J. Math. Soc. Japan
Vol. 33, No. 4, 1981

Eigenvalues of the Laplacian on Calabi-Eckmann manifolds

By Kazumi TSUKADA

(Received March 24, 1980)

1. Introduction.

On a compact Hermitian manifold M, we define two differential operators, i.e., the real Laplacian $\triangle = d\delta + \delta d$, and the complex Laplacian $\square = \bar{\partial}\theta + \theta\bar{\partial}$. In this note we deal with these operators acting on differentiable functions. We denote by Spec$(M, \triangle)$ (resp. Spec$(M, \square)$) the set of eigenvalues with multiplicity of $\triangle$ (resp. $\square$). It is an interesting problem to investigate the relationship between the geometry of a smooth manifold and the spectrum of its Laplacian.

A Hopf manifold is well-known as the first example of a compact complex manifold which does not admit Kaehler metrics. E. Bedford and T. Suwa ([**1**]) described explicitly the eigenvalues of $\triangle$ and $\square$ on Hopf manifolds. Some isospectral results were also given by them. In this note we will describe (in Theorem 5.7) the eigenvalues of $\triangle$ and $\square$ on Calabi-Eckmann manifolds which were discovered as the second example of non-Kaehler complex manifolds ([**4**]). Some isospectral results will be given in Theorem 6.4 and Theorem 6.5.

The author wishes to thank Professor K. Ogiue for helpful suggestions.

2. The complex Laplacian on Hermitian manifolds.

In this section we will find the relation between the complex and the real Laplacians, and give formulas for the asymptotic expansion of the eigenvalues of the complex Laplacian, making use of Gilkey's Theorem.

Let (M, g, J) be a compact connected Hermitian manifold with Hermitian metric g and complex structure J. By $C^\infty(M)$ we denote the space of complex-valued differentiable functions on M with a scalar product $\langle\varphi, \psi\rangle = \int_M \varphi\bar{\psi}dV_g$, where dV_g is the Riemannian volume form on M. For the definition and the fundamental properties of the complex Laplacian on Hermitian manifolds we refer to Morrow and Kodaira [**9**]. Let Ω be the 2-form defined by $\Omega(X, Y) = g(JX, Y)$, which is called the *Kaehler form* of (M, g, J). Let ζ be the vector field dual to $-\delta\Omega$, where δ denotes the codifferential operator.

PROPOSITION 2.1. *Let $\square$ be the complex Laplacian acting on differentiable functions on M. Then we have*

$$2\square=\triangle-iL_{\zeta},$$

where L_{ζ} is the Lie derivation with respect to the vector field ζ.

PROOF. We get the above result after straightforward computations (cf. [9]).

DEFINITION. A Hermitian manifold (M, g, J) is called *semi-Kaehler* if it satisfies $\delta\Omega=0$, or equivalently $\zeta=0$ (cf. K. Yano [11], p. 192).

REMARK 2.2. It is well-known that $\square=1/2\triangle$ holds as the operator acting on forms of type (p, q) on Kaehler manifolds (cf. [9]). Proposition 2.1 shows that $\square=1/2\triangle$ holds as the operator acting on differentiable functions on semi-Kaehler manifolds. The converse will be seen later in this section (cf. Corollary 2.8).

If (M, g, J) is a Hermitian manifold, then so is $(M, g, -J)$. Let $\bar{\square}$ be the complex Laplacian of $(M, g, -J)$. Then the following is easily seen from Proposition 2.1.

PROPOSITION 2.3. $\mathrm{Spec}(M, \square)=\mathrm{Spec}(M, \bar{\square})$.

Here we review the Gilkey's results ([6], [7]). Let M be an m-dimensional compact connected Riemannian manifold with Riemannian metric g and volume element dV_g. Let V be a smooth r-dimensional vector bundle over M and $D: C^{\infty}(V)\to C^{\infty}(V)$ a second order differential operator with leading symbol given by the metric tensor. Let $\tilde{\nabla}$ be any connection on the vector bundle V. We denote by $D_{\tilde{\nabla}}$ the Laplacian on V defined by $\tilde{\nabla}$ and g.

THEOREM 2.4. (Gilkey [6], [7]) *Given a second order differential operator $D: C^{\infty}(V)\to C^{\infty}(V)$ with leading symbol given by the metric tensor, there is a unique connection $\tilde{\nabla}$ on V such that $D_{\tilde{\nabla}}-D$ is 0-th order operator, i.e., an endomorphism of V.*

We assume that V has a smooth inner product and D is a self-adjoint operator. For D we consider only such a connection $\tilde{\nabla}$ on V as in Theorem 2.4. Let $\{\lambda_1, \lambda_2, \cdots\}$ be the spectrum of D and let

$$\sum_{j=1}^{\infty}\exp(-t\lambda_j)\underset{t\to+0}{\sim}(4\pi t)^{-m/2}(a_0(D)+a_1(D)t+a_2(D)t^2+\cdots)$$

be the asymptotic expansion. Then we have

THEOREM 2.5. (Gilkey [6], [7])

(1) $a_0(D)=r\,\mathrm{Vol}(M, g)$

(2) $a_1(D)=r/6\int_M \tau dV_g+\int_M \mathrm{Tr}(E)dV_g$

(3) $a_2(D)=r/360\int_M \{5\tau^2-2|\rho|^2+2|R|^2\}\,dV_g$

$$+1/360\int_M \{30\mathrm{Tr}(G(W^2))+\mathrm{Tr}(60\tau E+180E^2)\}\,dV_g\,,$$

where $E=D_{\tilde\nabla}-D$; R, ρ, *and* τ *denote the curvature tensor, the Ricci tensor, and the scalar curvature of* (M, g) *respectively;* W *is the curvature tensor of* $\tilde\nabla$ *and* $G(W^2)$ *is the endomorphism of* V *defined by* $G(W^2)=g^{jh}g^{kl}W_{jk}W_{hl}$.

Now we apply the Gilkey's results to our case and give formulas for the asymptotic expansion of the complex Laplacian. In our case we take $V=M\times \boldsymbol{C}$, the trivial complex line bundle over the Hermitian manifold M. We may naturally consider a complex-valued function φ on M as a cross-section of V and the complex Laplacian $\Box$ as a differential operator acting on cross-sections of V. Given a complex-valued 1-form ω on M, we can define a connection $\tilde\nabla$ on V by $\tilde\nabla_X\varphi=X\varphi+\omega(X)\varphi$ for any vector field X on M.

LEMMA 2.6. *Let* $\omega=-\frac{1}{2}i\delta\Omega$, $\tilde\nabla$ *the connection determined by* ω *and* $D_{\tilde\nabla}$ *the Laplacian defined by* $\tilde\nabla$ *and the Hermitian metric* g. *Then* $D_{\tilde\nabla}-2\Box$ *is a* 0-*th order operator and* $D_{\tilde\nabla}-2\Box=\frac{1}{4}|\delta\Omega|^2 I$, *where* I *denotes the identity transformation.*

PROOF. Applying Proposition 2.1 we have

$$\begin{aligned}-2\Box\varphi&=g^{jh}\nabla_j\nabla_h\varphi+i\zeta^k\partial_k\varphi\\&=g^{jh}(\partial_j\partial_h\varphi-\Gamma^k_{jh}\partial_k\varphi)+i\zeta^k\partial_k\varphi\,,\end{aligned}$$

where ∇ is the Riemannian connection and Γ^k_{jh} is the Christoffel's symbol with respect to the Hermitian metric g. Since $D_{\tilde\nabla}$ is the Laplacian defined by $\tilde\nabla$ and g, we have

$$\begin{aligned}-D_{\tilde\nabla}\varphi&=g^{jh}\{\tilde\nabla_j(\tilde\nabla_h\varphi)-\tilde\nabla_{\nabla_{\partial_j}\partial_h}\varphi\}\\&=g^{jh}\{\tilde\nabla_j(\partial_h\varphi+\varphi\omega_h)-\Gamma^k_{jh}\tilde\nabla_k\varphi\}\\&=g^{jh}\{\partial_j\partial_h\varphi+\partial_h\varphi\omega_j+\partial_j\varphi\omega_h+\varphi\partial_j\omega_h+\varphi\omega_h\omega_j-\Gamma^k_{jh}\partial_k\varphi-\Gamma^k_{jh}\varphi\omega_k\}\\&=g^{jh}(\partial_j\partial_h\varphi-\Gamma^k_{jh}\partial_k\varphi)+2g^{jh}\omega_j\partial_h\varphi+\varphi g^{jh}(\partial_j\omega_h-\Gamma^k_{jh}\omega_k)+\varphi g^{jh}\omega_j\omega_h\,.\end{aligned}$$

Therefore we get

$$(D_{\tilde\nabla}-2\Box)\varphi=(i\zeta^k-2g^{jk}\omega_j)\partial_k\varphi+\varphi\delta\omega-\varphi g^{jh}\omega_j\omega_h\,.$$

Since $\omega=-\frac{1}{2}i\delta\Omega$, we see that $\delta\omega=-\frac{1}{2}i\delta^2\Omega=0$ and $\omega_j=-\frac{1}{2}i(\delta\Omega)_j=\frac{1}{2}ig_{jk}\zeta^k$. Thus we have $(D_{\tilde\nabla}-2\Box)\varphi=-\varphi g^{jh}\omega_j\omega_h$ and hence $D_{\tilde\nabla}-2\Box=1/4|\delta\Omega|^2 I$.

THEOREM 2.7. *Let* $\{\lambda_1, \lambda_2, \cdots\}$ *be the spectrum of* $2\Box$ *on the Hermitian manifold* M *and*

$$\sum_{j=1}^{\infty}\exp(-t\lambda_j)\underset{t\to+0}{\sim}(4\pi t)^{-m/2}(a_0+a_1t+a_2t^2+\cdots)$$

be the asymptotic expansion. Then we have

(1) $a_0=\mathrm{Vol}(M, g)$

(2) $a_1=1/6\int_M \tau dV_g+1/4\int_M |\delta\Omega|^2 dV_g$

(3) $a_2=1/360\int_M \{5\tau^2-2|\rho|^2+2|R|^2\}dV_g$

$$+1/360\int_M \{-30|d\delta\Omega|^2+60/4\tau|\delta\Omega|^2+180/16|\delta\Omega|^4\}dV_g .$$

PROOF. By Theorem 2.4 and Lemma 2.6, we can apply Theorem 2.5 to the case where $D=2\Box$ and $\tilde{\nabla}$ is the connection defined by $\omega=-\frac{1}{2}i\delta\Omega$. It is clear that the curvature W of $\tilde{\nabla}$ is given by $W=-id\delta\Omega$. Q. E. D.

By Proposition 2.1, Theorem 2.7, and the asymptotic expansion for the real Laplacian, the following is easily seen.

COROLLARY 2.8. *Let M be a compact connected Hermitian manifold. Then* $\mathrm{Spec}(M, 2\Box)=\mathrm{Spec}(M, \triangle)$ *if and only if M is a semi-Kaehler manifold.*

REMARK 2.9. Gilkey has proved that the spectra of $\Box^{0,0}$, $\Box^{0,1}$, and $\Box^{1,0}$ ($\Box^{p,q}$ means the complex Laplacian acting on forms of type (p, q)) of a Hermitian manifold determine whether the metric is Kaehler ([5]). Corollary 2.8 shows that the spectra of $\Box$ and $\triangle$ of a Hermitian manifold determine whether the metric is semi-Kaehler.

3. Complex structures and Hermitian metrics of Calabi-Eckmann manifolds.

In this section, applying A. Morimoto's method ([8]), we introduce complex structures on the product of two odd-dimensional spheres and define Hermitian metrics compatible with the complex structures.

3.1. An almost contact structure on an odd-dimensional sphere.

DEFINITION. (cf. Blair [3]) Let M be a differentiable manifold of dimension $2n+1$. An *almost contact structure* on M is a triple $\Sigma=(\phi, \xi, \eta)$, where ϕ is a tensor field of type (1, 1) on M, ξ is a vector field on M, and η is a 1-form on M satisfying the following conditions:

(3.1) $$\eta(\xi)=1$$

(3.2) $$\phi^2X=-X+\eta(X)\xi \text{ for any tangent vector } X \text{ on } M.$$

Let $\boldsymbol{C}^{n+1}$ be a complex Euclidean $(n+1)$-space with the standard complex structure $\tilde{J}$. Let (S^{2n+1}, g) be a unit sphere in C^{n+1}, i.e., $S^{2n+1}=\{z\in C^{n+1}; \|z\|=1\}$ with the induced metric g. We define $\Sigma=(\phi, \xi, \eta)$ on S^{2n+1} by $\tilde{J}z=\iota_*\xi$ and $\tilde{J}(\iota_*X)=\iota_*\phi X-\eta(X)z$ for $z\in S^{2n+1}$ and $X\in T_z(S^{2n+1})$, where $\iota: S^{2n+1}\to C^{n+1}$ is the inclusion map. Then $\Sigma=(\phi, \xi, \eta)$ on S^{2n+1} is an almost contact structure. Moreover the almost contact structure Σ satisfies the following equations ([8]):

$$\begin{aligned}\Psi(X, Y)&=\phi([X, Y])-[\phi X, Y]-[X, \phi Y]-\phi[\phi X, \phi Y]\\&\quad+\{(\phi X)(\eta(Y))-(\phi Y)(\eta(X))\}\xi\\&=0. \end{aligned} \tag{3.3}$$

$$L_\xi\eta=0. \tag{3.4}$$

The induced metric g on S^{2n+1} is compatible with the almost contact structure Σ in the following sense:

(3.5) $g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y)$ for any tangent vectors X, Y of S^{2n+1}.

By (3.5) we immediately get the following:

(3.6) $\eta(X)=g(\xi, X)$ for any tangent vector X of S^{2n+1}

$$g(\xi, \xi)=1. \tag{3.7}$$

Moreover (ϕ, ξ, η, g) is *Sasakian*, namely, the following differential equations are satisfied (cf. [3]):

$$(\nabla_X\phi)Y=\eta(Y)X-g(X, Y)\xi \tag{3.8}$$

$$d\eta(X, Y)=g(\phi X, Y) \tag{3.9}$$

(3.10) $\nabla_X\xi=\phi X$ for any vector fields X, Y on S^{2n+1}.

3.2 Complex structures on the product of two odd-dimensional spheres.

Morimoto ([8]) introduced an almost complex structure J on the product of two almost contact manifolds M and M' with almost contact structures $\Sigma=(\phi, \xi, \eta)$ and $\Sigma'=(\phi', \xi', \eta')$ respectively by

$$J(X+X')=\phi X-\eta'(X')\xi+\phi'(X')+\eta(X)\xi'$$

for any tangent vector X of M and any X' of M'. He has proved that the induced almost complex structure J on $M\times M'$ is integrable if and only if $\Psi=0$ and $\Psi'=0$, where Ψ and Ψ' are the tensor fields defined by (3.3).

Now we shall introduce almost complex structures on the product of two odd-dimensional spheres $S^{2p+1}\times S^{2q+1}$. Let $\Sigma=(\phi, \xi, \eta)$ (resp. $\Sigma'=(\phi', \xi', \eta')$) be the almost contact structure on S^{2p+1} (resp. S^{2q+1}) given in 3.1. Then we define a family of almost complex structures $J_{a,b}$ (or simply J) on $S^{2p+1}\times S^{2q+1}$ as follows. For any tangent vector X of S^{2p+1} and any X' of S^{2q+1},

(3.11) $$J(X+X')=\phi X-\{a/b\eta(X)+(a^2+b^2)/b\eta'(X')\}\xi +\phi'(X')+\{1/b\eta(X)+a/b\eta'(X')\}\xi',$$

where $a, b\in \boldsymbol{R}$ and $b\neq 0$.
It is easily seen that $J^2=-I$, which shows that J is an almost complex structure on $S^{2p+1}\times S^{2q+1}$.

Noticing that $\Sigma=(\phi, \xi, \eta)$ and $\Sigma'=(\phi', \xi', \eta')$ satisfy $\Psi=0$ and $\Psi'=0$ respectively, we can prove the following by the same way as Morimoto [8].

PROPOSITION 3.1. *The almost complex structure J defined by* (3.11) *is integrable.*

We denote by $M_{a,b}^{p;q}$ the complex manifold $S^{2p+1}\times S^{2q+1}$ with the complex structure $J_{a,b}$ defined by (3.11).

3.3. A Hermitian metric on $M_{a,b}^{p;q}$.

We define a Riemannian metric $g_{a,b}$ (or simply g) on $M_{a,b}^{p;q}$ by

(3.12) $$g=g_1+a(\eta\otimes\eta'+\eta'\otimes\eta)+[a^2+b^2-1]\eta'\otimes\eta'+g_2,$$

where g_1 and g_2 are the canonical Riemannian metrics on S^{2p+1} and S^{2q+1} respectively.

The following is easily checked.

PROPOSITION 3.2. *The Riemannian metric g given in* (3.12) *is a Hermitian metric on the complex manifold $M_{a,b}^{p;q}$.*

From now on $M_{a,b}^{p;q}$ denotes the Hermitian manifold with the metric g given in (3.12).

We shall give several remarks on the Hermitian metric of $M_{a,b}^{p;q}$:

(1) In the case of $a=0$ and $b=1$, the Riemannian metric $g_{0,1}$ gives the Riemannian product of (S^{2p+1}, g_1) and (S^{2q+1}, g_2). For the sake of simplicity we denote $g_{0,1}$ by $\mathring{g}$.

(2) In the case of $q=0$, the Hermitian structure of $M_{a,b}^{p;q}$ coincides with that of Hopf manifold M_α in [1], where $\alpha=e^{-2\pi(b-ia)}$.

(3) $M_{-a,b}^{0;q}$ is biholomorphically isometric to the Hermitian manifold $(M_{a,b}^{0;q}, -J_{a,b})$.

Proposition 2.3 and this remark imply that $\mathrm{Spec}(M_{-a,b}^{0;q}, \square)=\mathrm{Spec}(M_{a,b}^{0;q}, \square)$.

(4) In the case of $q\neq 0$, $M_{a,-b}^{p;q}$ is biholomorphically isometric to the Hermitian manifold $(M_{a,b}^{p;q}, -J_{a,b})$.

By Proposition 2.3 and this remark, we have $\mathrm{Spec}(M_{a,-b}^{p;q}, \square)=\mathrm{Spec}(M_{a,b}^{p;q}, \square)$. So from now on *we assume that b is positive.*

(5) $M_{-a,b}^{p;q}$ is isometric to $M_{a,b}^{p;q}$ as a Riemannian manifold, but not biholomorphic in general.

To show (3), (4), and (5) we prepare the following lemma.

LEMMA. *The diffeomorphism φ_p of S^{2p+1} defined by $\varphi_p(z)=(\bar{z}_0, \bar{z}_1, \cdots, \bar{z}_p)$ for $z=(z_0, z_1, \cdots, z_p)\in S^{2p+1}\subset C^{p+1}$ has the following properties:*

$$\varphi_p^* g_1 = g_1\,, \quad (\varphi_p)_*\xi = -\xi\,, \quad \varphi_p^*\eta = -\eta\,, \quad \phi\circ(\varphi_p)_* = -(\varphi_p)_*\circ\phi\,.$$

Using the above Lemma, we can prove that (id, φ_q): $S^1\times S^{2q+1}\to S^1\times S^{2q+1}$ is a biholomorphic isometry from $M^{0,q}_{-a,b}$ to $(M^{0,q}_{a,b}, -J_{a,b})$, (φ_p, φ_q): $S^{2p+1}\times S^{2q+1}\to S^{2p+1}\times S^{2q+1}$ is a biholomorphic isometry from $M^{p,q}_{a,-b}$ to $(M^{p,q}_{a,b}, -J_{a,b})$, and (φ_p, id): $S^{2p+1}\times S^{2q+1}\to S^{2p+1}\times S^{2q+1}$ is an isometry from $M^{p,q}_{-a,b}$ to $M^{p,q}_{a,b}$.

4. Some formulas for a Riemannian submersion.

In [10] O'Neill studied fundamental equations of a Riemannian submersion. We review some formulas which are useful in the sequel. Given a Riemannian submersion $\pi: M\to B$, we denote by $\mathcal{V}E$ (resp. $\mathcal{H}E$) a vertical part (resp. a horizontal part) of a vector field E on M. Following O'Neill, we define two tensor fields T and A by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \qquad \text{and}$$

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

respectively, where ∇ denotes the Riemannian connection on M. The tensor field A is called an *integrability tensor* associated with the submersion.

DEFINITION. A *basic vector field* is a horizontal vector field X^* which is π-related to a vector field X on B, i.e., $\pi X^*_u = X_{\pi(u)}$ for all $u\in M$. We sometimes call X^* the *horizontal lift* of X.

LEMMA 4.1. *Suppose X^* and Y^* are the horizontal lifts of X and Y. Then*

(1) *$\mathcal{H}([X^*, Y^*])$ is the horizontal lift of $[X, Y]$.*

(2) *$\mathcal{H}\nabla_{X^*}Y^*$ is the horizontal lift of $\check{\nabla}_X Y$, where $\check{\nabla}$ is the Riemannian connection on B.*

LEMMA 4.2. (1) *At each point, A_E is a skew-symmetric linear operator on the tangent space on M.*

(2) *Let X and Y be horizontal vector fields on M. Then*

$$A_X Y = 1/2\mathcal{V}([X, Y])\,.$$

LEMMA 4.3. *Let X and Y be horizontal vector fields and V and W vertical vector fields. Moreover let $\hat{\nabla}$ be the Riemannian connection on the fibres. Then we have*

(1) $\nabla_V W = T_V W + \hat{\nabla}_V W$

(2) $\nabla_V X = \mathcal{H}(\nabla_V X) + T_V X$

(3) $\nabla_X V = A_X V + \mathcal{V}(\nabla_X V)$

(4) $\nabla_X Y = \mathcal{H}(\nabla_X Y) + A_X Y$.

Furthermore if X is basic, then $\mathcal{H}(\nabla_V X)=A_X V$.

Let R be the curvature tensor of M, $\check{R}$ the curvature tensor of B, and $\hat{R}$ the curvature tensor of the fibre. The horizontal lift $\check{R}^*$ of $\check{R}$ is defined by $\langle \check{R}^*_{h_1^* h_2^*} h_3^*, h_4^* \rangle = \langle \check{R}_{h_1 h_2} h_3, h_4 \rangle$, where h_j^* is the horizontal lift of h_j. We remark that the curvature tensor R is defined by

$$R_{XY}Z=\nabla_{[X,Y]}Z-\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z .$$

LEMMA 4.4. *If U, V, W, F, are vertical vector fields and X, Y, Z, H, are horizontal vector fields, then we have*

(0) $\langle R_{UV}W, F\rangle=\langle \hat{R}_{UV}W, F\rangle-\langle T_U W, T_V F\rangle+\langle T_U F, T_V W\rangle$

(1) $\langle R_{UV}W, X\rangle=\langle(\nabla_V T)_U W, X\rangle-\langle(\nabla_U T)_V W, X\rangle$

(2) $\langle R_{XV}Y, W\rangle=\langle(\nabla_X T)_V W, Y\rangle+\langle(\nabla_V A)_X Y, W\rangle-\langle T_V X, T_W Y\rangle+\langle A_X V, A_Y W\rangle$

(2′) $\langle R_{VW}X, Y\rangle=\langle(\nabla_V A)_X Y, W\rangle-\langle(\nabla_W A)_X Y, V\rangle+\langle A_X V, A_Y W\rangle$
$-\langle A_X W, A_Y V\rangle-\langle T_V X, T_W Y\rangle+\langle T_W X, T_V Y\rangle$

(3) $\langle R_{XY}Z, V\rangle=\langle(\nabla_Z A)_X Y, V\rangle+\langle A_X Y, T_V Z\rangle-\langle A_Y Z, T_V X\rangle-\langle A_Z X, T_V Y\rangle$

(4) $\langle R_{XY}Z, H\rangle=\langle \check{R}^*_{XY}Z, H\rangle-2\langle A_X Y, A_Z H\rangle+\langle A_Y Z, A_X H\rangle+\langle A_Z X, A_Y H\rangle$.

The proof of these results is found in [**10**].

5. The real Laplacian and the complex Laplacian of $M_{a;b}^{p;q}$ and their eigenvalues.

It is well-known that the fibration $\pi:(S^{2p+1}, g_1)\to \boldsymbol{C}P^p$ is a Riemannian submersion with totally geodesic fibres S^1, where $\boldsymbol{C}P^p$ is furnished with the Fubini-Study metric. Moreover ξ defined in 3.1 is a vertical vector field with respect to the submersion. For any tangent vector X of S^{2p+1}, $\eta(X)\xi$ is a vertical part of X.

LEMMA 5.1. (1) *Let X be a vector field on $\boldsymbol{C}P^p$ and X^* its horizontal lift. Then we have*

$$L_\xi X^*=[\xi, X^*]=0 .$$

(2) *Let A be the integrability tensor associated with the Riemannian submersion π. If X and Y are horizontal vectors on S^{2p+1}, then we have $A_X Y=\langle X, \phi Y\rangle\xi$.*

PROOF. (1) Since ξ is a vector field which is π-related to 0 on $\boldsymbol{C}P^p$, we get

$$\pi_*([\xi, X^*])=[\pi_*\xi, \pi_* X^*]=[0, X]=0 .$$

On the other hand by (3.4) we have

$$\eta([\xi, X^*])=\xi\eta(X^*)-(L_\xi\eta)(X^*)=0 .$$

Thus (1) is proved.

(2) The proof is found in [10].

Let π_1, and π_2 be the submersions $\pi_1: S^{2p+1} \to \boldsymbol{C}P^p$ and $\pi_2: S^{2q+1} \to \boldsymbol{C}P^q$ respectively. We put $\pi = (\pi_1, \pi_2): S^{2p+1} \times S^{2q+1} \to \boldsymbol{C}P^p \times \boldsymbol{C}P^q$. It is easily seen that $\pi: (S^{2p+1} \times S^{2q+1}, \mathring{g}) \to (\boldsymbol{C}P^p \times \boldsymbol{C}P^q, h)$ is a Riemannian submersion with totally geodesic fibres $T^2 = S^1 \times S^1$ and ξ and ξ' span the vertical subspaces with respect to the submersion π, where $\mathring{g}$ (resp. h) denotes the product metric on $S^{2p+1} \times S^{2q+1}$ (resp. $\boldsymbol{C}P^p \times \boldsymbol{C}P^q$). We consider the Laplacian Δ_{g_1} on (S^{2p+1}, g_1) as a differential operator acting on differentiable functions on $S^{2p+1} \times S^{2q+1}$ in the following way; For $\varphi \in C^\infty(S^{2p+1} \times S^{2q+1})$, $\Delta_{g_1}\varphi(x, x') = \Delta_{g_1}\iota_{x'}^*\varphi(x)$ at (x, x'), where $\iota_{x'}$ denotes the natural imbedding $\iota_{x'}: S^{2p+1} \to S^{2p+1} \times S^{2q+1}$ given by $\iota_{x'}(x) = (x, x')$. The Laplacian Δ_{g_2} on (S^{2q+1}, g_2) is considered similarly. For an arbitrary point $x \in \boldsymbol{C}P^p$, let U_1 be a neighborhood of x in $\boldsymbol{C}P^p$ and $\{e_1, e_2, \cdots, e_{2p}\}$ be a local field of orthonormal frames on U_1. We denote by e_j^* the horizontal lift of e_j on $\pi_1^{-1}(U_1)$ with respect to the submersion π_1. Similarly, for an arbitrary point $x' \in \boldsymbol{C}P^q$, we choose U_2 and a local field of orthonormal frames $\{e_1', \cdots, e_{2q}'\}$ on U_2. With respect to the Riemannian submersion π_2, $e_k'^*$ denotes the horizontal lift of e_k' on $\pi_2^{-1}(U_2)$. We put $U = U_1 \times U_2$. Then $\{e_1, \cdots, e_{2p}, e_1', \cdots, e_{2q}'\}$ is naturally considered as a local field of orthonormal frames on U, and $\{e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ is considered as a local field of orthonormal vectors on $\pi^{-1}(U) = \pi_1^{-1}(U_1) \times \pi_2^{-1}(U_2)$.

The following Lemma is easily seen.

LEMMA 5.2. (1) *e_j^* and $e_k'^*$ $(1 \leqq j \leqq 2p,\ 1 \leqq k \leqq 2q)$ on $\pi^{-1}(U)$ are horizontal lifts of e_j and e_k' with respect to the Riemannian submersion π.*

(2) *For $f \in C^\infty(S^{2p+1} \times S^{2q+1})$, Δ_{g_1} and Δ_{g_2} are expressed as follows;*

$$-\Delta_{g_1} f = (\mathrm{Hess}\, f)(\xi, \xi) + \sum_{j=1}^{2p} (\mathrm{Hess}\, f)(e_j^*, e_j^*)$$

$$-\Delta_{g_2} f = (\mathrm{Hess}\, f)(\xi', \xi') + \sum_{k=1}^{2q} (\mathrm{Hess}\, f)(e_k'^*, e_k'^*),$$

where Hess f *denotes the Hessian of f with respect to the metric $\mathring{g}$.*

Let ∇ and $\mathring{\nabla}$ be the Riemannian connections with respect to the metric tensors g and $\mathring{g}$ on $S^{2p+1} \times S^{2q+1}$ respectively. Moreover let $\check{\nabla}$ be the Riemannian connection with respect to h on $\boldsymbol{C}P^p \times \boldsymbol{C}P^q$.

LEMMA 5.3. *The Riemannian metric g has the following properties.*

(1) *The vector field ξ (resp. ξ') on $S^{2p+1} \times S^{2q+1}$ is a Killing vector field with constant length* 1 *(resp. $(a^2+b^2)^{1/2}$) with respect to the metric g.*

(2) $\nabla_\xi \xi = \nabla_{\xi'} \xi' = \nabla_\xi \xi' = \nabla_{\xi'} \xi = 0$.

(3) *$\pi: (S^{2p+1} \times S^{2q+1}, g) \to (\boldsymbol{C}P^p \times \boldsymbol{C}P^q, h)$ is a Riemannian submersion with totally geodesic fibres.*

(4) *Horizontal distributions associated with the Riemannian submersion*

$\pi:(S^{2p+1}\times S^{2q+1}, g)\rightarrow(\boldsymbol{C}P^{p}\times\boldsymbol{C}P^{q}, h)$ and the submersion $\pi:(S^{2p+1}\times S^{2q+1}, \mathring{g})\rightarrow(\boldsymbol{C}P^{p}\times\boldsymbol{C}P^{q}, h)$ are identical.

(5) *Let e_{j}^{*} and $e_{k}^{\prime*}$ be vector fields on $\pi^{-1}(U)$ defined in Lemma 5.2. Then we have*

$$\nabla_{e_{i}^{*}}e_{j}^{*}=\mathring{\nabla}_{e_{i}^{*}}e_{j}^{*} \quad \text{and} \quad \nabla_{e_{k}^{\prime*}}e_{l}^{\prime*}=\mathring{\nabla}_{e_{k}^{\prime*}}e_{l}^{\prime*}.$$

(6) *Volume elements with respect to g and $\mathring{g}$ have the following relations; $dV_{g}=b\,dV_{\mathring{g}}$ and $\mathrm{Vol}(S^{2p+1}\times S^{2q+1}, g)=b\,\mathrm{Vol}(S^{2p+1}\times S^{2q+1}, \mathring{g})$.*

PROOF. (1) Noticing that ξ is a Killing vector field of (S^{2p+1}, g_{1}) and $L_{\xi}\eta=0$, we see that ξ is a Killing vector field of $(S^{2p+1}\times S^{2q+1}, g)$. By the definition (3.12) we have $g(\xi, \xi)=g_{1}(\xi, \xi)=1$. Similarly for ξ'.

(2) Lemma 5.3 (1) implies that $\nabla_{\xi}\xi=0$ and $\nabla_{\xi'}\xi'=0$. Let $\{e_{1}^{*}, \cdots, e_{2p}^{*}, e_{1}^{\prime*}, \cdots, e_{2q}^{\prime*}\}$ be a local field on $\pi^{-1}(U)$ given in Lemma 5.2. Then $\{\xi, \xi', e_{1}^{*}, \cdots, e_{2p}^{*}, e_{1}^{\prime*}, \cdots, e_{2q}^{\prime*}\}$ is clearly a local field of linear frames. By Lemma 5.1 we have $L_{\xi}e_{j}^{*}=0$, $L_{\xi}e_{k}^{\prime*}=0$, $L_{\xi'}e_{j}^{*}=0$, and $L_{\xi'}e_{k}^{\prime*}=0$. Therefore we get

$$\begin{aligned}2g(\nabla_{\xi}\xi', e_{j}^{*})&=\xi g(\xi', e_{j}^{*})+\xi' g(\xi, e_{j}^{*})-e_{j}^{*}g(\xi, \xi')\\&\quad+g([\xi, \xi'], e_{j}^{*})+g([e_{j}^{*}, \xi], \xi')+g(\xi, [e_{j}^{*}, \xi'])\\&=0\end{aligned}$$

and

$$\begin{aligned}2g(\nabla_{\xi}\xi', e_{k}^{\prime*})&=\xi g(\xi', e_{k}^{\prime*})+\xi' g(\xi, e_{k}^{\prime*})-e_{k}^{\prime*}g(\xi, \xi')\\&\quad+g([\xi, \xi'], e_{k}^{\prime*})+g([e_{k}^{\prime*}, \xi], \xi')+g(\xi, [e_{k}^{\prime*}, \xi'])\\&=0.\end{aligned}$$

Moreover it is easily seen that $g(\nabla_{\xi}\xi', \xi)=0$ and $g(\nabla_{\xi}\xi', \xi')=0$. Hence we have $\nabla_{\xi}\xi'=0$, which, together with $[\xi, \xi']=0$, implies $\nabla_{\xi'}\xi=0$.

(3) and (4). Let X^{*} and Y^{*} be any two vector fields in $\{e_{1}^{*}, \cdots, e_{2p}^{*}, e_{1}^{\prime*}, \cdots, e_{2q}^{\prime*}\}$. Then the vector fields X^{*} and Y^{*} satisfy $\mathring{g}(X^{*}, \xi)=\mathring{g}(X^{*}, \xi')=\mathring{g}(Y^{*}, \xi)=\mathring{g}(Y^{*}, \xi')=0$, $\eta(X^{*})=\eta'(X^{*})=\eta(Y^{*})=\eta'(Y^{*})=0$ and $\mathring{g}(X^{*}, Y^{*})=h(X, Y)$, where $\pi X^{*}=X$ and $\pi Y^{*}=Y$. Similarly we have $g(X^{*}, \xi)=g(X^{*}, \xi')=g(Y^{*}, \xi)=g(Y^{*}, \xi')=0$ and $g(X^{*}, Y^{*})=\mathring{g}(X^{*}, Y^{*})=h(X, Y)$. These facts imply that $\pi:(S^{2p+1}\times S^{2q+1}, g)\rightarrow(\boldsymbol{C}P^{p}\times\boldsymbol{C}P^{q}, h)$ is a Riemannian submersion and that horizontal distributions associated with two Riemannian submersions are identical. Moreover Lemma 5.3 (2) shows that the fibres are totally geodesic submanifolds.

(5) It follows from (4) that e_{j}^{*} and $e_{k}^{\prime*}$ are basic vector fields with respect to the Riemannian submersion $\pi:(S^{2p+1}\times S^{2q+1}, g)\rightarrow(\boldsymbol{C}P^{p}\times\boldsymbol{C}P^{q}, h)$. By Lemma 4.1 (2) and Lemma 4.2 we have

$$\mathcal{H}(\nabla_{e_{i}^{*}}e_{j}^{*})=(\check{\nabla}_{e_{i}}e_{j})^{*}=\mathcal{H}(\mathring{\nabla}_{e_{i}^{*}}e_{j}^{*}) \quad \text{and}$$

$$\mathcal{V}(\nabla_{e_{i}^{*}}e_{j}^{*})=1/2\,\mathcal{V}([e_{i}^{*}, e_{j}^{*}])=\mathcal{V}(\mathring{\nabla}_{e_{i}^{*}}e_{j}^{*}).$$

Therefore we get $\nabla_{e_i^*} e_j^* = \mathring{\nabla}_{e_i^*} e_j^*$. Similarly we can prove $\nabla_{e_k'^*} e_l'^* = \mathring{\nabla}_{e_k'^*} e_l'^*$.

(6) For any point of $\pi^{-1}(U)$, $\{\xi, \xi', e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ is an orthonormal basis with respect to the metric $\mathring{g}$. The metric tensor g is represented with respect to this basis by

$$g = \begin{pmatrix} 1 & a & & & 0 & \\ a & a^2+b^2 & & & & \\ & & 1 & & & 0 \\ & 0 & & \ddots & & \\ & & 0 & & & 1 \end{pmatrix}.$$

Therefore (6) is proved. (We remark that b is assumed to be positive.)

Q. E. D.

We shall find the vector field ζ (defined in § 2) for the Hermitian manifold $M_{a,b}^{p,q}$ in the following lemma.

LEMMA 5.4. $\zeta = -2p\xi - 2q\xi'$.

PROOF. We use the local field of frames $\{\xi, \xi', e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ on $\pi^{-1}(U)$. It is easily seen that $\{\xi, (\xi' - a\xi)/b, e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ is a local field of orthonormal frames with respect to g. Therefore we have

$$\zeta = (\nabla_\xi J)\xi + (\nabla_{(\xi' - a\xi)/b} J)((\xi' - a\xi)/b) + \sum_{j=1}^{2p} (\nabla_{e_j^*} J) e_j^* + \sum_{k=1}^{2q} (\nabla_{e_k'^*} J) e_k'^* .$$

From the equation (3.8), Lemma 5.3 (5), and $\mathcal{V}(\nabla_{e_j^*} e_j^*) = 0$, we get

$$\begin{aligned}(\nabla_{e_j^*} J) e_j^* &= \nabla_{e_j^*}(J(e_j^*)) - J(\nabla_{e_j^*} e_j^*) \\ &= \mathring{\nabla}_{e_j^*}(\phi(e_j^*)) - \phi(\mathring{\nabla}_{e_j^*} e_j^*) \\ &= {}'\nabla_{e_j^*}(\phi(e_j^*)) - \phi({}'\nabla_{e_j^*} e_j^*) \\ &= ({}'\nabla_{e_j^*} \phi) e_j^* \\ &= \eta(e_j^*) e_j^* - g_1(e_j^*, e_j^*)\xi \\ &= -\xi ,\end{aligned}$$

where $'\nabla$ denotes the Riemannian connection of (S^{2p+1}, g_1). Similarly we see that $(\nabla_{e_k'^*} J) e_k'^* = -\xi'$.

By Lemma 5.3 (2) we have

$$\begin{aligned}(\nabla_\xi J)\xi &= \nabla_\xi(J\xi) - J(\nabla_\xi \xi) \\ &= \nabla_\xi(-a/b\xi + 1/b\xi')\end{aligned}$$

$$=-a/b\nabla_{\xi}\xi+1/b\nabla_{\xi}\xi'$$
$$=0.$$

Similarly we see that $(\nabla_{\xi}J)\xi'=(\nabla_{\xi'}J)\xi=(\nabla_{\xi'}J)\xi'=0$. Thus Lemma 5.4 is proved.

We are now in a position to calculate the real Laplacian $\triangle_{g}$ and the complex Laplacian $\square_{g}$ on $M_{a;b}^{p;q}$.

PROPOSITION 5.5. *For any complex-valued differentiable function f on $M_{a;b}^{p;q}$, we have*

$$\triangle_{g}f=\triangle_{g_1}f+\triangle_{g_2}f-a^2/b^2L_{\xi}L_{\xi}f+(b^2-1)/b^2L_{\xi'}L_{\xi'}f+2a/b^2L_{\xi}L_{\xi'}f$$

and

$$2\square_{g}f=\triangle_{g_1}f+\triangle_{g_2}f-a^2/b^2L_{\xi}L_{\xi}f+(b^2-1)/b^2L_{\xi'}L_{\xi'}f$$
$$+2a/b^2L_{\xi}L_{\xi'}f+2piL_{\xi}f+2qiL_{\xi'}f.$$

PROOF. We calculate $\triangle_{g}$ on $\pi^{-1}(U)$ using the local field of orthonormal frames $\{\xi, (\xi'-a\xi)/b, e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ in the same way as in the proof of Lemma 5.4. We denote by $\mathrm{Hess}_g f$ and $\mathrm{Hess}\, f$ the Hessian of f with respect to g and $\mathring{g}$ respectively. Then we have

$$-\triangle_{g}f=(\mathrm{Hess}_g f)(\xi, \xi)+(\mathrm{Hess}_g f)((\xi'-a\xi)/b, (\xi'-a\xi)/b)$$
$$+\sum_{j=1}^{2p}(\mathrm{Hess}_g f)(e_j^*, e_j^*)+\sum_{k=1}^{2q}(\mathrm{Hess}_g f)(e_k'^*, e_k'^*)$$
$$=(\mathrm{Hess}_g f)(\xi, \xi)+1/b^2(\mathrm{Hess}_g f)(\xi', \xi')$$
$$-2a/b^2(\mathrm{Hess}_g f)(\xi, \xi')+a^2/b^2(\mathrm{Hess}_g f)(\xi, \xi)$$
$$+\sum_{j=1}^{2p}(\mathrm{Hess}_g f)(e_j^*, e_j^*)+\sum_{k=1}^{2q}(\mathrm{Hess}_g f)(e_k'^*, e_k'^*).$$

Lemma 5.3 (2) implies that

$$(\mathrm{Hess}_g f)(\xi, \xi)=L_{\xi}L_{\xi}f-(\nabla_{\xi}\xi)f=L_{\xi}L_{\xi}f,$$
$$(\mathrm{Hess}_g f)(\xi', \xi')=L_{\xi'}L_{\xi'}f-(\nabla_{\xi'}\xi')f=L_{\xi'}L_{\xi'}f, \quad \text{and}$$
$$(\mathrm{Hess}_g f)(\xi, \xi')=L_{\xi}L_{\xi'}f-(\nabla_{\xi}\xi')f=L_{\xi}L_{\xi'}f.$$

By Lemma 5.3 (5) we have

$$(\mathrm{Hess}_g f)(e_j^*, e_j^*)=L_{e_j^*}L_{e_j^*}f-(\nabla_{e_j^*}e_j^*)f$$
$$=L_{e_j^*}L_{e_j^*}f-(\mathring{\nabla}_{e_j^*}e_j^*)f$$
$$=(\mathrm{Hess}\, f)(e_j^*, e_j^*).$$

Similarly we have $(\mathrm{Hess}_g f)(e_k'^*, e_k'^*)=(\mathrm{Hess}\, f)(e_k'^*, e_k'^*)$. By Lemma 5.2 (2) we see that

$$-\triangle_g f = L_\xi L_\xi f + \sum_{j=1}^{2p} (\mathrm{Hess}\, f)(e_j^*, e_j^*) + L_{\xi'} L_{\xi'} f$$

$$+ \sum_{k=1}^{2q} (\mathrm{Hess}\, f)(e_k'^*, e_k'^*) + a^2/b^2 L_\xi L_\xi f + (1-b^2)/b^2 L_{\xi'} L_{\xi'} f$$

$$-2a/b^2 L_\xi L_{\xi'} f$$

$$= -\triangle_{g_1} f - \triangle_{g_2} f + a^2/b^2 L_\xi L_\xi f + (1-b^2)/b^2 L_{\xi'} L_{\xi'} f$$

$$-2a/b^2 L_\xi L_{\xi'} f.$$

By Proposition 2.1 and Lemma 5.4 we see that $2\square_g$ is expressed in the form of this proposition. Q.E.D.

We shall describe the eigenvalues of the real Laplacian and the complex Laplacian on $M_{a,b}^{p,q}$. Let $\triangle_0 = -4 \sum_{j=0}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ be the standard Laplacian on $\boldsymbol{C}^{n+1}$. Let $\mathscr{H}_{k,l}$ be the space of harmonic polynomials of type (k, l), i.e., the polynomial f on $\boldsymbol{C}^{n+1}$ such that $\triangle_0 f = 0$ and $f(z) = \sum_{|\mu|=k, |\nu|=l} C_{\mu\nu} z^\mu \bar{z}^\nu$, where $\mu = (\mu_0, \mu_1, \cdots, \mu_n)$ and $\nu = (\nu_0, \nu_1, \cdots, \nu_n)$ are $(n+1)$-tuples of non-negative integers, $|\mu| = \sum_{j=0}^{n} \mu_j$, $|\nu| = \sum_{j=0}^{n} \nu_j$, and $z^\mu = z_0^{\mu_0} z_1^{\mu_1} \cdots z_n^{\mu_n}$, $\bar{z}^\nu = \bar{z}_0^{\nu_0} \bar{z}_1^{\nu_1} \cdots \bar{z}_n^{\nu_n}$.

LEMMA 5.6. *Let S^{2n+1} be the unit $(2n+1)$-sphere in $\boldsymbol{C}^{n+1}$ and ξ be the vector field on S^{2n+1} given in* 3.1. *Moreover let $\triangle_g$ be the Laplacian on S^{2n+1} with respect to the metric g induced from the standard Euclidean metric on $\boldsymbol{C}^{n+1}$. Then*

$$\triangle_g f = (k+l)(k+l+2n) f \quad \text{and}$$

$$L_\xi f = i(k-l) f,$$

for $f \in \mathscr{H}_{k,l}$ restricted to S^{2n+1}.

PROOF. The first equality is well-known ([2]). From the definition of ξ we see that the integral curve $\varphi(\theta)$ of ξ through $z \in S^{2n+1}$ is given by $\varphi(\theta) = e^{i\theta} z$. Therefore we have

$$L_\xi f(z) = \frac{d}{d\theta} f(e^{i\theta} z)|_{\theta=0}$$

$$= \frac{d}{d\theta} e^{i(k-l)\theta} f(z)|_{\theta=0}$$

$$= i(k-l) f(z).$$

THEOREM 5.7. (i) *The eigenvalues of $\triangle_g$ on $M_{a,b}^{p,q}$ are*

$$(k+l)(k+l+2p) + (s+t)(s+t+2q) + a^2(k-l)^2/b^2 - 2a(k-l)(s-t)/b^2$$

$$+(1-b^2)(s-t)^2/b^2 \qquad k, l, s, t \in \boldsymbol{Z}^+.$$

(ii) *The eigenvalues of* $2\square_g$ *on* $M_{a,b}^{p,q}$ *are*

$$(k+l)(k+l+2p)+(s+t)(s+t+2q)+a^2(k-l)^2/b^2-2a(k-l)(s-t)/b^2$$
$$+(1-b^2)(s-t)^2/b^2-2p(k-l)-2q(s-t) \qquad k, l, s, t\in \boldsymbol{Z}^+,$$

where $\boldsymbol{Z}^+$ *denotes the set of non-negative integers.*

PROOF. We restrict $\varphi\in\mathcal{H}_{k,l}(\boldsymbol{C}^{p+1})$ and $\phi\in\mathcal{H}_{s,t}(\boldsymbol{C}^{q+1})$ to S^{2p+1} and S^{2q+1} respectively and consider their product $\varphi\phi$ on $S^{2p+1}\times S^{2q+1}$. By Proposition 5.5 and Lemma 5.6 we see that $\varphi\phi$ is an eigenfunction of $\triangle_g$ with eigenvalue given in (i) and it is also an eigenfunction of $2\square_g$ with eigenvalue given in (ii). It is well-known that the functions of the form $\varphi\phi$, $\varphi\in\mathcal{H}_{k,l}(\boldsymbol{C}^{p+1})$, $\phi\in\mathcal{H}_{s,t}(\boldsymbol{C}^{q+1})$, $k, l, s, t\in\boldsymbol{Z}^+$, are dense in $C^\infty(S^{2p+1}\times S^{2q+1}, \mathring{g})$ ([**2**]), where $C^\infty(S^{2p+1}\times S^{2q+1}, \mathring{g})$ denotes the space of complex-valued differentiable functions on $S^{2p+1}\times S^{2q+1}$ with the scalar product $\langle\varphi, \phi\rangle=\int_{S^{2p+1}\times S^{2q+1}}\varphi\bar{\phi}dV_{\mathring{g}}$. By Lemma 5.3 (6) we see that the functions of this form are dense in $C^\infty(S^{2p+1}\times S^{2q+1}, g)$ as well. Therefore eigenvalues of $\triangle_g$ (resp. $2\square_g$) take the form of (i) (resp. (ii)).

REMARK (i). It does not seem easy to find the dimension (multiplicity) of each eigenspace. But the dimension of $\mathcal{H}_{k,l}(\boldsymbol{C}^n)$ is known ([**1**]):

$$\dim\mathcal{H}_{k,l}(\boldsymbol{C}^n)=\binom{n+k-1}{k}\binom{n+l-1}{l}-\binom{n+k-2}{k-1}\binom{n+l-2}{l-1}.$$

REMARK (ii). Theorem 5.7 for $q=0$ coincides with the results of Hopf manifolds in [**1**] up to a constant multiple (cf. 3.3 (2)).

6. Some isospectral results.

In the case of $q=0$ (i.e., Hopf manifolds), some isospectral results are given in [**1**]. We shall give some results in the case of $q\neq 0$, using the formulas on the coefficients of the asymptotic expansion.

Let A be the integrability tensor associated with the Riemannian submersion π: $(S^{2p+1}\times S^{2q+1}, g)\to(\boldsymbol{C}P^p\times\boldsymbol{C}P^q, h)$. There exist two direct sum decompositions of the tangent space of $S^{2p+1}\times S^{2q+1}$ at (x, x'):

$$T_{(x,x')}(S^{2p+1}\times S^{2q+1})=T_xS^{2p+1}\oplus T_{x'}S^{2q+1}$$
$$=\mathcal{V}\oplus\mathcal{H},$$

where $\mathcal{V}$ and $\mathcal{H}$ denote the vertical subspace and the horizontal subspace respectively with respect to the Riemannian submersion π.

LEMMA 6.1. *The integrability tensor* A *satisfy the following equations for* $X, Y, Z\in T_xS^{2p+1}\cap\mathcal{H}$ *and* $X', Y', Z'\in T_{x'}S^{2q+1}\cap\mathcal{H}$:

(1) $A_X Y=\langle X, \phi Y\rangle\xi$

$A_X Y'=0$

$A_{X'} Y'=\langle X', \phi' Y'\rangle\xi'$

(2) $A_X\xi=\phi X$

$A_{X'}\xi=a\phi' X'$

$A_X\xi'=a\phi X$

$A_{X'}\xi'=(a^2+b^2)\phi' X'$

(3) $\nabla_X\xi=\phi X$

$\nabla_X\xi'=a\phi X$

$\nabla_{X'}\xi=a\phi' X'$

$\nabla_{X'}\xi'=(a^2+b^2)\phi' X'$

(3)′ $\nabla_\xi X=\phi X$

$\nabla_\xi X'=a\phi' X'$

$\nabla_{\xi'} X=a\phi X$

$\nabla_{\xi'} X'=(a^2+b^2)\phi' X'$

(In (3)′ we assume X and X' are basic vector fields.)

(4) $(\nabla_X A)_Y Z=\langle Y, \phi Z\rangle\phi X-\langle X, \phi Z\rangle\phi Y$

$(\nabla_X A)_{Y'} Z=-a\langle X, \phi Z\rangle\phi' Y'$

$(\nabla_X A)_Y Z'=0$

$(\nabla_X A)_{Y'} Z'=a\langle Y', \phi' Z'\rangle\phi X$

$(\nabla_{X'} A)_Y Z=a\langle Y, \phi Z\rangle\phi' X'$

$(\nabla_{X'} A)_{Y'} Z=0$

$(\nabla_{X'} A)_Y Z'=-a\langle X', \phi' Z'\rangle\phi Y$

$(\nabla_{X'} A)_{Y'} Z'=(a^2+b^2)\{\langle Y', \phi' Z'\rangle\phi' X'-\langle X', \phi' Z'\rangle\phi' Y'\}$

(5) *For any horizontal vectors V, W*

$$(\nabla_\xi A)_V W=(\nabla_{\xi'} A)_V W=0\,,$$

where $\langle\ ,\ \rangle$ denotes the metric tensor g and for the other notations we refer to § 3.

PROOF. (1) is easily seen from Lemma 5.1 (2) and Lemma 5.3 (4).

(2) We use again the local field $\{e_1^*, \cdots, e_{2p}^*, e_1'^*, \cdots, e_{2q}'^*\}$ given in Lemma 5.2. Then we have

$$\begin{aligned} A_{X'}\xi' &= \sum_{j=1}^{2p}\langle A_{X'}\xi', e_j^*\rangle e_j^* + \sum_{k=1}^{2q}\langle A_{X'}\xi', e_k'^*\rangle e_k'^* \\ &= -\sum_{j=1}^{2p}\langle \xi', A_{X'}e_j^*\rangle e_j^* - \sum_{k=1}^{2q}\langle \xi', A_{X'}e_k'^*\rangle e_k'^* \\ &= -\sum_{k=1}^{2q}\langle \xi', \langle X', \phi' e_k'^*\rangle\xi'\rangle e_k'^* \\ &= (a^2+b^2)\sum_{k=1}^{2q}\langle \phi' X', e_k'^*\rangle e_k'^* \\ &= (a^2+b^2)\phi' X'\,. \end{aligned}$$

Similarly we can prove the other equations in (2).

(3) We shall prove $\nabla_X \xi = \phi X$. First of all we see that $\mathcal{V}(\nabla_X \xi) = 0$. In fact we have

$$g(\nabla_X \xi, \xi) = 1/2\, Xg(\xi, \xi) = 0 \qquad \text{and}$$

$$g(\nabla_X \xi, \xi') = Xg(\xi, \xi') - g(\xi, \nabla_X \xi') = -g(\xi, \nabla_{\xi'} X) = 0,$$

where we extend X to the basic vector field in a neighborhood of this point.

By Lemma 4.3 (3) and Lemma 6.1 (2) we see that

$$\nabla_X \xi = A_X \xi + \mathcal{V}(\nabla_X \xi) = \phi X.$$

Similarly we can prove the other equations in (3).

(3)′ Using Lemma 4.3 we can prove (3)′.

(4) and (5) We extend X, Y and Z to the basic vector fields in a neighborhood of this point to obtain

$$\begin{aligned}(\nabla_X A)_Y Z &= \nabla_X (A_Y Z) - A_{\nabla_X Y} Z - A_Y (\nabla_X Z)\\ &= \nabla_X (\langle Y, \phi Z\rangle \xi) - \langle \nabla_X Y, \phi Z\rangle \xi - A_Y(\mathcal{H}(\nabla_X Z) + \langle X, \phi Z\rangle \xi)\\ &= X\langle Y, \phi Z\rangle \xi + \langle Y, \phi Z\rangle \phi X - \langle \nabla_X Y, \phi Z\rangle \xi\\ &\qquad - \langle Y, \phi(\nabla_X Z)\rangle \xi - \langle X, \phi Z\rangle \phi Y\\ &= \langle Y, (\nabla_X \phi)(Z)\rangle \xi + \langle Y, \phi Z\rangle \phi X - \langle X, \phi Z\rangle \phi Y\\ &= \langle Y, \eta(Z) X - \langle X, Z\rangle \xi\rangle \xi + \langle Y, \phi Z\rangle \phi X - \langle X, \phi Z\rangle \phi Y\\ &= \langle Y, \phi Z\rangle \phi X - \langle X, \phi Z\rangle \phi Y.\end{aligned}$$

The other equations in (4) and (5) are proved similarly.

PROPOSITION 6.2. *Let τ, ρ, R be the scalar curvature, the Ricci tensor, and the curvature tensor of $M_{a;b}^{p;q}$ respectively. Then we have*

$$\mathrm{Vol}(M_{a;b}^{p;q}) = b\, \mathrm{Vol}(S^{2p+1} \times S^{2q+1}, \mathring{g})$$

$$\tau = 2p(2p+1) + 4q(q+1) - 2(a^2+b^2)q$$

$$|\rho|^2 = 4p^2(2p+1) + 8a^2pq + 4q(q+2)(a^2+b^2)^2 - 8q(2q+2)(a^2+b^2) + 2q(2q+2)^2$$

$$\begin{aligned}|R|^2 = {} & 8p + 8(a^2+b^2)^2 q + 4p(2p-1) + 4q(2q-1) + 48a^2pq\\ & + 24q[1-(a^2+b^2)] + [1-(a^2+b^2)]^2 12q(2q+1),\end{aligned}$$

where $(S^{2p+1} \times S^{2q+1}, \mathring{g})$ is the Riemannian product of two unit spheres and $|R|$ and $|\rho|$ denote the lengths of the curvature tensor and the Ricci tensor, respectively.

PROOF. The volume of $M_{a;b}^{p;q}$ is given in Lemma 5.3 (6). We calculate τ,

$|\rho|^2$, and $|R|^2$, using the formulas in Lemma 4.4 and the results in Lemma 6.1.

PROPOSITION 6.3. *The Kaehler form Ω of the Hermitian manifold $M_{a,b}^{p,q}$ satisfies*

$$\delta\Omega=2\{(p+aq)\eta+(ap+[a^2+b^2]q)\eta'\}$$

$$d\delta\Omega=2\{(p+aq)d\eta+(ap+[a^2+b^2]q)d\eta'\}$$

$$|\delta\Omega|^2=4\{p^2+2apq+[a^2+b^2]q^2\}$$

$$|d\delta\Omega|^2=8\{p(p+aq)^2+q(ap+[a^2+b^2]q)^2\},$$

where $|\delta\Omega|$ and $|d\delta\Omega|$ denote the lengths of the 1-form $\delta\Omega$ and the 2-form $d\delta\Omega$, respectively.

PROOF. Since $-\delta\Omega$ is the 1-form dual to the vector field ζ, Lemma 5.4 implies that $\delta\Omega$ is expressed as above. The other equations follow immediately.

THEOREM 6.4. *If* Spec$(M_{a,b}^{p,q}, \triangle)$=Spec$(M_{a',b'}^{p,q}, \triangle)$ *and q is not zero, then $M_{a,b}^{p,q}$ is isometric to $M_{a',b'}^{p,q}$.*

PROOF. If Spec$(M_{a,b}^{p,q}, \triangle)$=Spec$(M_{a',b'}^{p,q}, \triangle)$, then the volumes of $M_{a,b}^{p,q}$ and $M_{a',b'}^{p,q}$ are the same and the integrals of the scalar curvatures of $M_{a,b}^{p,q}$ and $M_{a',b'}^{p,q}$ are the same ([2]). By Proposition 6.2 we get $b'=b$ and $a'^2=a^2$. If $a'=-a$, then 3.3 (5) implies that $M_{a,b}^{p,q}$ is isometric to $M_{a',b'}^{p,q}$.

THEOREM 6.5. *Assume that* Spec$(M_{a,b}^{p,q}, \square)$=Spec$(M_{a',b'}^{p,q}, \square)$ *and q is not zero. Then we have $b=b'$.*

(1) *In the case of $p=0$ we have $a'=a$ or $a'=-a$, i.e., $M_{a',b}^{0,q}$ is biholomorphically isometric to $M_{a,b}^{0,q}$ or $(M_{a,b}^{0,q}, -J_{a,b})$.*

(2) *In the case of $p\neq 0$ unless $a=a'$, a and a' satisfy the following relations:*

$$(*)\qquad a+a'=-6p/(3q-1)$$

$$(**)\qquad aa'=b^2+4(q+1)/(15q^2-3)+(135q^2-39)p^2/(3q-1)^2(15q^2-3).$$

PROOF. By the formula for a_0 of the asymptotic expansion given in Theorem 2.7 and Proposition 6.2 we have $b=b'$. We calculate a_1 in Theorem 2.7 using Proposition 6.2 and Proposition 6.3 to get $(a-a')(a+a'+6p/(3q-1))=0$. Therefore in the case of $p=0$ we have $a'=a$ or $a'=-a$. If $a'=-a$, then 3.3 (3) implies that $M_{a',b}^{0,q}$ is biholomorphically isometric to the Hermitian manifold $(M_{a,b}^{0,q}, -J_{a,b})$.

Furthermore if p is not zero and a is not equal to a', calculating a_2 we get (**).

COROLLARY 6.6. *Assume that $p\neq 0$ and $3p^2\leqq(q+1)(3q-1)^2$. If* Spec$(M_{a,b}^{p,q}, \square)$ =Spec$(M_{a',b'}^{p,q}, \square)$, *then $a=a'$ and $b=b'$, i.e., $M_{a,b}^{p,q}$ is biholomorphic and isometric to $M_{a',b'}^{p,q}$.*

PROOF. Since the system of the equations (*) and (**) with respect to a

and a' in Theorem 6.5 does not have real solutions under the assumption of this Corollary, we conclude that $a=a'$.

REMARK (i). Theorem 6.5 shows that for most values of (a, b), $M_{a,b}^{p;q}$ is determined from Spec$(M_{a,b}^{p;q}, \square)$. We consider the case of $q\neq 0$ and $3p^2>(q+1)(3q-1)^2$, which is not refered to in Corollary 6.6. If Spec$(M_{a,b}^{p;q}, \square)=$ Spec$(M_{a',b'}^{p;q}, \square)$ and if (a, b) is not on the curve in figure, then $a=a'$ and $b=b'$, i.e., $M_{a,b}^{p;q}$ is biholomorphic and isometric to $M_{a',b'}^{p;q}$. But there is left the possibility that for (a, b) and (a', b) on the curve $M_{a,b}^{p;q}$ and $M_{a',b}^{p;q}$ have the same spectrum.

Figure.

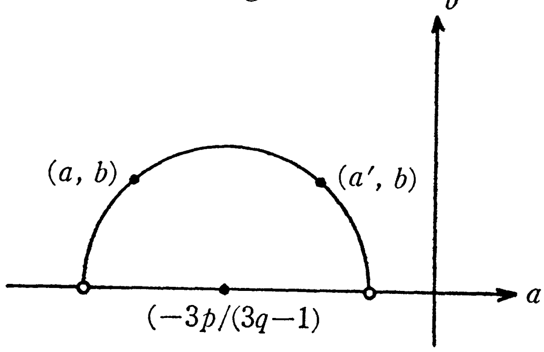

Figure.

a semicircle centered at $(-3p/(3q-1), 0)$ with the radius $2\sqrt{\{3p^2-(q+1)(3q-1)^2\}/(3q-1)^2(15q^2-3)}$.

REMARK (ii). By Corollary 6.6 we see that, in the case of $p\neq 0$, $3p^2\leqq(q+1)(3q-1)^2$, and $a\neq 0$, $M_{a,b}^{p;q}$ is isometric to $M_{-a,b}^{p;q}$ as a Riemannian manifold but Spec$(M_{a,b}^{p;q}, \square)\neq$Spec$(M_{-a,b}^{p;q}, \square)$. This shows that the complex Laplacian actually reflects the complex structures of Hermitian manifolds.

References

[1] B. Bedford and T. Suwa, Eigenvalues of Hopf manifolds, Proc. Amer. Math. Soc., **60** (1976), 259-264.

[2] M. Berger, P. Gauduchon et E. Mazet, Le spectre d'une variété riemanniennes, Lecture Notes in Math., Springer-Verlag, 194.

[3] D. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math., Springer-Verlag, 509.

[4] E. Calabi and B. Eckmann, A class of compact complex manifolds which are not algebraic, Ann. of Math., **58** (1953), 494-500.

[5] P. Gilkey, Spectral geometry and the Kaehler condition for complex manifolds, Invent. Math., **26** (1974), 231-258.

[6] P. Gilkey, The spectral geometry of real and complex manifolds, Proc. Sympos. Pure Math., **27** (1975), 265-280.

[7] P. Gilkey, The spectral geometry of a Riemannian manifold, J. Differential Geometry, **10** (1975), 601-618.
[8] A. Morimoto, On normal almost contact structures, J. Math. Soc. Japan, **15** (1963), 420-436.
[9] J. Morrow and K. Kodaira, Complex manifolds, Holt, Rinehart and Winston, New York.
[10] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J., **13** (1966), 459-469.
[11] K. Yano, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

Kazumi TSUKADA
Department of Mathematics
Tokyo Metropolitan University
Setagaya, Tokyo 158
Japan