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Hamiltonian circuits on simple 3-polytopes with up to 30 vertices

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§1. Introduction.

Klee in [5] asked what is the minimun number, n, of vertices for a simple 3-polytope with no Hamiltonian circuit, that is, no closed path on the edges of the polytope which goes through each vertex exactly once. The smallest known non-Hamiltonian simple 3-polytope has 38 vertices (see p. 359 in [5]), so $n \leqq 38$. Lederberg [6] proved $n \geqq 20$, Butler [2] and Goodey [4] proved $n \geqq 24$, Barnette and Wegner [1] proved $n \geqq 28$. In this paper we show $n \geqq 32$.

THEOREM. *Every simple* 3-*polytope of order* 30 *or less is Hamiltonian.*

By Steinitz's theorem [5, p. 235] a graph is the graph of a simple 3-polytope if and only if it is planar, 3-connected and 3-valent. A set S of edges of a graph is called a cut if the removal of these edges separates G into two connected components and no proper subset of S has this property. If the cardinality of the cut is k it will be called a k-cut. The components separated by a k-cut are called k-pieces. A cut will be called non-trivial if each of its k-pieces contains a circuit, trivial otherwise. A non-trivial k-cut will be called essential if each of its k-pieces contains more than k vertices, non-essential otherwise. A graph will be called cyclically k-connected if every l-cut with $l<k$ is trivial, it will be called cyclically exactly k-connected if it is cyclically k-connected but not cyclically $(k+1)$-connected. The order of a graph G will be denoted by $|G|$.

§2. Preliminaries.

In this section we prepare some lemmas. By [2] and [4] we have Lemma 1.

LEMMA 1. *In any simple* 3-*polytope of order* 22 *or less each edge is used by some Hamiltonian circuit.*

By [3] we have Lemma 2.

LEMMA 2. *Any minimal non-Hamiltonian simple* 3-*polytope of order* 34 *or less is cyclically exactly* 4-*connected and has no essential* 4-*cut.*

In what follows, let G be a minimal non-Hamiltonian simple 3-polytope of order 30 or less. By [1] we have $|G|=28$ or 30. By Lemma 2 we have Lemma 3.

LEMMA 3. *G can not contain adjacent quadrilaterals.*

The number of k-gons of G and edges of a face f will be denoted by p_k and $e(f)$ respectively. Then the following equation holds [5, p. 254].

$$3p_3+2p_4+p_5=12+\sum_{k\geq 7}(k-6)p_k. \qquad (1)$$

§3. Proof of Theorem.

LEMMA 4x. *G can not contain a part as illustrated in Figure* 1x (x=a, b, ⋯, f. *When* x=e, *let* $|G|=28$).

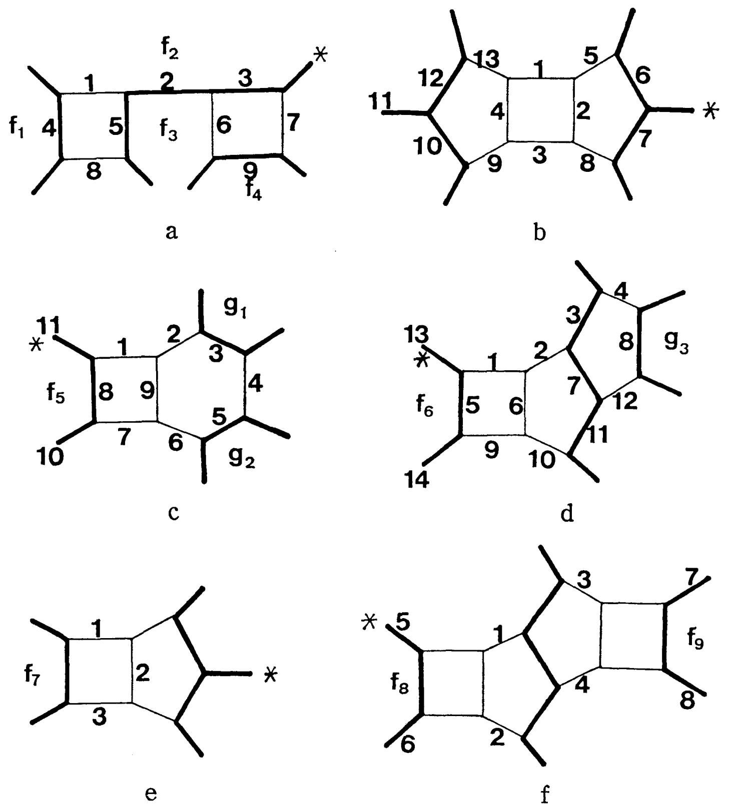

Figure 1.

PROOF. If G contains one of the parts as illustrated in Figure 1, then we replace this part by a part as indicated by heavy lines, producing a new graph G'. In Figure 1 we have $e(f_i)\geqq 5$ $(i=1, \cdots, 9)$ by Lemma 3. If $e(g_1)$ or $e(g_2)=4$ then G contains a part as illustrated in Figure 1a, thus we may assume that $e(g_1)$, $e(g_2)\geqq 5$. Similarly we may assume that $e(g_3)\geqq 5$ by Figure 1f.

First we will show that G' is 3-connected. Note that if G has a non-trivial 4-cut, then one of the 4-pieces is a quadrilateral, since G has no essential 4-cut by Lemma 2. In Figure 1a G has no non-trivial 4-cut with 1, 6, 7 or 8, since $e(f_i)\geqq 5$ $(i=1, 2, 3, 4)$; and G has no non-trivial 5-cut or 6-cut with three or four of 1, 6, 7, 8 respectively. Thus G' is 3-connected. Since $e(f_7)\geqq 5$, in Figure 1b, 1e $G-\{1, 3\}$ is 3-connected, and so G' is 3-connected. In Figure 1c the only non-trivial 4-cut with 2, 4 or 6 is $\{2, 6, 10, 11\}$, since $e(g_1)$, $e(g_2)\geqq 5$; and G has no cut with 2, 4, 6. Thus G' is 3-connected. In Figure 1f the non-trivial 4-cuts with 1, 2, 3 or 4 are $\{1, 2, 5, 6\}$ and $\{3, 4, 7, 8\}$, and G has no non-trivial 5-cut or 6-cut with three or four of 1, 2, 3, 4 respectively. Thus G' is 3-connected. In Figure 1d G' is similarly 3-connected.

Now $|G'|\leqq 22$ and by Lemma 1 G' has a Hamiltonian circuit H' using the edge marked by an asterisk. Then G is also Hamiltonian, since H' extends to a Hamiltonian circuit H in G. Indeed in Figure 1a if $H'\ni 4, 9$ then $H=H'$, if $H'\not\ni 4, 9$ then $H=(H'-\{3, 5\})\cup\{7, 9, 6, 1, 4, 8\}$, if $H'\ni 4$ and $H'\not\ni 9$ then $H=(H'-\{3\})\cup\{7, 9, 6\}$ and if $H\not\ni 4$ and $H\ni 9$ then $H=(H'-\{5\})\cup\{1, 4, 8\}$. In Figure 1b $H'\ni 6$ or 7, say 6. If $H'\ni 11$ then $H\ni 10$ or 12, say 10, and $H=(H'-\{6, 10\})\cup\{7, 8, 2, 5, 9, 4, 13, 12\}$. If $H'\not\ni 11$ then $H'\ni 10, 12$ and $H=(H'-\{6\})\cup\{7, 8, 3, 4, 1, 5\}$. In Figure 2c if $H'\ni 3$, $H'\not\ni 5$ then $H=(H'-\{3\})\cup\{4, 5, 6, 9, 2\}$, if $H'\not\ni 3, 5$ then $H=(H'-\{8\})\cup\{1, 2, 3, 4, 5, 6, 7\}$, for other cases similar. In Figure 1d if $H'\ni 7, 8$ then $H=(H'-\{5\})\cup\{1, 6, 9\}$, if $H'\ni 7$, $H'\not\ni 8$ then $H=(H'-\{3, 11\})\cup\{4, 8, 12, 2, 6, 10\}$, if $H'\not\ni 7$, $H'\ni 8$ then $H=(H'-\{8\})\cup\{4, 3, 2, 6, 10, 11, 12\}$, if $H'\not\ni 7, 8$ then $H=(H'-\{5\})\cup\{1, 2, 3, 4, 8, 12, 11, 10, 9\}$. For Figure 1e, 1f the proofs are similar to Figure 1b, 1d respectively.

We will show that G contains one of the parts as illustrated in Figure 1 to obtain a contradiction. By Lemma 2 $p_3=0$ and $p_4>0$. By Lemma 3, 4a every k-gon with $k\geqq 5$ of G is adjacent to at most $[k/3]$ (which is the greatest integer $\leqq k/3$) quadrilaterals.

We assume that $|G|=28$. It is obvious that G contains a part as illustrated in Figure 1c or 1e when $\sum_{k\geqq 7} p_k\leqq 3$, and when >3 if the following inequality (2) is valid.

$$4p_4>\sum_{k\geqq 7}[k/3]p_k\,. \tag{2}$$

By (1) and $\sum_{k\geqq 4} p_k=16$, we have

$$p_4=p_6+\sum_{k\geqq 7} p_k+\sum_{k\geqq 7}(k-6)p_k-4\,. \tag{3}$$

When $\sum_{k\geqq 7} p_k\geqq 4$, we have (2) from (3) as follows.

$$4p_4\geqq 4\sum_{k\geqq 7} p_k+4\sum_{k\geqq 7}(k-6)p_k-16\geqq \sum_{k\geqq 7}4(k-6)p_k>\sum_{k\geqq 7}[k/3]p_k\,.$$

Thus we have $|G|=30$.

We can not use Lemma 4e. If $\sum_{k\geqq 7} p_k\leqq 1$ or the following inequality (4) is valid, then G contains a part as illustrated in Figure 1b or 1c.

$$2p_4 > \sum_{k\geqq 7} [k/3] p_k . \tag{4}$$

The other cases are in Table 1. Here, since $\sum_{k\geqq 4} p_k = 17$, if $p_4 \geqq 6$ then $p_5 + p_7 \leqq 11$ and we have (4) from (1) as follows.

$$2p_4 = 12 - p_5 + \sum_{k\geqq 7} (k-6) p_k > p_7 + \sum_{k\geqq 7} (k-6) p_k \geqq \sum_{k\geqq 7} [k/3] p_k .$$

Table 1.

	p_4	p_5	p_6	p_7	p_8	p_9
A	1	12	2	2	0	0
B	1	13	0	3	0	0
C	1	13	1	1	1	0
D	1	14	0	0	2	0
E	1	14	0	1	0	1
F	2	*	*		2	
G	2	11	1	3	0	0
H	2	12	0	2	1	0
I	3	9	2	3	0	0
J	3	10	0	4	0	0
K	3	10	1	2	1	0
L	3	11	0	1	2	0
M	3	11	0	2	0	1
N	3	12	0	0	0	2
O	4	8	1	4	0	0
P	4	9	0	3	1	0
Q	5	7	0	5	0	0

Let G be one type in Table 1. When $p_4=1$, $p_7+p_8+p_9 \leqq 3$, and when $p_4 \geqq 2$, (2) is valid, and so G has a quadrilateral adjacent to a pentagon. By Lemma 4b, 4c G contains a part as illustrated in Figure 2, where $e(f_i) \geqq 7$ $(i=3, 4)$.

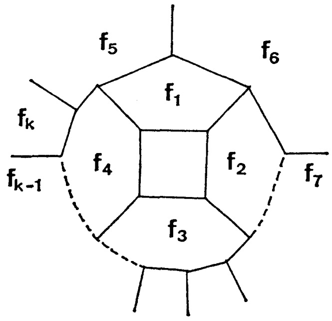

Figure 2.

By Lemma 2, $f_i \neq f_j$ $(5 \leqq i < j \leqq k)$. In G of Type (F), it is easy to see that G contains a part as illustrated in Figure 1a, 1b or 1c. When $\sum_{k \geqq 6} p_k \leqq 4$, if $e(f_2) = 5$ $(e(f_2) \geqq 7)$ then f_5, f_6 or f_7 (f_5 or f_6) must be a pentagon, contrary to Lemma 4d. In G of type (I), (O) or (Q), $e(f_2)=5$, $e(f_3)=7$ and f_8 or f_9 must be a quadrilateral, since $2p_4 = 2p_7 + 2p_8 + 3p_9$ and by Lemma 4a, 4b, 4c. If $e(f_9)=4$, then $e(f_i)=7$ $(i=8$ or $10)$; hence f_5, f_6 or f_7 must be a pentagon, contrary to Lemma 4d. Suppose that $e(f_8)=4$. If $e(f_i) \neq 5$ $(i=5, 6, 7)$ then $e(f_j)=5$ $(j=9, 10)$, contrary to Lemma 4d. This completes the proof of Theorem.

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