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Impossibility criterion of being an ample divisor*)

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In [**So 1**], Sommese gave many examples of manifolds that cannot be ample divisors in any manifold. His theory works also to construct non-smoothable singularities (see [**So 2**]). In this note we give the following criterion:

THEOREM. *Let A be a manifold such that $H^1(A, T[-L])=0$ for any ample line bundle L on A, where T is the tangent bundle of A. Then A cannot be an ample divisor in any manifold unless $A \cong \boldsymbol{P}^n$.*

As we shall see in §1, this result follows easily from a characterization theorem of projective spaces due to Mori-Sumihiro [**MS**]. In §2, we show that various types of manifolds, including many of those in [**So 1**], satisfy the above criterion. In §3, similarly as in [**So 2**], we construct examples of non-smoothable singularities.

Notation, convention and terminology.

Usually we employ the notation which is commonly used in algebraic geometry. We work in the category of $\boldsymbol{C}$-schemes of finite type. In most cases everything is assumed to be proper over Spec($\boldsymbol{C}$). *Point* means a closed point. *Variety* is an irreducible, reduced scheme. *Manifold* is a non-singular variety. Vector bundles are confused with locally free sheaves. Line bundles are regarded as linear equivalence classes of divisors, and their tensor products are denoted additively.

Now we list up some symbols.

$[D]$: The line bundle associated with a (Cartier) divisor D.

$\mathcal{F}[L]$: $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}$, where $\mathcal{F}$ is a coherent sheaf and $\mathcal{L}$ is the invertible sheaf corresponding to a line bundle L.

T^M: The tangent bundle of a manifold M.

E_X: The pull back of a vector bundle E on Y by a morphism $X \to Y$. Sometimes we write simply E instead of E_X, when there is no danger of confusion.

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For example, if $D \subset M$, T^M_D denotes the restriction of the tangent bundle of M to D.

§ 1. Proof of the criterion.

(1.1) Our starting point is the following result.

THEOREM (Mori-Sumihiro [**MS**]). *Let D be an effective ample divisor on a manifold M. Suppose that $H^0(M, T^M[-D]) \neq 0$. Then $M \cong \boldsymbol{P}^n$. Moreover, if $n \geqq 2$, D is a hyperplane on it.*

(1.2) THEOREM. *Let A be a smooth ample divisor on a manifold M with* $\dim M = n$ *such that $H^1(A, T^A[-tA]_A) = 0$ for any $t > 0$. Then $M \cong \boldsymbol{P}^n$ and A is a hyperplane on it.*

PROOF. We have a natural exact sequence $0 \to T^A \to T^M_A \to [A]_A \to 0$ on A. Taking cohomologies after tensoring $[-tA]$, we get $H^1(A, T^A[-tA]) \to H^1(A, T^M[-tA]_A) \to H^1(A, [(1-t)A])$. The first term vanishes for $t > 0$ by assumption and the third term vanishes for $t > 1$ by the vanishing theorem of Kodaira. So $H^1(A, T^M[-tA]_A) = 0$ for $t \geqq 2$. Hence $H^1(M, T^M[-2A]) = 0$ by [**F1**, (2.2)]. Therefore $H^0(M, T^M[-A]) \to H^0(A, T^M[-A]_A)$ is surjective. On the other hand, in view of the exact sequence $H^0(A, T^M[-A]_A) \to H^0(A, [0]) \to H^1(A, T^A[-A]_A) = 0$, we infer that $H^0(A, T^M[-A]) \neq 0$. Thus we obtain $H^0(M, T^M[-A]) \neq 0$, and we apply (1.1) to prove the assertion.

(1.3) COROLLARY. *Let A be a manifold such that $H^1(A, T^A[-L]) = 0$ for any ample line bundle L on it. Then A cannot be an ample divisor in any manifold unless $A \cong \boldsymbol{P}^n$.*

(1.4) DEFINITION. If $H^q(A, T^A[-tL]) = 0$ for any positive integer t and $q = 0, 1$, we say that A satisfies the *NS-condition* with respect to L. If A satisfies the *NS*-condition with respect to any ample line bundle on it, we say that A satisfies the *NS*-condition.

If $A \cong \boldsymbol{P}^n$, then $H^0(A, T^A[-L]) \neq 0$ for the tautological line bundle L. Hence if A satisfies the *NS*-condition, then A cannot be an ample divisor in any manifold by (1.3).

If A satisfies the *NS*-condition with respect to any line bundle L such that $\kappa(L, A) = \dim A$, then we say that A satisfies the strong *NS*-condition.

REMARK. "*NS*" means presumably "negatively stable", "non-smoothable" or something like this (cf. (3.11)).

§ 2. Manifolds that satisfy the *NS*-condition.

(2.1) PROPOSITION. *Let $G_{n,r}$ be the Grassmann variety parametrizing r-codimensional linear subspaces of an n dimensional vector space. Then $G_{n,r}$ satisfies the strong NS-condition unless $r = 1$, $r = n-1$ or $(n, r) = (4, 2)$.*

This fact is well known among experts. Here we present an outline of our proof. Let the notation be as in [**F2**, § 4]. We may assume $n-r\geqq 3$ and $r\geqq 2$. The tangent bundle of $G_{n,r}=F_{\{r\}}$ is isomorphic to $\mathcal{H}om(E_{n-r}^{*}, E_r)$ and $\mathrm{Pic}(F_{\{r\}})$ is generated by $H_r=\det E_r$. So, similarly as in [**F2**, (4.18)], it suffices to show that $H^1(F, E_{j/j-1}^{\vee}\otimes E_{i/i-1}\otimes[-tH_r])=H^1(F, -H_j+H_{j-1}-tH_r+H_i-H_{i-1})=0$ for any $t>0$, $n\geqq j\geqq r+1$, $r\geqq i\geqq 1$. If $n>j$ and $i>1$, we use natural morphism $F_{\{j,j-1,r,i,i-1\}}\to F_{\{j-1,r,i\}}$. Since H_j+H_{i-1} is relatively ample and since the relative dimension of the natural morphism is greater than one, applying [**F1**, Corollary A6], we have the assertion. If $j<n-1$, we use $F_R\to F_{R-\{j\}}$, where R is the set $\{j, j-1, r, i, i-1\}$. If $i\geqq 3$, we use $F_R\to F_{R-\{i-1\}}$. Thus, it suffices to consider the remaining cases in which $(i, j)=(1, n)$, $(1, n-1)$ or $(2, n)$. If $(i, j)=(1, n)$, we can apply [**F1**, Corollary A6] to $F_R\to F_{R-\{r\}}$, since $n-r\geqq 3$ and $r\geqq 2$. If $(i, j)=(1, n-1)$, we use $F_R\to F_{R-\{j,r\}}$. If $(i, j)=(2, n)$ and $r>2$, we use $F_R\to F_{R-\{r\}}$. If $(i, j)=(2, n)$, $r=2$ and $t>1$, we use $F_R\to F_{R-\{2,1\}}$. If $(i, j)=(2, n)$, $r=2$ and $t=1$, we use $F_{R-\{2\}}\to F_{R-\{2,1\}}$. In any case [**F1**, Corollary A6] proves the required assertion.

(2.2) PROPOSITION. *Any abelian variety A with* $\dim A\geqq 2$ *satisfies the strong NS-condition.*

This fact is well known since $\kappa(L, A)=\dim A$ implies that L is ample.

(2.3) PROPOSISION. *Let A be a non-trivial product $A_1\times\cdots\times A_k$ of manifolds $A_1, \cdots, A_k$. Suppose in addition that A_i is immersed in an abelian variety for each i such that* $\dim A_i=\dim A-1$. *Then A satisfies the NS-condition.*

REMARK 1. A manifold is immersed in an abelian variety if and only if its cotangent bundle is generated by global sections.

REMARK 2. The assumption of (2.3) is satisfied in either of the following cases: a) $k\geqq 3$. b) $A=A_1\times A_2$, both A_1 and A_2 are immersed in an abelian variety. c) $A=A_1\times A_2$, $\dim A_1\geqq 2$ and A_1 is immersed in an abelian variety. d) $A=A_1\times A_2$, $\dim A_1\geqq 2$ and $\dim A_2\geqq 2$.

Proof of the proposition. Let $\pi_i: A\to A_i$ be the projection and let T_i be the pull back of the tangent bundle of A_i to A. In order to show $H^q(A, T^A[-L])=0$, it suffices to prove $H^q(A, T_i[-L])=0$ for each i. When $\dim A_i\leqq\dim A-2$, we can apply [**F1**, Corollary A6]. Otherwise, $(T_i)^{\vee}$ is generated by global sections. Thus we have an exact sequence $0\to T_i\to[0]^{\oplus r}\to\pi_i{}^*Q\to 0$ for some vector bundle Q on A_i. For any ample line bundle L on A, we get $H^0(A, \pi_i{}^*Q[-L])=0$ by [**F1**, Corollary A6]. So, using the long exact sequence and the vanishing theorem of Kodaira, we easily infer that $H^1(A, T_i[-L])=0$. $H^0(A, T_i[-L])=0$ is proved by applying [**F1**, Corollary A6] to π_i.

(2.4) REMARK. Let $A, A_1, \cdots, A_k$ be as in (2.3) and let F_i be the fiber of π_i (Note: $F_i=A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k$). Suppose in addition that $H^q(F_i, [-L])=0$ for $q<\mathrm{Min}(2, \dim F_i)$ and for any $L\in\mathrm{Pic}(F_i)$ with $\kappa(L, F_i)=$

$\dim F_i$. Then A satisfies the strong NS-condition. Proof is similar to that in (2.3).

(2.5) PROPOSITION. *Let $f: A \to S$ be a surjective morphism everywhere of maximal rank between manifolds. (Hence $F_y = f^{-1}(y)$ is smooth for any $y \in S$). Let L be a line bundle on A and let T_y denote the tangent bundle of F_y. Suppose that $H^0(F_y, -tL) = 0 = H^0(F_y, T_y[-tL])$ for every $y \in S$ and any $t>0$, and that $H^1(F_x, -tL) = 0 = H^1(F_x, T_x[-tL])$ for some $x \in S$ and any $t>0$. Then A satisfies the NS-condition with respect to L.*

PROOF. By [**H**, p. 288, Corollary 12.9] we see $f_*(\mathcal{L}^{-t}) = 0$, where $\mathcal{L}^{-t} = \mathcal{O}_A[-tL]$. Moreover, in view of [**H**, p. 284, Proposition 12.4], we have a coherent sheaf $\mathcal{T}$ on S such that $R^1 f_*(\mathcal{L}^{-t} \otimes_{\mathcal{O}_S} \mathcal{M}) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{T}, \mathcal{M})$ for any coherent $\mathcal{O}_S$-module $\mathcal{M}$. Considering the case $\mathcal{M} = \mathcal{O}_x$, we see $\mathcal{T}_x = 0$. So $\mathcal{T}$ is a torsion sheaf. Hence $0 = \mathcal{H}om_S(\mathcal{T}, \mathcal{O}_S) = H^0(S, R^1 f_* \mathcal{L}^{-t})$. Using the Leray spectral sequence, we infer $H^q(A, [-tL] \otimes E) = 0$ for $q = 0, 1$ and for any vector bundle E on S. In particular, $H^q(A, T^S[-tL]) = 0$. On the other hand, by a similar argument as above, we infer $H^q(A, T^{A/S}[-tL]) = 0$ where $T^{A/S}$ is the relative tangent bundle. Now, using the exact sequence $0 \to T^{A/S} \to T^A \to T^S_A \to 0$, we prove the assertion.

(2.6) COROLLARY. *Let f, A and S be as above and suppose that $H^0(F_y, T_y[-L]) = 0$ for every $y \in S$ and for any ample line bundle L on A, and that F_x satisfies the NS-condition for some $x \in S$. Then A satisfies the NS-condition.*

(2.7) PROPOSITION. *Let $f: B \to A$ be a finite unramified covering of a manifold A. Let L be a line bundle on A. Then, A satisfies the NS-condition with respect to L if B satisfies the NS-condition with respect to L_B. In particular, if B satisfies the (strong) NS-condition, then so does A.*

By the assumption, $T^B \cong T^A_B$. Hence the proposition follows from [**F1**, (4.17)].

(2.8) PROPOSITION. *Let $f: N \to A$ be a k-sheeted finite branched cyclic covering with smooth branch locus B (cf. [**W**]). Let $R \subset N$ be the ramification locus endowed with the reduced structure. (Note: $f_R: R \to B$ is an isomorphism). Let L be a line bundle on A such that $H^q(R, [jR - tL]_R) = 0$ for any $2 \leqq j \leqq k$, $q = 0, 1$ and any $t > 0$, and suppose that N satisfies the NS-condition with respect to L_N. Then A satisfies the NS-condition with respect to L.*

PROOF. Note that $f^*[B] = [kR]$ in $\mathrm{Pic}(N)$. Let $\mathcal{J}$ be the ideal of R in N and let $\mathcal{F} = f^*(\mathcal{O}_B[B])$ (Note: $\mathcal{F}$ is an $\mathcal{O}_N/\mathcal{J}^k$-module, but is not an $\mathcal{O}_N/\mathcal{J}$-module). The cokernel of the natural monomorphism $\mathcal{O}_N[T^N] \to \mathcal{O}_N[f^*T^A]$ is isomorphic to $\mathcal{F}/\mathcal{J}^{k-1}\mathcal{F}$ as an $\mathcal{O}_N$-module. Set $\mathcal{F}_j = \mathcal{J}^j \mathcal{F}/\mathcal{J}^{j+1}\mathcal{F}$. Then $\mathcal{F}_j = \mathcal{O}_R[(k-j)R]$ as an $\mathcal{O}_N$-module. So $H^q(\mathcal{F}/\mathcal{J}^{k-1}\mathcal{F}[-tL]) = 0$ for $q = 0, 1$ and for any $t > 0$ by assumption. Combining them we infer $H^q(N, T^A_N[-tL]) = H^q(N, T^N[-tL]) = 0$. Now we apply [**F1**, (4.17)] to obtain $H^q(A, T^A[-tL]) = 0$.

(2.9) COROLLARY. *Let f, N, A, B and R be as above and suppose that* $\dim A$

$\geqq 3$ *and that the normal bundle of each connected component of* R *is seminegative. Then* A *satisfies the NS-condition if so does* N.

PROOF. $[R]_R$ is seminegative. If L is ample on A, then L_R is ample and so is $[tL-jR]_R$. So the condition in (2.8) is verified by the vanishing theorem of Kodaira.

(2.10) PROPOSITION. *Let* A *be the blowing up of a manifold* M *with non-singular center* C *such that* $\operatorname{codim} C_i \geqq 3$ *for each connected component* C_i *of* C. *Then* A *satisfies the NS-condition either if* M *satisfies the strong NS-condition, or if* $\dim C=0$ *and* M *satisfies the NS-condition.*

PROOF. Let E_i be the connected component of the exceptional divisor on A lying over C_i, with $\pi_i: E_i \to C_i$ being the natural morphism. Then $E_i=\boldsymbol{P}(N_i^\vee)$ where $N_i^\vee$ is the conormal bundle of C_i in M. Moreover $[E_i]_{E_i}=-H_i$, which corresponds to $\mathcal{O}(-1)$, the dual of the tautological line bundle of $E_i=\boldsymbol{P}(N_i^\vee)$.

We will show $H^q(A, T^A[-L])=0$ for $q=0, 1$ and for every ample line bundle L on A. Set $F=L+\sum_i d_i E_i$, where d_i is the degree of the restriction of L to a fiber of π_i (of course it is a projective space of dimension $\operatorname{codim} C_i-1$). Then the restriction of F to each fiber of π_i is trivial. Hence F is the pull back of a line bundle on M, which is denoted by F by abuse of notation. We have $\kappa(F, M)=\kappa(F, A) \geqq \kappa(L, A)=\dim A=\dim M$ since $d_i>0$. Moreover, F is ample on M if $\dim C=0$ (cf. [**F1**, (5.7)]). Hence $H^q(A, T_A^M[-F])=H^q(M, T^M[-F])=0$ by assumption.

We claim that $H^q(A, T_A^M[-F+\sum_i \mu_i E_i])=0$ for any non-negative integers μ_i. To prove this, we use the induction on $\mu=\sum_i \mu_i$. If $\mu=0$, then $\mu_i=0$ for each i and the assertion is true. If $\mu_i>0$ for some i, then $H^q(E_i, T^M[-F+\sum_j \mu_j E_j])=H^q(E_i, \mathcal{O}(-\mu_i) \otimes \pi_i^*(T^M[-F])_{C_i})=0$ since $R^q(\pi_i)_* \mathcal{O}(-\mu_i)=0$ for $q<\operatorname{codim} C_i-1$. Now, using the exact sequence $H^q(A, T^M[-F+\sum_j \mu_j E_j-E_i]) \to H^q(A, T^M[-F+\sum_j \mu_j E_j]) \to H^q(E_i, T^M[-F+\sum_j \mu_j E_j])$, we prove the claim by the induction. In particular, we get $H^q(A, T^M[-L])=0$ for $q=0, 1$.

Let $\mathcal{C}$ be the cokernel of the natural monomorphism $\mathcal{O}_A[T^A] \to \mathcal{O}_A[T_A^M]$. By the preceding argument it suffices to show $H^0(\mathcal{C}[-L])=0$. One easily sees that $\mathcal{C}=\oplus_j \mathcal{O}_{E_j}(T_j(-1))$, where T_i is the relative tangent bundle of π_i. Therefore $H^0(\mathcal{C}_i[-L])=H^0(E_i, T_i[E_i-L])=0$, since $H^0(E_x, T_i[E_i-L])=H^0(\boldsymbol{P}^{r-1}, T^P(-1-d_i))=0$ for every fiber $E_x \cong \boldsymbol{P}^{r-1}$ ($r=\operatorname{rank}(N_i)=\operatorname{codim} C_i$) of π_i. This completes the proof. q.e.d.

REMARK. By the above argument, we prove the following: If M satisfies the NS-condition with respect to an ample line bundle L on A, that is, $H^q(T^M A[-tL])=0$ for $q=0, 1$ and $t>0$, then A satisfies the NS-condition with respect to L.

(2.11) Combining the foregoing results we can find many manifolds which satisfy the NS-condition. For example, in view of (2.1) and (2.6), we infer that many flag manifolds satisfy the NS-condition. Hyperelliptic surfaces satisfy the NS-condition by (2.2) and (2.7). We don't list up all such examples. Before closing the section, we just present the following sample result.

(2.12) PROPOSITION. *Let A be a manifold with* $\dim A \geqq 3$ *satisfying the NS-condition. Let ι be a holomorphic involution of A with only isolated fixed points. Then the standard resolution M of A/ι satisfies the NS-condition.*

PROOF. Let C be the set of fixed points of ι. Let A' be the blowing up of A with center C. Then ι lifts to a holomorphic involution ι' of A'. The set of fixed points of ι' is the exceptional divisor E over C. Therefore A'/ι' is smooth, and this is nothing other than M. A' satisfies the NS-condition by (2.10). On the other hand, A' is a two-sheeted branched covering of M with ramification locus E. Each connected component of E is isomorphic to $\boldsymbol{P}^n$ with $n=\dim A-1$, and its normal bundle is $\mathcal{O}(-1)$. So M satisfies the NS-condition by (2.9).

(2.13) COROLLARY. *Kummer manifold of dimension $\geqq 3$ satisfies the NS-condition.*

§ 3. Non-smoothable singularities.

(3.1) DEFINITION. A *deformation family* of polarized manifolds is a quadruple (A, X, π, L), consisting of manifolds A and X which may not be complete, a proper surjective morphism $\pi: A \to X$ everywhere of maximal rank, and a line bundle L on A which is relatively ample with respect to π. Then, for each $x \in X$, $A_x = \pi^{-1}(x)$ is a projective manifold polarized by the restriction L_x of L to A_x.

(3.2) PROPOSITION. *Let (A, X, π, L) be a deformation family of polarized manifolds. Let q be a positive integer and let E be a vector bundle on A with E_x being the restriction to A_x. Suppose that there is $o \in X$ such that $H^q(A_o, E_o[tL])=0$ for any $t>0$. Then there is a neighbourhood U of o such that $H^q(A_x, E_x[tL_x])=0$ for any $t>0$ and any $x \in U$.*

PROOF. Let $P=\boldsymbol{P}(E)$ and $H=\mathcal{O}_P(1)$. Let P_x be the fiber over $x \in X$ and let H_x be the restriction of H to P_x. Let K be the canonical bundle of P. Then $K=-rH+\det E+K^A$, where $r=\mathrm{rank}(E)$. Moreover, the restriction K_x of K to P_x is the canonical bundle of P_x. Now, we easily see that $H-K$ is relatively ample with respect to $P \to A$. Therefore, $H+tL-K$ is relatively ample with respect to $P \to X$ for $t \gg 0$, or precisely speaking, for any $t \geqq c$ with c being a fixed constant. So, $H^q(A_x, E_x[tL])=H^q(P_x, H_x+tL_x)=0$ for any $t \geqq c$ and any $x \in X$ by the vanishing theorem of Kodaira. On the other hand, by the upper-semicontinuity theorem, there is a neighbourhood U of o such that $H^q(A_x, E_x[tL_x])=0$ for any $x \in U$ and any t with $0<t<c$. Clearly this U has

the required property.

(3.3) COROLLARY. *Let* A, X, π, L, E *and* o *be as above and let* q *be an integer such that* $q<\dim A_0$ *and that* $H^q(A_0, E_0[-tL_0])=0$ *for any* $t>0$. *Then there is a neighbourhood* U *of* o *such that* $H^q(A_x, E_x[-tL_x])=0$ *for any* $t>0$ *and any* $x\in U$.

For a proof, use the Serre duality on A_x.

(3.4) COROLLARY. *Let* A, X, π, L *and* o *be as above and suppose that* A_0 *satisfies the NS-condition with respect to* L_0. *Then, there is a neighbourhood* U *of* o *such that* A_x *satisfies the NS-condition* with respect to L_x *for any* $x\in U$.

For a proof, apply (3.3) to the case $E=T^{A/X}$.

(3.5) DEFINITION. A *Lefschetz pair* is a pair (V, A) consisting of a projective scheme V and an ample effective divisor A on V. Such a pair is said to be smooth if both V and A are non-singular.

A flat family of Lefschetz pairs is a quadruple (V, A, X, π) consisting of schemes V and X, a projective flat morphism $\pi: V\to X$, and an effective divisor A on V which is relatively ample with respect to π, such that the restriction $\pi_A: A\to X$ of π to A is flat. Moreover, X is assumed to be connected and smooth unless specifically stated to the contrary. Of course, (V_x, A_x) is a Lefschetz pair for any $x\in X$.

A Lefschetz pair (V, A) is said to be *smoothable* if there is a flat family $(\mathcal{V}, \mathcal{A}, X, \pi)$ of Lefschetz pairs and two points o, $x\in X$ such that $(V_0, A_0)\cong(V, A)$ and that (V_x, A_x) is smooth. (X is assumed to be connected as usual.)

(3.6) THEOREM. *Let* (V, A) *be a Lefschetz pair such that* A *is a manifold which satisfies the NS-condition with respect to* $[A]_A$. *Then* (V, A) *is not smoothable.*

PROOF. Suppose that there is a flat family of Lefschetz pairs $(\mathcal{V}, \mathcal{A}, X, \pi)$ such that $(V_0, A_0)\cong(V, A)$ and (V_x, A_x) is smooth for some o, $x\in X$. In view of (3.4), we infer that there is a neighbourhood U of o such that A_x is a manifold satisfying the NS-condition with respect to $[\mathcal{A}]_{A_x}$ for every $x\in U$. On the other hand, the set $\{x\in X\,|\,V_x \text{ is smooth}\}$ is dense in the manifold X since it is non-empty and Zariski-open. Therefore there is $y\in U$ such that V_y is smooth. Then A_y is an ample divisor in the manifold V_y. This contradicts (1.2), since A_y satisfies the NS-condition with respect to $[A_y]$.

REMARK. If (V, A) is a Lefschetz pair such that A is smooth, then V can have at most isolated singularities, because the singular locus of V does not meet A.

(3.7) COROLLARY. *Let* V *be a subscheme in* $\boldsymbol{P}^N$ *and suppose that there is a hypersurface section* A *of* V *such that* A *is a smooth manifold satisfying the NS-condition with respect to* $\mathcal{O}(1)$. *Then* V *is not smoothable in* $\boldsymbol{P}^N$ *in the sense of* [**So 2**].

PROOF. Let H be the hypersurface in $\boldsymbol{P}^N$ such that $A=V\cap H$. Suppose that there is a smoothing deformation $\{V_x\}$ of V in $\boldsymbol{P}^N$. Then the family $\{(V_x, A_x=V_x\cap H)\}$ would be a smoothing deformation of the Lefschetz pair (V, A). This contradicts (3.6).

(3.8) COROLLARY. *Let V be a projective cone in $\boldsymbol{P}^N$ over a base manifold A which satisfies the NS-condition with respect to $\mathcal{O}(1)$. Then V is not smoothable in $\boldsymbol{P}^N$.*

(3.9) Now, combining with the results in §2, we obtain many examples of non-smoothable singularities. In particular, the ones in (2.1.1) and (2.1.2) of [**So 2**] are obtained by our (2.2) and (2.3). (2.3) gives also a partial answer to Question (3.1) of [**So 2**]. However, the example (2.1.3) of [**So 2**] does not seem to be obtained by our method.

(3.10) Let us speculate here on the relation of our criterion to Schlessinger's theory (cf. [**Sc**] and [**P**]).

Let A be a submanifold of $\boldsymbol{P}^N$ contained in a hyperplane $\boldsymbol{P}^{N-1}$, and let V be the projective cone over A with vertex v off this hyperplane. Let $U=V-A-\{v\}$. Then U is a principal $\boldsymbol{C}^{\times}$-bundle over A. The infinitesimal deformations (meaning flat families over Artinian rings) of U are parametrized by $\tau_U=H^1(U, T^U)$, which is decomposed as $\bigoplus_{t\in\boldsymbol{Z}}\tau(t)$ according to the $\boldsymbol{C}^{\times}$-action. Set $\tau^+=\bigoplus_{t>0}\tau(t)$ and $\tau^-=\bigoplus_{t<0}\tau(t)$. Then the infinitesimal deformations of U coming from V are parametrized by $\tau^-\oplus\tau(0)$, and $\tau(0)$ corresponds to those which preserve the cone structure. Moreover, $\tau(t)$ can be related to cohomologies on A and $\tau(t)\cong H^1(A, T^A(t))$ if in addition $H^q(A, \mathcal{O}(t))=0$ for $q=1$, 2 and for any $t\in\boldsymbol{Z}$. Thus, it is natural to expect that V is not smoothable when $\tau^-=0$, like (3.8).

However, besides the formulations, there are slight differences between his viewpoint and ours. Namely, he considers all the deformations of V which may not be realized in $\boldsymbol{P}^N$. As a price, he needs some additional assumptions on the singularity of V at the vertex. Compare [**P**] also.

(3.11) We call attention to the following phenomena too (compare [**P**, p. 46]).

For a polarized manifold (A, L), let $P(A, L)=\boldsymbol{P}([0]\oplus L)$. The subbundle L defines a section A_0 of $\pi: P(A, L)\to A$ such that its normal bundle is $-\pi^*L$. So A_0 can be contracted to a normal point by Grauert's criterion. Let $P(A, L)\to P'(A, L)$ be the contraction morphism and let $v(A, L)$ be the image point of A_0. Note that, if A is a projectively normal submanifold of $\boldsymbol{P}^N$ with hyperplane section H, then $P'(A, H)$ is nothing other than the projective cone over A.

Suppose that A is an ample divisor on a manifold M and let $L=[A]_A$. Then $P'(A, L)$ is smoothable provided that $H^0(M, [tA])\to H^0(A, tL)$ is surjective for any $t>0$.

To see this, let M_∞ be a section of $\pi: P(M, [A])\to M$ such that $M_\infty\cap M_0=\emptyset$

and $[M_\infty]=\mathcal{O}(1)$. M_∞ is linearly equivalent to $M_0+\pi^*A$. Let Λ be the pencil containing them. Let Λ' be the image of Λ on $P'(M, [A])$. Any general member of Λ' is isomorphic to M. The member D corresponding to $M_0+\pi^*A$ is almost isomorphic to $P'(A, L)$, namely, $D/v(M, [A])\cong P'(A, L)/v(A, L)$. Moreover, the surjectivity of $H^0(M, [tA])\to H^0(A, tL)$ implies that this extends to an isomorphism $D\cong P'(A, L)$. Thus, Λ' gives rise to a smoothing family of $P'(A, L)$.

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