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Nonexistence of bounded functions on the homology covering surface of $P^1-\{3 \text{ points}\}$

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Introduction.

Let D be the Riemann sphere punctured at three points. We consider the existence problem of bounded functions on its homology covering surface R, that is, the normal covering surface of D determined by the commutator subgroup of its fundamental group. We shall prove the following

THEOREM I. *The homology covering surface of the Riemann sphere punctured at three points belongs to the class* O_{AB}.

A complex manifold is said to belong to the *class* O_{AB} if and only if it carries no bounded, holomorphic, uniform and nonconstant function.

The motivation of this study rose from the question if the universal covering manifold of the complement of $n+2$ hyperplanes in general position in the n dimensional complex projective space $\boldsymbol{P}^n$ belongs to the class O_{AB}. By means of Theorem I, we solved this question affirmatively as follows.

THEOREM II ([7]). *If* $n\geqq 2$, *the universal covering manifold of the complement of* $n+2$ *hyperplanes in general position in* $\boldsymbol{P}^n$ *belongs to the class* O_{AB}.

In our previous paper [7] we announced Theorem I and derived Theorem II from Theorem I. The purpose of the present paper is to give the proof of Theorem I.

The proof of Theorem I is based on the criterion of A. Pfluger [5] which asserts that a Riemann surface having an exhaustion with some suitable properties belongs to the class O_{AB}.

The proof will be carried out as follows. In order to study topological and analytic properties of the homology covering surface R, we divide it into an infinite number of triangles. Then we place all these triangles on the complex plane properly, and glue them according to a certain rule, and thus we reconstruct the surface R, realizing it on the complex plane. This enables us to have a visual image of R and to treat it with ease.

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To apply Pfluger's criterion to our surface R, first we construct an increasing sequence of polygons P_n consisting of a finite number of the above triangles. But the problem is very delicate, and the standard construction of P_n such as found in [1] p. 243 is not convenient for our purpose, even though it is convenient enough just to show that R belongs to the class O_{AD}. Therefore our first task is to choose proper P_n with great care. Secondly, since P_n is not relatively compact in R we must replace it by a relatively compact subdomain $\Omega_n \subset P_n$ and obtain an exhaustion of R. Then we construct annuli by enlarging the boundary components of Ω_n. In this process too, we must construct these Ω_n carefully enough to obtain sufficiently wide annuli and to be able to apply Pfluger's criterion.

In § 1, we study the topological and analytic structure of our homology covering surface. In § 2, we recall Pfluger's criterion. § 3 is consecrated to the construction of annuli on our surface. In § 4, we estimate the harmonic module of each annulus constructed in § 3, and we show that Pfluger's criterion is applicable to our surface.

I would like to express my sincere thanks to Professor Kôtaro Oikawa for his encouragement and many valuable suggestions during the course of this work. I am grateful also to the referee for his suggestion to use an appropriate linear density, which made the proof simpler.

After we finished this work, we were informed that J. P. Demailly [2], [3] proved Theorem I by a completely different method.

§ 1. Structure of R.

1.1. Triangulation of R and numbering of triangles.

DEFINITION 1. The *homology covering surface* of a Riemann surface is the normal covering surface determined by the commutator subgroup of its fundamental group.

This covering surface has an Abelian group of cover transformations, and is the strongest normal covering surface with this property (cf. [1]).

We shall divide the homology covering surface R of the complement D of three points $\{0, 1, \infty\}$ in the Riemann sphere into an infinite number of triangles. Then we study how they are arranged in R. In the following we consider triangles without vertices, so our triangulation differs slightly from the usual triangulation.

Let A be the segment $(0, 1)$, let B be the segment $(1, \infty)$ and let C be the segment $(\infty, 0)$ on the real axis. Let us set $\Delta_1=\{z\in \boldsymbol{C} \mid \operatorname{Im} z\geqq 0,\ z\neq 0,\ 1\}$ and $\Delta_2=\{z\in \boldsymbol{C} \mid \operatorname{Im} z\leqq 0,\ z\neq 0,\ 1\}$. We regard Δ_1 and Δ_2 as triangles with three sides A, B and C. We note that these triangles have no vertices. For $\nu=1, 2$, the inverse image of Δ_ν, $\pi^{-1}(\Delta_\nu)$, by the canonical projection $\pi: R\to D$ consists of an infinite number of connected components conformally equivalent to Δ_ν, which we call also triangles.

REMARK 1. To simplify the notation, we shall use the same letters A, B and C also for the sides of every triangle of $\pi^{-1}(\Delta_1)$ or of $\pi^{-1}(\Delta_2)$ which lie over the segments A, B and C respectively.

We give numbers to the triangles of $\pi^{-1}(\Delta_1)$ and to those of $\pi^{-1}(\Delta_2)$ in the following way.

Let us take an interior point p of Δ_1. Let $G=\pi_1(D, p)$ be the fundamental group of D with base point p, and let G' be the commutator subgroup of G. Let α be a closed curve in D from p which goes round the point 0 once in the positive sense. Likewise, let β be a closed curve in D from p which goes round the point 1 once in the positive sense. Then G is a free group generated by α and β. So each element of G can be expressed as $\alpha^{i_1}\beta^{j_1}\cdots\beta^{j_h}$, where $i_1, j_1, \cdots, j_h$ belongs to the set of integers $\boldsymbol{Z}$. We see easily that it belongs to the commutator subgroup G' if and only if $i_1+\cdots+i_h=0$ and $j_1+\cdots+j_h=0$. This implies that each element of the quotient group G/G' can be represented uniquely by an element of the form $\alpha^i\beta^j$ $(i, j\in\boldsymbol{Z})$. Furthermore, the group of cover transformations of the homology covering surface R is by definition none other than G/G'. Hence, if we choose once and for all a point p_{00} of $\pi^{-1}(p)$, then each element of G/G' determines a unique point of $\pi^{-1}(p)$. Namely, the element of G/G' represented by $\alpha^i\beta^j$ determines the end point of the curve in R from p_{00} lying over the curve $\alpha^i\beta^j$. On the other hand, for each point of $\pi^{-1}(p)$, there exists a unique element of G/G' which determines the point in that manner. So let us denote by p_{ij} the point of $\pi^{-1}(p)$ determined by the element of G/G' which is represented by $\alpha^i\beta^j$. Then, let us denote by Δ_1^{ij} the connected component of $\pi^{-1}(\Delta_1)$ which contains the point p_{ij}. Next we denote by Δ_2^{ij} the connected component of $\pi^{-1}(\Delta_2)$ which is adjacent to Δ_1^{ij} along the common side A. It is clear that if $(i, j)\neq(i', j')$, then $\Delta_\nu^{ij}\neq\Delta_\nu^{i'j'}$ for $\nu=1, 2$, and that $\pi^{-1}(\Delta_\nu)=\bigcup\Delta_\nu^{ij}$ $((i, j)\in\boldsymbol{Z}\times\boldsymbol{Z})$ for $\nu=1, 2$. Thus all the connected components of $\pi^{-1}(\Delta_1)$ and $\pi^{-1}(\Delta_2)$ are numbered.

1.2. Reconstruction of R. Let us study the topological and analytic structure of the surface R. To this aim we construct a Riemann surface in the following way.

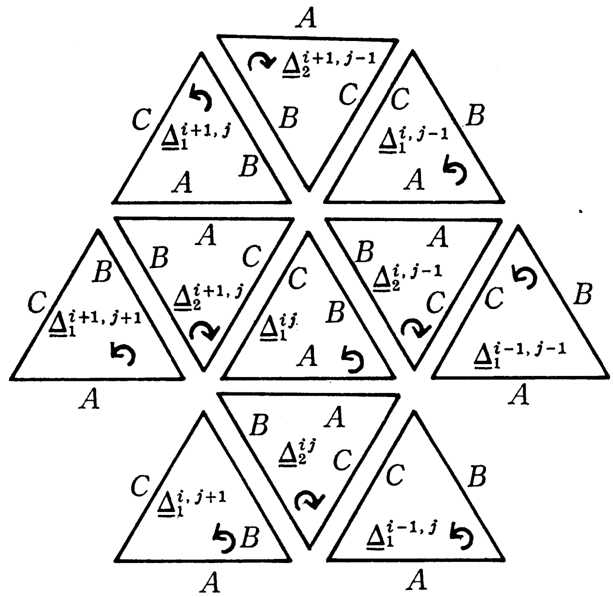
First let us prepare a countable number of equilateral triangles of the same size. We denote them by $\underline{\Delta}_1^{ij}$ and $\underline{\Delta}_2^{ij}$, where i and j range over $\boldsymbol{Z}$. Since later it will be convenient for us, we assume that the lengths of the sides of these triangles are equal to 1. We assume also that the vertices do not belong to $\underline{\Delta}_\nu^{ij}$, but that the sides belong to $\underline{\Delta}_\nu^{ij}$.

We denote by A, B and C the sides of $\underline{\Delta}_1^{ij}$ and those of $\underline{\Delta}_2^{ij}$ as illustrated in Figure 1. Let us note that the order of arrangement of the sides A, B and

C of Δ_1^{ij} is different from that of Δ_2^{ij}. We assume that in Figure 1 the triangles Δ_ν^{ij} are placed on the complex plane, and that they have the induced analytic structure. So each Δ_ν^{ij} has an orientation as Figure 1. Now we place first the triangles Δ_1^{ij} (i, $j\in\boldsymbol{Z}$) as indicated in Figure 2.

Figure 1.

Figure 2.



Then, for each (i, j), reversing the triangle Δ_2^{ij} face downward, we place it below the triangle Δ_1^{ij} so that we obtain Figure 2.

After having placed all the triangles like this, we now consider each pair of sides which are adjacent in Figure 2. Let us consider for example the side A of Δ_1^{ij} and the side A of Δ_2^{ij}. Let Q_0 be the middle point of A of Δ_1^{ij}, and let Q_0' be the middle point of A of Δ_2^{ij}. Let Q be any point on A of Δ_1^{ij}, and let Q' be the symmetric point of Q with respect to the middle point of the segment Q_0Q_0'; so Q' lies on the side A of Δ_2^{ij}, and the distance between Q and Q_0 is equal to the distance between Q' and Q_0', and the segments QQ_0 and $Q'Q_0'$ are situated in the symmetric position with respect to the middle point of the segment Q_0Q_0'. Now we identify the point Q with the point Q'. We do this identification for all the points on A of Δ_1^{ij} and A of Δ_2^{ij}. In the same manner as we did of the side A, we identify the side B of Δ_1^{ij} with the side B of $\Delta_2^{i,j-1}$, using the analogous symmetry, and also we identify the side C of

$\underline{\Delta}_1^{ij}$ with the side C of $\underline{\Delta}_2^{i+1,j}$ in the same way. We perform this identification for all i and j. Then of each pair of sides which are adjacent in Figure 2, the sides are identified. We verify easily that the orientations of $\underline{\Delta}_\nu^{ij}$ coincide after this identification. Furthermore it is clear that the analytic structures of $\underline{\Delta}_\nu^{ij}$ coincide on each side after this gluing, so the surface constructed in this way has an analytic structure. Thus we have obtained a Riemann surface, which we denote by R'.

PROPOSITION 1. *The Riemann surface R' constructed above is conformally equivalent to the Riemann surface R.*

PROOF. *Step* 1. First we wish to prove that the surfaces R and R' are homeomorphic. Since R is divided into the triangles Δ_ν^{ij}, and R' is also divided into the triangles $\underline{\Delta}_\nu^{ij}$, it is sufficient to show that Δ_ν^{ij} are arranged on R in the same manner as $\underline{\Delta}_\nu^{ij}$ are arranged on R'. So we show that Δ_ν^{ij} are arranged on R as follows.

(i) Δ_1^{ij} and Δ_2^{ij} are adjacent to each other, having the side A in common. From the way of our numbering, this is clear.

(ii) Δ_1^{ij} and $\Delta_2^{i,j-1}$ are adjacent, having the side B in common.

To show this, let us start from the point $p_{i,j-1}$ and let us go round the point 1 once, following the curve on R over the closed curve β. Then we arrive, by definition, at the point p_{ij}. This means that, starting from $\Delta_1^{i,j-1}$, we enter the triangle $\Delta_2^{i,j-1}$ across the side A as (i) shows, and then we arrive at Δ_1^{ij}, crossing the side B. Hence $\Delta_2^{i,j-1}$ and Δ_1^{ij} have the side B in common.

(iii) Δ_1^{ij} and $\Delta_2^{i+1,j}$ are adjacent, having the side C in common.

To show this, we start from $p_{i+1,j}$ and go round the point 0 once in the negative sense, following the curve on R over the closed curve α^{-1}. Then we arrive, by definition, at the point p_{ij}. This shows that, starting from $\Delta_1^{i+1,j}$, we enter $\Delta_2^{i+1,j}$ across the side A, and we arrive at Δ_1^{ij}, crossing the side C. Hence $\Delta_2^{i+1,j}$ and Δ_1^{ij} have the side C in common.

(i), (ii) and (iii), together with the way of construction of R' by means of Figure 2, show that Δ_ν^{ij} are arranged on R in the same manner as $\underline{\Delta}^{ij}$ are arranged on R', which implies that R and R' are homeomorphic, as required.

Step 2. Next we wish to prove that the Riemann surfaces R and R' are conformally equivalent. To this aim let us take the triangle $\Delta_1^{0,0}$ of R. As $\Delta_1^{0,0}$ is conformally equivalent to the upper half plane, there exists a unique conformal mapping ϕ of $\Delta_1^{0,0}$ onto the equilateral triangle $\underline{\Delta}_1^{0,0}$ such that the sides A, B and C of $\Delta_1^{0,0}$ are sent to the sides A, B and C of $\underline{\Delta}_1^{0,0}$ respectively. By the Schwarz principle of reflexion, ϕ is analytically continuable to $\Delta_2^{0,0}$ across the side A. Since $\Delta_1^{0,0}$ and $\Delta_2^{0,0}$ are symmetric with respect to A, and after the gluing $\underline{\Delta}_1^{0,0}$ and $\underline{\Delta}_2^{0,0}$ are symmetric with respect to A, the image $\phi(\Delta_2^{0,0})$ under the extended mapping ϕ coincides with $\underline{\Delta}_2^{0,0}$. Hence the extended mapping ϕ

is a conformal mapping of $\Delta_1^{0,0}\cup\Delta_2^{0,0}$ onto $\underline{\Delta}_1^{0,0}\cup\underline{\Delta}_2^{0,0}$. The same argument holds good for the other sides B and C, and for the other triangles. On the other hand the triangles Δ_ν^{ij} are arranged on R in the same way as the triangles $\underline{\Delta}_\nu^{ij}$ on R', so ϕ is extended to a conformal mapping of R onto R' by using the Schwarz principle of reflexion infinite times. This is what we wished to prove.

From Proposition 1 the Riemann surfaces R and R' are conformally equivalent, so we may take off the underline from the equilateral triangles $\underline{\Delta}^{ij}$, and write them as Δ^{ij} in the following.

REMARK 2. Our triangles Δ^{ij} contain no vertex. But when we regard Δ^{ij} as an equilateral triangle described in Figure 2, we permit us to say '0 of Δ_ν^{ij}', '1 of Δ_ν^{ij}' and '∞ of Δ_ν^{ij}' to indicate the vertex between the sides A and C, the vertex between the sides A and B and the vertex between the sides B and C respectively.

§2. Pfluger's criterion.

To prove Theorem I, we shall use Pfluger's criterion which gives a sufficient condition for a Riemann surface to carry no bounded, holomorphic, uniform and nonconstant functions. Pfluger stated and proved this criterion in terms of conformal metric. A. Mori [4] gave a variant of the criterion in terms of harmonic module. For domains in the complex plane $\boldsymbol{C}$, N. Suita [6] gave an alternative proof of this criterion. In this section we shall recall Pfluger's criterion in the form stated by Mori.

2.1. Statement of Pfluger's criterion. Let W be an open Riemann surface, and let A_n^k, $k=1, 2, \cdots, k(n)<\infty$, $n=1, 2, \cdots$, be a collection of doubly connected domains of W satisfying the following conditions:

(1.1) each A_n^k is bounded by two piecewise analytic, closed curves γ_n^k and γ'^k_n,

(1.2) any two of A_n^k are disjoint,

(1.3) the complement of $\bigcup_{k=1}^{k(n)} A_n^k$ in W has a unique compact connected component B_n,

(1.4) B_n is bounded by $k(n)$ closed curves γ'^k_n, $k=1, 2, \cdots, k(n)$, and contains all $A_{n'}^{k'}$ such that $n'<n$.

In this paper a doubly connected domain in a Riemann surface will be called an *annulus*.

We denote by μ_n^k the harmonic module of A_n^k. It is well known that the harmonic module of an annulus conformally equivalent to $\{z\in\boldsymbol{C}\mid r<|z|<R\}$ is $\log R/r$ (cf. [4]).

We set
$$\mu_n=\operatorname*{Min}_k \mu_n^k\,, \quad \text{and} \quad K(N)=\operatorname*{Max}_{n\leq N} k(n)\,.$$

PFLUGER'S CRITERION. *If*

$$\overline{\lim_{N\to\infty}}\left\{\sum_{n=1}^{N}\mu_n-\frac{1}{2}\log K(N)\right\}=\infty, \tag{1}$$

then the Riemann surface W belongs to the class O_{AB}.

2.2. Extremal length. The harmonic module $\mu(V)$ of an annulus V can be calculated also by the extremal length. Let Γ be the family of all the curves γ in V joining its two boundary components. Let $\lambda(\Gamma)$ be the extremal length of the family Γ. For simplicity's sake, we shall often denote it by $\lambda(V)$. Actually the *extremal length* of Γ is given by

$$\lambda(\Gamma)=\lambda(V)=\sup_{\rho}\frac{\inf_{\gamma\in\Gamma}\left(\int_{\gamma}\rho|dz|\right)^2}{\iint_{V}\rho^2dxdy}$$

where ρ ranges over the set of all linear densities in V. A *linear density* is by definition an invariant form $\rho|dz|$ which is nonnegative and Borel measurable. An easy calculation shows (cf. [1], p. 224) that, if V is conformally equivalent to $\{z\in\boldsymbol{C}|r<|z|<R\}$, then

$$\lambda(V)=\frac{1}{2\pi}\log\frac{R}{r}.$$

So, for any linear density ρ in V, we have

$$\mu(V)=2\pi\times\lambda(V)\geqq 2\pi\times\frac{\inf_{\gamma\in\Gamma}\left(\int_{\gamma}\rho|dz|\right)^2}{\iint_{V}\rho^2dxdy}. \tag{2}$$

Therefore, if the quantity of the right hand side of (2) is sufficiently large for a suitably chosen linear density ρ, then the harmonic module μ itself is large. In the following we shall estimate by means of (2) the harmonic module μ_n^k of A_n^k which we shall construct in §3.

§3. Construction of annuli A_n^k.

To apply Pfluger's criterion to our Riemann surface R, we shall construct on R the annuli A_n^k introduced in §2. In our construction the number $k(n)$ of the family A_n^k, $k=1, 2, \cdots, k(n)$, increases as an arithmetic progression. So to apply Pfluger's criterion to our Riemann surface, it is sufficient, from (1), to construct A_n^k so that their harmonic modules μ_n^k are greater than c/n for a certain constant $c>1/2$ which is independent of n and k.

3.1. Construction of polygons P_n. We shall construct on the surface R, a sequence of polygons P_n such that

(a) each P_n consists of a finite number of triangles Δ_ν^{ij},

(b) each P_n is contained in P_{n+1}, and

(c) the union $\bigcup_n^\infty P_n$ is equal to R.

We note that for our purpose it will be sufficient to construct these polygons P_n only for sufficiently large integers. We shall use later the polygons P_n to construct an exhaustion $\{\Omega_n\}$ of R.

Now let us define P_n to be the union of the two families of triangles given below:

the triangles Δ_1^{ij} such that the indices i and j satisfy one of the following conditions

$(\mathrm{i})_n$ $\quad -n-1\leqq i\leqq n, \quad -n\leqq j\leqq n-1,$

$(\mathrm{ii})_n$ $\quad -n\leqq i\leqq n, \quad j=n,$ and

$(\mathrm{iii})_n$ $\quad -n+1\leqq i\leqq n, \quad j=n+1,$

and the triangles Δ_2^{ij} such that the indices i and j satisfy one of the following conditions

$(\mathrm{i}')_n$ $\quad -n\leqq i\leqq n+1, \quad -n+1\leqq j\leqq n,$

$(\mathrm{ii}')_n$ $\quad -n\leqq i\leqq n, \quad j=-n,$ and

$(\mathrm{iii}')_n$ $\quad -n\leqq i\leqq n-1, \quad j=-n-1.$

The polygons defined in this way satisfy the three conditions (a), (b) and (c). The shape of P_n is illustrated in Figure 3.

DEFINITION 2. A triangle of P_n is a *border triangle* if two of its sides are not adjacent to any sides of other triangles of P_n in Figure 3, and a triangle which is adjacent to a border triangle is also a *border triangle*. A triangle of P_n which is not a border triangle is an *inner triangle*.

3.2. Construction of an exhaustion $\{\Omega_n\}$. By using the polygons P_n constructed in 3.1, we shall construct an exhaustion of R, namely a sequence $\{\Omega_n\}$ of relatively compact subdomains of R such that $\Omega_n\Subset\Omega_{n+1}$ and $\bigcup_n^\infty \Omega_n=R$. We shall later construct annuli A_n^k by enlarging the boundary components of Ω_n.

Now in order to construct Ω_n, suppose Figure 3 is placed on the complex z-plane. Regarding Δ_ν^{ij} of P_n as the equilateral triangle described in Figure 3, we denote by a_0, a_1 and a_2 the coordinates of the vertices 0, 1 and ∞ of Δ_ν^{ij} respectively. For example we may suppose $a_0=-1/2$, $a_1=1/2$ and $a_2=i\sqrt{3}/2$ for $\Delta_1^{0,0}$, and $a_0=1/2$, $a_1=-1/2$ and $a_2=-i\sqrt{3}/2$ for $\Delta_2^{0,0}$. To simplify the notation we shall use the same letters a_0, a_1 and a_2 for all triangles Δ_ν^{ij}, even though the values may be different.

Let ε be a positive number such that

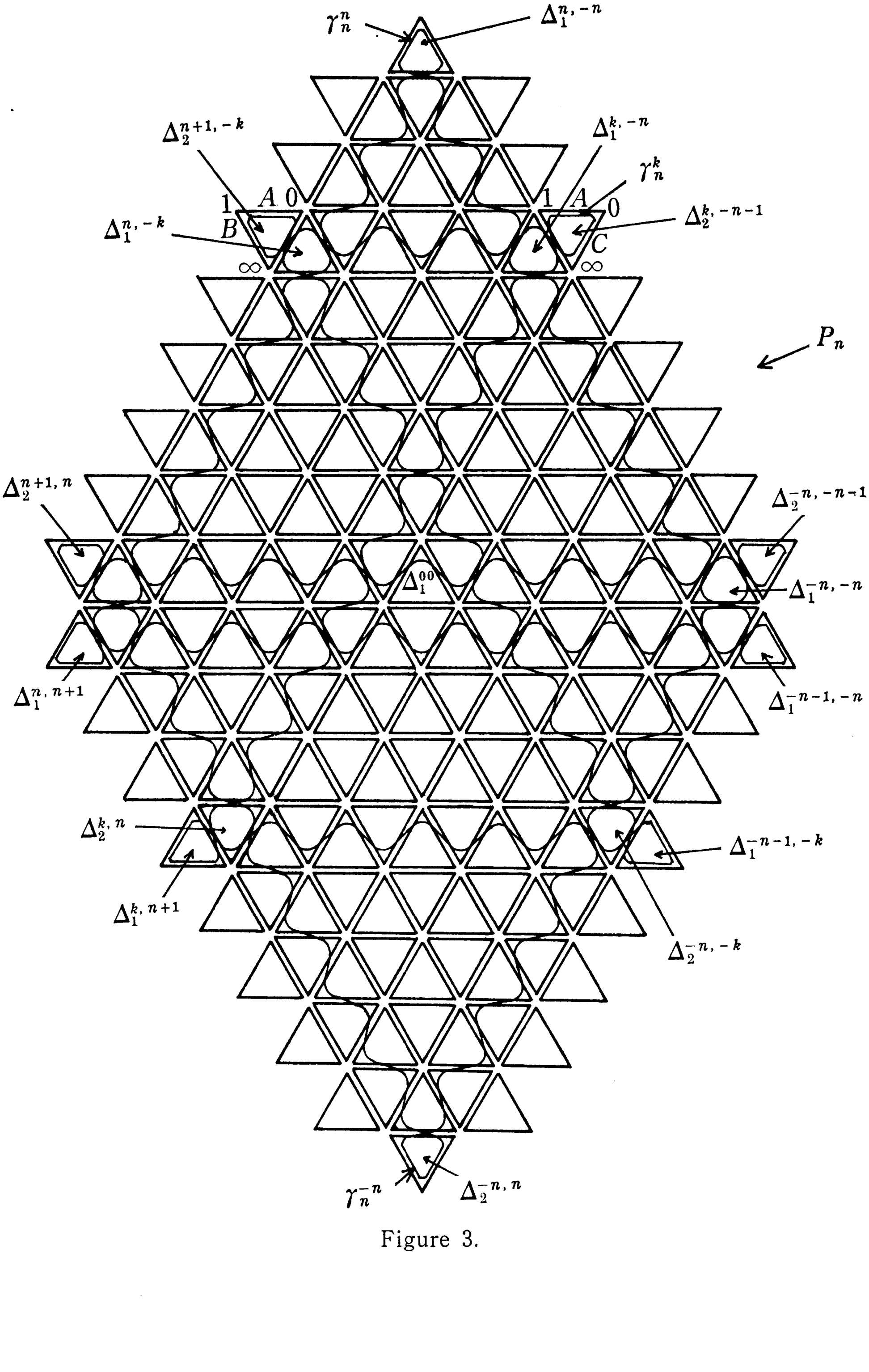

γ_n^n
$\Delta_1^{n,-n}$
$\Delta_2^{n+1,-k}$
$\Delta_1^{k,-n}$
γ_n^k
1
A
0
1
A
0
$\Delta_1^{n,-k}$
B
$\Delta_2^{k,-n-1}$
C
∞
∞
P_n
$\Delta_2^{n+1,n}$
$\Delta_2^{-n,-n-1}$
Δ_1^{00}
$\Delta_1^{-n,-n}$
$\Delta_1^{n,n+1}$
$\Delta_1^{-n-1,-n}$
$\Delta_2^{k,n}$
$\Delta_1^{-n-1,-k}$
$\Delta_1^{k,n+1}$
$\Delta_2^{-n,-k}$
γ_n^{-n}
$\Delta_2^{-n,n}$

Figure 3.

$$\varepsilon < \frac{1}{2} e^{-2\pi/3}, \tag{3}$$

and we set

$$r_t = \varepsilon e^{-\pi t/3}. \tag{4}$$

Now let us define Ω_n to be the remaining part of P_n after having taken off from the triangles of P_n the sectors as mentioned in the following (a), (b) and (c).

(a) We take off from Δ_1^{ij} the sectors $\{|z-a_0| \leqq r_{2n+3}\}$, $\{|z-a_1| \leqq r_{2n+3}\}$ and $\{|z-a_2| \leqq r_{2n+3-|i-j|}\}$.

(b) We take off from Δ_2^{ij} the sectors $\{|z-a_0| \leqq r_{2n+3}\}$, $\{|z-a_1| \leqq r_{2n+3}\}$ and $\{|z-a_2| \leqq r_{2n+3-|i-j-1|}\}$.

(c) If two of the sides of Δ_ν^{ij} are not adjacent to any sides of other triangles in Figure 3, and a_h is the coordinate of the vertex between the two sides, then we take off furthermore from Δ_ν^{ij} the sector $\{|z-a_h| \leqq \varepsilon\}$.

Then it is verified by aid of Figure 3 that the remaining part Ω_n is a relatively compact subdomain of R such that $\Omega_n \Subset \Omega_{n+1}$ and $\bigcup_n^{\infty} \Omega_n = R$.

REMARK 3. The sizes of the sectors which are taken off depend not only on n but also on the indices i and j. We shall later construct A_n^k using these Ω_n, and it will be shown that owing to this choice of sectors the area of the part of A_n^k contained in the border triangles is reduced small enough for us to apply Pfluger's criterion to our surface R. If we took off certain sectors such that their sizes depend only on n, then A_n^k would be too large to satisfy the condition (1) of Pfluger's criterion.

3.3. Boundary of Ω_n. Let us try to find the boundary components of Ω_n. We denote by $\partial\Omega_n$ the boundary of Ω_n.

Among the triangles of P_n, let us consider first a border triangle $\Delta_2^{k,-n-1}$ such that $-n+1 \leqq k \leqq n-1$. This triangle and $\Delta_1^{k,-n}$ have the side B in common, and $\Delta_1^{k,-n}$ belongs to P_n by definition. Hence this side B does not belong to the boundary of P_n. Therefore by the definition of Ω_n, it is easy to see that this side B does not contain any part of $\partial\Omega_n$ except their two intersection points. But on account of the shape of P_n and by the definition of Ω_n, we see that the side A contains some part of a boundary component of Ω_n. Let us denote this boundary component by γ_n^k (see Figures 3 and 4). In $\Delta_2^{k,-n-1}$, γ_n^k consists of two segments contained in the sides A and C of $\Delta_2^{k,-n-1}$, and of three arcs near 0, 1 and ∞. Here we mean by an arc near 0, 1 or ∞, the arc of the sector which is given in (a), (b) or (c) of 3.2 and whose vertex is 0, 1 or ∞.

Let us follow γ_n^k, starting from a point of γ_n^k on the side A of $\Delta_2^{k,-n-1}$. We go along A to the arc near 0, and follow this arc, and arrive at the side C.

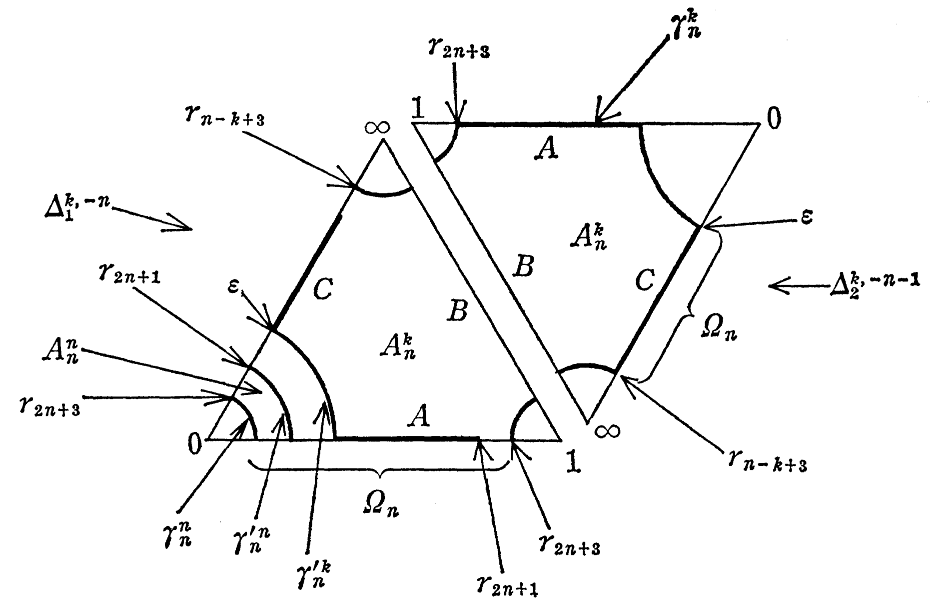


Figure 4.

We go along C, follow the arc near ∞ and arrive at the side B. As the triangles $\Delta_2^{k,-n-1}$ and $\Delta_1^{k,-n}$ are adjacent along B, we then enter $\Delta_1^{k,-n}$. Note that, since the side B of $\Delta_2^{k,-n-1}$ and the side B of $\Delta_1^{k,-n}$ are glued by use of the symmetry described in 1.2, we enter $\Delta_1^{k,-n}$ at a point near ∞ of $\Delta_1^{k,-n}$. We further follow the arc near ∞ and enter $\Delta_2^{k+1,-n}$. Going on in this way, we arrive at $\Delta_1^{n,-k}$. We follow the arc near ∞ of $\Delta_1^{n,-k}$. Then we enter $\Delta_2^{n+1,-k}$. In $\Delta_2^{n+1,-k}$ we follow successively the arc near ∞, the side B, the arc near 1, the side A and the arc near 0. We then enter $\Delta_1^{n,-k}$, follow the arc near 0 and enter $\Delta_2^{n,-k}$. We continue to follow γ_n^k in this way. Then after having passed a finite number of triangles, we finally return to the starting point. Thus we know completely the boundary component γ_n^k. To sum up, we write down below the triangles which the boundary component γ_n^k passes successively:

$$\begin{aligned}\gamma_n^k :\ & \Delta_2^{k,-n-1},\ \Delta_1^{k,-n},\ \Delta_2^{k+1,-n},\ \Delta_1^{k+1,-n+1},\ \cdots,\ \Delta_2^{k+i,-n-1+i},\\ & \Delta_1^{k+i,-n+i},\ \cdots,\ \Delta_2^{n,-k-1},\ \Delta_1^{n,-k},\ \Delta_2^{n+1,-k},\ \Delta_1^{n,-k},\\ & \Delta_2^{n,-k},\ \cdots,\ \Delta_1^{n-i,-k},\ \Delta_2^{n-i,-k},\ \cdots,\ \Delta_1^{-n,-k},\ \Delta_2^{-n,-k},\\ & \Delta_1^{-n-1,-k},\ \Delta_2^{-n,-k},\ \Delta_1^{-n,-k+1},\ \cdots,\ \Delta_2^{-n+i,-k+i},\\ & \Delta_1^{-n+i,-k+1+i},\ \cdots,\ \Delta_1^{k-1,n},\ \Delta_2^{k,n},\ \Delta_1^{k,n+1},\ \Delta_2^{k,n},\\ & \Delta_1^{k,n},\ \cdots,\ \Delta_2^{k,n-i},\ \Delta_1^{k,n-i},\ \cdots,\ \Delta_2^{k,-n},\ \Delta_1^{k,-n},\ \Delta_2^{k,-n-1}.\end{aligned}$$

To each integer k such that $-n+1\leqq k\leqq n-1$ there corresponds one boundary component γ_n^k. Hence Ω_n has $2n-1$ boundary components of this kind.

Now let us consider the border triangle $\Delta_1^{n,-n}$. We find that another boundary component of Ω_n passes it. Let us denote it by γ_n^n. Following γ_n^n exactly in the same manner as above, we observe that the triangles which γ_n^n passes successively are the following:

$$\begin{aligned}\gamma_n^n:\ &\Delta_1^{n,-n},\ \Delta_2^{n,-n},\ \Delta_1^{n-1,-n},\ \cdots,\ \Delta_2^{n-i,-n},\ \Delta_1^{n-1-i,-n},\ \cdots,\\ &\Delta_1^{-n,-n},\ \Delta_2^{-n,-n},\ \Delta_1^{-n-1,-n},\ \Delta_2^{-n,-n},\ \Delta_1^{-n,-n+1},\ \cdots,\\ &\Delta_2^{-n+i,-n+i},\ \Delta_1^{-n+i,-n+1+i},\ \cdots,\ \Delta_1^{n-1,n},\ \Delta_2^{n,n},\ \Delta_1^{n,n+1},\\ &\Delta_2^{n,n},\ \Delta_1^{n,n},\ \cdots,\ \Delta_2^{n,n-i},\ \Delta_1^{n,n-i},\ \cdots,\ \Delta_1^{n,-n+1},\\ &\Delta_2^{n,-n},\ \Delta_1^{n,-n}.\end{aligned}$$

Finally let us consider the border triangle $\Delta_2^{-n,n}$. Another boundary component of Ω_n passes it. Let us denote it by γ_n^{-n}. The triangles which γ_n^{-n} passes successively are the following:

$$\begin{aligned}\gamma_n^{-n}:\ &\Delta_2^{-n,n},\ \Delta_1^{-n,n},\ \Delta_2^{-n,n-1},\ \cdots,\ \Delta_1^{-n,n-i},\ \Delta_2^{-n,n-1-i},\ \cdots,\\ &\Delta_2^{-n,-n},\ \Delta_1^{-n,-n},\ \Delta_2^{-n,-n-1},\ \Delta_1^{-n,-n},\ \Delta_2^{-n+1,-n},\ \cdots,\\ &\Delta_1^{-n+i,-n+i},\ \Delta_2^{-n+1+i,-n+i},\ \cdots,\ \Delta_2^{n,n-1},\ \Delta_1^{n,n},\\ &\Delta_2^{n+1,n},\ \Delta_1^{n,n},\ \Delta_2^{n,n},\ \cdots,\ \Delta_1^{n-i,n},\ \Delta_2^{n-i,n},\ \cdots,\\ &\Delta_2^{-n+1,n},\ \Delta_1^{-n,n},\ \Delta_2^{-n,n}.\end{aligned}$$

Thus we have obtained $2n+1$ boundary components γ_n^k, $k=-n,\ -n+1,\ \cdots,\ n$, in all. We see easily by means of Figure 3 that there is no other boundary component of Ω_n, which concludes the following

PROPOSITION 2. *The boundary of Ω_n consists of the above mentioned $2n+1$ boundary components γ_n^k, $k=-n,\ -n+1,\ \cdots,\ n$.*

REMARK 4. By Figure 3 we find that the number of the triangles which the boundary component γ_n^k passes is almost equal to $12n$. The number depends on the shape of the polygon. It may increase if the shape is different from the diamond. It is a standard way to take, for the n-th polygon, the union of all the triangles of generation smaller than a certain number that depends on n, say $2n$ (cf. [1] p. 243). The union in this case is hexagonal, and each boundary component passes $18n$ triangles. Even though this choice is just sufficient to show that our Riemann surface R belongs to the class O_{AD}, this is not sufficient to show that R belongs to the class O_{AB}. Namely, as we shall see later in the calculation of the area of A_n^k, the value $18n$ is too large to obtain the relation $\mu_n^k \geqq c/n$ with $c>1/2$, and the condition (1) of Pfluger's criterion would not probably be satisfied.

3.4. Construction of annuli A_n^k. We have seen in 3.3 that Ω_n has $2n+1$ boundary components γ_n^k, $k=-n,\ -n+1,\ \cdots,\ n$. We shall construct annuli A_n^k

by enlarging γ_n^k. To this aim, in every equilateral triangle Δ_ν^{ij}, we define once and for all a linear density ρ given by

$$\rho|dz| = \underset{0 \leq i \leq 2}{\mathrm{Max}} \frac{|dz|}{|z-a_i|}, \tag{5}$$

where a_0, a_1 and a_2 are the coordinates of the vertices 0, 1 and ∞ of Δ_ν^{ij} respectively. The collection of such ρ defines evidently a linear density on R, which will be denoted also by ρ. The distance between two sets E and F measured by ρ will be called the *ρ-distance* between E and F and denoted by $d(E, F)$.

Roughly speaking, A_n^k will be defined to be the set of all the points z of Ω_n such that $d(z, \gamma_n^k) < 2\pi/3$. In inner triangles Δ_ν^{ij} we shall adopt this definition, but in border triangles Δ_ν^{ij} we need a slight modification.

(a) *Case of A_n^k with $-n+1 \leqq k \leqq n-1$.* In order to define A_n^k, let us consider in the triangles which γ_n^k passes, the sets as follows:

(a-1) in each inner triangle Δ_ν^{ij} (also in $\Delta_2^{n,-n}$ for $k=n-1$, and in $\Delta_1^{-n,n}$ for $k=-n+1$), the set $\{z \in \Omega_n | d(z, \gamma_n^k) < 2\pi/3\}$,

(a-2) in each of the border triangles $\Delta_2^{k,-n-1}$, $\Delta_2^{n+1,-k}$, $\Delta_1^{-n-1,-k}$ and $\Delta_1^{k,n+1}$, the set of all points of Ω_n, and

(a-3) in the border triangles $\Delta_1^{k,-n}$, $\Delta_1^{n,-k}$, $\Delta_2^{-n,-k}$ and $\Delta_2^{k,n}$, the sets $\{z \in \Omega_n | |z-a_0| > \varepsilon\}$, $\{z \in \Omega_n | |z-a_1| > \varepsilon\}$, $\{z \in \Omega_n | |z-a_1| > \varepsilon\}$ and $\{z \in \Omega_n | |z-a_0| > \varepsilon\}$ respectively.

Now we define A_n^k $(-n+1 \leqq k \leqq n-1)$ to be the union of all the sets given by (a-1), (a-2) and (a-3) (see Figure 4).

(b) *Case of A_n^n.* In order to define A_n^n, let us consider in the triangles which γ_n^n passes, the sets as follows:

(b-1) in each of the border triangles $\Delta_1^{n,-n}$, $\Delta_1^{-n-1,-n}$ and $\Delta_1^{n,n+1}$, the set of all points of Ω_n,

(b-2) in the border triangles $\Delta_2^{n,-n}$, $\Delta_2^{-n,-n}$ and $\Delta_2^{n,n}$, the sets $\{z \in \Omega_n | |z-a_2| > \varepsilon\}$, $\{z \in \Omega_n | |z-a_1| > \varepsilon\}$ and $\{z \in \Omega_n | |z-a_0| > \varepsilon\}$ respectively, and

(b-3) in each triangle other than (b-1) and (b-2), the set $\{z \in \Omega_n | d(z, \gamma_n^n) < 2\pi/3\}$.

We define A_n^n to be the union of all the sets given by (b-1), (b-2) and (b-3).

(c) *Case of A_n^{-n}.* In order to define A_n^{-n}, let us consider in the triangles which γ_n^{-n} passes, the sets as follows:

(c-1) in each of the border triangles $\Delta_2^{-n,n}$, $\Delta_2^{-n,-n-1}$ and $\Delta_2^{n+1,n}$, the set of all points of Ω_n,

(c-2) in the border triangles $\Delta_1^{-n,n}$, $\Delta_1^{-n,-n}$ and $\Delta_1^{n,n}$, the sets $\{z \in \Omega_n | |z-a_2| > \varepsilon\}$, $\{z \in \Omega_n | |z-a_0| > \varepsilon\}$ and $\{z \in \Omega_n | |z-a_1| > \varepsilon\}$ respectively, and

(c-3) in each triangle other than (c-1) and (c-2), the set $\{z \in \Omega_n | d(z, \gamma_n^{-n}) < 2\pi/3\}$.

We define A_n^{-n} to be the union of all the sets given by (c-1), (c-2) and (c-3).

3.5. Conditions for A_n^k. Let us show that the family of A_n^k constructed in 3.4 satisfies the conditions (1.1), $\cdots$, (1.4) given in 2.1. It is easily verified by using Figures 3 and 4 that A_n^k is an annulus bounded by two piecewise analytic and closed curves of which one is γ_n^k. The other will be denoted by γ'^k_n. Hence (1.1) is satisfied.

Next let us show that the annuli A_n^k are disjoint. It is clear by definition that A_n^k is contained in Ω_n. We wish to show that A_n^k is in fact contained in $\Omega_n - \Omega_{n-1}$. If Δ_ν^{ij} is a border triangle of P_n, then it is not contained in P_{n-1}. Hence $\Delta_\nu^{ij} \cap A_n^k \subset \Omega_n - \Omega_{n-1}$. If Δ_ν^{ij} is an inner triangle of P_n, then from (a) and (b) of 3.2, $\partial\Omega_n \cap \Delta_\nu^{ij}$ consists of arcs of the form $\{|z-a_h|=r_{2n+3+s}\}$, and $\partial\Omega_{n-1} \cap \Delta_\nu^{ij}$ consists of arcs of the form $\{|z-a_h|=r_{2n+1+s}\}$ with certain h and s. But by (4) and (5) the ρ-distance between the arcs $\{|z-a_h|=r_{2n+3+s}\}$ and $\{|z-a_h|=r_{2n+1+s}\}$ is equal to

$$\int_{r_{2n+3+s}}^{r_{2n+1+s}} \frac{dx}{x} = \frac{2\pi}{3}.$$

So in an inner triangle Δ_ν^{ij}, the points z of Δ_ν^{ij} such that $d(z, \gamma_n^k) < 2\pi/3$ do not belong to Ω_{n-1}. From this and (a-1), (b-3) and (c-3) in 3.4, we see that $\Delta_\nu^{ij} \cap A_n^k \subset \Omega_n - \Omega_{n-1}$. Hence $A_n^k \subset \Omega_n - \Omega_{n-1}$. It follows that if $n \neq n'$, then $A_n^k \cap A_{n'}^{k'} = \emptyset$. We wish to show furthermore that if $k \neq k'$, then $A_n^k \cap A_n^{k'} = \emptyset$. In fact, in inner triangles of P_n which both γ_n^k and $\gamma_n^{k'}$ pass, we see from the same argument as above that not only γ_n^k and $\gamma_n^{k'}$ but also A_n^k and $A_n^{k'}$ are separated by Ω_{n-1}. In border triangles Δ_ν^{ij} we have only to show that A_n^n is disjoint from other A_n^k, and so is A_n^{-n}. But this holds by the definitions of Ω_n and A_n^k, and by the fact that the ρ-distance between the arcs $\{|z-a_h|=r_2\}$ and $\{|z-a_h|=\varepsilon\}$ is $2\pi/3$ (see Figure 4). Consequently (1.2) holds.

We defined Ω_n to be the remaining part of P_n after having taken off certain sectors with vertices 0, 1 and ∞. Hence any point of the complement of Ω_n in R can be joined to one of the vertices 0, 1 and ∞ by a curve in the complement. Since the vertices 0, 1 and ∞ do not belong to R actually, this implies that the complement of Ω_n in R has no compact connected component. Therefore, by the definition of A_n^k, the complement of $\bigcup_{k=-n}^{n} A_n^k$ in R has clearly a unique compact connected component, which is denoted by B_n. Hence (1.3) holds.

Since A_n^k is contained in $\Omega_n - \Omega_{n-1}$, B_n contains Ω_{n-1}. This implies that B_n contains all $A_{n'}^{k'}$ such that $n' < n$. Hence (1.4) holds. Thus all the four conditions (1.1), $\cdots$, (1.4) are satisfied.

§ 4. Harmonic module of A_n^k.

We shall evaluate the harmonic module of our annulus A_n^k by calculating the extremal length of the family Γ of all the curves joining γ_n^k and $\gamma_n'^k$. We shall use to this aim the linear density ρ given by (5), and show that the width of A_n^k measured by this ρ is $2\pi/3$ and that its area measured by ρ is almost equal to $(2\pi/3)\times(\pi/3)\times 24n$.

This area is obtained roughly as follows. The contribution to it of the area of the part contained in inner triangles is $(2\pi/3)\times(\pi/3)\times 12n$, since the number of the inner triangles intersecting A_n^k is almost equal to $12n$. The contribution of the area of the part contained in border triangles is $(2\pi/3)\times 4n\pi$, since the total length of this part is $4n\pi$.

Thus our estimate will show that the harmonic module of A_n^k is greater than

$$2\pi\times\frac{(2\pi/3)^2}{16\pi^2 n/3}=\frac{\pi}{6n}\sim 0.52\times\frac{1}{n}.$$

In the following we shall estimate it precisely.

4.1. Lengths of curves of Γ. Let γ be a curve in A_n^k joining its boundary components γ_n^k and $\gamma_n'^k$. We wish to establish the following inequality

$$\int_\gamma \rho\,|dz| \geqq \frac{2\pi}{3} \tag{6}$$

where ρ is the linear density given by (5). We divide the proof into cases.

(i) Suppose γ is contained in an inner triangle. Then by definition $A_n^k \cap \Delta_\nu^{ij} = \{z\in\Delta_\nu^{ij}\cap\Omega_n \mid d(z, \gamma_n^k)<2\pi/3\}$. Hence (6) holds.

(ii) Suppose γ is contained in the union of a finite number of inner triangles. In this case also we obtain (6) with no additional difficulty. If necessary, it is sufficient to note that $\{z\in\Delta_\nu^{ij}\cap\Omega_n \mid d(z, \gamma_n^k)<2\pi/3\}$ is nothing but the set $\{r_{2n+3+s}<|z-a_h|<r_{2n+1+s}\}$ with certain h and s, and that the intersection of A_n^k with the union of a finite number of inner triangles is mapped by $w=\log(z-a_h)$ onto a rectangle of width $2\pi/3$.

(iii) Suppose γ is contained in the union of two border triangles. Since the proof is just the same, we suppose for example that $\gamma\subset\Delta_1^{k,-n}\cup\Delta_2^{k,-n-1}$, and that γ starts from $\gamma_n'^k$ and ends at γ_n^k (see Figure 4). This case is further divided into three cases.

(iii-1) Suppose γ starts from the side A of $\Delta_1^{k,-n}$. If γ ends at the arc $\{|z-a_1|=r_{2n+3}\}$ in $\Delta_1^{k,-n}$ or $\Delta_2^{k,-n-1}$, then (6) holds, because the ρ-distance between the arcs $\{|z-a_1|=r_{2n+3}\}$ and $\{|z-a_1|=r_{2n+1}\}$ is $2\pi/3$. If γ ends at the arc $\{|z-a_2|=r_{n-k+3}\}$ in $\Delta_1^{k,-n}$ or $\Delta_2^{k,-n-1}$, then γ crosses the arc $\{|z-a_2|=r_{n-k+1}\}$ in $\Delta_1^{k,-n}$ or $\Delta_2^{k,-n-1}$ too, and again (6) holds for the same reason. If

γ ends at the side A of $\Delta_2^{k,-n-1}$, then (6) holds, because the angle between the sides A and B of $\Delta_1^{k,-n}$ is $\pi/3$, and that of $\Delta_2^{k,-n-1}$ is also $\pi/3$. If necessary, replace γ, using the reflexion, by a curve with the same length. If γ ends at the side C of $\Delta_2^{k,-n-1}$, then (6) holds according to the same argument. If γ ends at the arc $\{|z-a_0|=\varepsilon\}$ in $\Delta_2^{k,-n-1}$, then γ crosses the arc $\{|z-a_0|=1/2\}$ in $\Delta_2^{k,-n-1}$. As $\varepsilon<(e^{-2\pi/3})/2$ by (3), the ρ-distance between these arcs is greater than $2\pi/3$. Hence (6) holds.

(iii-2) Suppose γ starts from the side C of $\Delta_1^{k,-n}$. Then by symmetry and (iii-1), (6) holds.

(iii-3) Suppose γ starts from the arc $\{|z-a_0|=\varepsilon\}$ in $\Delta_1^{k,-n}$. Then γ crosses the arc $\{|z-a_0|=1/2\}$ in $\Delta_1^{k,-n}$, and (6) holds, because the ρ-distance between these arcs is larger than $2\pi/3$.

(iv) Suppose γ is contained in the union of border triangles and inner triangles. Then we obtain (6) by an analogous argument to (ii) and (iii). Consequently (6) holds for all curves of Γ.

4.2. Area of A_n^k. For a set E in the homology covering surface R and the linear density ρ given by (5), we write $z=x+iy$ and set

$$S(E)=\iint_E \rho^2 dxdy$$

and call it simply the *area* of E. We wish to show the inequality

$$S(A_n^k)\leqq\frac{16\pi^2}{3}n+c \tag{7}$$

where c is a constant independent of n and k. In the following we shall denote by c all constants which are independent of n and k. By an easy calculation we have the relation

$$S(\{r_t<|z-a_h|<1/2\}\cap\Delta_\nu^{ij})=\frac{\pi^2}{9}t+c \tag{8}$$

which will be used below.

(a) *Case of A_n^k with $-n+1\leqq k\leqq n-1$.* We set

$$S_1=\sum S(A_n^k\cap\Delta_\nu^{ij})$$

where the sum is over all inner triangles Δ_ν^{ij}, and set

$$S_2=\sum S(A_n^k\cap\Delta_\nu^{ij})$$

where the sum is over all border triangles, so that we have $S(A_n^k)=S_1+S_2$.

First let Δ_ν^{ij} be an inner triangle of P_n such that $A_n^k\cap\Delta_\nu^{ij}\neq\emptyset$. Then, since $A_n^k\cap\Delta_\nu^{ij}$ is of the form $\{r_{2n+3+s}<|z-a_h|<r_{2n+1+s}\}$ with certain h and s, we have $S(A_n^k\cap\Delta_\nu^{ij})=2\pi^2/9$ by (8). By using Figure 3, we find that there are

$12n-2$ such inner triangles. Hence

$$S_1=\frac{2\pi^2}{9}(12n-2).$$

Next we calculate the area of the part of A_n^k in border triangles. Since $A_n^k\cap\Delta_1^{k,-n}=\{|z-a_0|>\varepsilon\}\cap\{|z-a_1|>r_{2n+3}\}\cap\{|z-a_2|>r_{n-k+3}\}$, and $A_n^k\cap\Delta_2^{k,-n-1}$, $A_n^k\cap\Delta_1^{n,-k}$ and $A_n^k\cap\Delta_2^{n+1,-k}$ have the same shape as $A_n^k\cap\Delta_1^{k,-n}$, we have, by (8),

$$S(A_n^k\cap\Delta_1^{k,-n})=S(A_n^k\cap\Delta_2^{k,-n-1})=S(A_n^k\cap\Delta_1^{n,-k})=S(A_n^k\cap\Delta_2^{n+1,-k})$$
$$=\frac{\pi^2}{9}(3n-k)+c.$$

On the other hand, since $A_n^k\cap\Delta_1^{-n-1,-k}=\{|z-a_0|>r_{2n+3}\}\cap\{|z-a_1|>\varepsilon\}\cap\{|z-a_2|>r_{n+k+2}\}$, and $A_n^k\cap\Delta_2^{-n,-k}$, $A_n^k\cap\Delta_2^{k,n}$ and $A_n^k\cap\Delta_1^{k,n+1}$ have the same shape as $A_n^k\cap\Delta_1^{-n-1,-k}$, we have

$$S(A_n^k\cap\Delta_1^{-n-1,-k})=S(A_n^k\cap\Delta_2^{-n,-k})=S(A_n^k\cap\Delta_2^{k,n})=S(A_n^k\cap\Delta_1^{k,n+1})$$
$$=\frac{\pi^2}{9}(3n+k)+c.$$

Summing these eight areas together, we have

$$S_2=\frac{8\pi^2}{3}n+c.$$

Therefore in this case we have

$$S(A_n^k)=\frac{16\pi^2}{3}n+c.$$

(b) *Case of* A_n^n. By using Figure 3, we find that there are $12n-3$ triangles Δ_ν^{ij} such that $A_n^n\cap\Delta_\nu^{ij}$ is of the form $\{r_{2n+3+s}<|z-a_h|<r_{2n+1+s}\}$. Let S_1 be the area of the part of A_n^n in the union of such triangles. Then clearly we have

$$S_1=\frac{2\pi^2}{9}(12n-3).$$

Let S_2 be the area of the part of A_n^n in the union of $\Delta_1^{n,-n}$, $\Delta_2^{n,-n}$, $\Delta_2^{-n,-n}$, $\Delta_1^{-n-1,-n}$, $\Delta_2^{n,n}$ and $\Delta_1^{n,n+1}$. Since $A_n^n\cap\Delta_1^{n,-n}$ and $A_n^n\cap\Delta_2^{n,-n}$ are of the form $\{|z-a_0|>r_{2n+3}\}\cap\{|z-a_1|>r_{2n+3}\}\cap\{|z-a_2|>\varepsilon\}$, we have, by (8),

$$S(A_n^n\cap\Delta_1^{n,-n})=S(A_n^n\cap\Delta_2^{n,-n})=\frac{4\pi^2}{9}n+c.$$

Since $A_n^n\cap\Delta_2^{-n,-n}=\{|z-a_0|>r_{2n+3}\}\cap\{|z-a_1|>\varepsilon\}\cap\{|z-a_2|>r_{2n+2}\}$, and $A_n^n\cap\Delta_1^{-n-1,-n}$, $A_n^n\cap\Delta_2^{n,n}$ and $A_n^n\cap\Delta_1^{n,n+1}$ have the same shape as $A_n^n\cap\Delta_2^{-n,-n}$, we have

$$S(A_n^n \cap \Delta_2^{-n,-n}) = S(A_n^n \cap \Delta_1^{-n-1,-n}) = S(A_n^n \cap \Delta_2^{n,n}) = S(A_n^n \cap \Delta_1^{n,n+1})$$
$$= \frac{4\pi^2}{9} n + c .$$

Summing these six areas together, we obtain

$$S_2 = \frac{8\pi^2}{3} n + c .$$

Hence in this case too, we have

$$S(A_n^n) = \frac{16\pi^2}{3} n + c .$$

(c) *Case of* A_n^{-n}. We calculate the area of A_n^{-n} exactly in the same way as A_n^n, and we obtain

$$S(A_n^{-n}) = \frac{16\pi^2}{3} n + c .$$

From the results of (a), (b) and (c), noting that the constant c in the three cases may be different, we obtain the inequality (7), which is what we wished to show.

4.3. Application of Pfluger's criterion. We obtain, by (2), (6) and (7), the following inequality for the harmonic module μ_n^k of A_n^k, $k=-n, -n+1, \cdots, n$:

$$\mu_n^k \geqq 2\pi \frac{(2\pi/3)^2}{16\pi^2 n/3 + c} .$$

It follows that $\mu_n = \operatorname*{Min}_k \mu_n^k$ is greater than or equal to the right hand side. As to the number $k(n)$ of the annuli A_n^k, we have $k(n) = 2n+1$, and so

$$K(N) = \operatorname*{Max}_{n \leq N} k(n) = 2N+1 .$$

Therefore, from the relation

$$\frac{2\pi(2\pi/3)^2}{16\pi^2/3} = \frac{\pi}{6} = 0.52\cdots > \frac{1}{2},$$

we obtain the relation (1). Consequently, by grace of Pfluger's criterion, the proof of Theorem I is entirely achieved.

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