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Classes on ZF models

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Let $\mathcal{A}=(A, E)$ be a model of ZF where A is a set and $E\subseteqq A\times A$, and $\bar{K}$ a new predicate letter. We say that a subset K of A is a class of $\mathcal{A}$ if and only if $[\mathcal{A}, K]$ is a model of $ZF(\bar{K})$ where $\bar{K}$ is interpreted by K and the replacement scheme holds for all formulae involving both $\in$ and the new predicate letter $\bar{K}$. In this paper we prove some results about classes.

A class K of $\mathcal{A}$ is definable if and only if for some formula $\phi(v_0, v_1, \cdots, v_n)$ not involving $\bar{K}$ and some elements $a_1, a_2, \cdots, a_n$ of A, $K=\{x\in A\mid\mathcal{A}\models\phi(x, a_1, a_2, \cdots, a_n)\}$. We denote by $\mathrm{def}(\mathcal{A})$ the set of all definable classes of $\mathcal{A}$, and say that a class K of $\mathcal{A}$ is undefinable if and only if $K\notin\mathrm{def}(\mathcal{A})$. Let κ be a strongly inaccessible cardinal. Then V_κ is a model of ZF and every subset of V_κ is a class of V_κ. Since $|\mathrm{def}(V_\kappa)|=|V_\kappa|<2^{|V_\kappa|}$, there exist undefinable classes of V_κ. In section 1, we prove the following:

THEOREM. *If $\mathcal{A}$ is a standard model of ZF, then there exists an undefinable class of $\mathcal{A}$.*

If $\mathcal{A}$ is a model of ZF, then $[\mathrm{def}(\mathcal{A}), A]$ is a model of GB (Gödel Bernays set theory). Theorem means that if $\mathcal{A}$ is standard, then there exists $N\supsetneqq\mathrm{def}(\mathcal{A})$ such that $[N, A]$ is a model of GB.

Let K and K' be classes of $\mathcal{A}$. K and K' are incompatible if and only if $[\mathcal{A}, K, K']\not\models ZF(\bar{K}, \bar{K}')$ where $\bar{K}$ and $\bar{K}'$ are new predicate letters and $ZF(\bar{K}, \bar{K}')$ are axioms of ZF in the language $(\in, \bar{K}, \bar{K}')$. There are many incompatible classes in countable models of ZF (Mostowski [7]). The existence of incompatible classes means that $ZF(\bar{K}, \bar{K}')$ and $ZF(\bar{K})+ZF(\bar{K}')$ are not equivalent, in other words, there exists a sentence Φ such that $ZF(\bar{K}, \bar{K}')\vdash\Phi$ but $ZF(\bar{K})+ZF(\bar{K}')\nvdash\Phi$. In section 2, we present such a sentence Φ explicitly under some assumption.

1. Undefinable classes.

We begin with some definitions from model theory. Let $\mathcal{L}$ be a first order language and P a class of structures of $\mathcal{L}$. P is inductive if and only if the union of any chain $M_0\subseteqq M_1\subseteqq\cdots\subseteqq M_\alpha\subseteqq\cdots(\alpha<\lambda)$ of structures from P is again in P. Let $\phi(v_1, v_2, \cdots, v_n)$ be a formula of $\mathcal{L}$. ϕ is said to be P-persistent when it holds that for every pair of structures $M \subseteqq M'$ of P and any elements $a_1, a_2, \cdots, a_n$ of M, if M satisfies $\phi(a_1, a_2, \cdots, a_n)$, then M' also satisfies it. For example, all existential formulae are P-persistent. A structure M of P is P-persistently complete if and only if for every extension M' of M in P, every P-persistent formula $\phi(a_1, a_2, \cdots, a_n)$ true in M' is already true in M where $a_1, a_2, \cdots, a_n$ are elements of M. The class of all P-persistently complete structures is denoted by P'. The chain $P \supsetneqq P^1 \supseteqq P^2 \supseteqq \cdots \supseteqq P^n \supseteqq \cdots (P^{n+1}=(P^n)')$ is called Cherlin chain. A subclass Q of P is said to be cofinal with P if and only if every structure in P has an extension in Q.

LEMMA 1 [Cherlin]. *Let P be a inductive class of structures. Then for each $n<\omega$, P^n is inductive and cofinal with P.*

For the proof of this Lemma, refer to Cherlin [3] or Hirschfeld-Wheeler [5].

LEMMA 2. *Every Σ_{n+1}-formula is P^n-persistent.*

PROOF. We prove by induction on n. If ϕ is P^n-persistent, then so is $\exists x\phi$ and $\neg\phi$ is P^{n+1}-persistent. Obviously every existential formula (Σ_1-formula) is P-persistent. Assume that every Σ_n-formula is P^{n-1}-persistent. Then every Π_n-formula is P^n-persistent, so every Σ_{n+1}-formula is P^n-persistent.

The proof of Lemma 2 is included in [5]. However, we gave it because of its brevity.

We write $M \prec_n M'$ if and only if M is Σ_n-elementary substructure of M', namely for any elements $a_1, a_2, \cdots, a_m$ of M and any Σ_n-formula $\phi(v_1, v_2, \cdots, v_m)$, M satisfies $\phi(a_1, a_2, \cdots, a_m)$ if and only if M' satisfies it.

LEMMA 3. *For every M, M' of P^n, if $M \subseteqq M'$, then $M \prec_n M'$.*

This Lemma follows immediately from Lemma 2.

A set A is said to be κ-complete if and only if for any subset B of A such that the cardinality of B is less than κ, $\cup B \in A$.

PROPOSITION. *Let $\mathcal{A}$ be a standard transitive model of ZF. If A is* cf *$(On^{\mathcal{A}})$-complete, then there exists an undefinable class of $\mathcal{A}$.*

PROOF. Let $\mathcal{L}=(\in, \bar{K})$ where $\in$ is binary and $\bar{K}$ is unary predicates, $P=\{(a, b)\in A \mid a \text{ is transitive}, b\subseteqq a,\ a, b\in A\}$ and $\lambda=\mathrm{cf}(On^{\mathcal{A}})$. Obviously P is a inductive class of $\mathcal{A}$. Let f be a cofinal function from λ to $On^{\mathcal{A}}$, namely $\mathrm{dom}(f)=\lambda$ and $\bigcup_{\alpha<\lambda} f(\alpha)=On^{\mathcal{A}}$. We define $(a, b)\subseteqq(a', b')$ if and only if (a, b) is a substructure of (a', b'), in other words $a\subseteqq a'$ and $b=a\cap b'$.

We define $(a_\alpha, b_\alpha)\in P$ for $\alpha<\lambda$ by induction. Let $(a_0, b_0)\in P$ be arbitrary. Let $(a_{\alpha+1}, b_{\alpha+1})$ be such that $(a_\alpha, b_\alpha)\subseteqq(a_{\alpha+1}, b_{\alpha+1})$, $a_\alpha\in a_{\alpha+1}$, $V^{\mathcal{A}}_{f(\alpha)}\subseteqq a_{\alpha+1}$ and $(a_{\alpha+1}, b_{\alpha+1})\in P^n$ where $\alpha+1=\beta+n$ for some limit ordinal β. If α is a limit ordinal, let $(a_\alpha, b_\alpha)=(\bigcup_{\beta<\alpha} a_\beta, \bigcup_{\beta<\alpha} b_\beta)$. Since A is λ-complete, $a_\alpha, b_\alpha\in A$ for all $\alpha<\lambda$. (The condition of λ-completeness is used only in this place, cf. the proof of Theorem) $V^{\mathcal{A}}_{f(\alpha)}\subseteqq a_{\alpha+1}$ and f is a cofinal function from λ to $On^{\mathcal{A}}$, therefore

$\bigcup_{\alpha<\lambda} a_\alpha = A$. Let $K=\bigcup_{\alpha<\lambda} b_\alpha$. We prove that K satisfies the Proposition.

First we show that $[\mathcal{A}, K]$ is a model of $ZF(\bar{K})$. We prove that $[\mathcal{A}, K]$ satisfies replacement scheme. Let $\phi(v_1, v_2)$ be a formula of $\mathcal{L}$ with constants in A and for some $a \in A$,

$$[\mathcal{A}, K] \models \forall x \in a \, \exists y \phi(x, y).$$

By Lemma 3, for sufficiently large n and α, $a \in a_{\alpha+n}$ and

$$(a_{\alpha+n}, b_{\alpha+n}) \models \forall x \in a \, \exists y \phi(x, y).$$

Then for every $x \in a$, there is a $y \in a_{\alpha+n}$,

$$(a_{\alpha+n}, b_{\alpha+n}) \models \phi(x, y).$$

By Lemma 3,

$$(a_{\alpha+n+1}, b_{\alpha+n+1}) \models \phi(x, y).$$

Since $a_{\alpha+n} \in a_{\alpha+n+1}$,

$$(a_{\alpha+n+1}, b_{\alpha+n+1}) \models \forall x \in a \exists y \in a_{\alpha+n} \phi(x, y),$$

$$(a_{\alpha+n+1}, b_{\alpha+n+1}) \models \exists z \forall x \in a \exists y \in z \phi(x, y).$$

Again by Lemma 3,

$$[\mathcal{A}, K] \models \exists z \forall x \in a \, \exists y \in z \phi(x, y).$$

Hence it suffices to show that $[\mathcal{A}, K]$ satisfies the separation. Let $\phi(v)$ be a formula of $\mathcal{L}$ with constants in A and a be an element of A. We must find $b \in A$ such that

$$[\mathcal{A}, K] \models b = \{x \in a \mid \phi(x)\}.$$

By Lemma 3, for sufficiently large n and α, $a \in a_{\alpha+n}$ and for any $x \in a$,

(*) $\quad [\mathcal{A}, K] \models \phi(x)$ if and only if $(a_{\alpha+n}, b_{\alpha+n}) \models \phi(x)$.

On the other hand, since $\mathcal{A}$ is a model of ZF and "$(a_{\alpha+n}, b_{\alpha+n}) \models \phi(x)$" can be described in ZF, there exists a set $b \in A$ such that

$$\mathcal{A} \models b = \{x \in a \mid (a_{\alpha+n}, b_{\alpha+n}) \models \phi(x)\}.$$

Then by (*)

$$[\mathcal{A}, K] \models b = \{x \in a \mid \phi(x)\}.$$

Second, we prove that K is undefinable in $\mathcal{A}$. If not, for some formula $\phi(v)$ not containing $\bar{K}$ with constants in A,

$$[\mathcal{A}, K] \models K = \{x \mid \phi(x)\}.$$

By Lemma 3, for sufficiently large n and α,

$$(a_{\alpha+n},\ b_{\alpha+n})\models b_{\alpha+n}=\{x\mid\phi(x)\}.$$

Pick $d\notin a$ and assume $\mathcal{A}\models\phi(d)$. (Similarly for $\mathcal{A}\models\neg\phi(d)$.) By the definition of P, we can take $(a', b')\in P$ such that $d\in a'$, $d\notin b'$ and $(a_{\alpha+n}, b_{\alpha+n})\subseteq(a', b')$. Since P^n is cofinal with P, there is $(a, b)\in P^n$ such that $(a', b')\subseteq(a, b)$. Thus $d\notin b$. On the other hand, $\mathcal{A}\models\phi(d)$ and $(a, b)\in P^n$, so $(a, b)\models\phi(d)$ by Lemma 3. Therefore

$$(a,\ b)\models b\neq\{x\mid\phi(x)\}.$$

By Lemma 3,

$$(a_{\alpha+n},\ b_{\alpha+n})\models b_{\alpha+n}\neq\{x\mid\phi(x)\},$$

$$[\mathcal{A},\ K]\models K\neq\{x\mid\phi(x)\}.$$

This is a contradiction.

COROLLARY. *Let $\mathcal{A}$ be a standard transitive model of ZF. If $V_\alpha\models ZF$, then $V_\alpha^{\mathcal{A}}$ has an undefinable class in $\mathcal{A}$.*

PROOF. $V_\alpha^{\mathcal{A}}$ is cf(α)-complete in $\mathcal{A}$.

We write $M\prec M'$ if M is an elementary submodel of M'.

LEMMA 4. *Let $\mathcal{A}$ be a standard transitive model of ZF and $cf(On^{\mathcal{A}})>\omega$. Then $\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\prec\mathcal{A}\}$ is closed unbounded in $On^{\mathcal{A}}$.*

PROOF. Let $P=\{V_\alpha^{\mathcal{A}}\mid\alpha\in On^{\mathcal{A}}\}$, then P is an inductive class of $\mathcal{A}$. Since P^n is cofinal with P, $\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\in P^n\}$ is unbounded in $On^{\mathcal{A}}$. $V_\alpha^{\mathcal{A}}\in P^n$ implies $V_\alpha^{\mathcal{A}}\prec_n\mathcal{A}$, thus $\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\prec_n\mathcal{A}\}$ is unbounded in $On^{\mathcal{A}}$. Closedness is obvious by definition. Since cf$(On^{\mathcal{A}})>\omega$ and $\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\prec\mathcal{A}\}=\bigcap_{n<\omega}\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\prec_n\mathcal{A}\}$, the result follows.

A set X is said to be ordinal definable if and only if it is definable by some formula $\phi(v_0, v_1, \cdots, v_m)$ of ZF with ordinal parameters, i.e. $X=\{x\mid\phi(x, \alpha_1, \cdots, \alpha_m)\}$ where $\alpha_1, \cdots, \alpha_m$ are ordinals. We denote by OD the class of all ordinal definable sets and by $<_{OD}$ the definable wellordering of OD.

Let $\mathcal{L}$ be a first order language and P a class of structures of $\mathcal{L}$. A class P is OD-inductive if and only if every element of P is ordinal definable i.e. $P\subseteq OD$, P itself is an ordinal definable class i.e. for some formula ϕ of ZF and some ordinals $\alpha_1, \alpha_2, \cdots, \alpha_m$, $P=\{x\mid\phi(x, \alpha_1, \alpha_2, \cdots, \alpha_m)\}$, and for every sequence $M_0\subseteq M_1\subseteq\cdots\subseteq M_\alpha\subseteq\cdots(\alpha<\lambda)$ of P such that $\langle M_\alpha\mid\alpha<\lambda\rangle\in OD$, $\bigcup_{\alpha<\lambda}M_\alpha$ is an element of P.

LEMMA 5. *Let P be an OD-inductive class. Then for each $n<\omega$, P^n is OD-inductive and cofinal with P.*

PROOF. The proof is by induction on n. First we show that P^n is cofinal with P. It suffices to prove the case $n=1$. Let $M\in P$, $\kappa=\max\{\text{card}(M), \text{card}(\mathcal{L}), \aleph_0\}$ and $\langle\phi_\alpha\mid\alpha<\kappa\rangle$ be an enumeration of all P-persistent sentences with constant in M. Obviously we can take $\langle\phi_\alpha\mid\alpha<\kappa\rangle\in OD$.

Let $M_0=M$. We define a sequence $\langle M_\alpha\mid\alpha<\kappa\rangle\in OD$ of members of P inductively as follows. For $\alpha=\beta+1$, if M_β satisfies ϕ_β or every extension $M'\in P$ of M_β does not satisfy ϕ_β, then we let $M_\alpha=M_\beta$, otherwise for some extension $M'\in P$ of M_β, M' satisfies ϕ_β, then we take M_α to be the $<_{OD}$-least of such M'. For limit ordinal α, let $M_\alpha=\bigcup_{\beta<\alpha} M_\beta$. $M_\alpha\in P$ because P is OD-inductive and $\langle M_\beta\mid\beta<\alpha\rangle\in OD$. (In fact it is definable from $M_0\in OD$ and α.) Now we let $M^1=\bigcup_{\alpha<\kappa} M_\alpha$. Obviously $M^1\in P$ by the same reason as above. By construction of M^1, any P-persistent sentence defined in M and true in some extension of M^1 is also true in M^1. Iterating this procedure we get a sequence $M\subseteqq M^1\subseteqq M^2\subseteqq\cdots$ such that $M^i\in P$ and every P-persistent sentence defined in M^i and true in some extension of M^{i+1} is also true in M^{i+1}. Let $M^\omega=\bigcup_{i<\omega} M^i$. Then $M^\omega\in P$ is P-persistently complete. Now we get $M^\omega\in P^1$ such that $M\subseteqq M^\omega$.

The proof that P^1 is OD-inductive is the same as Lemma 1, so we omit it (cf. [5], 79-81).

Let $\mathcal{B}=(B,\in)$ be a standard transitive model of ZF and $P[B]=\{(V_\alpha^{\mathcal{B}}, b)\mid b\subseteqq V_\alpha^{\mathcal{B}}, b\in OD^B\}$. Then $P[\mathcal{B}]$ is a class of structures of $L=(\in,\bar{K})$ and OD-inductive in $\mathcal{B}$. We call a chain $C\subseteqq P[\mathcal{B}]$ to be perfect if and only if

(1) for any M, $M'\in C$, $M\subseteqq M'$ or $M'\subseteqq M$,

(2) for each $n<\omega$ and any $M\in C$, there exists a $M'\in C\cap P^n[\mathcal{B}]$ such that $M\subseteqq M'$,

(3) $\bigcup\{V_\alpha^{\mathcal{B}}\mid(V_\alpha^{\mathcal{B}}, b)\in C\}=\mathcal{B}$.

By the proof of Proposition, the existence of a perfect chain of $P[\mathcal{B}]$ implies the existence of an undefinable class of $\mathcal{B}$ (In the proof of Proposition, we constructed a perfect chain by using cf$(On^{\mathcal{A}})$-completeness).

THEOREM. *If $\mathcal{A}$ is a standard model of ZF, then there exists an undefinable class of $\mathcal{A}$.*

PROOF. We may assume A is transitive. Since any standard transitive model of ZF is ω-complete, it suffices to prove the case cf$(On^{\mathcal{A}})=\mu>\omega$ by Proposition. By Lemma 4, there is a strictly increasing function F such that

$$F:\lambda\xrightarrow[\text{onto}]{1;1}\{\alpha\in On^{\mathcal{A}}\mid V_\alpha^{\mathcal{A}}\prec\mathcal{A}\}.$$

Remark that F is not a class of A but for any $\alpha<\lambda$, $F\restriction\alpha$ is a set of $\mathcal{A}$ i.e. $F\restriction\alpha\in A$. For $F\restriction\alpha$ is a strictly increasing function from $\alpha\in A$ one-to-one onto $\{\beta\in On^{\mathcal{A}}\mid V_\beta^{\mathcal{A}}\prec V_{F(\alpha)}^{\mathcal{A}}\}\in A$ which is definable in $\mathcal{A}$. It follows from the inspection of the proof of Proposition and Corollary that there exists a perfect chain C_0 of $V_{F(0)}^{\mathcal{A}}$ in A which is ordinal definable in $\mathcal{A}$. We take the $<_{OD}$-least of such C_0 in $\mathcal{A}$. Similarly, we can take the $<_{OD}$-least perfect chain $C_{\alpha+1}$ of $V_{F(\alpha+1)}^{\mathcal{A}}$ in $\mathcal{A}$ which extends the perfect chain C_α of $V_{F(\alpha)}^{\mathcal{A}}$, because C_α is in $\mathcal{A}$

i.e. $C_\alpha \in A$. If α is a limit ordinal, we take the union C_α of the perfect chains C_β of $V^{\mathcal{A}}_{F(\beta)}$ for $\beta<\alpha$. The set of the perfect chains C_β of $V^{\mathcal{A}}_{F(\beta)}$ for $\beta<\alpha$ is definable from $F\upharpoonright\alpha$ in $\mathcal{A}$, then the union C_α is in $\mathcal{A}$ and a perfect chain of $V^{\mathcal{A}}_{F(\alpha)}$ in A because $V^{\mathcal{A}}_{F(\beta)} \prec V^{\mathcal{A}}_{F(\alpha)}$ means $P^n[V^{\mathcal{A}}_{F(\beta)}] = V^{\mathcal{A}}_{F(\beta)} \cap P^n[V^{\mathcal{A}}_{F(\alpha)}]$. Therefore for any $\alpha<\lambda$, we can take the perfect chain C_α of $V^{\mathcal{A}}_{F(\alpha)}$ in $\mathcal{A}$ such that $C_\beta \subsetneqq C_\alpha$ for $\beta<\alpha$. Let $C=\bigcup_{\alpha<\lambda} C_\alpha$. By the same reason as above, C is a perfect chain of $\mathcal{A}$. Then the Theorem follows.

2. Incompatible classes.

Let $\mathcal{L}$ be the language of ZF and F be a new function letter. Let $\sigma(F)$ be the following sentence.

"F is a function from On to On, closed unbounded and strictly increasing,
$\wedge \forall\alpha\forall\beta(\alpha<\beta \rightarrow V_{F(\alpha)} \prec_{\mathcal{L}} V_{F(\beta)})$
$\wedge \forall\alpha(V_\alpha \prec_{\mathcal{L}} V_{F(0)} \rightarrow \alpha=F(0))$
$\wedge \forall\alpha\forall\beta(V_{F(\alpha)} \prec_{\mathcal{L}} V_\beta \prec_{\mathcal{L}} V_{F(\alpha+1)} \rightarrow \beta=F(\alpha)$ or $\beta=F(\alpha+1))$"

where $A \prec_{\mathcal{L}} B$ means that A is an elementary substructure of B in the language $\mathcal{L}$. Let $\Phi \equiv \sigma(F) \wedge \sigma(F') \rightarrow \forall\alpha(F(\alpha)=F'(\alpha))$. We prove that *if* $ZF+\sigma(F)$ *is consistent, then* $ZF(F, F') \vdash \Phi$ *but* $ZF(F)+ZF(F') \nvdash \Phi$.

Leaving aside the proof for the moment, we define some notations and prove lemmas in model theory. Let $\mathcal{L}$ be a first order language and $\sigma(\bar{K})$ be a set of sentences of $\mathcal{L}$ which contain the predicate letter, $\bar{K}$. $\bar{K}$ is said to be explicitly definable with respect to $\sigma(\bar{K})$ if and only if there is a formula ϕ not containing $\bar{K}$ such that $\sigma(\bar{K}) \vdash \forall\vec{v}(\bar{K}(\vec{v}) \leftrightarrow \phi(\vec{v}))$ where $\vec{v}$ is a finite sequence of variables. $\bar{K}$ is said to be implicitly definable with respect to $\sigma(\bar{K})$ if and only if $\sigma(\bar{K})+\sigma(\bar{K}') \vdash \forall\vec{v}(\bar{K}(\vec{v}) \leftrightarrow \bar{K}'(\vec{v}))$ where $\bar{K}'$ is a new predicate letter. $\bar{K}$ is said to be implicitly ZF-definable with respect to $\sigma(\bar{K})$ if and only if $ZF(\bar{K}, \bar{K}')+\sigma(\bar{K})+\sigma(\bar{K}') \vdash \forall\vec{v}(\bar{K}(\vec{v}) \leftrightarrow \bar{K}'(\vec{v}))$.

BETH'S THEOREM [refer to Bell-Slomson [1]]. *$\bar{K}$ is implicitly definable with respect to $\sigma(\bar{K})$ if and only if $\bar{K}$ is explicitly definable with respect to $\sigma(\bar{K})$.*

First we prove $ZF(F, F') \vdash \Phi$.

LEMMA 6. *F is implicitly ZF-definable with respect to $\sigma(F)$, in other words, $ZF(F, F')+\sigma(F)+\sigma(F') \vdash \forall\alpha(F(\alpha)=F'(\alpha))$ where F' is a new function letter.*

PROOF. The proof is by transfinite induction. Let $\alpha=0$. We may assume $F(0) \geqq F'(0)$. We define β_n and γ_n such that $\cdots \leqq F'(\gamma_n) \leqq F(\beta_n) \leqq F'(\gamma_{n+1}) \cdots$. Since F and F' are closed, $F(\beta_\omega)=\bigcup_{n<\omega} F(\beta_n)=\bigcup_{n<\omega} F'(\gamma_n)=F'(\gamma_\omega)$, where $\beta_\omega=\bigcup_{n<\omega}\beta_n$ and $\gamma_\omega=\bigcup_{n<\omega}\gamma_n$. $V_{F(0)} \prec_{\mathcal{L}} V_{F(\beta_\omega)} = V_{F'(\gamma_\omega)} \succ_{\mathcal{L}} V_{F'(0)}$, then $V_{F'(0)} \prec_{\mathcal{L}} V_{F(0)}$. By the definition of $\sigma(F)$, $F'(0)=F(0)$. Let $\alpha=\beta+1$. Both $F(\beta+1)$ and $F'(\beta+1)$ are defined as the least γ such that $V_{F(\beta)}=V_{F'(\beta)}\prec_{\mathcal{L}} V_\gamma$, then $F(\beta+1)=F'(\beta+1)$. For limit ordinal α, $\forall\beta<\alpha(F(\beta)=F'(\beta))\rightarrow F(\alpha)=F'(\alpha)$ because F and F' are closed.

By Lemma 6, $ZF(F, F')\vdash\Phi$. Next we show $ZF(F)+ZF(F')+\sigma(F)+\sigma(F')\nvdash\Phi$.

LEMMA 7. *F is not implicitly (explicitly) definable with respect to $ZF(F)+\sigma(F)$.*

PROOF. If not, there is a function G which is definable in the language $\mathcal{L}$ such that $ZF(F)+\sigma(F)\vdash\forall\alpha(F(\alpha)=G(\alpha))$. By the definition of $\sigma(F)$, if ϕ is a sentence of L, then $\phi\leftrightarrow V_{F(0)}\models\phi$. Let $\alpha_0=F(0)=G(0)$. Since $\exists\alpha(G(0)=\alpha)$ is a sentence of L, $V_{\alpha_0}\models\exists\alpha(G(0)=\alpha)$. Then for some $\alpha<\alpha_0$, $V_{\alpha_0}\models G(0)=\alpha$. By the definition of $\sigma(F)$, for some $\alpha<\alpha_0$, $G(0)=\alpha$, this is a contradiction.

By Lemma 7, $ZF(F)+ZF(F')+\sigma(F)+\sigma(F')\nvdash\forall\alpha(F(\alpha)=F'(\alpha))$. Therefore $ZF(F)+ZF(F')\nvdash\Phi$.

REMARK. If Cons(ZF+there is an inaccessible cardinal), then Cons($ZF+\sigma(F)$) where Cons$(\cdots)$ means $\cdots$ is consistent.

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