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Segal-Becker theorem for KR-theory

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§ 1. Introduction.

The purpose of this paper is to prove the following result.

THEOREM 1.1. *Let X be a based finite $\boldsymbol{Z}_2$-complex in the sense of* [5]. *Then there exists a natural split epimorphism*

$$\lambda_*: \{X, \boldsymbol{C}P^{\infty}\}_{Z_2} \longrightarrow \tilde{K}_R(X).$$

As corollaries of this theorem we deduce the results of G. Segal [12] and J. C. Becker [4].

First we fix our notation.

Let X be a compact based $\boldsymbol{Z}_2$-space and let

$$\tilde{K}_R(X)=K_R(X, *)$$

be the reduced KR-group of Atiyah [2]. If X and Y are based $\boldsymbol{Z}_2$-spaces, then $[X, Y]_{Z_2}$ denotes the set of $\boldsymbol{Z}_2$-homotopy classes of based $\boldsymbol{Z}_2$-maps from X to Y. Let $\boldsymbol{R}^{p,q}$ be the representation of $\boldsymbol{Z}_2$ on $\boldsymbol{R}^{p+q}$ given by

$$g(x_1, \cdots, x_{p+q})=(-x_1, \cdots, -x_p, x_{p+1}, \cdots, x_{p+q}), \qquad g\in \boldsymbol{Z}_2,$$

and let $\Sigma^{p,q}=(\boldsymbol{R}^{p,q})^c$ be the one-point compactification of $\boldsymbol{R}^{p,q}$. Then we define the stable $\boldsymbol{Z}_2$-homotopy group $\{X, Y\}_{Z_2}$ to be $\varinjlim_n [\Sigma^{n,n}\wedge X, \Sigma^{n,n}\wedge Y]_{Z_2}$. Let $\boldsymbol{C}P^n$ be the complex projective space with the involution σ given by

$$\sigma[z_0, \cdots, z_n]=[\bar{z}_0, \cdots, \bar{z}_n].$$

The construction of λ_* is given as follows.

Let BR denote a classifying space for stable Real vector bundles so that $\tilde{K}_R(X)=[X, BR]_{Z_2}$. (We shall give a specific model for BR in § 3.) For a $\boldsymbol{Z}_2$-space X,

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$$Q_{Z_2}(X)=\varinjlim_n F(\Sigma^{n,n}, \Sigma^{n,n}\wedge X)$$

denotes the equivariant infinite loop space, where $F(\ ,\ )$ denotes the based function space with the natural $\boldsymbol{Z}_2$-action. Then by use of the Bott periodicity of KR-groups [**2**], we can define an infinite loop space structure

$$\xi: Q_{Z_2}(BR) \longrightarrow BR$$

similarly as for BU.

Note that the canonical line bundle over $\boldsymbol{C}P^n$ is a Real line bundle in the canonical way. Let

$$j: \boldsymbol{C}P^\infty \longrightarrow BR$$

be a $\boldsymbol{Z}_2$-map which classifies the stable class of the canonical Real line bundle. Then put

$$\lambda=\xi\circ Q_{Z_2}(j): Q_{Z_2}(\boldsymbol{C}P^\infty) \longrightarrow BR\,.$$

Then using the natural isomorphism

$$\{X, \boldsymbol{C}P^\infty\}_{Z_2}\cong[X, Q_{Z_2}(\boldsymbol{C}P^\infty)]_{Z_2},$$

we obtain a natural homomorphism

$$\lambda_*: \{X, \boldsymbol{C}P^\infty\}_{Z_2} \longrightarrow \tilde{K}_R(X)\,.$$

By forgetting $\boldsymbol{Z}_2$-action in the above argument we obtain a natural homomorphism

$$\lambda'_*: \{X, \boldsymbol{C}P^\infty\} \longrightarrow \tilde{K}(X)\,.$$

Then we have

COROLLARY 1.2 (Segal [**12**]). *Let X be a finite complex. Then λ'_* is a split epimorphism.*

PROOF. We have a commutative diagram

$$\begin{array}{ccc}
\{X\times S^0/*\times S^0, \boldsymbol{C}P^\infty\}_{Z_2} & \xrightarrow{\lambda_*} & \tilde{K}_R(X\times S^0/*\times S^0) \\
\phi'\downarrow & & \downarrow\phi \\
\{X, \boldsymbol{C}P^\infty\} & \xrightarrow{\lambda'_*} & \tilde{K}(X)
\end{array}$$

where S^0 is the zero-sphere with free involution and ϕ', ϕ are the restriction maps on the subspace $X=X\times\{0\}\subset X\times S^0$, which are known [**3**] to be natural isomorphisms.

Next consider the case that X is a trivial $\boldsymbol{Z}_2$-space. Then clearly $\tilde{K}_R(X)=\widetilde{KO}(X)$. The following proposition will be proved in §4.

PROPOSITION 1.3. *Let X be a finite complex with trivial $\boldsymbol{Z}_2$-action. Then there exists a natural isomorphism*

$$\{X, \boldsymbol{C}P^{\infty}\}_{Z_2} \cong \{X, BO(2)\} \oplus \{X, BO(1)\}.$$

Moreover, we have via this isomorphism

$$\lambda_* = \mu_{2*} \oplus \mu_{1*} : \{X, \boldsymbol{C}P^{\infty}\}_{Z_2} \longrightarrow \tilde{K}_R(X) = \widetilde{KO}(X),$$

where $\mu_2 : QBO(2) \to BO$ *and* $\mu_1 : QBO(1) \to BO$ *are the extensions of the standard inclusions* $BO(2) \to BO$ *and* $BO(1) \to BO$ *respectively.*

COROLLARY 1.4 (Becker [4]). *Let* X *be a finite complex. Then* μ_{2*}: $\{X, BO(2)\} \to \widetilde{KO}(X)$ *is a split epimorphism.*

§ 2. The equivariant transfer homomorphism.

In this section we recall the definition of the transfer homomorphism in equivariant cohomology theories [8].

Let G be a finite group and Γ a compact Lie group. Let $\alpha : G \to \mathrm{Aut}\,\Gamma$ be a homomorphism and let $\Gamma \times_{\alpha} G$ be the semi-direct product. In [8] we have defined the notion of a principal (Γ, α, G)-bundle $\tilde{p} : \tilde{E} \to B$, namely, $\tilde{E}$ is a $\Gamma \times_{\alpha} G$-space such that free as a Γ-space, B is a G-space and $B \cong \tilde{E}/\Gamma$ via the projection $\tilde{p}$. Let F be a compact smooth $\Gamma \times_{\alpha} G$-manifold and suppose that B is compact. Then we obtain a (Γ, α, G)-bundle

$$F \xrightarrow{i} E = \tilde{E} \times_{\Gamma} F \xrightarrow{p} B.$$

Choose a $\Gamma \times_{\alpha} G$-embedding $F \subset W$ into a Euclidean $\Gamma \times_{\alpha} G$-space W. Then since B is compact, there exists a G-vector bundle η over B and a G-vector space V such that

$$(\tilde{E} \times_{\Gamma} W) \oplus \eta \cong B \times V.$$

Now if E_1 and E_2 are fibre bundles over B, then $E_1 \times_B E_2$ denotes the fibre product. Note that $X \mapsto (\tilde{E} \times_{\Gamma} X) \times_B \eta$ is a functor on $\Gamma \times_{\alpha} G$-spaces. Let $\nu(F)$ be the normal bundle of F in W. Then we have $\Gamma \times_{\alpha} G$-maps

$$W \xleftarrow{i} \nu(F) \xrightarrow{j} \nu(F) \oplus \tau(F) \cong F \times W$$

where $j(v) = (v, 0)$. Then applying the above functor we obtain G-maps

$$(\tilde{E} \times_{\Gamma} W) \times_B \eta \xleftarrow{i'} (\tilde{E} \times_{\Gamma} \nu(F)) \times_B \eta \xrightarrow{j'} (\tilde{E} \times_{\Gamma} (F \times W)) \times_B \eta.$$

By definition, $(\tilde{E} \times_{\Gamma} W) \times_B \eta \cong (\tilde{E} \times_{\Gamma} W) \oplus \eta \cong B \times V$, and by [8, Lemma 2.2], $(\tilde{E} \times_{\Gamma} (F \times W)) \times_B \eta \cong E \times V$. We remark that if p is a differentiable (Γ, α, G)-bundle then the G-embedding

$$E = \tilde{E} \times_{\Gamma} F \longrightarrow (\tilde{E} \times_{\Gamma} W) \times_B \eta \cong B \times V$$

is G-homotopic to the projection $p: E\rightarrow B$ and the normal bundle $\nu(E, B\times V)$ is just $(\tilde{E}\times_{\Gamma}\nu(F))\times_{B}\eta$.

Now clearly $i': (\tilde{E}\times_{\Gamma}\nu(F))\times_{B}\eta\rightarrow B\times V$ is an embedding onto an open set and $j': (\tilde{E}\times_{\Gamma}\nu(F))\times_{B}\eta\rightarrow E\times V$ is a proper map. Hence by taking the one-point compactification we obtain a G-map $B_{+}\wedge V^{c}\rightarrow E_{+}\wedge V^{c}$, where $+$ means a disjoint base point. By taking the limit over V we obtain a stable G-map

$$t=t(p): B_{+} \longrightarrow E_{+}$$

which is independent of choices of the embedding $F\subset W$ and η ([8]), and is called a trace of the bundle $p=(F\rightarrow E\rightarrow B)$.

Now let M be a closed G-manifold. Then the unique map $p: M\rightarrow *$ is a $(\{e\}, G)$-bundle, and we obtain a trace $t: S^{0}\rightarrow M_{+}$. We define the equivariant Euler characteristic $\chi_{G}(M)$ by $\pi\circ t\in\{S^{0}, S^{0}\}_{G}\cong A(G)$, where $\pi: M_{+}\rightarrow S^{0}$ is the projection and $A(G)$ is the Burnside ring of G (see [11]). Let H be a subgroup of G and let $\phi_{H}: A(G)\rightarrow \boldsymbol{Z}$ be the ring homomorphism given by $\phi_{H}(S)=\#\{S^{H}\}$ for a finite G-set S.

LEMMA 2.1. *Let M be a closed G-manifold and let H be a subgroup of G. Then we have $\phi_{H}(\chi_{G}(M))=\chi(M^{H})$, where $M^{H}=\{x\in M;\ h(x)=x$ for any $h\in H\}$.*

PROOF. Let $\psi_{H}: \{S^{0}, S^{0}\}_{G}\rightarrow \boldsymbol{Z}$ be the ring homomorphism given by $\psi_{H}(\{f\})=\deg(f^{H})$ for a G-map $f: V^{c}\rightarrow V^{c}$. By the result of [11], we can replace $A(G)$ and ϕ_{H} with $\{S^{0}, S^{0}\}_{G}$ and ψ_{H}. Now $\chi_{G}(M)$ is represented by a G-map

$$V^{c} \xrightarrow{t} M_{+}\wedge V^{c} \xrightarrow{\pi} V^{c}$$

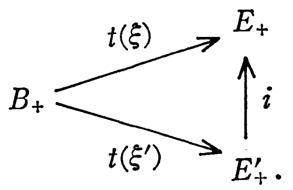
and by the definition of the trace we see easily that

$$t^{H}: (V^{H})^{c} \longrightarrow M_{+}^{H}\wedge(V^{H})^{c}$$

is a trace of the manifold M^{H}. This shows the lemma.

Let now F be a closed $\Gamma\times_{\alpha}G$-manifold and F' a $\Gamma\times_{\alpha}G$-submanifold. Let N be a closed $\Gamma\times_{\alpha}G$-invariant tubular neighborhood of F' in F. We assume that there exists a non-singular $\Gamma\times_{\alpha}G$-invariant vector field Δ on $\overline{F-N}$ whose restriction to ∂N lies in the tangent space of ∂N. Let $\tilde{p}: \tilde{E}\rightarrow B$ be a principal (Γ, α, G)-bundle. Let $\xi=(F\rightarrow E\rightarrow B)$ and $\xi'=(F'\rightarrow E'\rightarrow B)$ be the associated bundles with fibre F and F', respectively. Let $i: E'\rightarrow E$ be the natural inclusion. Then we obtain

PROPOSITION 2.2. *With the above assumptions the following diagram is G-homotopy commutative*



PROOF. Let X and Y be locally compact G-spaces. We define a locally proper G-map from X to Y to be a proper G-map $f: U \to Y$ for some open subset U of X. Let f_0 and f_1 be locally proper G-maps from X to Y. A G-homotopy H of f_0 and f_1 is a locally proper G-map H from $X \times I$ to Y whose restriction to $X \times \{i\}$ is f_i, $i=0, 1$. It is clear that a locally proper G-map defines a G-map $f: X^c \to Y^c$ and similarly for homotopy.

Now we choose a $\Gamma \times_\alpha G$-embedding $F \subset W$ into a Euclidean $\Gamma \times_\alpha G$-space. Then F' $(\subset F) \subset W$. Then we obtain locally proper $\Gamma \times_\alpha G$-maps

$$W \supset \nu(F) \xrightarrow{k} \nu(F) \oplus \tau(F) \cong F \times W$$

$$W \supset \nu(F') \xrightarrow{k'} \nu(F') \oplus \tau(F') \cong F' \times W .$$

We first show that the locally proper $\Gamma \times_\alpha G$-maps k and $(j \times id) \circ k'$ are homotopic, where $j: F' \to F$ is the inclusion.

We may assume the following conditions:

(i) By the collaring theorem, we can identify

$$N \cong N' \cup (\partial N \times I), \qquad \partial N' = \partial N \times \{0\}, \qquad \partial(\overline{F-N}) = \partial N \times \{1\}$$

$$\text{and} \quad F = N' \cup (\partial N \times I) \cup \overline{F-N} .$$

Let $\pi: \nu(F) \to F$, $\pi': N \to F'$ and $\pi'': \partial N \times I \to I$ be the projections.

(ii) $\nu(F) = \{x \in W ; \|\overrightarrow{\pi(x)x}\| < 1\}$ and $N' = \{z \in F ; \|\overrightarrow{\pi'(z)z}\| < 1\}$, where $\| \ \|$ is the $\Gamma \times_\alpha G$-invariant norm in W.

(iii) The non-singular $\Gamma \times_\alpha G$-invariant vector field Δ is extended on $(\partial N \times I) \cup \overline{F-N}$ so that Δ is tangent to $\partial N \times \{s\}$ for any $s \in I$.

(iv) $\|\Delta(x)\| = 1$ in W for any $x \in (\partial N \times I) \cup \overline{F-N}$.

Note that we can identify $\nu(F)|_{N'-\partial N'}$ with the tubular neighborhood of F' in W. We deform a normal vector in $\nu(F)|_{N'-\partial N'}$ to a vector normal to F' in a canonical way, and deform $k: \nu(F)|_{F-N'} \to F \times W$ to the trivial map $\emptyset \to F \times W$.

Recall that $k(x) = (\pi(x), \overrightarrow{\pi(x)x})$ for $x \in \nu(F)$ and that $(j \times id) \circ k'(x) = (\pi'\pi(x), \overrightarrow{\pi'\pi(x)x})$ for $x \in \nu(F)|_{N'-\partial N'}$.

We define a locally proper $\Gamma \times_\alpha G$-map H from $\nu(F) \times I$ to $F \times W$ by

$$H(x, t) = (H_1(x, t), H_2(x, t)) \in F \times W ,$$

where $H_1(x, t)$ is the canonical path connecting $\pi(x)$ and $\pi'\pi(x)$ in N' if $\pi(x)\in N'-\partial N'$, equal to $\pi(x)$ if $\pi(x)\in\overline{F-N}$, and can be canonically extended on all $\nu(F)$ to be a continuous $\Gamma\times_{\alpha}G$-map, and

$$H_2(x, t)=\begin{cases}\overrightarrow{\pi(x)x}+t\overrightarrow{\pi'\pi(x)\pi(x)} & \text{if } \pi(x)\in N'-\partial N' \\ \overrightarrow{\pi(x)x}+t\Delta(\pi(x)) & \text{if } \pi(x)\in F-N \\ \overrightarrow{\pi(x)x}+t\{1-2s+2s\|\overrightarrow{\pi'\pi(x)\pi(x)}\|^{-1}\}\overrightarrow{\pi'\pi(x)\pi(x)} & \\ \qquad \text{if } \pi(x)\in\partial N\times I,\ s=\pi''\pi(x) \text{ and } 0\leqq s\leqq 1/2 & \\ \overrightarrow{\pi(x)x}+tK(2s-1;\|\overrightarrow{\pi'\pi(x)\pi(x)}\|^{-1}\overrightarrow{\pi'\pi(x)\pi(x)},\ \Delta(\pi(x))) & \\ \qquad \text{if } \pi(x)\in\partial N\times I,\ s=\pi''\pi(x) \text{ and } 1/2\leqq s\leqq 1, & \end{cases}$$

where $K(s;v,w)=\|(1-s)v+sw\|^{-1}\{(1-s)v+sw\}$ which is well-defined when v and w are linearly independent. The domain of H is the open set

$$\{(x, t)\in\nu(F)\times I;\ \|H_2(x, t)\|<1\}.$$

H_2 is well-defined because $\overrightarrow{\pi'\pi(x)\pi(x)}$ and $\Delta(\pi(x))$ are linearly independent and $\|\Delta(\pi(x))\|=1$ for any $\pi(x)\in\partial N\times I$. If $\pi(x)\in\overline{F-N}$ and $t=1$, then the domain is empty because $\|\Delta\|=1$ and Δ is normal to $\overrightarrow{\pi(x)x}$. If $\pi(x)\in\partial N\times I$ and $t=1$, then the domain is empty because $\|\overrightarrow{\pi'\pi(x)\pi(x)}\|\geqq 1$, $\|K\|=1$ and $\overrightarrow{\pi(x)x}$ is normal to both $\overrightarrow{\pi'\pi(x)\pi(x)}$ and K.

Thus H is the required locally proper $\Gamma\times_{\alpha}G$-homotopy of k and $(j\times id)\circ k'$.

Now to define the homotopy between the traces of ξ and ξ', consider the functor $(\tilde{E}\times_{\Gamma}(\ ))\times_B\eta$. Then clearly we obtain a homotopy between the locally proper G-maps

$$(\tilde{E}\times_{\Gamma}W)\times_B\eta\supset(\tilde{E}\times_{\Gamma}\nu(F))\times_B\eta\longrightarrow(\tilde{E}\times_{\Gamma}(F\times W))\times_B\eta$$

and

$$(\tilde{E}\times_{\Gamma}W)\times_B\eta\supset(\tilde{E}\times_{\Gamma}\nu(F'))\times_B\eta$$
$$\longrightarrow(\tilde{E}\times_{\Gamma}(F'\times W))\times_B\eta\longrightarrow(\tilde{E}\times_{\Gamma}(F\times W))\times_B\eta\,.$$

This completes the proof.

Let h_G^* be a generalized G-cohomology theory in the sense of [7]. For a $(\Gamma;\alpha, G)$-bundle $\xi=(F\to E\to B)$ where F is a closed smooth manifold and B is compact, we define the transfer homomorphism

$$p_!:\ h_G^*(E)\longrightarrow h_G^*(B)$$

by $p_!=(\sigma^V)^{-1}t(\xi)^*\sigma^V$, where $t(\xi):\ B_+\wedge V^c\to E_+\wedge V^c$ is a trace of ξ and $\sigma^V:\tilde{h}_G^{\alpha}(X)\to\tilde{h}_G^{\alpha+V}(X\wedge V^c)$ is the suspension isomorphism. It is known [8] that $h_G^*(X)$ is a $\pi_G^*(pt.)=\{S^0, S^0\}_G$-module and $p_!$ is a $\pi_G^*(pt.)$-module homomorphism. Let $\omega\in$

$\pi_G^0(pt.)$ and let M be a $\pi_G^0(pt.)$-module. We denote by $M[\omega^{-1}]$ the localization of M with respect to the multiplicative set $\{\omega^n\}_{n=1,2,\dots}$. Now we need the following generalization of [8, Theorem 4.7].

PROPOSITION 2.3*. *Let* $\xi=(F \to E \xrightarrow{p} B)$ *be a* (Γ, α, G)*-bundle, where* F *is a closed smooth manifold. Suppose that* B *is a finite* G*-complex in the sense of* [5]. *Then the composition of* $\pi_G^*(pt.)[\chi_G(F)^{-1}]$*-module homomorphisms*

$$p_! p^* : h_G^*(B)[\chi_G(F)^{-1}] \longrightarrow h_G^*(B)[\chi_G(F)^{-1}]$$

is an isomorphism.

PROOF. Let $B^{(0)}$ be the 0-skeleton of B. Then $B^{(0)}$ is a union of G-orbits G/H. Let $\tilde{E} \xrightarrow{\tilde{p}} B$ be the principal (Γ, α, G)-bundle associated with ξ. Let $G/H \subset B^{(0)}$. The principal bundle $\tilde{p}^{-1}(G/H) \to G/H$ is clearly identified with the canonical projection

$$\Gamma \times_\alpha G/\{e\} \times_\alpha H \longrightarrow G/H .$$

Then the associated bundle with fibre F over G/H is easily seen to be the product bundle

$$G/H \times F \longrightarrow G/H .$$

Now let $\omega(\xi)=p_! p^*(1) \in \pi_G^0(B)$, where $1 \in \pi_G^0(B)$ is the unit. Let $\pi : B \to pt.$ be the unique G-map. Then by the naturality ([8, Proposition 4.4]) of the transfer we have

$$i^*(\omega(\xi)-\pi^*\chi_G(F))=0 ,$$

where $i : B^{(0)} \to B$ is the inclusion. Then the proposition follows from the similar argument of [8, Theorem 4.7].

Now let K_R^* $(* \in RO(\boldsymbol{Z}_2))$ be the Real K-theory of Atiyah [2]. K_R^* is a generalized $\boldsymbol{Z}_2$-cohomology theory, and $K_R^0(X)$ is the Grothendieck group $K_R(X)$ of Real vector bundles over X. A $(\Gamma, \alpha, \boldsymbol{Z}_2)$-bundle $F \to E \xrightarrow{p} B$ is called a finite $\boldsymbol{Z}_2$-covering if F is a finite set. In this case, just alike for the usual K-group (Atiyah [1]), we can define the geometric transfer homomorphism ("direct image")

$$\tau : K_R(E) \longrightarrow K_R(B) .$$

PROPOSITION 2.4.

$$\tau = p_! : K_R^0(E) \longrightarrow K_R^0(B) .$$

PROOF. As F is of zero-dimensional, the normal bundle $\nu(F)$ in the construction of the trace is isomorphic to $F \times W$, and the trace

$$t(p) : B_+ \wedge V^c \longrightarrow E_+ \wedge V^c$$

is by definition the one-point compactification of the inclusion

*) Added in proof: The proof of Proposition 2.3 works only in the case $G=\boldsymbol{Z}_2$, $\Gamma=U(n)$, α; $\boldsymbol{Z}_2 \to \mathrm{Aut}\, U(n)$ is the conjugation. This was pointed out to us by K. Iriye.

$$B\times V\cong(\tilde{E}\times_{\Gamma}W)\times_{B}\eta \overset{i}{\hookleftarrow} (\tilde{E}\times_{\Gamma}\nu(F))\times_{B}\eta\cong E\times V\,.$$

We regard the real $\boldsymbol{Z}_2$-module V as a complex vector space with the usual $\boldsymbol{Z}_2$-action given by the conjugation of complex numbers. Then the suspension isomorphism σ^V in the K_R^*-theory is defined by the composition

$$K_R(X) \xrightarrow{\otimes\Lambda} K_R(X\times V) \underset{f}{\overset{\cong}{\longleftarrow}} \tilde{K}_R(X_+\wedge V^c)$$

where Λ is the Thom element and f is the canonical isomorphism (see [**10**]).

We shall use the following lemma.

LEMMA 2.5. *Let $i: X\hookrightarrow Y$ be the inclusion of an open subset. Assume that Y is locally compact and that every Real vector bundle over Y has an inverse bundle. Let*

$$\xi=[\xi_0 \xrightarrow{\partial} \xi_1]\in K_R(X)$$

and

$$\zeta=[\zeta_0 \xrightarrow{\partial'} \zeta_1]\in K_R(Y)$$

be complexes of Real vector bundles over X and Y respectively. Namely ∂ is a bundle homomorphism which is isomorphic outside a compact subset supp (ξ) *of X, similarly for ∂'.*

Assume that

$$\text{supp}\,(\zeta)\subset X \quad and \quad \zeta|_X=\xi$$

as an element of $K_R(X)$. Then

$$\zeta=f\circ(i^c)^*\circ f^{-1}(\xi)\,,$$

where $(i^c)^*: \tilde{K}_R(X^c)\to\tilde{K}_R(Y^c)$ is induced by the one-point compactification of i.

PROOF. This follows easily from the excision property of K_R-groups (see [**10**]).

We return to the proof of Proposition 2.4.

Take a Real vector bundle α over E. We have to show that

$$\tau(\alpha)\otimes\Lambda=f\circ(i^c)^*\circ f^{-1}(\alpha\otimes\Lambda)$$

in $K_R(B\times V)$, where

$$(i^c)^*=t(p)^*: \tilde{K}_R(E_+\wedge V^c) \longrightarrow \tilde{K}_R(B_+\wedge V^c)\,.$$

Here

$$\alpha\otimes\Lambda=[\alpha\otimes\lambda_0 \xrightarrow{id\otimes\partial^E} \alpha\otimes\lambda_1]\in K_R(E\times V)$$

and

$$\tau(\alpha)\otimes\Lambda=[\tau(\alpha)\otimes\lambda_0 \xrightarrow{id\otimes\partial^B} \tau(\alpha)\otimes\lambda_1]\in K_R(B\times V)\,,$$

where λ_0 and λ_1 are trivial bundles and $\operatorname{supp}(\alpha\otimes\Lambda)=E\times\{0\}$, $\operatorname{supp}(\tau(\alpha)\otimes\Lambda)=B\times\{0\}$.

Let $\{U_\mu\}$ be a $\boldsymbol{Z}_2$-invariant finite open covering of B such that each $p|_{U_\mu}$ is trivial, say

$$p^{-1}(U_\mu)=\coprod_{j=1}^{k} W_\mu^j .$$

Then by definition of the geometric transfer,

$$\tau(\alpha)|_{U_\mu}=\bigoplus_{j=1}^{k}\alpha|_{W_\mu^j} .$$

We define a complex vector bundle homomorphism

$$\psi_\mu^j(s):\ \alpha|_{W_\mu^j}\otimes\lambda_0 \longrightarrow \alpha|_{W_\mu^j}\otimes\lambda_1 \qquad (s\in I)$$

over $W_\mu^j\times V\cong U_\mu\times V$ by

$$\psi_\mu^j(s)_{(x,v)}(y\otimes w)=y\otimes\partial^B_{v-s\pi\circ i(x,0)}(w)$$

for $x\in W_\mu^j$, $y\in\alpha_x$, v, $w\in V$, $s\in I$, where $\pi: B\times V\to V$ is the projection.

Then $\bigoplus\limits_{j=1}^{k}\psi_\mu^j$ defines a bundle homomorphism

$$\Delta_\mu:\ (\bigoplus_j \alpha|_{W_\mu^j}\otimes\lambda_0)\times I \longrightarrow (\bigoplus_j \alpha|_{W_\mu^j}\otimes\lambda_1)\times I$$

over $U_\mu\times V\times I$. Since π and i are $\boldsymbol{Z}_2$-maps, Δ_μ is a Real vector bundle homomorphism.

Now $\{\Delta_\mu\}$ defines a bundle homomorphism

$$\Delta:\ (\tau(\alpha)\otimes\lambda_0)\times I \longrightarrow (\tau(\alpha)\otimes\lambda_1)\times I$$

because $\pi\circ i(x, 0)\in V$ does not depend on the choice of μ with $x\in W_\mu^j\subset E$. It is clear that $\Delta|_{B\times V\times\{0\}}=id\otimes\partial^B$ and $\operatorname{supp}(\Delta|_{B\times V\times\{1\}})=i(E\times\{0\})\subset B\times V$. Thus if we define

$$\beta=[\tau(\alpha)\otimes\lambda_0 \xrightarrow{\Delta|_{B\times V\times\{1\}}} \tau(\alpha)\otimes\lambda_1]\in K_R(B\times V),$$

we have $\beta=\tau(\alpha)\otimes\Lambda$ in $K_R(B\times V)$.

If $\beta|_{E\times V}=\alpha\otimes\Lambda$ in $K_R(E\times V)$, then Lemma 2.5 applies, and the proposition follows.

Consider the canonical injection

$$\alpha \hookrightarrow p^*\tau(\alpha)$$

of bundles over E. This injection splits:

$$p^*\tau(\alpha)=\alpha\oplus\gamma, \qquad \text{where } \gamma|_{W_\mu^j}=\bigoplus_{k\neq j}\alpha|_{W_\mu^k} .$$

We apply this splitting to

$$\beta|_{E\times V}=[p^*\tau(\alpha)\otimes\lambda_0 \xrightarrow{\Delta|_{E\times V\times\{1\}}} p^*\tau(\alpha)\otimes\lambda_1].$$

The restriction to $\alpha\otimes\lambda_0$ of Δ is equal to $id\otimes\iota^*\partial^E$, where ι denotes the inclusion of an open neighborhood of zero, which is identified with V, because if we restrict Δ to $\alpha\otimes\lambda_0|_{W^j_\mu\times V}$, then

$$\Delta_{(x,v)}(y\otimes w)=\phi^j_\mu(1)_{(x,\pi i(x,v))}(y\otimes w)$$
$$=y\otimes\partial^B_{\pi i(x,v)-\pi i(x,0)}(w).$$

On the other hand, the restriction to $\gamma\otimes\lambda_0$ of Δ is an isomorphism, because the restriction to $\gamma\otimes\lambda_0|_{W^j_\mu\times V}$ is $\bigoplus_{k\neq j}\phi^k_\mu(1)$, the domain of which is out of supp (∂^B).

Now clearly

$$[\alpha\otimes\lambda_0 \xrightarrow{id\otimes\iota^*\partial^E} \alpha\otimes\lambda_1]=[\alpha\otimes\lambda_0 \xrightarrow{id\otimes\partial^E} \alpha\otimes\lambda_1]$$
$$=\alpha\otimes\Lambda\in K_R(E\times V)$$

by Lemma 2.5. Thus

$$\beta|_{E\times V}=\alpha\otimes\Lambda\oplus[\gamma\otimes\lambda_0 \xrightarrow{\cong} \gamma\otimes\lambda_1]=\alpha\otimes\Lambda$$

in $K_R(E\times V)$. This completes the proof.

REMARK 2.6. The same proof applies to the K^*_G-theory for a finite group G, and

$$\tau=p_!:\ K^0_G(E)\longrightarrow K^0_G(B).$$

§ 3. Some homogeneous spaces.

Let $U(n)$ be the unitary group with the involutive automorphism $\alpha: \boldsymbol{Z}_2\to \operatorname{Aut} U(n)$ given by $\alpha(g)(A)=\bar{A}$, where $\bar{A}$ is the conjugate of $A\in U(n)$. If H is a closed $\boldsymbol{Z}_2$-subgroup of $U(n)$, then the homogeneous space $U(n)/H$ is a $\boldsymbol{Z}_2$-manifold.

Let T^n be the maximal torus of $U(n)$ and let $N(T^n)$ be the normalizer of T^n. Then we have a $\boldsymbol{Z}_2$-manifold $U(n)/N(T^n)$. In order to determine the $\boldsymbol{Z}_2$-Euler characteristic of $U(n)/N(T^n)$, we consider the fixed point submanifold $(U(n)/N(T^n))^{\boldsymbol{Z}_2}$.

Given a matrix $A\in U(n)$, $[A]$ denotes the class in $U(n)/N(T^n)$. If $[A]\in (U(n)/N(T^n))^{\boldsymbol{Z}_2}$, then $\bar{A}=A\cdot S\cdot P$ for some $S\in\Sigma_n\subset U(n)$ and $P\in T^n$. Since $\bar{\bar{A}}=A$, we see easily that $S^2=I$, $S\cdot P=P\cdot S$, and that the conjugacy class of S in Σ_n is independent of the choice of A in the class $[A]$. Then clearly $(U(n)/N(T^n))^{\boldsymbol{Z}_2}$ is the disjoint union $\coprod_{k=0}^{[n/2]} C_k$, where the open set C_k consists of $[A]$ whose S is conjugate to

$$S_{n,k}=\begin{pmatrix} D & & & & & \\ & \ddots & & & 0 & \\ & & D & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}\begin{matrix} \}k \\ \\ \}n-2k \end{matrix}, \quad D=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in Σ_n.

By the multiplication from the left, we consider $(U(n)/N(T^n))^{Z_2}$ as an $O(n)$-space.

LEMMA 3.1. *C_k is the homogeneous space $O(n)/N(T^k\times \boldsymbol{Z}_2^{n-2k})$.*

PROOF. Let Q be the (2×2)-matrix of the form

$$Q=1/\sqrt{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and let

$$Q_{n,k}=\begin{pmatrix} Q & & & & & \\ & \ddots & & & 0 & \\ & & Q & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}\begin{matrix} \}k \\ \\ \}n-2k \end{matrix}.$$

Clearly $[Q_{n,k}]\in C_k$. Let $[A]\in C_k$ be given, and let $\bar{A}=A\cdot S\cdot P$, where $S\in\Sigma_n$ and $P\in T^n$. We can now assume that

$$S=S_{n,k}.$$

Note that

$$\bar{Q}_{n,k}=Q_{n,k}\cdot S\cdot\begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix}.$$

Then put

$$X=A\cdot S\cdot P'\cdot Q_{n,k}^{-1},$$

where $P'\in T^n$ satisfies $P\cdot\begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix}=P'^2$. Then one can easily check that $X\in O(n)$, and hence the action of $O(n)$ on C_k is transitive.

Let H be the isotropy subgroup of $[Q_{n,k}]$. Then $H\ni X$ if and only if $X\cdot Q_{n,k}=Q_{n,k}\cdot S\cdot P$ for some $S\in\Sigma_n$ and $P\in T^n$. Since $X=Q_{n,k}\cdot S\cdot P\cdot Q_{n,k}^{-1}$ satisfies $\bar{X}=X$, a direct calculation shows that

$$S\cdot S_{n,k}=S_{n,k}\cdot S \quad\text{and}\quad S_{n,k}\cdot P=\bar{P}\cdot S_{n,k}.$$

This means that

$$S\in\left(\Sigma_k\int\Sigma_2\right)\times\Sigma_{n-2k}$$

and

$$P=\mathrm{diag}\,(\lambda_1, \bar{\lambda}_1, \cdots, \lambda_k, \bar{\lambda}_k, \pm 1, \cdots, \pm 1),$$

namely,

$$X=S\cdot Q_{n,k}\cdot P\cdot Q_{n,k}^{-1}\in N(SO(2)^k\times \boldsymbol{Z}_2^{n-2k}).$$

This completes the proof.

LEMMA 3.2.

$$\chi_{Z_2}(U(n)/N(T^n))=1\in A(\boldsymbol{Z}_2).$$

PROOF. By Lemma 2.1, it is sufficient to show that $\chi(U(n)/N(T^n))=1$ and $\chi((U(n)/N(T^n))^{Z_2})=1$. The first equality is well known ([6], p. 27). Now

$$\chi((U(n)/N(T^n))^{Z_2})=\sum_k \chi(C_k).$$

We have a fibre bundle

$$O(2k)/N_{O(2k)}(T^k) \longrightarrow O(n)/N_{O(n)}(T^k\times \boldsymbol{Z}_2^{n-2k})$$
$$\longrightarrow O(n)/O(2k)\times N_{O(n-2k)}(\boldsymbol{Z}_2^{n-2k}).$$

It is also known [6] that $\chi(O(2k)/N(T^k))=1$. If $k=[n/2]$, then the base space is either one point or $\boldsymbol{R}P^{n-1}=O(n)/O(n-1)\times \boldsymbol{Z}_2$. Hence the Euler characteristic is 1. Now if $k<[n/2]$, then $\chi(O(n)/O(2k))=0$ and $N_{O(n-2k)}(\boldsymbol{Z}_2^{n-2k})$ is finite. Hence $\chi(O(n)/O(2k)\times N_{O(n-2k)}(\boldsymbol{Z}_2^{n-2k}))=0$. This completes the proof.

Now let $F_n=U(2n)/N(T^{2n})$. Then F_n is a $U(2n)\times_\alpha \boldsymbol{Z}_2$-space. Let $U(2n)\subset U(2n)\times U(2)\subset U(2n+2)$ be the standard inclusion and we regard F_{n+1} as a $U(2n)\times_\alpha \boldsymbol{Z}_2$-space via the standard inclusion. We define a $U(2n)\times_\alpha \boldsymbol{Z}_2$-embedding

$$j: F_n \longrightarrow F_{n+1}$$

by $j([A])=[A\oplus Q]$ for $[A]\in U(2n)/N(T^{2n})$, where Q is the (2×2)-matrix defined in Lemma 3.1. Let

$$S^1=SO(2)\cong \{e\}\times SO(2)\subset U(2n)\times U(2)\subset U(2n+2).$$

Then S^1 is a subgroup of $U(2n+2)$ with the trivial $\boldsymbol{Z}_2$-action. Therefore F_{n+1} is a $(U(2n)\times_\alpha \boldsymbol{Z}_2)\times S^1$-manifold.

LEMMA 3.3. $F_n=(F_{n+1})^{S^1}$.

PROOF. Remark that $[Q]\in F_1^{Z_2}$ lies in $C_1=C_1(F_1)$ in Lemma 3.1 and $C_1(F_1)=*$. Hence $[X\cdot Q]=[Q]$ for any $X\in O(2)$. Let $[A]\in F_n$ and let $X\in S^1$. Then

$$X\cdot[A\oplus Q]=[A\oplus X\cdot Q]=[A\oplus Q]\in F_{n+1},$$

hence $F_n\subset(F_{n+1})^{S^1}$. Next let $[B]\in(F_{n+1})^{S^1}$. Then

$$B^{-1}\cdot S^1\cdot B\subset N(T^{2n+2})\subset U(2n+2).$$

This implies that $[B]$ is of the form $[A\oplus Y]$ for some $Y\in U(2)$, and since $Y^{-1}\cdot S^1\cdot Y\subset N(T^2)$, we see that $Y\in[Q]$ and $[B]=[A\oplus Q]$. This completes the proof.

Now we define $\boldsymbol{Z}_2$-equivariant homogeneous spaces of $U(4n)$:

$$B_n=U(4n)/U(2n)\times U(2n)$$

$$E_n=U(4n)/N_{U(2n)}(T^{2n})\times U(2n)$$

$$\bar{E}_n=U(4n)/T^1\times N_{U(2n-1)}(T^{2n-1})\times U(2n)$$

$$P_n=U(4n)/T^1\times U(4n-1)\,.$$

There are canonical projections:

$$p_n:\ E_n\longrightarrow B_n$$

$$\pi_n:\ \bar{E}_n\longrightarrow E_n$$

$$q_n:\ \bar{E}_n\longrightarrow P_n\,,$$

among which p_n is a $(U(2n),\ \alpha,\ \boldsymbol{Z}_2)$-bundle with fibre $F_n=U(2n)/N(T^{2n})$ associated with the principal $(U(2n),\ \alpha,\ \boldsymbol{Z}_2)$-bundle $U(4n)/U(2n)\to U(4n)/U(2n)\times U(2n)$, and π_n is a $(\Sigma_{2n},\ \textit{trivial},\ \boldsymbol{Z}_2)$-bundle with fibre $[2n]=\Sigma_{2n}/\Sigma_{2n-1}$ associated with the principal $(\Sigma_{2n},\ \textit{trivial},\ \boldsymbol{Z}_2)$-bundle $U(4n)/T^{2n}\times U(2n)\to U(4n)/N(T^{2n})\times U(2n)$.

Let $\boldsymbol{C}^{4n}\to\boldsymbol{C}^{4n+4}$ be the inclusion given by

$$(z_1,\ \cdots,\ z_{4n})\longmapsto(z_1,\ \cdots,\ z_{2n},\ 0,\ 0,\ z_{2n+1},\ \cdots,\ z_{4n},\ 0,\ 0)\,.$$

Then associated with this inclusion, we can define an inclusion

$$\oplus:\ U(4n)\times U(2)\times U(2)\longrightarrow U(4n+4)\,.$$

Let $I\in U(2)$ be the unit matrix and $Q\in U(2)$ as before. Then we define $\boldsymbol{Z}_2$-embeddings

$$B_n\longrightarrow B_{n+1}\quad\text{and}\quad P_n\longrightarrow P_{n+1}$$

by $[A]\to[A\oplus I\oplus I]$ for $[A]\in B_n$ or P_n, and

$$E_n\longrightarrow E_{n+1}\quad\text{and}\quad \bar{E}_n\longrightarrow \bar{E}_{n+1}$$

by $[A]\mapsto[A\oplus Q\oplus I]$ for $[A]\in E_n$ or $\bar{E}_n$, respectively. Note that $Q^2\in N(T^2)$ and the latter ones are $\boldsymbol{Z}_2$-embeddings.

We define $B=BR=\varinjlim B_n$, $P=\varinjlim P_n$, $E=\varinjlim E_n$ and $\bar{E}=\varinjlim \bar{E}_n$. It is clear that P is the complex projective space $\boldsymbol{C}P^\infty$ with the canonical involution given in § 1. Note that B_n is the Grassmann manifold with the involution given by the conjugation. Then it is easy to see the following

LEMMA 3.4. *Let X be a based compact $\boldsymbol{Z}_2$-space. Then there is a natural isomorphism*

$$[X,\ BR]_{Z_2}\cong\tilde{K}_R(X).$$

§ 4. Proof of the theorem.

Consider the commutative diagram

$$\begin{array}{ccccc} E_n & \overset{j}{\hookrightarrow} & i^*E_{n+1} & \overset{i}{\hookrightarrow} & E_{n+1} \\ {\scriptstyle p_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_{n+1}} \\ B_n & = & B_n & \overset{i}{\hookrightarrow} & B_{n+1} \end{array}$$

where i^*E_{n+1} is the bundle induced from $p_{n+1}: E_{n+1} \to B_{n+1}$. Then i^*E_{n+1} is a $(U(2n), \alpha, \boldsymbol{Z}_2)$-bundle with fibre F_{n+1} associated with the principal $(U(2n), \alpha, \boldsymbol{Z}_2)$-bundle $U(4n)/U(2n) \to B_n$. The inclusion $j: E_n \hookrightarrow i^*E_{n+1}$ in the above diagram is the map induced from the $U(2n) \times_\alpha \boldsymbol{Z}_2$-embedding

$$F_n \hookrightarrow F_{n+1} .$$

By Lemma 3.3, F_{n+1} is a $(U(2n) \times_\alpha \boldsymbol{Z}_2) \times S^1$-manifold and F_n is the submanifold of S^1-fixed points. Then the S^1-flow determines a non-singular $U(2n) \times_\alpha \boldsymbol{Z}_2$-invariant vector field Δ outside F_n satisfying the condition of Proposition 2.2. Therefore by Proposition 2.2 and by the naturality of the trace [8, Proposition 4.4], we see that the diagram of stable $\boldsymbol{Z}_2$-maps

$$\begin{array}{ccc} (E_n)_+ & \longrightarrow & (E_{n+1})_+ \\ {\scriptstyle t(p_n)}\uparrow & & \uparrow{\scriptstyle t(p_{n+1})} \\ (B_n)_+ & \longrightarrow & (B_{n+1})_+ \end{array}$$

is $\boldsymbol{Z}_2$-homotopy commutative. Therefore we can define a stable $\boldsymbol{Z}_2$-map

$$t: B_+ \longrightarrow E_+ .$$

Let $p = \varinjlim p_n : E \to B$ and let

$$\tilde{p}: Q_{Z_2}(E_+) \xrightarrow{Q_{Z_2}(p_+)} Q_{Z_2}(B_+) \xrightarrow{r} Q_{Z_2}(B) \xrightarrow{\xi} B = BR$$

be the extension of p, where r is the projection.

LEMMA 4.1. *$\tilde{p}_*: \{X, E_+\}_{Z_2} \to \tilde{K}_R(X)$ is a split epimorphism.*

PROOF. Let

$$p_{n!}: \pi^*_{Z_2}(E_n) \longrightarrow \pi^*_{Z_2}(B_n)$$

be the transfer homomorphism of the $(U(2n), \alpha, \boldsymbol{Z}_2)$-bundle

$$F_n \longrightarrow E_n \xrightarrow{p_n} B_n ,$$

where $\pi^*_{Z_2}(\ )$ is the stable $\boldsymbol{Z}_2$-cohomotopy theory (see [8]). Then by Proposition 2.3 and Lemma 3.2, the composition

$$p_{n!}\circ p_n^*:\ \pi^*_{Z_2}(B_n) \longrightarrow \pi^*_{Z_2}(B_n)$$

is an isomorphism. Therefore the limit of this composition

$$t^*\circ p^*:\ \pi^*_{Z_2}(B) \longrightarrow \pi^*_{Z_2}(B)$$

is an isomorphism, and hence by the equivariant S-duality theorem ([**13**]) the composition of stable $\boldsymbol{Z}_2$-maps

$$p_+\circ t:\ B_+ \xrightarrow{t} E_+ \xrightarrow{p_+} B_+$$

is a $\boldsymbol{Z}_2$-homotopy equivalence.

Let

$$\gamma:\ \{X, B_+\}_{Z_2} \longrightarrow \{X, E_+\}_{Z_2}$$

be the homomorphism induced by the stable map $t: B_+\to E_+$. Then the composition

$$p_*\circ\gamma:\ \{X, B_+\}_{Z_2} \longrightarrow \{X, B_+\}_{Z_2}$$

is an isomorphism, and hence

$$p_*:\ \{X, E_+\}_{Z_2} \longrightarrow \{X, B_+\}_{Z_2}$$

is a split epimorphism.

On the other hand

$$(\xi\circ r)_*:\ \{X, B_+\}_{Z_2} \longrightarrow K_R(X)$$

is a split epimorphism because the composition

$$\xi\circ r\circ k\circ i:\ B \xrightarrow{i} Q_{Z_2}(B) \xrightarrow{k} Q_{Z_2}(B_+) \xrightarrow{r} Q_{Z_2}(B) \xrightarrow{\xi} B$$

is $\boldsymbol{Z}_2$-homotopic to the identity, where i is the canonical inclusion and k is the right adjoint of the projection r.

This completes the proof.

Next consider the $2n$-fold $\boldsymbol{Z}_2$-covering

$$\pi_n:\ \bar{E}_n \longrightarrow E_n\,.$$

The following diagram is commutative:

$$\begin{array}{ccccc}
E_n & \xleftarrow{\pi_n} & \bar{E}_n & \xrightarrow{q_n} & P_n \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
E_{n+1} & \xleftarrow{\pi_{n+1}} & \bar{E}_{n+1} & \xrightarrow{q_{n+1}} & P_{n+1}
\end{array}$$

Let the stable map $t(\pi_n):(E_n)_+\to(\bar{E}_n)_+$ be the trace of π_n. Then we have the following

LEMMA 4.2. *The diagram of stable maps*

$$\begin{array}{ccccc} (E_n)_+ & \xrightarrow{t(\pi_n)} & (\bar{E}_n)_+ & \xrightarrow{q_n\circ r} & P_n \\ \downarrow & & & & \downarrow \\ (E_{n+1})_+ & \xrightarrow{t(\pi_{n+1})} & (\bar{E}_{n+1})_+ & \xrightarrow{q_{n+1}\circ r} & P_{n+1} \end{array}$$

is Z_2-homotopy commutative.

PROOF. An easy calculation shows that $\pi_{n+1}^{-1}(E_n)-\bar{E}_n$ is a disjoint union of two copies of E_n, which is mapped by q_{n+1} to a free Z_2-orbit (=two points) in P_{n+1}.

Since P_{n+1} is arcwise connected, any stable Z_2-map

$$(Z_2)_+ \longrightarrow P_{n+1}$$

is Z_2-homotopic to the trivial map, and the lemma follows.

By Lemma 4.2, we obtain a stable Z_2-map

$$\rho:\ E_+ \longrightarrow P\,.$$

Now Theorem 1.1 follows immediately from Lemma 4.1 and the following

LEMMA 4.3. *The diagram*

$$\begin{array}{ccc} & \overset{\rho_*}{\nearrow} & \{X, P\}_{Z_2} \\ \{X, E_+\}_{Z_2} & & \downarrow \lambda_* \\ & \underset{\tilde{p}_*}{\searrow} & \tilde{K}_R(X) \end{array}$$

is commutative.

PROOF. Proposition 2.4 gives the geometric description of the trace of π_n, and the lemma is proved similarly to the proof of (4.4) of [4].

Finally we prove Proposition 1.3. By the result of [9], we have a homotopy equivalence

$$Q_{Z_2}(CP_+^\infty)^{Z_2}\simeq Q((CP_+^\infty)^{Z_2})\times Q((EZ_2\times_{Z_2}CP^\infty)_+)\,.$$

Now clearly $(CP^\infty)^{Z_2}=BO(1)$ and $EZ_2\times_{Z_2}CP^\infty=BO(2)$. Hence

$$\{X, CP^\infty\}_{Z_2}\cong\{X, BO(1)\}\oplus\{X, BO(2)\}$$

for a CW-complex X with trivial Z_2-action.

Let $C_n^{Z_2}(X; B)$ be the set of isomorphism classes of pairs (π, f), where $\pi: \tilde{X}\to X$ is an n-fold $\boldsymbol{Z}_2$-covering and $f: \tilde{X}\to B$ is a $\boldsymbol{Z}_2$-map. In [**9**] it is shown that the functor $C_n^{Z_2}(X; B)$ is classified by the $\boldsymbol{Z}_2$-space $(E_{Z_2}\Sigma_n\times B^n)/\Sigma_n$ where $E_{Z_2}\Sigma_n$ is the universal Σ_n-free $\boldsymbol{Z}_2$-contractible $\Sigma_n\times\boldsymbol{Z}_2$-space, and that there is the equivariant Barratt-Quillen map

$$\omega_n: (E_{Z_2}\Sigma_n\times B^n)/\Sigma_n \longrightarrow Q_{Z_2}(B_+).$$

Now let $B=P=\boldsymbol{C}P^\infty$. The map which gives the above homotopy equivalence is given as follows.

Let

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & P^{Z_2} \\ {\scriptstyle\pi}\downarrow & & \\ X & & \end{array}$$

be in $C_n(X; P^{Z_2})$; then we take

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{i\circ f} & P \\ {\scriptstyle\pi}\downarrow & & \\ X & & \end{array}$$

in $C_n^{Z_2}(X; P)$, where $i: P^{Z_2}\hookrightarrow P$ is the inclusion and X is considered as a trivial $\boldsymbol{Z}_2$-space. Thus the map

$$Q((P_+)^{Z_2}) \longrightarrow Q_{Z_2}(P_+)$$

is the one induced by the canonical inclusion and we see that

$$\lambda_*|_{[X, BO(1)]}=\mu_{1*}.$$

On the other hand let

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & E\boldsymbol{Z}_2\times_{Z_2}P \\ {\scriptstyle\pi}\downarrow & & \\ X & & \end{array}$$

be in $C_n(X; E\boldsymbol{Z}_2\times_{Z_2}P)$; then we take

$$\begin{array}{ccc} \tilde{\tilde{X}} & \xrightarrow{\tilde{f}} & E\boldsymbol{Z}_2\times P\simeq P \\ {\scriptstyle\pi\circ f^*(q)}\downarrow & & \\ X & & \end{array}$$

in $C_{2n}^{Z_2}(X; P)$, where $f^*(q): \tilde{\tilde{X}}\to\tilde{X}$ is the 2-fold covering induced from the canonical covering

$$q: E\boldsymbol{Z}_2\times P \longrightarrow E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2} P.$$

Let $\xi\to P$ be the canonical line bundle. Then

$$\eta=E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}\xi \longrightarrow E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}P=BO(2)$$

coincides with the canonical $\boldsymbol{R}^2$-bundle.

In order to show that $\lambda_*|_{\{X,BO(2)\}}=\mu_{2*}$, we have only to show that the canonical line bundle $E\boldsymbol{Z}_2\times\xi$ over $E\boldsymbol{Z}_2\times P\simeq P$ is transferred by the covering q to the complexification $\eta\otimes\boldsymbol{C}$ over $E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}P=BO(2)$.

Let

$$\phi: E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}P \longrightarrow (E_{\boldsymbol{Z}_2}\Sigma_2\times P^2)/\Sigma_2$$

be the map which classifies the 2-fold $\boldsymbol{Z}_2$-covering

$$q\in C_2^{\boldsymbol{Z}_2}(E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}P; P).$$

Then a direct calculation shows that $((E_{\boldsymbol{Z}_2}\Sigma_2\times P^2)/\Sigma_2)^{\boldsymbol{Z}_2}$ is a disjoint union of $((E_{\boldsymbol{Z}_2}\Sigma_2)^{\boldsymbol{Z}_2}\times(P^{\boldsymbol{Z}_2})^2)/\Sigma_2$ and $E\times_{\Sigma_2}P$, where

$$E=\{e\in E_{\boldsymbol{Z}_2}\Sigma_2;\ \tau e=eg,\ \langle\tau\rangle=\boldsymbol{Z}_2,\ \langle g\rangle=\Sigma_2\}.$$

Since q is the canonical free $\boldsymbol{Z}_2$-covering, we can show that ϕ maps $E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}P$ onto $E\times_{\Sigma_2}P\subset((E_{\boldsymbol{Z}_2}\Sigma_2\times P^2)/\Sigma_2)^{\boldsymbol{Z}_2}$, and that $\phi(e, x)=(e, x, \bar{x})$. Therefore

$$\phi^*((E_{\boldsymbol{Z}_2}\Sigma_2\times\xi^2)/\Sigma_2)=E\boldsymbol{Z}_2\times_{\boldsymbol{Z}_2}(\xi\oplus\bar{\xi}),$$

which is easily seen to be $\eta\otimes\boldsymbol{C}$. This completes the proof.

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