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Some simple cases of Poincaré conjecture

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§ 1. Introduction.

In this paper we shall treat special cases of genus 2 Poincaré conjecture.

Let M be a closed orientable 3-manifold of genus $\leqq 2$. Then M is obtained by a Heegaard splitting of genus 2. Let $(M; V, W)$ be a Heegaard splitting of genus 2, where V and W are solid tori of genus 2 such that $M=V\cup W$ and $V\cap W=\partial V=\partial W$*). We set $T=V\cap W$. (See the Figure 1.)

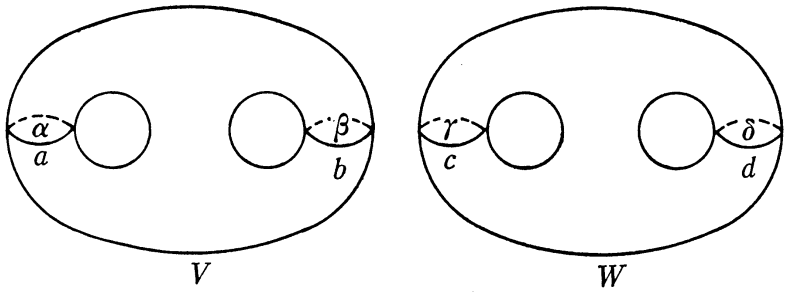


Figure 1.

We assume that M (and hence V and W) has a fixed orientation.

Let $\{\alpha, \beta\}$ be a system of meridian disks of V and let $\{\gamma, \delta\}$ be a system of meridian disks of W. We set $a=\partial\alpha$, $b=\partial\beta$, $c=\partial\gamma$ and $d=\partial\delta$. a, b, c, d are loops on T and $\{a, b\}$ is called a system of meridian loops of V and $\{c, d\}$ is called a system of meridian loops of W. We assume that $\alpha, \beta, \gamma, \delta$ and hence a, b, c, d have fixed orientation. Without loss of generality we can assume that a, b and c, d intersect transversely only in a finite number of points. Moreover we can eliminate the intersections of the type as shown in Figure 2.

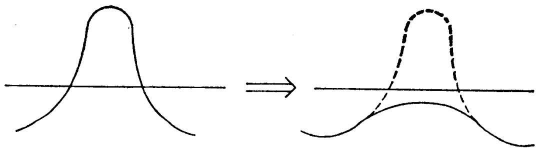


Figure 2.

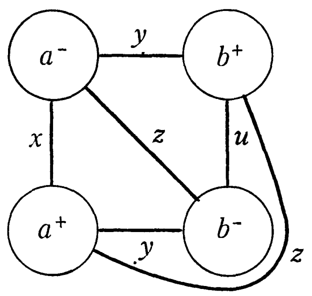
*) ∂V means the boundary of V.

So we assume that there are no such intersections.

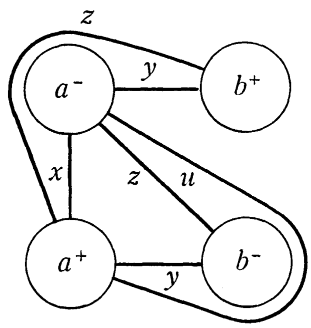
We cut V along α and β. Then we obtain a 3-disk, on the surface of which four circles obtained by cutting α and β (we name these a^+, a^-, b^+, b^-), and several fragments of the loops c and d are drawn. We call the chart drawn on the 3-disk a Whitehead graph or simply a graph. If we ignore the loop d then we obtain the graph of c, and if we ignore the loop c then we obtain the graph of d.

If we interchange the role of a, b and c, d then we obtain another graph. This is called the dual graph.

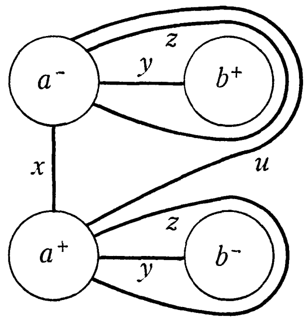
A graph is one of the three types (Type I$\sim$III) shown in Figure 3-1$\sim$3-3 (up to interchanging a, b or a, a^{-1} or b, b^{-1}).



Type I
Figure 3-1.



Type II
Figure 3-2.



Type III
Figure 3-3.

In these figures each of x, y, z, u means the number of parallel paths.

If a graph is of the type III, then we can decrease the number of intersection points by changing meridian loops a, b to a', b as shown in Figure 4. (a' is clearly a meridian loop.)

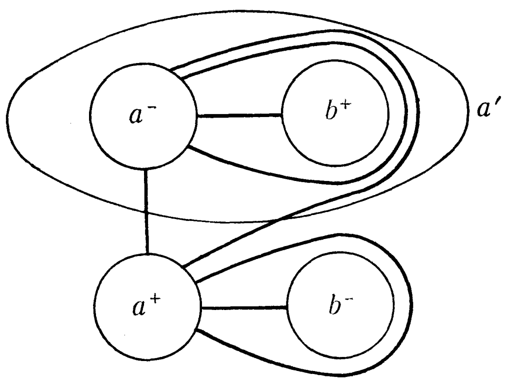


Figure 4.

So we mainly treat Type I and Type II.*)

*) Recently Ochiai [14] proved that a graph of Type II can be reduced to Type I without increasing the number of intersection points provided the manifold M is a homology sphere.

Let f be an arc on ∂V connecting two loops a and b. (We assume that one end point P of f is on a and another end point Q is on b and any other point of f is not on the loops a or b.)

Let U be a regular neighborhood (in ∂V) of $a \cup b \cup f$. Then ∂U consists of three loops and one of these is isotopic to a and one is isotopic to b. Let $\bar{f}$ be the third loop of these three loops.

Then $\bar{f}$ is also a meridian loop since there is a meridian disk η in V such that $\bar{f}=\partial\eta$. Moreover $\{a, \bar{f}\}$ (and also $\{b, \bar{f}\}$) is a system of meridian loops of V.

Changing the system from $\{a, b\}$ to $\{a, \bar{f}\}$ (or to $\{b, \bar{f}\}$) is called a band operation by f. Similarly a band operation by an arc on ∂W connecting two loops c and d can be defined.

Since a band operation does not affect a Heegaard splitting, the resulting manifold M does not change. (But the graph is changed.)

The fundamental group $\pi_1(M)$ of M can be presented in the following way. We take a and b as generators and two relations $\tilde{c}=1$ and $\tilde{d}=1$ are defined as follows*⁾: When a point P is moving on the loop c (in the direction of the orientation) from a fixed base point, if it crosses a or b as shown in Figure 5-1, 5-2, 5-3, 5-4, then we read a, a^{-1}, b, b^{-1}, respectively.

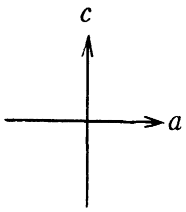


Figure 5-1.

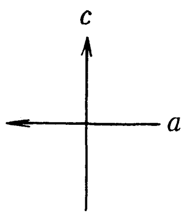


Figure 5-2.

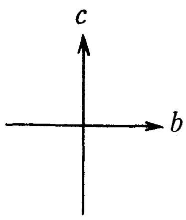


Figure 5-3.

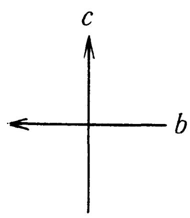


Figure 5-4.

When P moves once around c, we get a word $\tilde{c}(a, b)$. $\tilde{d}(a, b)$ is defined similarly.

REMARK. More generally, if l is a (not necessarily simple) path on T from P to Q, where P and Q are not on the loops a and b, then l defines a word $\tilde{l}(a, b)$ by the same way as above.

If l' is another path from P to Q and if l and l' is homotopic in T with P, Q fixed, then $\tilde{l}(a, b)$ and $\tilde{l}'(a, b)$ are the same word as elements of the free group generated by a and b.

Now we have a presentation of the fundamental group of M:

$$\pi_1(M) \cong \langle a, b : \tilde{c}(a, b)=\tilde{d}(a, b)=1 \rangle .$$

If we interchange the role of a, b and c, d we get another presentation of $\pi_1(M)$:

*⁾ The proof is by van Kampen theorem.

$$\pi_1(M)\cong\langle c,\ d:\ \tilde{a}(c,\ d)=\tilde{b}(c,\ d)=1\rangle .$$

This is called the dual presentation.

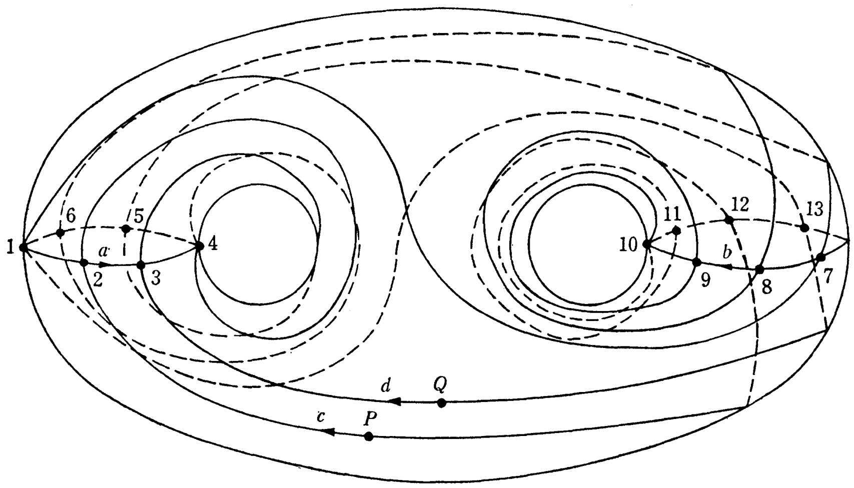


Figure 6-1.

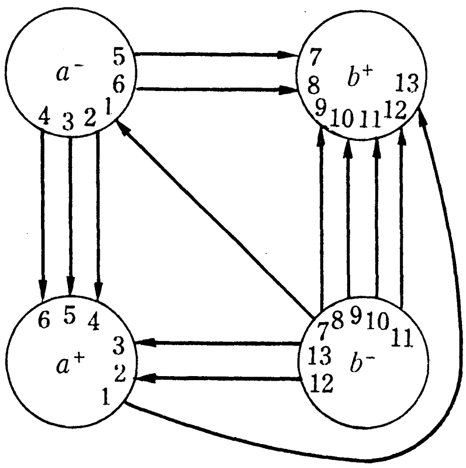


Figure 6-2.

For example, if a, b, c, d are as shown in Figure 6 and if we start along c from P and along d from Q, then the resulting word $\tilde{c}(a, b)$ is a^3b^5 and the word $\tilde{d}(a, b)$ is $a^2ba^{-1}b$. Similarly $\tilde{a}(c, d)=cdcdcd^{-1}$ and $\tilde{b}(c, d)=d^2c^5$. So

$$\pi_1(M)\cong\langle c,\ d:\ cdcdcd^{-1}=d^2c^5=1\rangle$$

is the dual presentation of

$$\pi_1(M)\cong\langle a,\ b:\ a^3b^5=a^2ba^{-1}b=1\rangle .$$

§ 2. The reverse development and Homma's theorem.

Let M, V, W, a, b, c, d, etc. be as in the introduction. Let Γ be the closure of a regular neighborhood of γ in W and let $N=V\cup\Gamma$. Then N is the complement of a regular neighborhood of a knot k in M. (See Figure 7.)

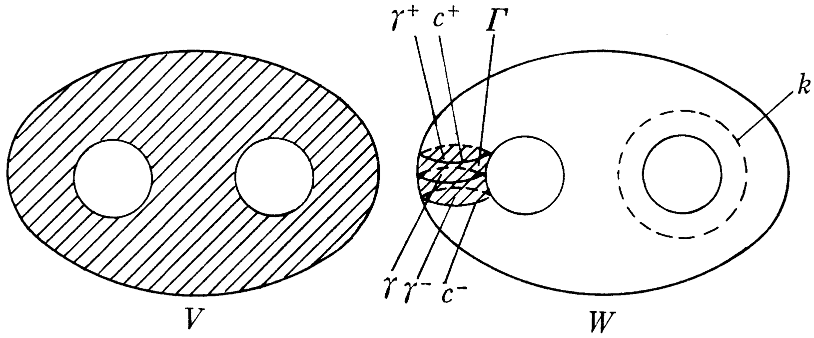


Figure 7.

By van Kampen theorem,

$$\pi_1(N)\cong\pi_1(M-k)\cong\langle a,\, b:\, \tilde{c}(a,\, b)=1\rangle\,.$$

We call this group the knot group of c.

Now the boundary of N is a torus T' on which two circles c^+ and c^- and several fragments of the loops a and b are drawn. We call this chart the reverse graph of c.

THEOREM 1 (Homma [**11**]). *Suppose that c crosses a or b at least once. Then the reverse graph of c consists of* 6 *groups W_0, W_0', W_1, W_2, W_3, W_4 of parallel paths (some of which may be empty) as shown in Figure* 8. *Suppose p is an arc on T' which crosses W_0 once to the direction as shown in Figure* 8 *and does not cross any other paths of the graph and let $w_0=\tilde{p}(a, b)$. Let w_0', w_1, w_2, w_3, w_4 be similarly defined. Then w_1, w_2, w_3, w_4 are symmetric, that is, $w_1=w_1^*$, $w_2=w_2^*$, $w_3=w_3^*$, $w_4=w_4^*$, where w^* is the word obtained by reading w from the last to the first. Moreover $w_0'=w_0^*$. Hence (if starting from a proper point) we have*

$$\tilde{c}(a,\, b)=w_0w_1w_2w_0^{-1}w_3w_4\,.$$

*Moreover $\tilde{c}(a, b)$ is freely symmetric.**)

Now we consider the universal covering space U of the torus T'. U is a plane. Let p be the covering map $U\to T'$. We introduce an orthogonal coordinate system (x, y) to U in such a way that

$$p(x,\, y)=p(x',\, y') \Leftrightarrow x-x'\in \boldsymbol{Z} \quad\text{and}\quad y-y'\in\boldsymbol{Z}\,.$$

Now the reverse graph of c drawn on T' induces an infinite graph on U. We call this infinite graph the reverse development of c. By Homma's theorem,

*) A word is said to be freely symmmetric if there are words u and v such that $w=uv$ and $w^*=vu$. The last part of the Theorem is due to [4].

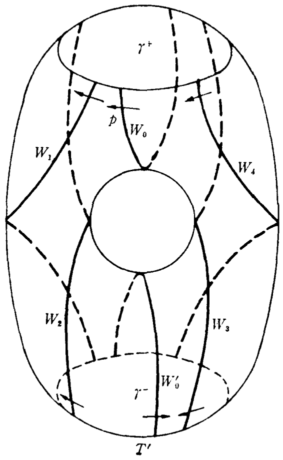


Figure 8.

the reverse development of c takes the form showm in Figure 9, where $p^{-1}(\gamma^+)=\{\gamma_{i,j}^+\}$, $p^{-1}(\gamma^-)=\{\gamma_{i,j}^-\}$, $c_{i,j}^+=\{\partial\gamma_{i,j}^+\}$ and $c_{i,j}^-=\{\partial\gamma_{i,j}^-\}$.

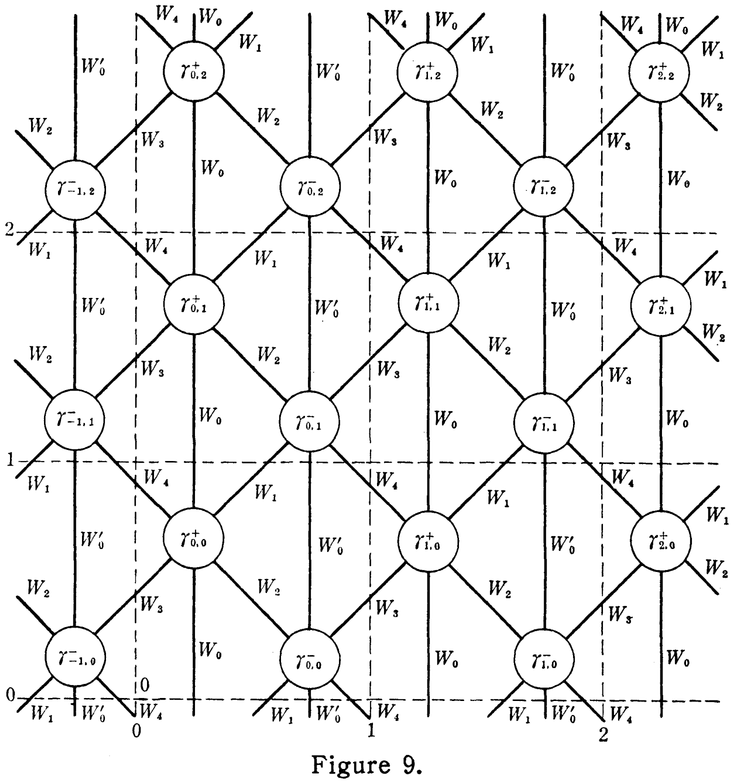
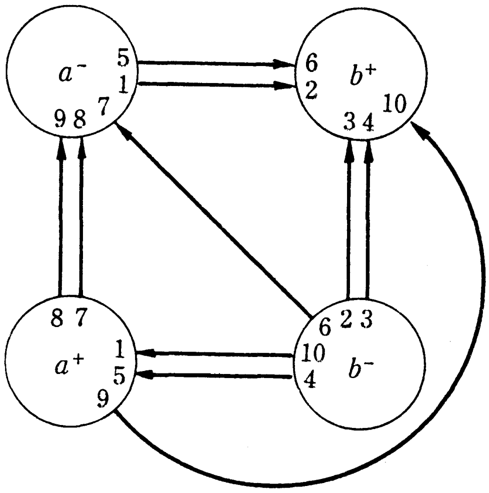


Figure 9.

Moreover we can assume

$$\{p^{-1}(\gamma^+)\cup p^{-1}(\gamma^-)\}\cap\{(x,\ y)\mid x\in \boldsymbol{Z} \text{ or } y\in\boldsymbol{Z}\}=\emptyset.$$

For example, if the Whitehead graph of c is as shown in Figure 10, then the reverse development of c is as shown in Figure 11.



$\tilde{c}(a, b)=ab^3aba^{-3}b$

Figure 10.

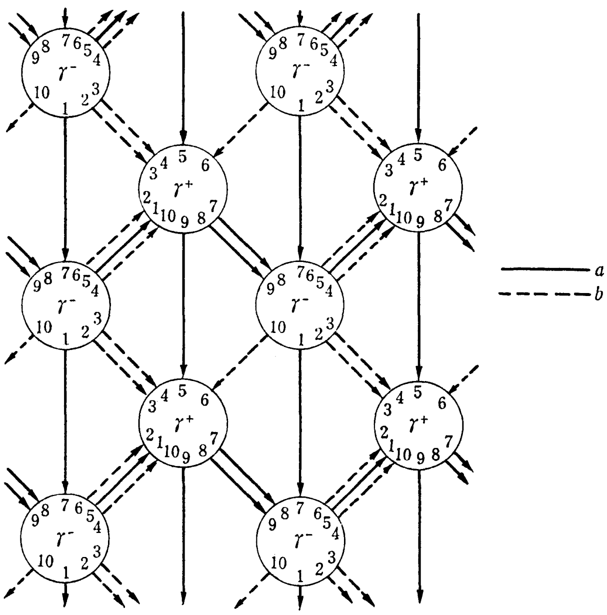


Figure 11.

Let d be another meridian loop on T. Then d is a loop on T' not intersecting c. For simplicity's sake we assume that the point $O=p(0, 0)$ is on d, and consider it as the base point of d.

Now d induces a curve $\bar{d}$ on U starting from $(0, 0)$ and ending at some (m, n), where $m, n\in \boldsymbol{Z}$. Since d is not homologous to 0 in T', $(m, n)\neq(0, 0)$. Since d is a simple loop, m and n must be relatively prime.

Let $U'=U-p^{-1}(\gamma^+)-p^{-1}(\gamma^-)$. By the same way as in the introduction, if l is a (not necessarily simple) path on U', then l reads a word $\tilde{l}(a, b)$.

We easily see that $\tilde{\tilde{d}}(a, b)=\tilde{d}(a, b)$, where $\tilde{\tilde{d}}$ is defined now and $\tilde{d}$ is defined in the introduction. (We assume that d starts from O.)

LEMMA. *Let l and l' be paths on U' from* $(0, 0)$ *to* (m, n). *Then* $\tilde{l}(a, b)=\tilde{l}'(a, b)$ *is a consequence of* $\tilde{c}(a, b)=1$.

PROOF. If l and l' are homotopic in U' then the result is obvious. If l and l' are not homotopic in U', then l' is obtained from l by crossing over some $\gamma_{i,j}^{+}$'s and $\gamma_{i,j}^{-}$'s in addition to homotopic deformation in U'.

Suppose, for instance, l' is obtained from l as shown in Figure 12.

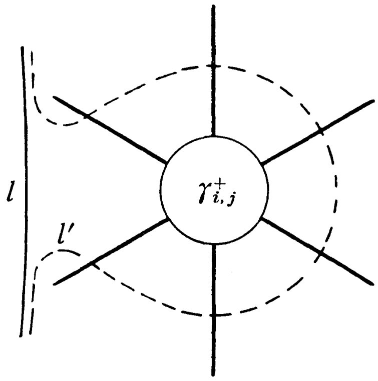


Figure 12.

Then $\tilde{l}(a, b)$ and $\tilde{l}'(a, b)$ are of the forms $\xi\eta$ and $\xi\tilde{c}(a, b)\eta$, respectively. Hence $\tilde{l}(a, b)=\tilde{l}'(a, b)$ is a consequence of $\tilde{c}(a, b)=1$. q. e. d.

Now let l be the path composed of the segment joining $(0, 0)$ to $(m, 0)$ and the segment joining $(m, 0)$ to (m, n). If the reverse development of c is as shown in Figure 9, then

$$\tilde{l}(a, b)=(w_0w_1w_0^*w_4)^m(w_3w_4)^n .$$

Since

$$\tilde{\tilde{d}}(a, b)=\tilde{l}(a, b)$$

is a consequence of $\tilde{c}(a, b)=1$ by the lemma,

$$\tilde{d}(a, b)=\tilde{l}(a, b)$$

is also a consequence of $\tilde{c}(a, b)=1$.

Hence

$$\begin{aligned}\pi_1(M)&\cong\langle a, b : \tilde{c}(a, b)=\tilde{d}(a, b)=1\rangle\\ &\cong\langle a, b : \tilde{c}(a, b)=\tilde{l}(a, b)=1\rangle\\ &=\langle a, b : w_0w_1w_2w_0^{-1}w_3w_4=(w_0w_1w_0^*w_4)^m(w_3w_4)^n=1\rangle .\end{aligned}$$

Thus we have proved the following:

THEOREM 2 (Homma). *If M is a closed orientable 3-manifold of genus $\leqq 2$, then the fundamental group of M is of the form*

$$\langle a, b : w_0w_1w_2w_0^{-1}w_3w_4=(w_0w_1w_0^*w_4)^m(w_3w_4)^n=1\rangle ,$$

for some relatively prime integers m and n.

§ 3. Main theorems.

In this chapter we shall prove our main theorems (theorem 3, 4 and 5).

THEOREM 3. *If $\tilde{c}(a, b)$ is of the form $a^p b^q$, then the manifold M cannot be a counterexample to Poincaré conjecture.**)

PROOF. Without loss of generality we can assume that $p \geqq 0$, $q \geqq 0$. If $\tilde{c}(a, b)$ is of the form $a^p b^q$, then the graph of c is one of the forms as shown in Figure 13-1 (case 1) or Figure 13-2 (case 2) (or 13-2 with a, b interchanged) and the

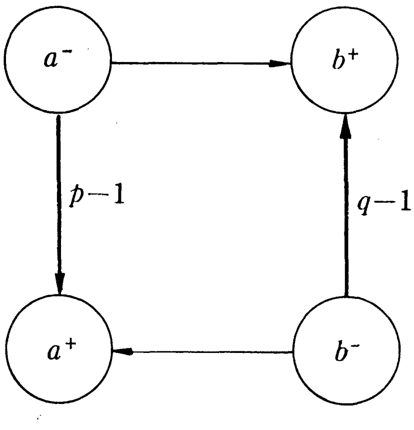


Figure 13-1.

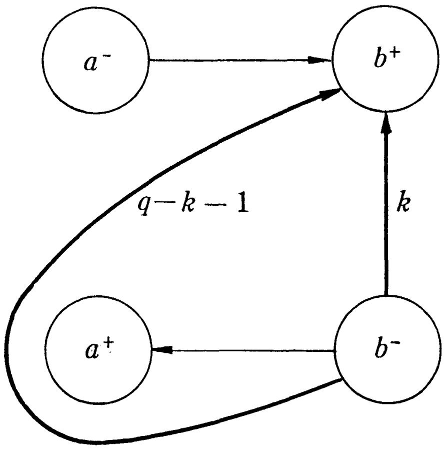


Figure 13-2.

reverse development of c is one of the forms

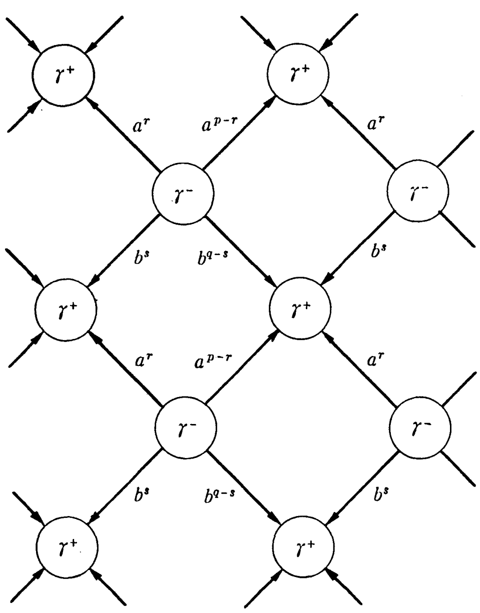


Figure 14-1.

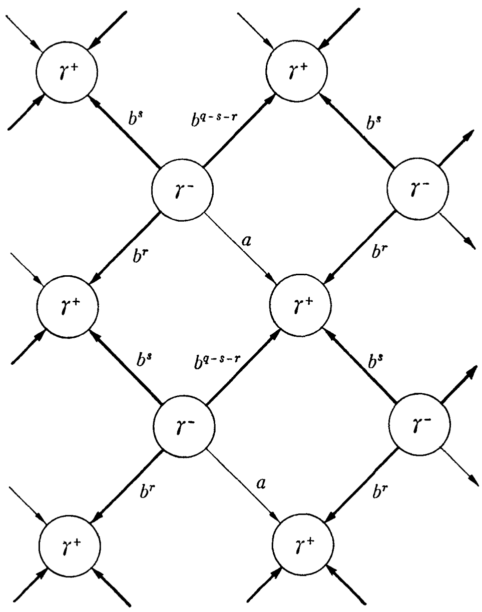


Figure 14-2.

*) Theorem 3 and Theorem 4 below are also obtained in [20] by a different method.

as shown in Figure 14-1 (case 1) or Figure 14-2 (case 2), for some r and s with $p \geqq r \geqq 0$, $q \geqq s \geqq 0$.

In the case 1, p and r are relatively prime since a is a single loop. Similarly, q and s are relatively prime.

Now by theorem 2, we have a presentation of the fundamental group for M:

$$\pi_1(M) = \langle a, b : a^p b^q = 1, a^{pm}(a^r b^s)^n = 1 \rangle \tag{1}$$

for some m and n with $(m, n) = 1$.

We prove the theorem by the induction on $p+q$.

If M is not a homology sphere, then M is not a counterexample to Poincaré conjecture. So we first examine the homology sphere condition for M. Since $H_1(M)$ is the abelianization of $\pi_1(M)$, the homology sphere condition is

$$\begin{vmatrix} p & pm+rn \\ q & sn \end{vmatrix} = \pm 1,$$

or,

$$(ps-qr)n - pqm = \pm 1. \tag{2}$$

From this it follows that $(p, q) = 1$ and $(m, n) = 1$.

First assume $p=0$. Then by the homology sphere condition we have $q=1$. Then by repeating the band operation as shown in Figure 15,

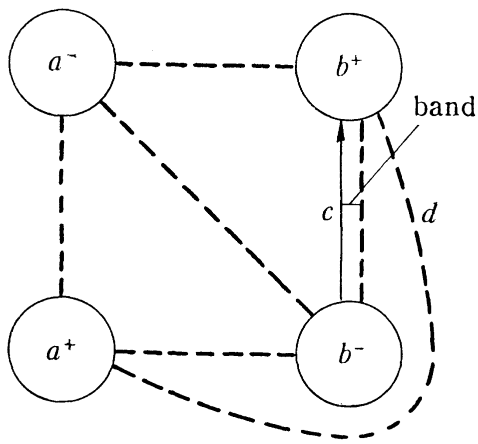


Figure 15.

the graph is changed to the one in Figure 16. Hence

$$\pi_1(M) \cong \langle a, b : a^u = 1, b = 1 \rangle,$$

for some u. But by the homology sphere condition $u = \pm 1$. In this case M is obviously a sphere.

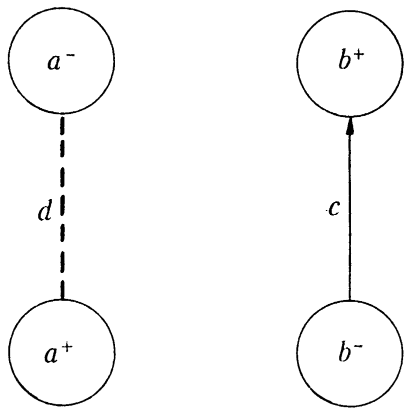


Figure 16.

Next assume $p=1$. Then we consider the graph of the loop c. Since $\tilde{c}=ab^q$, the graph is as shown in Figure 17-1 (case 1.1) or Figure 13-2 (case 1.2). Then we change the system $\{a, b\}$ of meridian loops to $\{a, b'\}$ as shown in Figure 17-2. Then $\tilde{c}=ab'^{q-1}$. By the induction hypothesis M is not a counterexample to Poincaré conjecture. Similarly for the case 1.2.

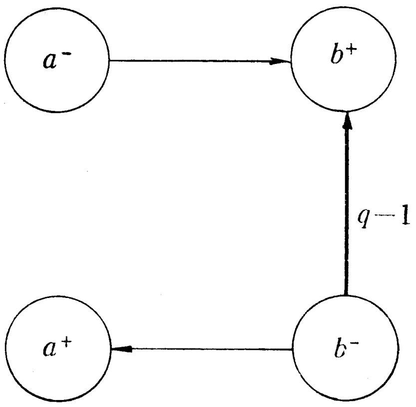


Figure 17-1.

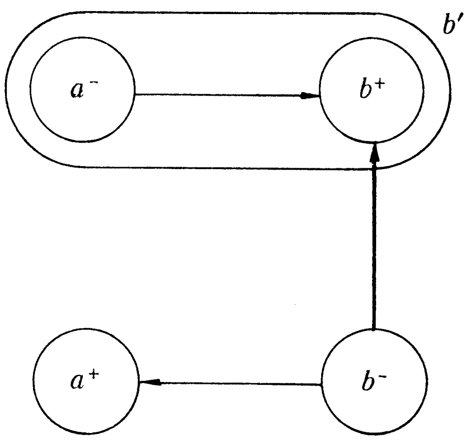


Figure 17-2.

Since the cases $q=0$ and $q=1$ can be treated similarly, we next assume $p\geqq 2$ and $q\geqq 2$.

Let

$$G\cong\langle a, b: a^p=1, b^q=1, (a^r b^s)^n=1\rangle .$$

Then by comparing it with the presentation (1), there is an epimorphism $\pi_1(M)\rightarrow G$. Hence if G is non-trivial so is $\pi_1(M)$.

Now G is generated by $A=a^r$ and $B=b^s$ since $(p, r)=1$ and $(q, s)=1$, and we have

$$G\cong\langle A, B: A^p=1, B^q=1, (AB)^n=1\rangle.$$

But this group is well-known to be non-trivial if $p\geqq 2$, $q\geqq 2$ and $n\geqq 2$ or $n=0$. (See e.g. [**9**]).

Now the only remaining case is $n=1$. In this case by the homology sphere condition we have

$$ps-qr-pqm=\pm 1$$

or,

$$m=\frac{s}{q}-\frac{r}{p}\mp\frac{1}{pq}.$$

From this it follows that $-1<m<1$ and hence that $m=0$.

Then (1) becomes

$$\pi_1(M)\cong\langle a, b: a^pb^q=1, a^rb^s=1\rangle.$$

Since $p+q>r+s$ and the second relation is obtained from some loop d' such that $\{c, d'\}$ constitutes the system of meridian loops, by the induction hypothesis M is not a counterexample to Poincaré conjecture. In the case 2, the proof is the same as in the case 1.2. This completes the proof of the theorem.

REMARK. In the theorem 1, if $ps-qr=\pm 1$, then M is obtained by a surgery along the torus knot of type (p, q).

If $r=s=m=1$, then M is homeomorphic to the Brieskorn manifold of type (p, q, n).

THEOREM 4. *If $\tilde{c}(a, b)$ is of the form $a^pb^qa^lb^k$, then the manifold M cannot be a counterexample to Poincaré conjecture.*

PROOF. By the free symmetricity of the word $a^pb^qa^lb^k$, we must have $p=l$ or $q=k$.

So without loss of generality we can assume $q=k>0$. There are two cases where $pl>0$ and where $pl<0$. Without loss of generality we only treat the case where $p>l>0$ (case 1) and the case where $p>0>l$ (case 2).

It is easily proved that, in the case 1, if the graph of c is of Type I, then the graph is as in Figure 18-1 (case 1.1) and if the graph is of Type II, then the graph is as in Figure 18-2 (case 1.2).

Similarly in the case 2, the graph of c is one of the forms as shown in Figure 19-1 (case 2.1) and Figure 19-2 (case 2.2).

In the case 1.2, we have $q=k=1$ and we change the system of meridian loops from $\{a, b\}$ to $\{a', b\}$ where a' is the loop shown in Figure 20. Then, the graph becomes the case 1.1 type or the case 1.2 type with $p+l$ decreased, or the case of theorem 3.

Next we treat the case 1.1. In this case the reverse development of c

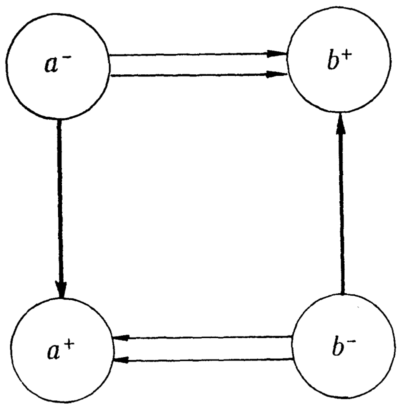


Figure 18-1.

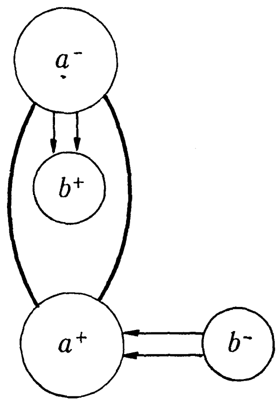


Figure 18-2.

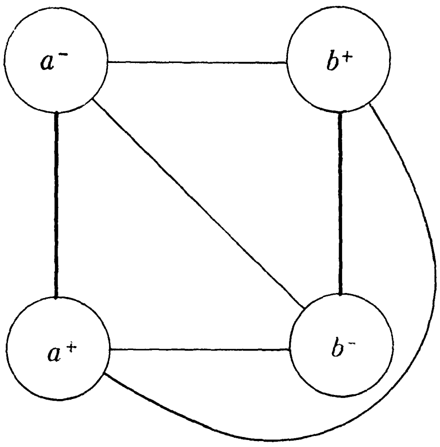


Figure 19-1.

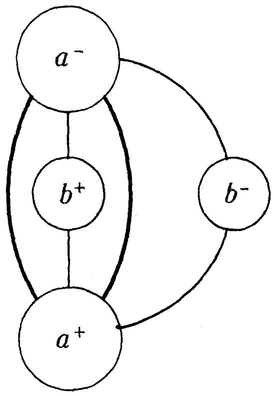


Figure 19-2.

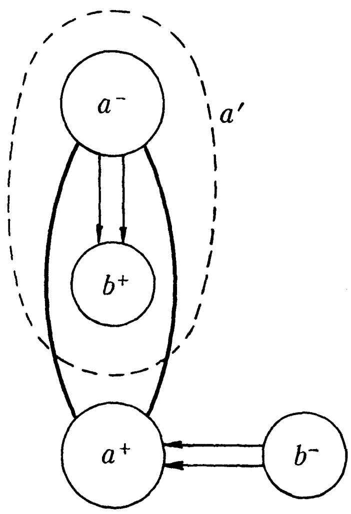


Figure 20.

must be as shown in Figure 21, for some t $(0<t<q)$.

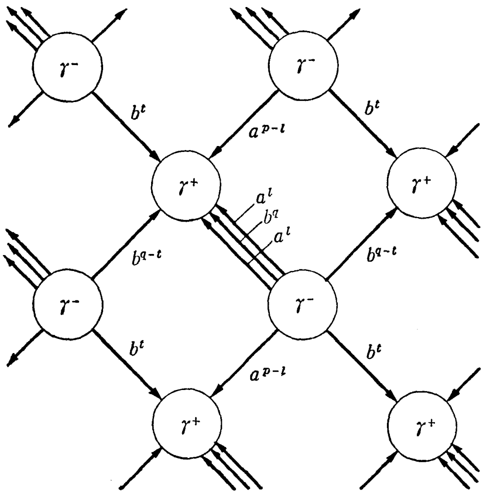


Figure 21.

Then by Homma's theorem we have

$$\pi_1(M)=\langle a,\, b:\ a^p b^q a^l b^q=1,\ (a^{p-l}b^t)^m b^{nq}=1\rangle\,.$$

The homology sphere condition is

$$((p+l)t-2(p-l)q)m+(pq+lq)n=\pm 1\,.$$

From this it follows that $(p,\, l)=1$ and $(q,\, t)=1$.

Similarly to the proof of theorem 1, let

$$G=\langle a,\, b:\ a^{p+l}=1,\ b^q=1,\ (a^{p-l}b^t)^m=1\rangle\,.$$

There is an epimorphism $\pi_1(M)\rightarrow G$.

Now $p+l\geqq 2$. So if $q\geqq 2$ and $m\neq\pm 1$, then G is non-trivial and so is $\pi_1(M)$. Let $q=1$. Then by some band operation the relation $a^p b a^l b=1$ reduces to $a'^{p-1}ba'^{l-1}b=1$. Repeating this process we have that M is not a counterexample to Poincaré conjecture. Next let $m=\pm 1$. We can assume $m=1$. Then,

$$\pi_1(M)=\langle a,\, b:\ a^p b^q a^l b^q=1,\ a^{p-l}b^{t+nq}=1\rangle\,.$$

The second relation is obtained from some loop d' such that $\{c,\, d'\}$ constitutes the system of meridian loops. So by the theorem 3 we have that M is not a counterexample to Poincaré conjecture. This proves the theorem 4 for the case 1.

In the case 2.2, we have $q=1$ or $p=-l=1$. But if $p=-l=1$, then the homology sphere condition is never satisfied. If $q=1$, then, as in 1.2, by a suitable choice of meridian loop a', the relation $a^p b a^l b=1$ reduces to

$a'^{p-1}ba'^{l-1}b=1$. Repeating this process we have finally the relation $ba'^{l-p}b=1$ or $(a'^{-1})^{p-l}b^2=1$. So by the previous theorem M is not a counterexample to Poincaré conjecture.

In the case 2.1, we can assume $p+l>0$. (Otherwise, change the orientations of a and b.) Then the reverse development of c is as shown in Figure 22, for some t $(0\leqq t\leqq q)$.

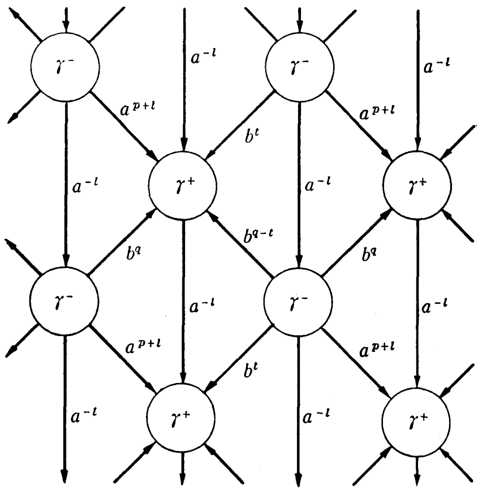


Figure 22.

Then by Homma's theorem, we have

$$\pi_1(M)\cong\langle a,\, b:\, a^pb^qa^lb^q=1,\, (a^{p-l}b^t)^mb^{qn}=1\rangle\,.$$

The homology sphere condition is

$$\{(p+l)t-2q(p-l)\}\,m+(p+l)qn=\pm 1\,.$$

Now let

$$G\cong\langle a,\, b:\, a^{p+l}=1,\, b^q=1,\, (a^{p-l}b^t)^m=1\rangle\,,$$

and

$$G'\cong\langle a,\, b:\, a^{p-l}=1,\, b^{tm+qn}=1,\, (a^pb^q)^2=1\rangle\,.$$

G is non-trivial if $p+l\geqq 2$, $q\geqq 2$ and $m\geqq 2$. In this case $\pi_1(M)$ is non-trivial since there is an epimorphism $\pi_1(M)\rightarrow G$. Similarly G' is non-trivial if $p-l\geqq 2$ and $|tm+qn|\geqq 2$, and also in this case $\pi_1(M)$ is non-trivial. But always we have

$$p-l=p+|l|\geqq 2\,.$$

So the remaining cases are

(i) $p+l=1$ and $|tm+qn|=1$

(ii) $q=1$ and $|tm+qn|=1$

(iii) $m=1$ and $|tm+qn|=1$.

In either case $|tm+qn|=1$ and so, by the homology sphere condition,

$$(p+l)(tm+qn)-2q(p-l)m=\pm(p+l)-2q(p-l)m=\pm 1\,.$$

So in the case (i) we have

$$2q(p-l)m=0 \quad \text{or} \quad \pm 2\,.$$

But $2q(p-l)m=\pm 2$ is impossible since $p-l\geqq 2$. So $2q(p-l)m=0$ and we have $m=0$ since $q\neq 0$ and $p-l\neq 0$. Then the second relation of (3) becomes $b=1$, and hence by the preceding theorem M is not a counterexample to Poincaré conjecture.

In the case (ii) we must have

$$m=\frac{\pm(p+l)\pm 1}{2(p-l)}\,. \tag{4}$$

Since m is an integer and $p>0>l$, (4) is possible only when $m=0$ and $p+l=1$. But this case was already treated.

In the case (iii) we also have $p+l=1$.

This completes the proof of theorem 4.

THEOREM 5. *If $\tilde{c}(a, b)$ is of the length $\leqq 9$, then M is not a counterexample to Poincaré conjecture.*

PROOF. If $\tilde{c}(a, b)$ is of the length $\leqq 5$, then $\tilde{c}(a, b)$ is of the form $a^p b^q$ or $a^p b^q a^l b^k$. So by the theorems 3 and 4, M is not a counterexample to Poincaré conjecture.

Next suppose that $\tilde{c}(a, b)$ is of the length 6 or 7 and not of the form $a^p b^q$ or $a^p b^q a^l b^k$. Then the graph of c must be one of the forms as shown in Figure 23. In each case we change the system of meridian loops from $\{a, b\}$ to $\{a', b\}$. Then the length of $\tilde{c}(a', b)$ is less than that of $\tilde{c}(a, b)$. So M is not a counterexample to Poincaré conjecture.

Now suppose that (i) $\tilde{c}(a, b)$ is of the length 8 or 9, (ii) $\tilde{c}(a, b)$ is not of the form $a^p b^q$ or $a^p b^q a^l b^k$, and (iii) the method used in the previous case does not apply. Then the graph of c must be one of the forms as shown in Figure 24. In each case we examine all the possibility of pasting a^+ to a^- and b^+ to b^- to obtain the simple loop c. This is shown in Figure 25. ($\tilde{c}(a, b)$ is written under each graph.)

By considering the abelianization of $\tilde{c}(a, b)$ we see that except for cases 3-3, 3-4, 3-6, 4-1, 4-2, a homology sphere cannot be obtained.

Next we show that by a change of meridian system, 3-4 is reduced to 4-1, and 4-2 is reduced to 3-3. This is shown in Figure 26 and 27.

Moreover by a change of meridian system, 4-1 and 3-6 are reduced to the case of $a^p b^q a^l b^k$. This is shown in Figure 28 and 29.

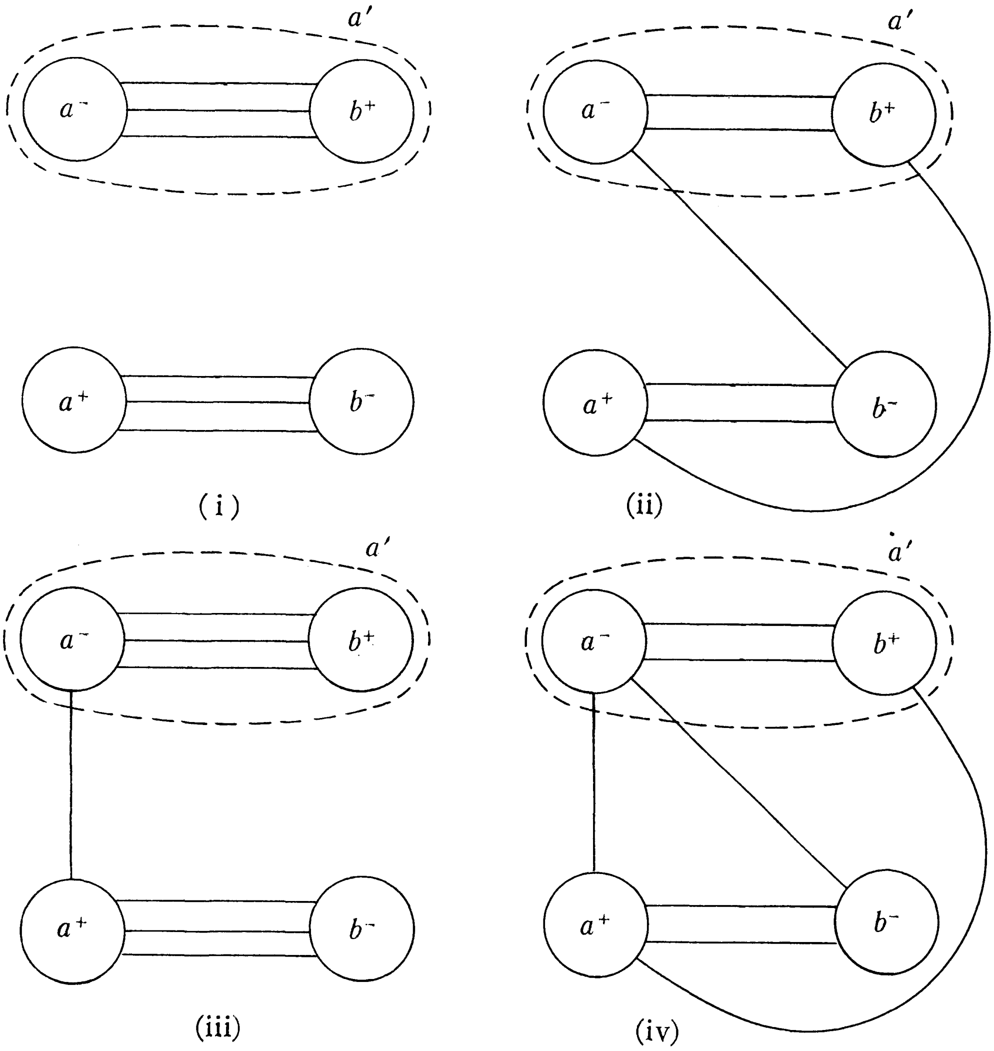


Figure 23.

Hence it remains only the case 3-3.

In this case we consider the reverse development of c. This is shown in Figure 30. Therefore,

$$\pi_1(M) \cong \langle a, b : a^3ba^{-1}b^{-2}a^{-1}b=1, (ba^4ba^{-1})^m(b^{-1}a^{-1})^n=1 \rangle .$$

The homology sphere condition is $2m-n=\pm 1$.

Now, in this group we add the relations $a^4=1$ and $b^3=1$. Then we have

$$G=\langle a, b : a^4=b^3=(a^{-1}b)^3=(b^{-1}a^{-1})^{m+n}=1 \rangle .$$

So there is an epimorphism $\pi_1(M) \to G$.

If we put $a^{-1}b=f^{-1}$, then

$$G \cong \langle b, f : b^3=f^3=(bf)^4=(b^{-1}f)^{m+n}=1 \rangle .$$

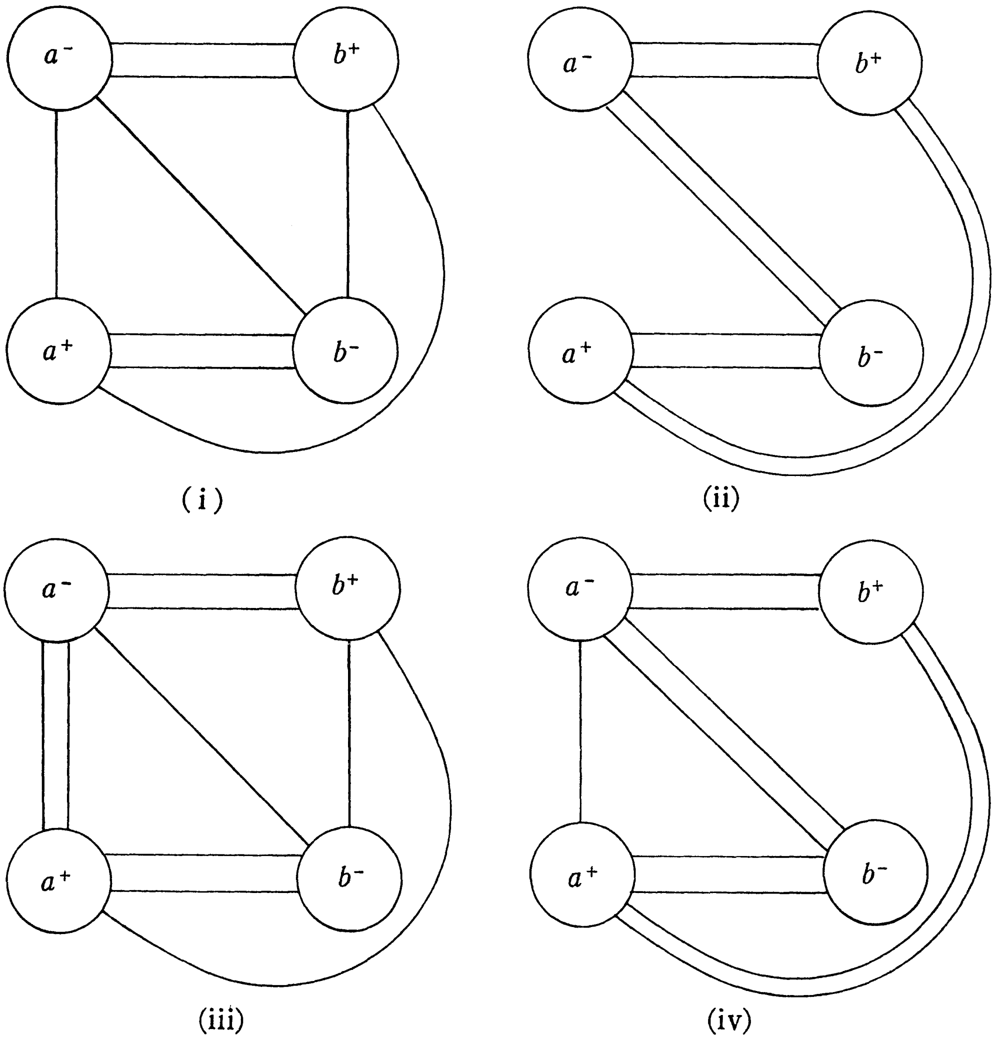
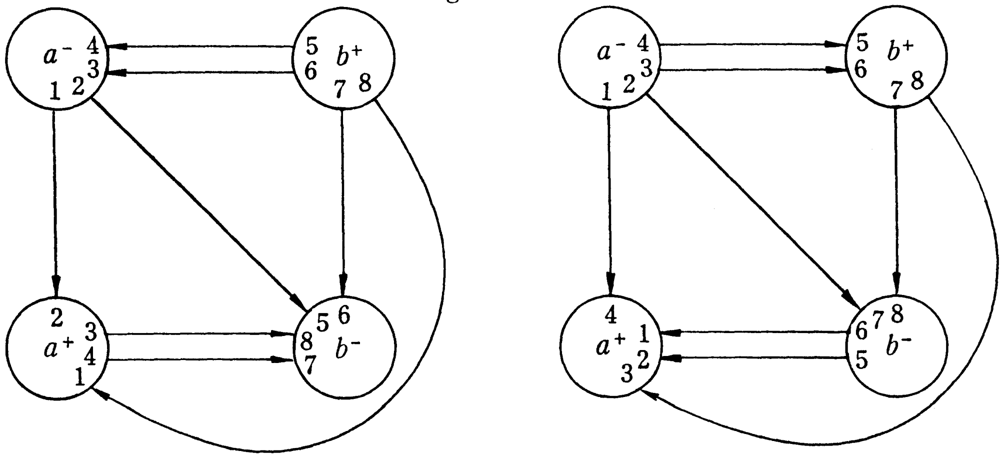


Figure 24.



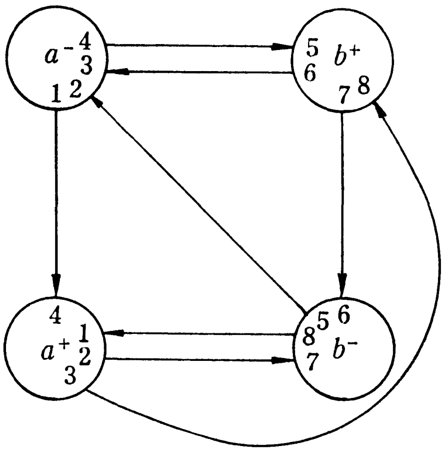
$a^2b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}$

1-1.

$a^2bab^{-2}ab$

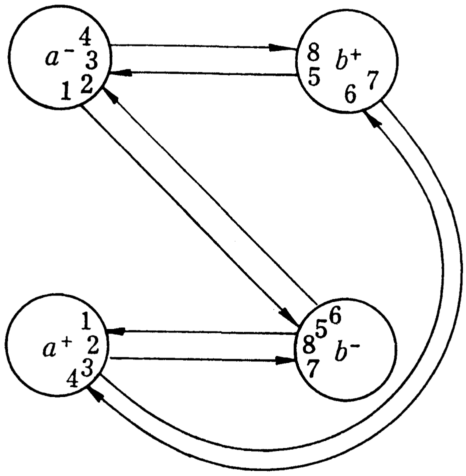
1-2.

Figure 25.



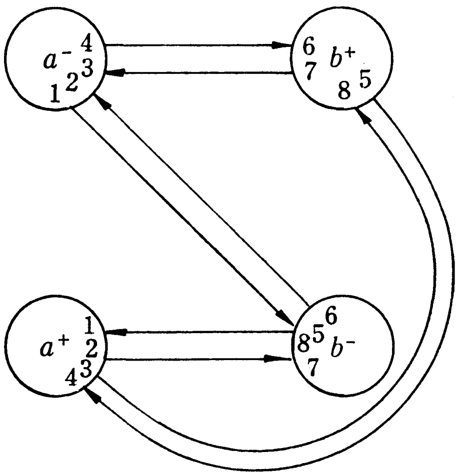
$a^2ba^{-1}b^{-2}a^{-1}b$

1-3.



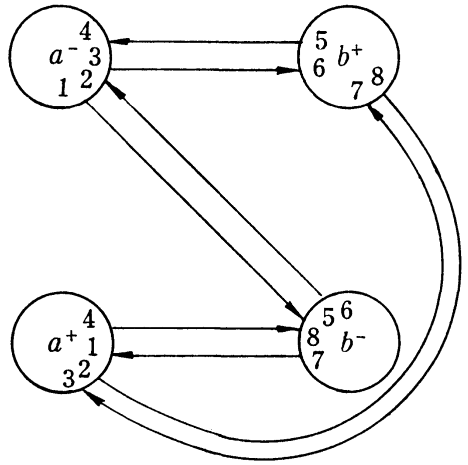
$ab^{-1}a^{-1}ba^{-1}b^{-1}ab$

2-1.



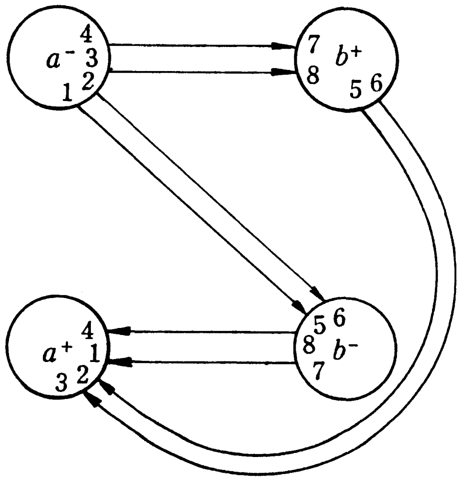
$ab^{-1}aba^{-1}b^{-1}a^{-1}b$

2-2.



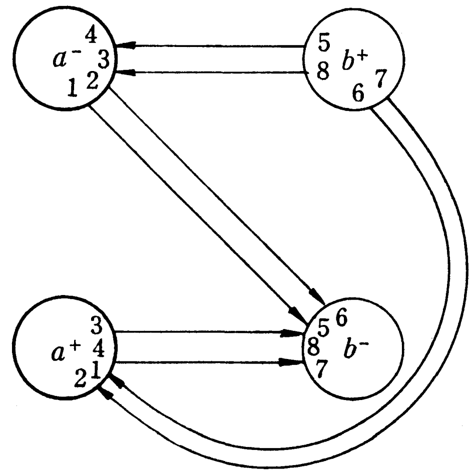
$ab^{-1}a^{-1}b^{-1}aba^{-1}b$

2-3.



$ab^{-1}ab^{-1}abab$

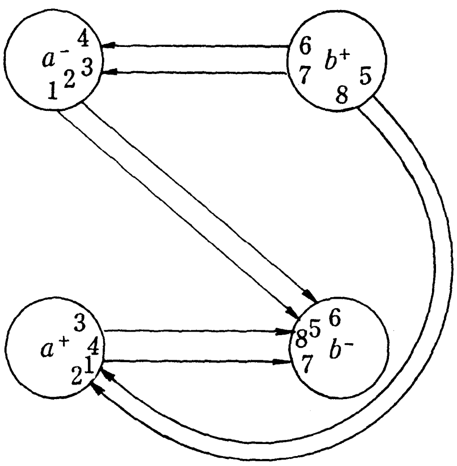
2-4.



$ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}ab^{-1}$

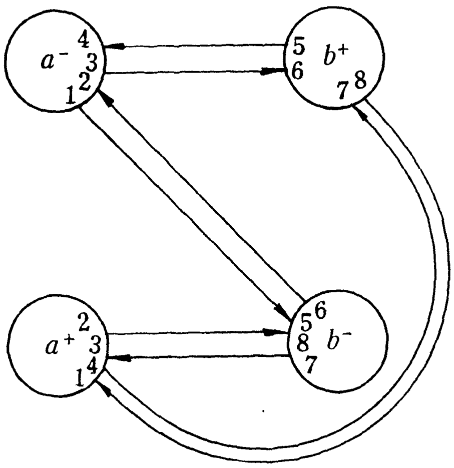
2-5.

Figure 25.



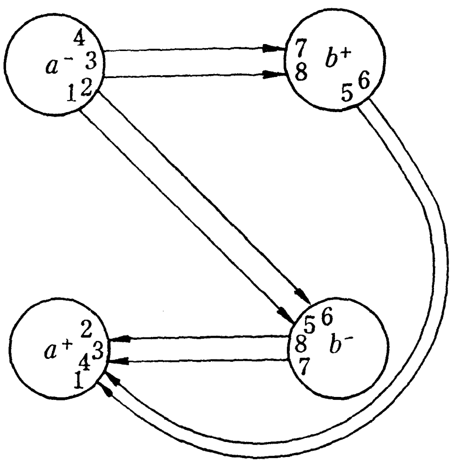
$ab^{-1}ab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$

2-6.



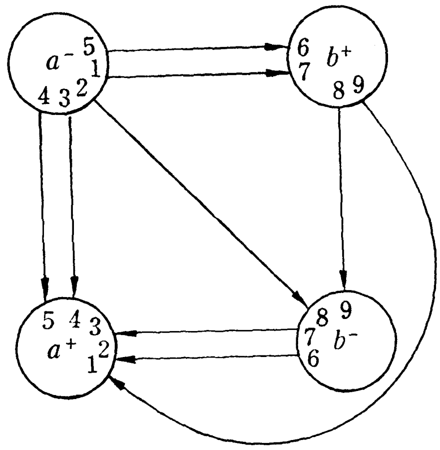
$ab^{-1}a^{-1}baba^{-1}b^{-1}$

2-7.



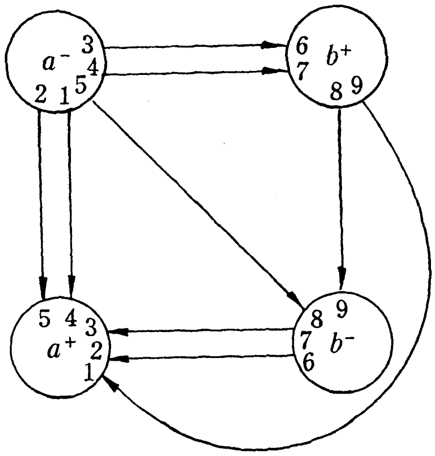
$ab^{-1}ababab^{-1}$

2-8.



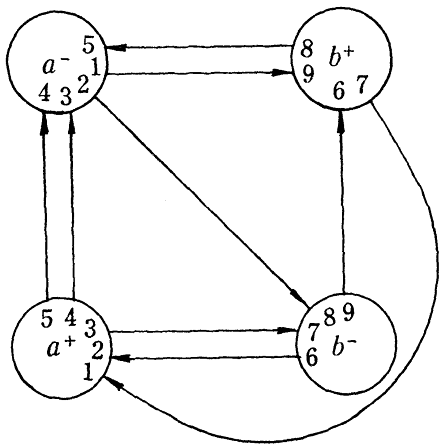
aba^3bab^{-2}

3-1.



$a^2baba^2b^{-2}$

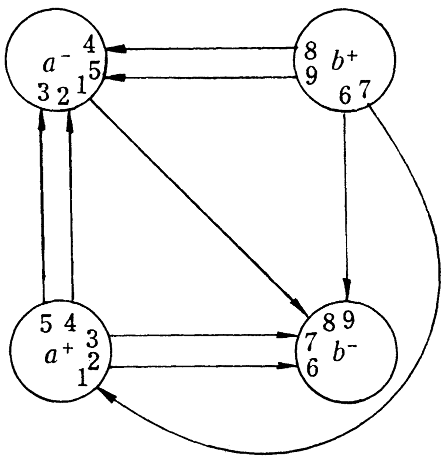
3-2.



$ab^2ab^{-1}a^3b^{-1}$

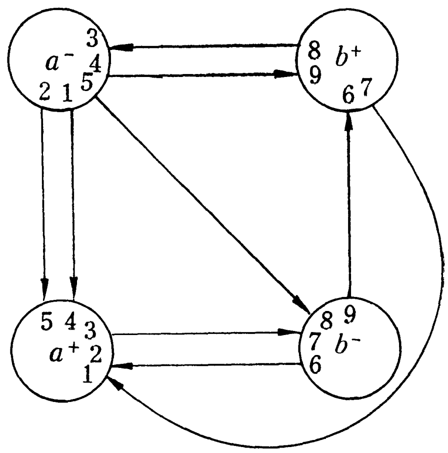
3-3.

Figure 25.



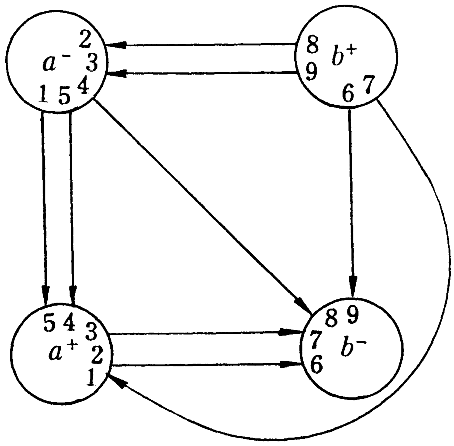
$ab^{-1}a^{-2}b^{-2}a^{-2}b^{-1}$

3-4.



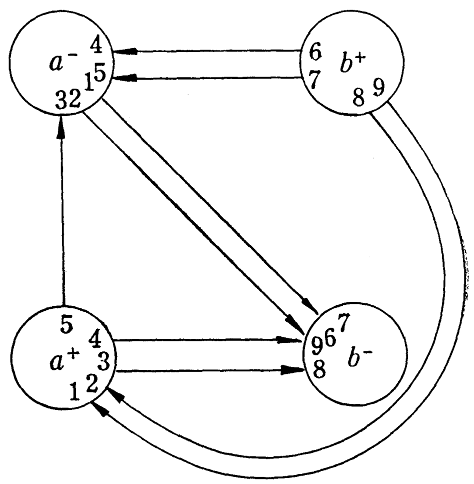
$a^2b^2a^2b^{-1}a^{-1}b^{-1}$

3-5.



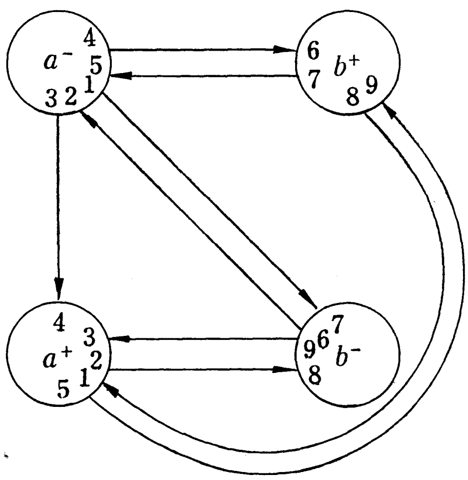
$a^3b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}$

3-6.



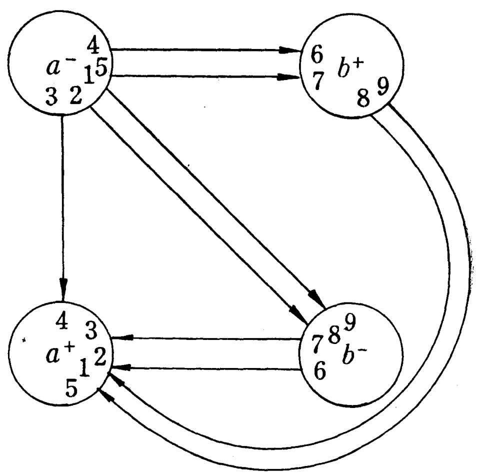
$ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b^{-1}$

4-1.



$ab^{-1}a^{-1}ba^2ba^{-1}b^{-1}$

4-2.



$ab^{-1}aba^2bab^{-1}$

4-3.

Figure 25.

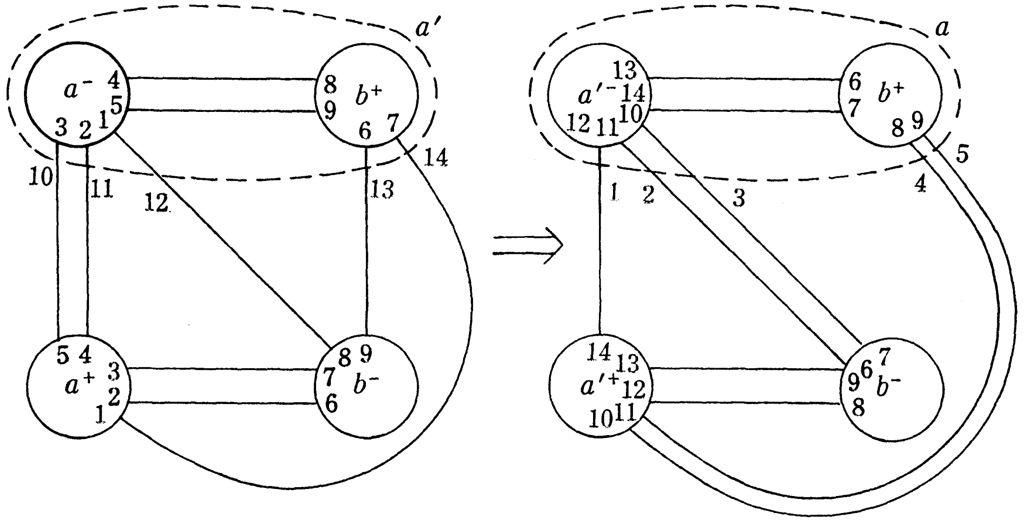


Figure 26.

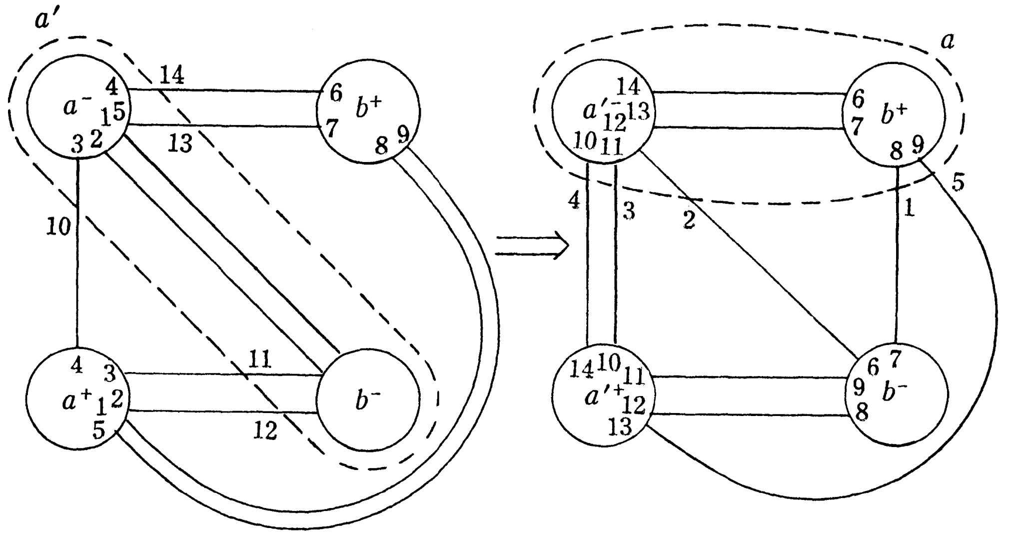


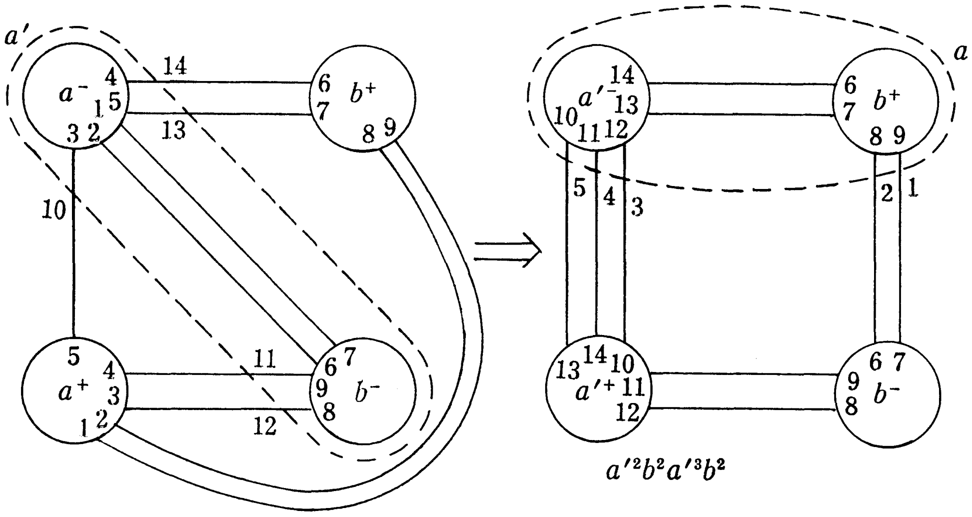
Figure 27.

This group is non-trivial if $|m+n| \geqq 3$ by [3], [8]. In this case $\pi_1(M)$ is also non-trivial.

The remaining case is $|m+n| \leqq 2$. We can assume $m \geqq 0$. So there are two cases (i) $m=0$, $n=1$, (ii) $m=1$, $n=1$, on account of the homology sphere condition $2m-n=\pm 1$.

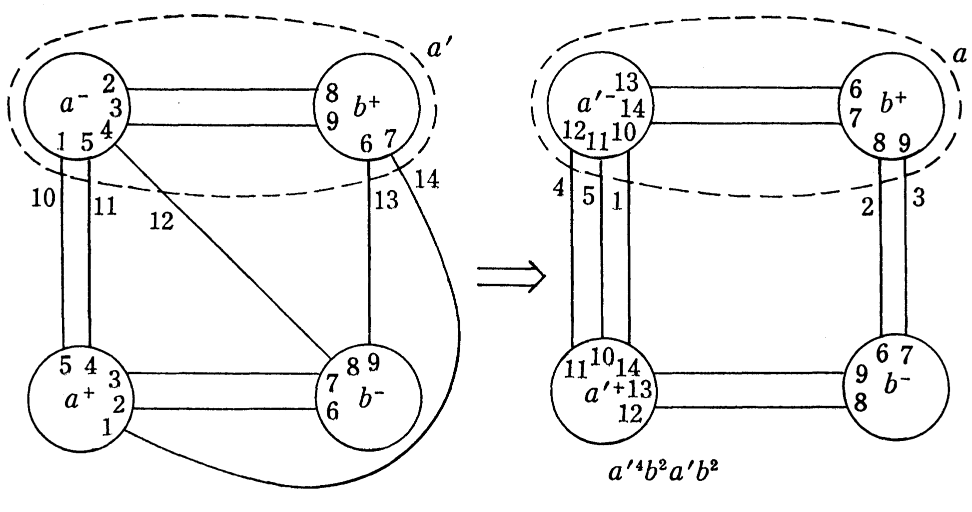
If $m=0$ and $n=1$, then M is a sphere, for it can be reduced to the canonical form by some changes of meridian system.

If $m=1$ and $n=1$, then



4-1.

Figure 28.



3-6.

Figure 29.

$$\pi_1(M)\cong\langle a,\ b:\ a^3ba^{-1}b^{-2}a^{-1}b=1,\ ba^4ba^{-1}b^{-1}a^{-1}=1\rangle .$$

From $a^3ba^{-1}b^{-2}a^{-1}b=1$ it follows that $a^3ba^{-1}b^{-1}=b^{-1}ab$ and from $ba^4ba^{-1}b^{-1}a^{-1}=1$ it follows that $a^3ba^{-1}b^{-1}=a^{-1}b^{-1}a$. So we have $b^{-1}a^{-1}b=a^{-1}ba$. Let $x=ab^{-1}a=b^{-1}ab^{-1}$, and $y=ab^{-1}$. Then $x^2=y^3$, and $a=y^{-1}x$, $b=yx^{-1}$.

Hence

$$\pi_1(M)=\langle x,\ y:\ x^2=y^3=(y^{-1}x)^7\rangle .$$

This is non-trivial and the proof of theorem 5 is complete.

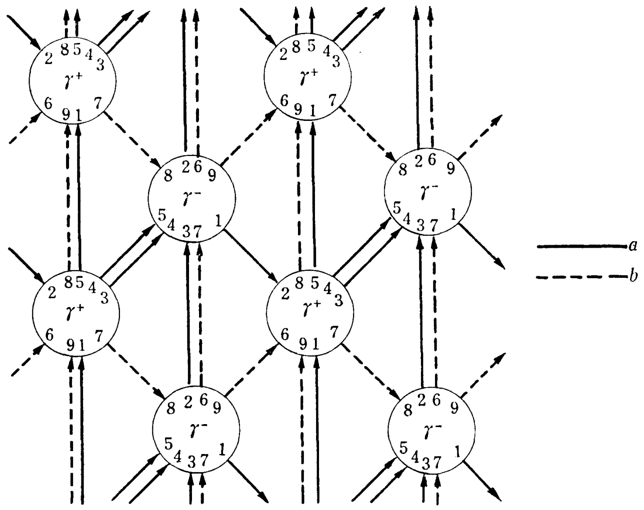


Figure 30.

REMARK. The length 10 case remains open. Indeed, if we apply the method used to prove theorem 5 to the length 10 case, we see that the theorem holds except the following fundamental groups. We are unable to prove that they are non-trivial.

$$\text{(i)}\quad \begin{cases} a^3b^{-1}ab^3ab^{-1}=1 \\ (ab^{-1}a^2b^{-1})^{9t+2}(ab^2)^{13t+3}=1\,, \end{cases}$$

$$\text{(ii)}\quad \begin{cases} a^3b^2ab^{-1}ab^2=1 \\ (ab^{-1}ab^2ab^{-1})^{2t+1}(ab)^{9t+4}=1\,, \end{cases}$$

$$\text{(iii)}\quad \begin{cases} a^2ba^2b^2a^{-1}b^2=1 \\ (b^{-2}ab^{-1}a)^t(ab^2)^{19t+1}=1\,,\quad (t\neq 0) \end{cases}$$

$$\text{(iv)}\quad \begin{cases} a^3b^2a^{-1}ba^{-1}b^2=1 \\ (a^3ba)^{8t+3}(a^2b^2)^{-19t-7}=1\,, \end{cases}$$

where $t\in \boldsymbol{Z}$.

Note added in proof. Recently we obtain the non-triviality of the above four classes of fundamental groups by using representations to PGL(2, $\boldsymbol{C}$).

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