_n \right).$$

Then  $\|\varphi(\alpha_1 s^{m(\Delta)_1}, \dots, \alpha_n s^{m(\Delta)_n}, s^{d(\Delta)}) - \varphi_\Delta(\alpha_1 s^{m(\Delta)_1}, \dots, \alpha_n s^{m(\Delta)_n}, s^{d(\Delta)})\| \leq M s^{d(\Delta)+1}$  for some  $M > 0$  and  $z = (\alpha_1 s^{m(\Delta)_1}, \dots, \alpha_n s^{m(\Delta)_n}) \in D_j(\Delta, t)$ . Thus  $\|\rho \varphi(z, t) + (1 - \rho) \varphi_\Delta(z, t)\| \neq 0$  for  $0 \leq \rho \leq 1$  on  $\partial D_j(\Delta, t)$  and using the “Rouché’s Principle” ([5])

we get

$$\begin{aligned} \mu(\varphi=0, D_j(\mathcal{A}, t)) &= \mu(\varphi_{\mathcal{A}}=0, D_j(\mathcal{A}, t)) \\ &= 1 \quad \text{by Lemma 4.3.} \end{aligned}$$

PROOF OF THEOREM 4.2. (modulo Lemma 4.3). By Lemma 4.3 and Lemma 4.4, we get

$$\begin{aligned} \mu\left(z_1 \frac{\partial f}{\partial z_1} = \cdots = z_n \frac{\partial f}{\partial z_n} = 0, 0\right) \\ = \text{Bf. } \mu(\varphi(z, t) = 0, 0) \\ = n! \text{ volume } \Gamma_-(f). \end{aligned}$$

The other statement is already proved.

**5. Proof of Lemma 4.3.**

Let  $\nu^1, \nu^2, \dots, \nu^m$  be the integral points of  $\mathcal{A}$  and let  $c_j (=c_{\nu^j})$  be the coefficient of  $z^{\nu^j}$ . (4.2.6;  $\mathcal{A}$ ) can be written as:

$$(5.1) \quad \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^m \\ \nu_2^1 & \cdots & \nu_2^m \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^m \end{pmatrix} \begin{pmatrix} c_1 \alpha^{\nu^1} \\ c_2 \alpha^{\nu^2} \\ \vdots \\ c_m \alpha^{\nu^m} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

We prove Lemma 4.3 by the induction on  $m$ . The beginning step is:

LEMMA 5.1. *Suppose  $m=n$ . Then (5.1) has  $n!$  volume  $\mathcal{A}(0)$  simple solutions.*

PROOF. By the regularity of  $\gamma$  and the convenience assumption for  $f$ ,  $\gamma_j \neq 0$  ( $j=1, 2, \dots, n$ ). Therefore (5.1) has no solution in  $\mathbb{C}^n - (\mathbb{C}^*)^n$ . (5.1) can be rewritten as

$$(5.1)' \quad \begin{pmatrix} c_1 \alpha^{\nu^1} \\ c_2 \alpha^{\nu^2} \\ \vdots \\ c_n \alpha^{\nu^n} \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}$$

where

$$\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^n \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^n \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

and by the regularity of  $\gamma$ ,  $\delta_j \neq 0$  ( $j=1, \dots, n$ ). Let  $M$  be the matrix:

$$(\nu^1, \dots, \nu^n) = \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^n \\ \nu_2^1 & \cdots & \nu_2^n \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^n \end{pmatrix}.$$

Let  $d = \det M$ .  $d$  is a non-zero integer. Let  $N = d \cdot M^{-1}$ . Then  $N = (b_j^i)$  is an integral matrix and  $\det N = d^{n-1}$ . Consider the transformation

$$h: \beta = (\beta_1, \dots, \beta_n) \longrightarrow \alpha = \beta^N \\ = (\beta_1^{b_1^1} \beta_2^{b_2^1} \dots \beta_n^{b_n^1}, \dots, \beta_1^{b_1^n} \beta_2^{b_2^n} \dots \beta_n^{b_n^n}).$$

Then  $h$  defines a  $|d|^{n-1}$ -fold covering map from  $(\mathbb{C}^*)^n$  to itself. The lifted equation of (5.1)' is:

$$(5.1)'' \quad \begin{pmatrix} c_1 \beta_1^d \\ c_2 \beta_2^d \\ \vdots \\ c_n \beta_n^d \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}.$$

Thus (5.1)'' has obviously  $|d|^n$  simple solutions in  $(\mathbb{C}^*)^n$  which implies immediately that (5.1)' has  $|d|^n / |d|^{n-1} = |d|$  simple solutions. Now that  $|d| = n!$  volume  $\Delta(0)$  completes the proof.

PROOF FOR THE GENERAL CASE. Suppose  $m > n$ . We can assume that  $\text{rank}(\nu^1, \dots, \nu^{m-1}) = n$ . Instead of (5.1), we consider the following equation:

$$(5.2; t): \quad \begin{pmatrix} \nu_1^1 & \dots & \nu_1^m \\ \vdots & & \vdots \\ \nu_n^1 & \dots & \nu_n^m \end{pmatrix} \begin{pmatrix} c_1 \alpha^{\nu^1} \\ \vdots \\ c_{m-1} \alpha^{\nu^{m-1}} \\ t c_m \alpha^{\nu^m} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

We count the number of solutions  $z(t)$  of (5.2;  $t$ ) for which either  $\|z(t)\| \rightarrow \infty$  or  $z_j(t) \rightarrow 0$  for some  $j$  when  $t \rightarrow 0$ . Considering the homogenized equation of (5.2;  $t$ ) and using Curve Selection lemma, such a solution lies on a solution curve parametrized as:

$$\alpha(s) = (\tilde{\alpha}_1 s^{a_1} + \text{higher}, \dots, \tilde{\alpha}_n s^{a_n} + \text{higher})$$

and

$$t(s) = s^b.$$

In this case,  $b$  is a positive integer but  $a_j$  ( $j=1, \dots, n$ ) are possibly negative. Thus  $\alpha(s)$  is a Laurent series. Let  $L (=L(\Delta))$  be the  $(n-1)$ -dimensional hyperplane containing  $\Delta$  and let  $\mathcal{S}(\Delta, \nu^m)$  be the set of the subset  $J \subset \{1, 2, \dots, m-1\}$  such that  $\text{rank}\{\nu_j; j \in J\} = n-1$  and the hyperplane  $H(\Delta_J)$  of  $L$  containing  $\{\nu_j; j \in J\}$  enjoys the following property:  $\nu^m$  is contained in an open half plane  $L_1$  of  $L - H(\Delta_J)$  and  $L_1$  contains no other vertex of  $\Delta$ . By the regularity of  $\gamma$ , we can assume that  $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_n) \in (\mathbb{C}^*)^n$ . Let  $e = \text{minimum}\{(\nu^1, a), \dots, (\nu^{m-1}, a), (\nu^m, a) + b\}$  where  $(\nu^j, a) = \sum_{k=1}^n \nu_k^j a_k$ . We consider the equation of the lowest terms of:

$$(5.3) \quad \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^m \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^m \end{pmatrix} \begin{pmatrix} c_1 \alpha(s)^{\nu^1} \\ \vdots \\ c_{m-1} \alpha(s)^{\nu^{m-1}} \\ t(s) c_m \alpha(s)^{\nu^m} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

The case  $e > 0$  is trivially impossible. The case  $e < 0$  is also impossible: Let  $J = \{1 \leq j \leq m; (\nu^j, a) = e \ (1 \leq j \leq m-1) \text{ or } (\nu^m, a) + b = e\}$  and let  $\Delta_1$  be the convex polyhedron spanned by  $\nu^j, j \in J$ . Then the leading term equation is nothing but  $\tilde{\alpha}_j \frac{\partial f_{\Delta_1}}{\partial z_j}(\tilde{\alpha}) = 0 \ (j=1, \dots, n)$  which has no solution in  $(\mathbf{C}^*)^n$  if the coefficients are generically chosen (Kouchnirenko [3]). Thus  $e=0$  is the only possible case. Let  $J$  and  $\Delta_1$  as above.

(i) Assume  $a = (a_1, \dots, a_n) \neq 0$  and  $e=0$ . If  $\text{rank}\{\nu^j; j \in J\} \leq n-1$  and the leading term equation is  $\tilde{\alpha}_j \frac{\partial f_{\Delta_1}}{\partial z_j}(\tilde{\alpha}) = \gamma_j \ (j=1, \dots, n)$ . This has no solution by the regularity of  $\gamma$ . Thus  $\dim \Delta_1 = n-1$ . Suppose  $m \in J$ . Then  $J = \{1, 2, \dots, m-1\}$  by the assumption  $\dim \Delta_1 = n-1$ . However this is a contradiction to the assumption  $e=0$  because the hyperplane  $L$  does not pass through the origin. Thus  $m \in J$ . Then clearly  $J' = J - \{m\} \in \mathcal{S}(\Delta, \nu^m)$ . The leading equation reduces to:

$$(5.4) \quad \alpha_j \frac{\partial f_{\Delta_1}}{\partial z_j}(\alpha) = \gamma_j \quad (j=1, \dots, n)$$

and  $b = -(a, \nu^m) = -\sum_{j=1}^n a_j \nu_j^m$  where  $a$  is determined up to a scalar multiplication by the hypersurface of  $\mathbf{R}^n$  passing through  $\{\nu^j; j \in J'\}$  and the origin. By the induction's assumption, (5.4) has  $n!$  volume  $\Delta_1(0)$  simple solutions. By exactly the same argument as Lemma 4.4 these solutions correspond bijectively to the bifurcating solutions  $z(t)$  of (5.3) such that  $\|z(t)\| \rightarrow \infty$ .

(ii) Assume  $a=0$ . Then the leading terms give the following equation:

$$(5.5) \quad \begin{pmatrix} \nu_1^1 & \cdots & \nu_1^{m-1} \\ \vdots & & \vdots \\ \nu_n^1 & \cdots & \nu_n^{m-1} \end{pmatrix} \begin{pmatrix} c_1 \alpha^{\nu^1} \\ \vdots \\ c_{m-1} \alpha^{\nu^{m-1}} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}.$$

Thus by the induction we find  $n!$  volume  $\Delta_2(0)$  simple solutions where  $\Delta_2$  is the convex polyhedron spanned by  $\nu^1, \nu^2, \dots, \nu^{m-1}$ . The correspondence to the bifurcating solutions of (5.3) is similar. These are solutions which remain to be solutions at  $t=0$ . For  $t$  sufficiently small, the solutions of types (i) and (ii) are mutually distinct. This completes the proof of Lemma 4.3, because it is easy to see that  $\Delta_2$  and various  $\Delta_1$ , which is derived from  $\mathcal{S}(\Delta, \nu^m)$  as in (i), is a convex polyhedral decomposition of  $\Delta$ .

**6. Bifurcation of the vanishing cycles.**

In this section, we consider the following family: Let  $f(z)$  be as before and let  $\mu$  be a fixed vertex on  $\Gamma(f)$ . Let  $f(z) = \sum_{\nu \neq \mu} c_\nu z^\nu$  and let  $g(z) = \sum_{\nu \neq \mu} c_\nu z^\nu$ . We assume that  $f$  and  $g$  are convenient and have non-degenerate Newton principal parts respectively. Let  $f_t(z) = g(z) + t c_\mu z^\mu$  ( $0 \leq t \leq 1$ ). Let  $\mathcal{S}(\mu)$  be the set of the faces of dimension  $n-1$   $\Delta$  of  $\Gamma(g)$  which is not contained in  $\Gamma(f)$  and let  $\Delta(\mu)$  be the cone over  $\Delta$  with the vertex  $\mu$  and let  $|\mathcal{S}(\mu)| = \bigcup_{\Delta \in \mathcal{S}(\mu)} \Delta(\mu)$ .

Then we have:

THEOREM 6.1. (i)  $\text{Bf.}\mu\left(z_1 \frac{\partial f_t}{\partial z_1} = \dots = z_n \frac{\partial f_t}{\partial z_n} = 0, 0\right) = n! \text{ volume } |\mathcal{S}(\mu)|.$

(ii)  $\text{Bf.}\mu\left(\frac{\partial f_t}{\partial z_1} = \dots = \frac{\partial f_t}{\partial z_n} = 0, 0\right) = \nu(|\mathcal{S}(\mu)|).$

PROOF. This is an immediate corollary of Theorem 4.2, Corollary 4.2.2 and Theorem 2.1.

REMARK 6.2. Suppose that one of the following conditions is satisfied.

(a)  $\mu$  is regular with respect to  $\Gamma(g, \mu) = \bigcup_{\Delta \in \mathcal{S}(\mu)} \Delta$

or

(b)  $\mathcal{S}(\mu)$  consists of only one face  $\Delta$ .

Then Theorem 6.1 can be proved directly. The case (a) can be treated in the exactly same way as in the proof of Theorem 4.2. For the case (b), we only need to prove:

( $\mu$ ): 
$$z_j \frac{\partial g_\Delta}{\partial z_j} + \mu_j c_\mu z^\mu = 0 \quad (j=1, 2, \dots, n)$$

and

( $\delta$ ): 
$$z_j \frac{\partial g_\Delta}{\partial z_j} + \delta_j c_\mu z^\mu = 0 \quad (j=1, 2, \dots, n)$$

have the same number of solutions such that  $z_j \neq 0$  if  $\mu_j \neq 0$  for a generic  $\delta = (\delta_1, \dots, \delta_n)$ . After proving this, we come to the similar situation as in Lemma 4.3. Theorem 4.2 can be also proved by iterated uses of Theorem 6.1 (in the case of (a) or (b)) and the "lifting" technique reducing to a Brieskorn polynomial case for which Theorem 4.2 is clear.

**Appendix to Lemma 4.3.**

Let  $\Delta$  be a maximal face of  $\Gamma(f)$  and let  $\nu^1, \dots, \nu^m$  be the integral points of  $\Delta$  i.e.  $\{\nu^1, \dots, \nu^m\} = \Delta \cap \mathbf{Z}^n$ . For a vector  $c = (c_1, \dots, c_m) \in \mathbf{C}^m$ , let  $f_c(z) = \sum_{j=1}^m c_j z^{\nu^j}$  and let  $U = \{c \in \mathbf{C}^m; \Gamma(f_c(z)) = \Delta \text{ and } f_c \text{ is non-degenerate in the sense of the}$

Newton boundary}. By the appendix below and Theorem 6.1 of [3],  $U$  is a Zariski open set. Let  $\gamma=(\gamma_1, \dots, \gamma_n)$  be a regular vector for  $\Delta$ . Then Lemma 4.3 can be sharpened as follows.

LEMMA 4.3'. *The following equation*

$$(A.1): \quad \begin{pmatrix} \nu_1^1 & \dots & \nu_1^m \\ \vdots & & \vdots \\ \nu_n^1 & \dots & \nu_n^m \end{pmatrix} \begin{pmatrix} c_1 z^{\nu_1} \\ \vdots \\ c_m z^{\nu_m} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \quad (c \in U)$$

has only isolated solutions in  $\mathbf{C}^n$  and the sum of the multiplicity of the solutions of (A.1) is independent of  $c \in U$ . (Thus it is equal to  $n!$  volume  $(\Delta(0))$  by Lemma 4.3.)

PROOF. Assume that (A.1) has non isolated solutions. As the set of solutions is an algebraic set, we can use the Curve Selection lemma to find a Laurent series  $z(s)=(\alpha_1 s^{a_1} + \dots, \dots, \alpha_n s^{a_n} + \dots) (0 \leq s \leq 1)$  such that  $\alpha=(\alpha_1, \dots, \alpha_n) \neq 0$  and (i)  $z_1(s) \frac{\partial f_c}{\partial z_i}(z(s)) = \gamma_i (i=1, \dots, n)$  and (ii)  $a_0 = \text{minimum}\{a_1, \dots, a_n\} < 0$  i.e.  $\lim_{s \rightarrow 0} \|z(s)\| = \infty$ . Let  $\Delta_a$  be the face of  $\Delta$  on which the function  $l_a(x) = \sum_{i=1}^n a_i x_i$  defined on  $\Delta$  takes its minimal value, say  $d(a)$ . We assume  $\alpha \in (\mathbf{C}^*)^n$ . (If  $\alpha_i = 0$  for some  $i$ , the following argument is available for the corresponding subspace of  $\mathbf{C}^n$ .) Comparing the leading terms of (i), we see that  $d(a) > 0$  is obviously impossible. Assume that  $d(a) < 0$ . Then the leading terms give the equation

$$\alpha_i \frac{\partial f_{c, \Delta_a}}{\partial z_i}(\alpha) = 0 \quad (i=1, \dots, n)$$

which is absurd by the non-degeneracy assumption. Thus  $d(a) = 0$  and by the regularity of  $\gamma$   $\dim \Delta_a$  must be  $n-1$ . This implies  $\Delta_a = \Delta$  and the hyperplane  $\sum_{i=1}^n a_i x_i = 0$  contains  $\Delta$ . This is again impossible from the geometric assumption of  $\Delta$ . Now we prove the latter assertion. By virtue of Rouché's principle ([5]), the local sum of the multiplicity is constant when  $c$  moves in a neighborhood of  $c_0 \in U$ . Thus the only possibility is that some solution of (A.1) might disappear in the infinity when  $c$  approaches  $c_0$ . In such a case we can use again the Curve Selection lemma to obtain Laurent series  $z(s)=(\alpha_1 s^{a_1} + \dots, \dots, \alpha_n s^{a_n} + \dots)$  and analytic curve  $c(s) (0 \leq s \leq 1)$  such that  $\alpha \neq 0, c(0) = c_0 \in U$  and (i)  $z_i(s) \frac{\partial f_{c(s)}}{\partial z_i}(z(s)) = \gamma_i (i=1, \dots, n)$  and (ii)  $a_0 = \text{minimum}\{a_1, \dots, a_n\} < 0$ . Let  $\Delta_a$  and  $d(a)$  be as above. By the exact same argument as above, we get a contradiction. This completes the proof.

**Appendix about the non-degeneracy condition.**

Let  $\Delta$ ,  $F(z, c) = f_c(z) = \sum_{j=1}^m c_j z^{\nu_j}$  and  $U$  be as in the above appendix to Lemma 4.3. In this appendix, we shall show that  $U$  is a non-empty Zariski open set. As  $f_c(z)$  is a weighted homogeneous polynomial, we can take positive integers  $q_1, \dots, q_n$  so that  $g_c(z) := f_c(z_1^{q_1}, \dots, z_n^{q_n})$  is a homogeneous polynomial. However it is an easy to see that  $f_c$  is non-degenerate if and only if  $g_c$  is non-degenerate. Thus we may assume that  $f_c$  is homogeneous. For any closed face  $\mathcal{E}$  of  $\Delta$  (of any dimension), we define sets  $V(\mathcal{E})$ ,  $V(\mathcal{E})^*$  and  $V(\mathcal{E})'$  by

$$V(\mathcal{E}) = \left\{ (z, c) \in \mathbf{P}^{n-1} \times \mathbf{P}^{m-1}; \frac{\partial F_{\mathcal{E}}}{\partial z_1}(z, c) = \dots = \frac{\partial F_{\mathcal{E}}}{\partial z_n}(z, c) = 0 \right\}$$

$$V(\mathcal{E})^* = V(\mathcal{E}) \cap \{z_1 z_2 \dots z_n \neq 0\} \quad \text{and}$$

$$V(\mathcal{E})' = \text{the closure of } V(\mathcal{E})^* \text{ in } \mathbf{P}^{n-1} \times \mathbf{P}^{m-1}.$$

It is well known that  $V(\mathcal{E})'$  is an algebraic set of dimension  $\dim V(\mathcal{E})^*$ . (Lemma 3.9, [10]). Let  $\pi: \mathbf{P}^{n-1} \times \mathbf{P}^{m-1} \rightarrow \mathbf{P}^{m-1}$  be the projection and let  $V^* = \bigcup_{\mathcal{E}} V(\mathcal{E})^*$ ,  $V' = \bigcup_{\mathcal{E}} V(\mathcal{E})'$  and  $W^* = \pi(V^*)$ ,  $W' = \pi(V')$ . By the definition,  $U$  is the complement of the affine cone of  $W^*$ . By the proper mapping theorem (p. 162, [9]),  $W'$  is an algebraic set which contains  $W^*$ . Now we claim that  $W^* = W'$  and thus  $U$  is a Zariski open set: Assume that  $W'$  properly contains  $W^*$ . We can take a point  $(\bar{z}, d)$  of  $V'$  such that  $d \in W' - W^*$ . We apply the Curve Selection lemma to the pair  $(V', V^*)$  at  $(\bar{z}, d)$  to find an analytic curve  $p(s) = (z(s), c(s))$  for  $0 \leq s \leq 1$  such that for some face  $\mathcal{E}$  of  $\Delta$   $p(s) \in V(\mathcal{E})^*$  for  $s > 0$  and  $p(0) = (\bar{z}, d)$ . Write  $p(s)$  explicitly as

$$z(s) = (\alpha_1 s^{a_1} + \dots, \dots, \alpha_n s^{a_n} + \dots) \quad \text{and}$$

$$c(s) = d + d_1 s + \dots.$$

By the assumption, we have that each  $\alpha_i$  is non-zero complex number for  $i = 1, 2, \dots, n$  and  $\{a_i\}_{i=1}^n$  are non-negative integers such that  $\max\{a_1, a_2, \dots, a_n\} > 0$ . Consider the linear function  $l_a(x) = \sum_{i=1}^n a_i x_i$  on  $\mathcal{E}$  and let  $\mathcal{E}(a)$  be the face of  $\mathcal{E}$  where  $l_a$  takes its minimal value, say  $q$ . We consider the identity

$$\frac{\partial F_{\mathcal{E}}}{\partial z_i}(z(s), c(s)) \equiv 0 \quad i = 1, 2, \dots, n.$$

The leading terms of the left hand are  $\frac{\partial F_{\mathcal{E}(a)}}{\partial z_i}(\alpha, d) s^{q-a_i}$  and thus we get  $\frac{\partial F_{\mathcal{E}(a)}}{\partial z_i}(\alpha, d) = 0$  for  $i = 1, \dots, n$ . This implies  $(\alpha, d) \in V(\mathcal{E}(a))^*$  and  $d \in W^*$  which is a contradiction to the assumption  $d \in W' - W^*$ .

REMARK.  $U$  is non-empty by Kouchnirenko, [3]. However it is easy to prove directly that  $\dim W' \leq m-2$  using the equality:

$$\text{rank} \left\{ \frac{\partial^2 F_{\mathcal{E}}}{\partial z_i \partial c_j} \right\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m \\ \nu_j \in \mathcal{E}}} = \text{rank} \{ \nu^k \}_{\nu^k \in \mathcal{E}} = \dim \mathcal{E} + 1 \text{ on } \{z_1 z_2 \cdots z_n \neq 0\}.$$

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