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Free group actions of $Z_{p,q}\times Z_h$ on homotopy spheres

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Introduction.

Let $Z_{p,q}$ be the metacyclic group with presentation

$$\{x,\ y \mid x^p=y^q=1,\ yxy^{-1}=x^\sigma\}\ ,$$

where p is an odd integer, q an odd prime, $(\sigma-1, p)=1$, and σ is a primitive q^{th} root of 1 mod p. Denote by Θ_n the group of homotopy spheres, and by $\Theta_n(\partial\pi)$ the group of homotopy spheres which bound parallelizable manifolds. Then Petrie [5] proved that for each $\Sigma\in\Theta_{2q-1}(\partial\pi)$ there is a free smooth action of $Z_{p,q}$ on Σ. This theorem will be generalized as follows in this paper.

THEOREM. *Let Z_h denote a cyclic group of order h and assume $h=2^n h'$, $(2, h')=1$. If n takes 0, 1, or 2 and $(h', pq)=1$, then for each $\Sigma\in\Theta_{2q-1}(\partial\pi)$ there is a free smooth action $Z_{p,q}\times Z_h$ on Σ.*

Our theorem follows immediately from the following two propositions.

PROPOSITION 5.7. *There exists a free smooth action of $Z_{p,q}\times Z_h$ on some homotopy sphere $\Sigma\in\Theta_{2q-1}(\partial\pi)$. Here $(h, pq)=1$.*

PROPOSITION 6.1. *Let m be any integer $\geqq 1$. Assume $h=2^n h'$ where $n=0$, 1, or 2 and $(h', pq)=1$. If $\Sigma\in\Theta_{4m+1}$ admits a free $Z_{p,q}\times Z_h$-action, then $\Sigma\,\sharp\,\Sigma_0$ admits a free $Z_{p,q}\times Z_h$-action, where Σ_0 generates $\Theta_{4m+1}(\partial\pi)$.*

Our methods are analogous to those in Petrie [5]. §§ 1-4 are preliminaries for Proposition 5.7 which is proved in § 5. In § 6, we prove Proposition 6.1 by applying a theorem of Browder [1].

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1. Construction of a $Z_{p,q}\times Z_h$-action.

We set $\pi=Z_{p,q}\times Z_h$ for the groups $Z_{p,q}$ and Z_h in Introduction, where $(h, pq)=1$. We denote by π_p, π_q the cyclic subgroups generated by x, y respectively. Let Z_{ph} be a cyclic group of order ph. Since $(p, h)=1$, there exist integers m and n such that $mp+nh=1$, and an isomorphism of Z_{ph} to

$\pi_p \times \pi_h$ is given by

$$g \longmapsto x^n z^m ,$$

where z, g are generators of Z_h, Z_{ph} respectively. There is an integer k such that $x^{\sigma n} z^m = (x^n z^m)^k$, $k - \sigma \equiv 0$ (p), and $k - 1 \equiv 0$ (h). Such m, n, and k will be fixed throughout this paper.

First we construct a linear action of π on C^{q+1}.

LEMMA 1.1. *Let ξ_{ph} be a primitive ph^{th} root of unity so that $\xi_p = (\xi_{ph})^h$, $\xi_h = (\xi_{ph})^p$ are primitive p^{th}, h^{th} roots of unity respectively, and let ξ_q be a primitive q^{th} root of unity. If $z_1, \cdots, z_{q+1}$ are complex coordinates for C^{q+1}, an action of π on C^{q+1} is given as follows:*

$$x(z_1, \cdots, z_q, z_{q+1}) = (\xi_p z_1, \xi_p^k z_2, \cdots, \xi_p^{k^{q-1}} z_q, z_{q+1}) ,$$

$$z(z_1, \cdots, z_q, z_{q+1}) = (\xi_h z_1, \xi_h z_2, \cdots, \xi_h z_q, z_{q+1}) ,$$

$$y(z_1, \cdots, z_q, z_{q+1}) = (z_2, z_3, \cdots z_q, z_1, \xi_q z_{q+1}) .$$

PROOF. We dissect the group π by means of the exact sequence

$$1 \longrightarrow Z_{ph} \xrightarrow{\ i\ } \pi \xrightarrow{\ j\ } \pi_q \longrightarrow 1.$$

Let V_{ph} and V_q be one dimensional complex representations of Z_{ph} and π_q given by ξ_{ph} and ξ_q respectively. Then

$$V = i_* V_{ph} + j^* V_q$$

is a complex $q+1$ dimensional representation of π. Here $i_* V_{ph} = C(\pi) \underset{C(Z_{ph})}{\otimes} V_{ph}$ is the induced representation, $C(\pi)$ the complex group ring of π and $j^* V_q$ the one dimensional complex reprerentation of π defined by V_q viewed as a $C(\pi)$-module via j. We may take $z_1, \cdots, z_q$ as complex coordinates for $i_* V_{ph}$ and z_{q+1} as a complex coordinate for $j^* V_q$.

By V we mean C^{q+1} with the action of π in Lemma 1.1. Putting $l = q^r$, we consider a polynomial

$$f_l(z_1, \cdots, z_{q+1}) = z_1^{ph} + z_2^{ph} + \cdots + z_q^{ph} + z_{q+1}^l$$

which is invariant under the action of π.

PROPOSITION 1.2. *π acts freely on the $(2q-1)$-dimensional Brieskorn manifold*

$$K_{f_l} = \{ v \in V \mid f_l(v) = 0, \ \|v\| = \eta \} ,$$

where η is a small positive number.

PROOF. If $g \in \pi$ fixes a point, then g^k fixes a point, for any k and all conjugates of g fixes a point. Hence it is enough to select for each prime

t dividing the order of π one Sylow t subgroup S_t and show that no element of order t in S_t fixes a point.

Case 1. $t=q$. Let S_q be the subgroup generated by y. Then if $y\vec{z}=\vec{z}$ for $\vec{z}=(z_1, \cdots, z_{q+1})\in V$, it follows that $\vec{z}=(z_1, \cdots, z_1, 0)$ and $f_l(\vec{z})=qz_1^{ph}=0$. Therefore we get $z_1=0$ but we have $|\vec{z}|^2=q|z_1|^2=\eta^2>0$. Hence y fixes no point.

Case 2. $t|p$. Let S_t be the subgroup generated by $\delta=x^d$, where $p=t^n d$ and $(t, d)=1$. Set $u=x^{t^{n-1}d}$. If $u\vec{z}=\vec{z}$, it follows that $\vec{z}=(0, ., 0, z_{q+1})$ and $f_l(\vec{z})=z_{q+1}^l=0$. Therefore we have $z_{q+1}=0$, and u fixes no point.

Case 3. $t|h$. Let S_t be the subgroup generated by $\delta'=z^e$, where $h=t^{n'}e$ and $(t, e)=1$. Set $u'=z^{t^{n'-1}e}$. Similarly to Case 2, we see that u' fixes no point.

2. Description of $H_{q-1}(K_{f_l})$.

The $(2q-1)$-dimensional smooth manifold K_{f_l} is $(q-2)$-connected (see [4, p. 45]). We shall describe the $Z(\pi)$-module $H_{q-1}(K_{f_l})$ following Milnor [4]. Here $Z(\pi)$ is the group ring of π over Z.

PROPOSITION 2.1. *There is an exact sequence of* $Z(\pi)$-*modules*

$$0\longrightarrow H_q(K_{f_l})\longrightarrow H_q(F)\xrightarrow{1-\phi}H_q(F)\longrightarrow H_{q-1}(K_{f_l})\longrightarrow 0$$

where

(i) $F=\{v\in V|f_l(v)=\text{const.}\}$,

(ii) $H_q(F)$ *is the tensor product* $I_{ph}^q\otimes I_{qr}$ *as a* $Z(\pi)$-*module (Two* $Z(\pi)$-*modules* I_{ph}^q *and* I_{qr} *are settled in Proof),*

(iii) $1-\phi$ *is a* $Z(\pi)$-*homomorphism.*

PROOF. From the Milnor fibering $S_\eta-K_{f_l}\longrightarrow S^1$, we have the above exact sequence and it follows that

$$H_q(F)\cong\tilde{H}_0(\Omega_{ph})\otimes\cdots\otimes\tilde{H}_0(\Omega_{ph})\otimes\tilde{H}_0(\Omega_{q^r})\qquad (q+1 \text{ fold}),$$

Ω_{ph} and Ω_{q^r} being the finite cyclic groups consisting of all ph^{th} and $q^{r\,\text{th}}$ roots of unity respectively (see [4]).

Denote by I_{ph} the ideal $(1-x^n z^m)Z(Z_{ph})$ of $Z(Z_{ph})$, and I_{qr} the ideal $(1-s)Z(\pi_{qr})$ of $Z(\pi_{qr})$. Here π_{qr} is the cyclic subgroup of order q^r generated by s. Obviously we see that

$$\tilde{H}_0(\Omega_{ph})=I_{ph}\quad\text{and}\quad\tilde{H}_0(\Omega_{q^r})=I_{qr}.$$

The q-fold tensor product of I_{ph} over Z with itself is denoted by I_{ph}^q. So, $H_q(F)$ is the tensor product $I_{ph}^q\otimes I_{qr}$. Moreover, $H_q(F)$ inherits the action of π on V. By Lemma 1.1, the action on I_{ph}^q is given by

$$x(\lambda_1\otimes\cdots\otimes\lambda_q)=(x\lambda_1\otimes x^k\lambda_2\otimes\cdots\otimes x^{k^{q-1}}\lambda_q),$$

$$z(\lambda_1\otimes\cdots\otimes\lambda_q)=(z\lambda_1\otimes z\lambda_2\otimes\cdots\otimes z\lambda_q),$$

$$y(\lambda_1\otimes\cdots\otimes\lambda_q)=(\lambda_2\otimes\lambda_3\otimes\cdots\otimes\lambda_q\otimes\lambda_1),$$

and the action of π on I_{q^r} is given by

$$x\lambda=\lambda,\qquad z\lambda=\lambda,\quad\text{and}\quad y\lambda=s^{l/q}\lambda.$$

Put

$$A=I_{ph}^q\otimes I_{q^r}.$$

The diagonal action on the tensor product A makes $H_q(F)$ a module over $Z(\pi)$. This proves (ii).

From the fact that

$$\phi(\lambda_1\otimes\cdots\otimes\lambda_q\otimes\lambda_{q+1})=(x^nz^m\lambda_1\otimes\cdots\otimes x^nz^m\lambda_q\otimes s\lambda_{q+1}),$$

it follows that the homomorphism $\phi: A\to A$ is a $Z(\pi)$-homomorphism. This complets the proof of Proposition 2.1.

By Proposition 2.1 and its proof, our present task is to know the cokernel of $1-\phi$. Let C denote the cokernel of $1-\phi$.

LEMMA 2.2. *C is annihilated by a power of q, so that $C\otimes Z_{(q)}=C$, where $Z_{(q)}$ denotes the integers localized at the prime ideal $(q)\subset Z$.*

PROOF. If $1-\phi$ is viewed as a linear transformation over Z, it suffices to show that the determinant of $1-\phi$ is a power of q. By the proof of (iii) in Proposition 2.1,

$$(1-\phi^{ph})(\lambda_1\otimes\cdots\otimes\lambda_{q+1})=\lambda_1\otimes\cdots\otimes\lambda_q\otimes(1-s^{ph})\lambda_{q+1}.$$

If we choose $\{(1-s^{ph})s^{phi},\ i=0, 1, \cdots, q^r-2\}$ as a basis of I_{q^r}, the homomorphism $1-s^{ph}$ of I_{q^r} to itself is given by the $(q^r-1)\times(q^r-1)$ matrix:

$$\begin{bmatrix} 1 & & & & 0 & 1 \\ -1 & 1 & & & & 1 \\ & -1 & \ddots & & & \vdots \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & 1 & 1 \\ & 0 & & & -1 & 2 \end{bmatrix}.$$

The determinant of this matrix is q^r. Hence

$$\det(1-\phi^{ph})=q^{r(ph-1)^q}.$$

Since $1-\phi^{ph}=(1-\phi)(1+\phi+\cdots+\phi^{ph-1})$ and q is prime, $\det(1-\phi)$ is a power of q. This proves the lemma.

In view of Lemma 2.2, C is also the cokernel of

$$(1-\phi)\otimes 1:\ A\otimes Z_{(q)}\longrightarrow A\otimes Z_{(q)}\,.$$

Let ω be a primitive ph^{th} root of 1. We can describe the $Z(\pi)$-module structure $C\otimes Z_{(q)}(\omega)$ rather than C itself. $C\otimes Z_{(q)}(\omega)$ is the cokernel of

$$(1-\phi)\otimes 1:\ A\otimes Z_{(q)}(\omega)\longrightarrow A\otimes Z_{(q)}(\omega)\,.$$

Let $I=(s_1, s_2, \cdots, s_q)$ be a q-tuple with $s_j\in Z_{ph}-\{0\}$ and put

$$|I|=\sum_{j=1}^{q} s_j\,.$$

Set

$$\chi_{s'}=\sum_{i=0}^{ph-1}\omega^{-s'i}(x^n z^m)^i \qquad s'\in Z_{ph}-\{0\}$$

and

$$\chi_I=\chi_{s_1}\otimes\chi_{s_2}\otimes\cdots\otimes\chi_{s_q}\,.$$

PROPOSITION 2.3. $C\otimes Z_{qr}(\omega)=\sum_{|I|\equiv 0(ph)} Z_{qr}(\omega)\chi_I\,.$
The $Z(\pi)$-module structure of $C\otimes Z_{qr}(\omega)$ is given by

(i) $x\chi_I=\omega^{h\alpha(I)}\chi_I$ *with* $\alpha(I)=\sum_{j=1}^{q} k^{j-1}s_j$,

(ii) $z\chi_I=\chi_I$ *(trivial action)*,

(iii) $y\chi_I=\chi_{y(I)}$,

where $y(I)=y(s_1, s_2, \cdots, s_q)=(s_2, s_3, \cdots, s_q, s_1)$.

PROOF. By definition of $\chi_{s'}$, $\{\chi_{s'},\ s'\in Z_{ph}-\{0\}\}$ is a basis of $I_{ph}(\omega)$. Thus

$$A\otimes Z_{(q)}(\omega)=\sum_{I\subset S'} Z_{(q)}(\omega)\chi_I\otimes(1-s)Z_{(q)}(\pi_{qr})\,.$$

S' is the set of all I with $s_j\in Z_{ph}-\{0\}$. Since $(x^n z^m)\chi_{s'}=\omega^{s'}\chi_{s'}$, we have

$$\phi(\chi_I\otimes(1-s))=\omega^{|I|}\chi_I\otimes s(1-s)\,.$$

Set

$$\eta_I=1-s\omega^{|I|}$$

and

$$N=(1-s)Z_{(q)}(\omega)(\pi_{qr})\,.$$

Then we get

$$C\otimes Z_{(q)}(\omega)=\sum_{I\subset S'}[N/\eta_I N]\chi_I\,.$$

For $|I|\not\equiv 0\ (ph)$, $\eta_I N=N$ and if $|I|\equiv 0\ (ph)$, $N/\eta_I N=Z_{qr}(\omega)$. Hence it follows that

$$C\otimes Z_{(q)}(\omega)=\sum_{|I|\equiv 0(ph)} Z_{qr}(\omega)\chi_I\,.$$

By Lemma 2.2, $C\otimes Z_{q^r}(\omega)\cong C\otimes Z_{(q)}(\omega)$. This proves the first assertion. The second follows since π acts on χ_I as in the proof of Proposition 2.1 and $x=(x^n z^m)^h$, $z=(x^n z^m)^p$.

3. Some representations of $Z_{p,q}\times Z_h$.

We discuss certain representations of $\pi=Z_{p,q}\times Z_h$ over the ring Z_l where $l=q^r$. For each integer i, let S_i denote the set of primitive i^{th} roots of unity and let

$$\Phi_i(t)=\prod_{\omega\in S_i}(t-\omega)$$

denote the i^{th} cyclotomic polynomial.

LEMMA 3.1. *For each integer d dividing ph, the ring*

$$\theta_d=Z[t]/(\Phi_d(t))$$

is a $Z(\pi)$-module if we set

(i) $x(\sum a_i t^i)=\sum a_i t^{i+h}$,

(ii) $z(\sum a_i t^i)=\sum a_i t^{i+p}$,

(iii) $y(\sum a_i t^i)=\sum a_i t^{ki}$.

PROOF. Note that $\Phi_d(t^k)=\Phi_d(t)\psi(t)$ where $\psi(t)$ is an integral polynomial. It is easily seen that the action of π on θ_d is well defined.

For $l=q^r$, let (l) be the principal ideal defined by l in θ_d, and we set

$$M_d=\theta_d/(l)\,. \tag{3.2}$$

From the identities

$$t^p-1=\prod_{d\mid p}\Phi_d(t)\,, \tag{3.3}$$

$$\Sigma=1+t+\cdots+t^{p-1}=\prod_{d\mid p,\ d\neq 1}\Phi_d(t)\,, \tag{3.4}$$

and the fact that the polynomials $\Phi_d(t)$ are relatively prime over $Z_l[t]$, we have

LEMMA 3.5. $\Gamma_l=Z_l[t]/(\Sigma)\cong\sum_{d\mid p,\ d\neq 1}M_d$, *where (Σ) denotes the ideal generated by Σ.*

For the group $Z_{ph}\cong\pi_p\times Z_h$, we also need following representations over $Z_l(\omega)$, ω being a primitive ph^{th} root of 1. N_{sh} is the free $Z_l(\omega)$-module with one generator χ_{sh} for $s\in Z_p$ and relations

$$x\chi_{sh}=\omega^{(sh)h}\chi_{sh}\,,$$

$$z\chi_{sh}=\chi_{sh} \quad (z \text{ acts trivially}).$$

From the $Z_l(\omega)(Z_{ph})$-modules N_{sh} for $s\in Z_p$ and $d\mid p$, we can form the sums

$$L_d=\sum_{s\in S_d} N_{sh}\,. \tag{3.6}$$

These L_d are $Z_l(\omega)(\pi)$-modules if we set

$$y\chi_{sh}=\chi_{k^{-1}sh}\,.$$

If S is a $Z_l(\pi)$-module, we denote the $Z_l(\omega)(\pi)$-module $S\underset{Z_l}{\otimes}Z_l(\omega)$ by $E_*(S)$. The Krull-Schmidt Theorem [2] implies that

$$E_*(S)\cong E_*(S') \quad \text{if and only if} \quad S\cong S'.$$

PROPOSITION 3.7.
$$E_*(M_d)=L_d\,,$$
$$E_*(\Gamma_l)=\sum_{d\mid p,\, d\neq 1} L_d\,.$$

PROOF. A correspondence of L_d onto $E_*(M_d)$ is defined by

$$\chi_{sh}\longmapsto\bar{\chi}_{sh}=\sum_{i=0}^{ph-1}[t^i]\otimes\omega^{-shi}\,.$$

It is easily seen that this correspondence is an equivariant isomorphism. The second follows from Lemma 3.5.

4. $Z(\pi)$-module structure of $H_{q-1}(K_{f_l})$.

In this section we determine $H_{q-1}(K_{f_l})$ as a $Z(\pi)$-module. Set

$$T=\sum_{|I|\equiv 0(ph)} Z_l(\omega)\chi_I\,,$$

where $I=(s_1, s_2, \cdots, s_q)$ is a q-tuple with $s_j\in Z_{ph}-\{0\}$ and $|I|=\sum_{j=1}^{q} s_j$.

Recall that the action of π on T. From Proposition 2.3, we have

$$(x^n z^m)\chi_I=\omega^{\alpha(I)}\chi_I\,, \qquad y\chi_I=\chi_{y(I)}\,,$$

where $\alpha(I)=\sum_{j=1}^{q} k^{j-1}s_j$, $y(I)=y(s_1, s_2, \cdots, s_q)=(s_2, s_3, \cdots, s_q, s_1)$. From the relation $k-1\equiv 0\ (h)$, we have $\alpha(I)\equiv 0\ (h)$. And $x=(x^n z^m)^h$, $z=(x^n z^m)^p$ show that

$$x\chi_I=\omega^{h\alpha(I)}\chi_I\,,$$
$$z\chi_I=\chi_I \quad (\text{the action of } z \text{ is trivial}).$$

For $i\in Z_p$, we set

$$A_i=\{I \mid |I|\equiv 0\ (ph),\ \alpha(I)=hi\in Z_{ph}\}.$$

Then we obtain

$$T=\sum_{i\in Z_p}\sum_{I\in A_i} Z_l(\omega)\chi_I.$$

LEMMA 4.1. *The cardinalities n_i of the sets A_i satisfy:*

(i) $n_i\equiv 0\ (2)$, $\quad i\in Z_p$,

(ii) $n_i=n_j$ *if* i, j *are not zero in* Z_p,

(iii) $n_0\equiv 0\ (2q)$.

PROOF. Consider tuples $I=(s_1, s_2, \cdots, s_q)$ with the properties

(1) $s_j\in Z_{ph}$ admits zero,

(2) $|I|\equiv 0\ (ph)$,

(3) $\alpha(I)=hi\in Z_{ph}$.

Let P_j be the "property" such that $s_j=0$, and R be the set of "properties" $\{P_j \mid 1\leqq j\leqq q\}$. For each subset $X\subset R$, let $N(X)$ be the number of tuples satisfying (1), (2), and (3), and at least the properties of X. Then it follows that

$$n_i=\sum_{X\subset R}(-1)^{|X|}N(X),$$

where $|X|$ is the number of elements in X and $|X|=0$ if $X=\emptyset$.

If $|X|=q$ or $q-1$, we have

$$N(X)=\begin{cases} 0 & (i\neq 0) \\ 1 & (i=0). \end{cases}$$

Assume $|X|<q-1$ and suppose $P_\alpha, P_\beta\notin X$. Solving (2) and (3) for s_α, s_β, we have

$$\begin{cases} s_\alpha+s_\beta=* \\ k^{\alpha-1}s_\alpha+k^{\beta-1}s_\beta=*, \end{cases} \tag{4}$$

$*$ being terms given by conditions (2) and (3). The determinant of the coefficients of s_α, s_β in the equation (4) is $k^{\alpha-1}(k^{\beta-\alpha}-1)$ (say $\beta>\alpha$). Since $k-1\equiv 0\ (h)$, this determinant is not invertible over Z_{ph}. But in the image of the projection $\pi: Z_{ph}\to Z_p$, we have from (4)

$$\begin{cases} \bar{s}_\alpha+\bar{s}_\beta=\bar{*} \\ k^{\alpha-1}\bar{s}_\alpha+k^{\beta-1}\bar{s}_\beta=\bar{*}, \end{cases} \tag{5}$$

where $^-$ is the image of π. The determinant of the coefficients of $\bar{s}_\alpha$, $\bar{s}_\beta$ in (5) is $\overline{k^{\alpha-1}(k^{\beta-\alpha}-1)}$. This is invertible over Z_p, because the relations $\sigma-k\equiv 0\ (p)$, $(\sigma-1, p)=1$, and $k^q\equiv 1\ (p)$ imply that $(k^\alpha-1, p)=1$ $(1\leqq\alpha<q)$. There are $(ph)^{|R-X|-2}$ choices of $*$ in (4), and $\bar{s}_\alpha$, $\bar{s}_\beta$ are unique by (5). Therefore, from the exact sequence

$$0\longrightarrow Z_h\xrightarrow{\ xp\ }Z_{ph}\xrightarrow{\ \pi\ }Z_p\longrightarrow 0\,, \tag{6}$$

it follows that

$$N(X)=(ph)^{|R-X|-2}\cdot h\,.$$

Thus we obtain

$$n_i=(ph)^{-2}\Big(\sum_{|X|<q-1,\ X\subset R}(-1)^{|X|}(ph)^{|R-X|}h\Big)+\delta_{0i}((-1)^{q-1}q+(-1)^q)$$

$$=(ph)^{-2}h((ph-1)^q+(-1)^q(phq-1))+\delta_{0i}((-1)^{q-1}q+(-1)^q)\,,$$

which implies (i) and (ii).

π acts on $\sum_{I\in A_0}Z_l(\omega)\chi_I$ invariantly, since

$$\alpha(y(I))=k^{-1}\alpha(I)=0 \quad (I\in A_0)$$

and

$$|y(I)|=|I|=0\,.$$

We can form $\sum Z_l(\omega)\chi_I$ as the sum of disjoint union

$$\sum Z_l(\omega)\chi_I=Z_l(\omega)(\pi(\chi_{I_1}))\oplus\cdots\oplus Z_l(\omega)(\pi(\chi_{I_\mu}))\,,$$

$\pi(\chi_{I_i})$ being orbits $(i=1,\cdots,\mu)$. Set

$$\pi_{I_i}=\{g\in\pi\,|\,g\chi_{I_i}=\chi_{I_i}\}\,.$$

Since $\alpha(I_i)=0$, $x\chi_{I_i}=\omega^{h\alpha(I_i)}\chi_{I_i}=\chi_{I_i}$ and also $z\chi_{I_i}=\chi_{I_i}$. But we have $y^j(I_i)\neq I_i$ for $1\leqq j<q$, q prime. Thus

$$\pi_{I_i}\cong\pi_p\times Z_h\,.$$

The orbits $\pi(\chi_{I_i})$ consist of q elements. Hence the cardinality n_0 of A_0 is divisible by q. This completes the proof.

THEOREM 4.2. *$H_{q-1}(K_{f_l})\cong n_0/qZ_l(\pi_q)+n_1\Gamma_l$ as a $Z(\pi)$-module.*

PROOF. When we regard $Z_l(\pi_q)$ as a $Z_l(\pi)$-module such that x and z act trivially, it has been seen in the proof of Lemma 4.1 that

$$\sum_{I\in A_0}Z_l(\omega)\chi_I=n_0/qE_*(Z_l(\pi_q))\,. \tag{4.3}$$

In view of Proposition 3.7, there is an isomorphism of $\sum_{i\in Z_p-0}\sum_{A_i}Z_l(\omega)\chi_I$ onto $n_1E_*(\Gamma_l)$, i.e.,

$$\psi:\ \sum_{A_i}Z_l(\omega)\chi_I\longrightarrow n_iZ_l(\omega)\chi_{\alpha(I)}\,.$$

Here we use (ii) of Lemma 4.1. It is easily checked that ϕ is an isomorphism of modules, and so

$$\sum_{i\in Z_p-0,}\ \sum_{I\in A_i} Z_l(\omega)\chi_I = n_1 E_*(\Gamma_l)\,. \tag{4.4}$$

By (4.3) and (4.4), it follows that

$$T = E_*(n_0/qZ_l(\pi_q)+n_1\Gamma_l)\,. \tag{4.5}$$

By Proposition 2.3, $E_*(C)=T$. Applying the Krull-Schmidt Theorem, we have

$$C \cong n_0/qZ_l(\pi_q)+n_1\Gamma_l\,. \tag{4.6}$$

Since C is the cokernel of $1-\phi$ (see Proposition 2.1), the result follows from (4.6).

5. Surgery on K_{f_l}.

Since q is an odd prime, we try to apply the Theorem 5.6 [8] (or equivalently §5 in [5]) to the $(2q-1)$-smooth manifold K_{f_l}. Here is the Theorem stated in §5 of [5] which also holds for $\pi=Z_{p,q}\times Z_h$.

THEOREM 5.6 OF [8]. *A necessary and sufficient condition that surgery is possible on K_{f_l}/π yielding a manifold N with $\pi_1(N)=\pi$ and such that the universal cover Σ of N is a homotopy sphere is that there exist a free module F of finite rank s over $Z(\pi)$ and an $(s\times s)$-matrix B such that $B^*=-B$ and an exact sequence*

$$0\longrightarrow F\xrightarrow{\ B\ }F\xrightarrow{\ \omega\ }H_{q-1}(K_{f_l})\longrightarrow 0$$

such that the linking number form

$$\phi:\ H_{q-1}(K_{f_l})\longrightarrow \mathrm{Hom}_{Z(\pi)}(H_{q-1}(K_{f_l}),\ Q(\pi)/Z(\pi))$$

is given by

$$\phi(y)(x)=\sum x_i B_{ij}^{-t}\bar{y}_j \qquad \mathrm{mod}\ Z(\pi)$$

when $x=(x_1,\cdots,x_s)$, $y=(y_1,\cdots,y_s)\in F$.

To apply this theorem, we must know a homological property of Γ_l and $Z_l(\pi_q)$ in view of Theorem 4.2. The homological dimension of both constructed as above is the same as that of §3 [5].

Fix $l=q^r$ where r is the order of the projective class group $K_0(Z(\pi))$ which is finite (see [6],[7]). Moreover, from the fact that n_0, n_1 are even by Lemma 4.1, we have the following theorem similar to the theorem 3.2 [5].

THEOREM 5.1. *There exists a free $Z(\pi)$-module F of rank s, an $(s\times s)$-matrix A over $Z(\pi)$ with $A^*=-A$, and the following exact sequence*

$$0\longrightarrow F\xrightarrow{\ A\ }F\longrightarrow n_0/qZ_l(\pi_q)+n_1\Gamma_l\longrightarrow 0\,.$$

Since the linking number form (see [8], p 248)

$$\phi: H_{q-1}(K_{f_l})\longrightarrow \mathrm{Hom}_{Z(\pi)}(H_{q-1}(K_{f_l}),\ Q(\pi)/Z(\pi))$$

is a non-singular skew Hermitian form, we have only to classify skew Hermitian forms on $H_{q-1}(K_{f_l})$ over $(Z_l(\pi), -)$ where $-$ denotes an involution.

LEMMA 5.2. *If $\phi: H_{q-1}(K_{f_l})\times H_{q-1}(K_{f_l})\to Z_l(\pi)$ is a skew Hermitian form, ϕ can be reduced "uniquely" to a skew Hermitian form*

$$\phi': H_{q-1}(K_{f_l})\times H_{q-1}(K_{f_l})\longrightarrow Z_l(Z_{p,q})$$

on the $Z(Z_{p,q})$-module $H_{q-1}(K_{f_l})$ over $Z_l(Z_{p,q})$.

PROOF. We shall recall the action of π on $H_{q-1}(K_{f_l})$. On $Z_l(\pi_q)$, x and z act trivially and y acts cyclotomically. We have $\Gamma_l=Z_l[t]/(\Sigma)=\sum_{d\mid p,\, d\neq 1} M_d$ by Lemma 3.5. π acts on Γ_l as follows:

$$x[\sum a_i t^i]=[\sum a_i t^{i+h}],$$

$$z[\sum a_i t^i]=[\sum a_i t^{i+p}]=[\sum a_i t^i],$$

$$y[\sum a_i t^i]=[\sum a_i t^{ki}].$$

From the relation $mp+nh=1$, we may replace a generator of π_p by x^n. Then we have

$$x^n[\sum a_i t^i]=[\sum a_i t^{i+1}].$$

From $k-\sigma\equiv 0\ (p)$, it follows that

$$y[\sum a_i t^i]=[\sum a_i t^{\sigma i}].$$

Note that $y(x^n)y^{-1}=(x^n)^{\sigma}$. Comparing this action with that of $Z_{p,q}$ in § 2 [5], we see that the $Z(\pi)$-module $H_{q-1}(K_{f_l})=n_0/qZ_l(\pi_q)+n_1\Gamma_l$ is the same as the one stated in § 2 [5] if we ignore the action of Z_h. Since the action of Z_h is trivial on $H_{q-1}(K_{f_l})$, $\mathrm{Im}\,\phi$ is contained in $Z_l(\pi)^{Z_h}\subset Z_l(\pi)$ and we have an isomorphism $Z_l(\pi)^{Z_h}\cong Z_l(Z_{p,q})$ preserving the involution $-$. Hence ϕ can be reduced uniquely to a skew Hermitian form

$$\phi': H_{q-1}(K_{f_l})\times H_{q-1}(K_{f_l})\longrightarrow Z_l(Z_{p,q})$$

on the $Z(Z_{p,q})$-module $H_{q-1}(K_{f_l})$ over $Z_l(Z_{p,q})$.

As for the uniqueness of skew Hermitian forms on the $Z(Z_{p,q})$-module $H_{q-1}(K_{f_l})$ over $Z_l(Z_{p,q})$, there is a proposition (see Corollary 4.18 [5]).

PROPOSITION 5.3. *Every "skew Hermitian Form" on*

$$C=H_{q-1}(K_{f_l})=n_0/qZ_l(\pi_q)+n_1\Gamma_l$$

over $(Z_l(Z_{p,q}), -)$ is hyperbolic.

From Lemma 5.2 and Proposition 5.3, we have

PROPOSITION 5.4. *There is a unique skew Hermitian form on the* $Z(\pi)$-*module* $H_{q-1}(K_{f_l})$ *over* $(Z_l(\pi), -)$.

Since $l\cdot H_{q-1}(K_{f_l})=0$, we have the following corollary in view of the Remark of [5, p. 123].

COROLLARY 5.5. *There is a unique skew Hermitian form*

$$\Phi: H_{q-1}(K_{f_l}) \longrightarrow \mathrm{Hom}_{Z(\pi)}(H_{q-1}(K_{f_l}), Q(\pi)/Z(\pi))$$

with

$$\Phi(y)(x)=-\overline{\Phi(x)(y)}.$$

Let A be the matrix given by Theorem 5.1. A induces a skew Hermitian form ψ_A on $H_{q-1}(K_{f_l})$ by setting

$$\psi_A(y)(x)=\sum x_i A_{ij}^{-t}\bar{y}_j \quad \mathrm{mod}\, Z(\pi)$$

where $x=(x_1, x_2, \cdots, x_s)$, $y=(y_1, y_2, \cdots, y_s)\in F$.

COROLLARY 5.6. *The skew Hermitian form* ψ_A *coincides with the linking number form on* $H_{q-1}(K_{f_l})$.

Theorem 5.1 and Corollary 5.6 provide us a sufficient condition in Theorem 5.6 of [8], and we have

PROPOSITION 5.7. *There exists a free smooth action of* $Z_{p,q}\times Z_h$ *on some homotopy sphere* $\Sigma^{2q-1}\in\Theta_{2q-1}(\partial\pi)$.

6. Application of the generalized Kervaire invariant.

In [1], Browder defined the generalized Kervaire invariant. In particular, in the case that a "B-orientation" of 'Browder' is the orientation induced by a normal map, this invariant is well defined. And if a surgery obstruction of the normal map is the Kervaire invariant, these generalized Kervaire invariant and Kervaire invariant agree.

Let M be a smooth (or p. l.) manifold with $\pi_1(M)=Z_2$, $p: \tilde{M}\to M$ the double covering map and $f: M'\to M$ a normal map covered by $b: \nu_{M'}\to\nu_M$. For $n=2s+1$, form the manifold

$$M'(n)=(nM')\cup(-s\tilde{M})$$

where $nM'=M'\cup\cdots\cup M'$, n times, $-s\tilde{M}=(-\tilde{M})\cup\cdots\cup(-\tilde{M})$, s times, $-\tilde{M}$ being $\tilde{M}$ with the opposite orientation. Then there is a normal map

$$nf: M'(n) \longrightarrow M$$

defined to be f on each copy of M' and p on each copy of $-\tilde{M}$, and covered by b on each $\nu_{M'}$ and by a fixed map d covering p on each $\nu_{\tilde{M}}$, where d is the inverse of $dp: \tau_{\tilde{M}}\to\tau_M$. Clearly nf has degree 1.

The Kervaire invariants of $M'(n)$ have been computed by Browder in terms of the invariants of M (see [1, p. 212]). As an application of this result, we prove:

PROPOSITION 6.1. *Let m be any integer $\geqq 1$. Assume $h=2^n h'$ where $n=0$, 1 or 2 and $(h', pq)=1$, and h' odd. If $\Sigma\in\Theta_{4m+1}$ admits a free $Z_{p,q}\times Z_h$-action, then $\Sigma\#\Sigma_0$ admits a free $Z_{p,q}\times Z_h$-action, where Σ_0 generates $\Theta_{4m+1}(\partial\pi)$.*

PROOF. Set $\pi=Z_{p,q}$ and $|\pi|=$the order of π.

Case 1. $n=0$, $h=h'=$odd. Since the order of $\Theta_{4m+1}(\partial\pi)$ is at most 2, the universal cover of $\Sigma/\pi\times Z_h\#\Sigma_0$ is $\Sigma\#\Sigma_0$ which admits a free $\pi\times Z_h$-action.

Case 2. $n=1$, $h=2h'$ and h' odd. Let Z_2 be a subgroup of order 2 in $\pi\times Z_{2h'}$. First we construct a non zero surgery obstruction in $L_{4m+2}(Z_2, -)$ $=Z_2$. We shall prove the following (see [3, p. 68]).

LEMMA 6.2. *Let η (resp. $E(\eta)$) be the canonical line (resp. disk) bundle over Σ/Z_2, so that $\partial E(\eta)=\Sigma$. Then there is a normal map $g: N\to E(\eta)$ which is a homotopy equivalence on the boundary $\partial N=\Sigma$ and $\theta(g)$ is not zero in $L_{4m+2}(Z_2, -)$ $=Z_2$ (i.e., the obstruction to making g a homotopy equivalence rel. boundary is not zero).*

PROOF OF LEMMA 6.2. Take $\Sigma/Z_2(5)=(5\Sigma/Z_2)\cup(-2\Sigma)$ and $5id=(5id)\cup(-2p)$. The normal map $5id: \Sigma/Z_2(5)\to\Sigma/Z_2$ induces a normal map

$$E(5id): E(5id^*\eta)\longrightarrow E(\eta).$$

Here $E(5id^*\eta)$ is the disk bundle induced by $5id$. The boundary $\partial E(5id^*\eta)$ is $5\Sigma\cup -4\Sigma$, and so connecting the components of $E(5id^*\eta)$ along the boundary we obtain a normal map

$$g: \overline{E(5id^*\eta)}\longrightarrow E(\eta)$$

so that $\overline{\partial E(5id^*\eta)}=\Sigma$ and $g|\overline{\partial E(5id^*\eta)}$ is a homotopy equivalence. Attaching cone to the boundary of $\overline{E(5id^*\eta)}$, we have

$$\overline{E(5id^*\eta)}\cup(\text{cone})=(\Sigma(\Sigma/Z_2))(5),$$

where $\Sigma(\Sigma/Z_2)$ is the suspension of the (p. l.) manifold Σ/Z_2. Similarly we have

$$E(\eta)\cup(\text{cone})=\Sigma(\Sigma/Z_2).$$

Then g can be extended to a normal map

$$\bar{g}: (\Sigma(\Sigma/Z_2))(5)\longrightarrow\Sigma(\Sigma/Z_2).$$

Applying Theorem 4.1 and 4.8 of [1] to $\bar{g}$, we see that the surgery obstruction $\theta(\bar{g})$ is the Kervaire invariant $k((\Sigma(\Sigma/Z_2))(5))$, and $k((\Sigma(\Sigma/Z_2))(5))=1$ in $Z_2\cong$

$L_{4m+2}(Z_2, -)$, therefore the obstruction $\theta(g)$ is not zero. This completes the proof of Lemma 6.2.

Form a manifold

$$M_{\pi\times Z_{2h'}}=(5\Sigma/\pi\times Z_{2h'})\cup(-2\Sigma/\pi\times Z_{h'}),$$

where $Z_{h'}$ is a subgroup of order h' in $Z_{2h'}$. There is a normal map

$$f=(5id)\cup(-2p):\ M_{\pi\times Z_{2h'}}\longrightarrow\Sigma/\pi\times Z_{2h'},$$

where id is the identity map on $\Sigma/\pi\times Z_{2h'}$ and $p:\ \Sigma/\pi\times Z_{h'}\to\Sigma/\pi\times Z_{2h'}$ is the projection. The normal map $g:\overline{E(5id^*\eta)}\to E(\eta)$ in Lemma 6.2 is transverse regular to Σ/Z_2 and $g^{-1}(\Sigma/Z_2)=\Sigma/Z_2(5)$, $g|\Sigma/Z_2(5)=5id$. We have a commutative diagram of normal maps

$$(1)\qquad \begin{array}{ccc} E(5(id)^*\eta) & \xrightarrow{\quad g\quad} & E(\eta) \\ \cup & & \cup \\ \Sigma/Z_2(5) & \xrightarrow{\quad 5id\quad} & \Sigma/Z_2 \\ \big\downarrow p & & \big\downarrow p \\ M_{\pi\times Z_{2h'}} & \xrightarrow{\quad f\quad} & \Sigma/\pi\times Z_{2h'} \end{array}$$

where $p:\ \Sigma/Z_2\to\Sigma/\pi\times Z_{2h'}$ is the $h'|\pi|$-fold covering map. By Theorem 12 of [**9**] (see also [**10**]), the surgery obstruction $\theta(f)$ is zero in $L_1(\pi\times Z_{2h'})$. Chasing the diagram (1), we have a normal cobordism $F:\ V\to\Sigma/Z_2\times I$, $\partial_- V=\Sigma/Z_2(5)$, $F|\partial_- V=5id$ and $\partial_+ V$ is an $h'|\pi|$-fold cover of a manifold which is homotopy equivalent to $\Sigma/\pi\times Z_{2h'}$. By the normal cobordism extension property (see [**3**, IV 3.3]), there is a normal map

$$g':\ W\longrightarrow E(\eta)$$

normally cobordant to g, rel. boundary. g' is transverse regular to Σ/Z_2, $g'^{-1}(\Sigma/Z_2)=\partial_+ V$, and $g'|\partial_+ V$ is a homotopy equivalence.

Let N be a normal bundle of $\partial_+ V$ in W, and put $X=W-\text{int}\,N$. Then we have a normal map $g'|X:\ X\to\Sigma\times I$ which is a homotopy equivalence on the boundary. Therefore X is a framed cobordism between Σ and the universal cover Σ' of $\partial_+ V$ which admits a free $\pi\times Z_{2h'}$-action. Since $\theta(g')\neq 0$, the Kervaire obstruction of X is not zero in $Z_2\cong L_{4m+2}(1)$. Hence $\Sigma'=\Sigma\,\sharp\,\Sigma_0$.

Case 3. $n=2$, $h=4h'$ and h' odd. Form a manifold

$$M_{\pi\times Z_{4h'}}=(5\Sigma/\pi\times Z_{4h'})\cup(-\Sigma/\pi\times Z_{h'})$$

and a normal map

$$f'=(5id)\cup(-p):\ M_{\pi\times Z_{4h'}}\longrightarrow \Sigma/\pi\times Z_{4h'}$$

where $p:\ \Sigma/\pi\times Z_{h'}\to\Sigma/\pi\times Z_{4h'}$ is the projection. Then we have a commutative diagram of normal maps

$$(2)\qquad \begin{array}{ccc} M_{\pi\times Z_{2h'}} & \xrightarrow{\ \ f\ \ } & \Sigma/\pi\times Z_{2h'} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ M_{\pi\times Z_{4h'}} & \xrightarrow{\ \ f'\ \ } & \Sigma/\pi\times Z_{4h'} \end{array}$$

where $p:\ \Sigma/\pi\times Z_{2h'}\to\Sigma/\pi\times Z_{4h'}$ is 2-fold covering map. The surgery obstruction $\theta(f')$ is zero in $L_{4m+1}(\pi\times Z_{4h'})$ by Theorem 12 of [**9**]. It follows similarly to the case 2 that $\Sigma\,\#\,\Sigma_0$ admits a free $\pi\times Z_{4h'}$-action. This completes the proof of Proposition 6.1.

NOTE. In the case $n\geqq 3$, the proof of proposition 6.1 fails because of the following reason. The obstruction $\theta(g)$ of a normal map g constructed from $((2^n+1)id,\ \Sigma/Z_2(2^n+1))$ as in the proof of Lemma 6.2 is zero for $n\geqq 3$, since $k((\Sigma(\Sigma/Z_2)(2^n+1))=0$.

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