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Scattering of solutions of nonlinear Klein-Gordon equations in three space dimensions

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1. Introduction.

Let $h(u)$ be a C^1 function on $\boldsymbol{R}$ such that $h(0)=0$,

(H1) $$|h'(u)|\leqq c|u|^{p-1}, \qquad \forall u\in\boldsymbol{R},$$

and

(H2) $$H(u)=\int_0^u h(s)ds\geqq 0 \qquad \forall u\in\boldsymbol{R}$$

where $p>1$.

Consider the nonlinear Klein-Gordon equation (the "perturbed" equation)

(NLKG) $$\frac{\partial^2}{\partial t^2}u-\Delta u+m^2u+h(u)=0 \qquad (x\in\boldsymbol{R}^3,\ t\in\boldsymbol{R})$$

together with the "free" equation

(KG) $$\frac{\partial^2}{\partial t^2}v-\Delta v+m^2v=0, \qquad (x\in\boldsymbol{R}^3,\ t\in\boldsymbol{R})$$

where m is a positive constant.

In [10]-[13] W. Strauss developed the theory of nonlinear scattering at low energy, in which one looks for conditions under which solutions u of (NLKG) are related to free solutions $u_\pm$ of (KG) by the asymptotic condition $\|u(t)-u_\pm(t)\|_e \to 0$ as $t\to\pm\infty$, where $\|\cdot\|_e$ denotes the energy norm:

$$\|w(t)\|_e^2=\int\left[\left(\frac{\partial w}{\partial t}\right)^2+|\nabla w|^2+m^2w^2\right]dx \quad \left(=\left\|\left(w(t),\ \frac{\partial w}{\partial t}(t)\right)\right\|_e\right).$$

It has been shown that under the least regularity assumption on solutions, that is, under the assumption that solutions are of finite energy, the theory of nonlinear scattering at low energy holds in case of $1+\frac{4}{3}\leqq p\leqq 3$. If we require more regularity of solutions, the theory holds for $2<p<5$. In [3] Glassey showed that the theory does not hold if $1<p<1+\frac{2}{3}$.

Our aim of this paper is to show that in case of $3<p<5$ the theory holds valid under the assumption that solutions are of finite energy. Thus we have the nonlinear scattering theory at low energy in the range $1+\frac{4}{3}\leqq p<5$, which is the conjectured optimal one (see [11], [12]). Although our argument relies deeply on Strauss, our approach to construct solutions is different from his and does not seem to fit his abstract framework in which the contraction mapping theorem plays an important role. In order to develop the theory in the range $3<p<5$, we make use of Besov space estimates of free solutions and of nonlinearity, which has been used by Brenner and von Wahl [2], Pecher [6], and Tsutsumi [15].

In a forthcoming paper the case of higher space dimension will be considered.

Let $u_-(t)$ be a free solution of finite energy:

$$u_-(t)=\frac{d}{dt}E(t)f_-+E(t)g_-$$

for some $f_-\in H^1(\boldsymbol{R}^3)$ and $g_-\in L^2(\boldsymbol{R}^3)$, where

$$E(t)=\mathcal{F}^{-1}\frac{1}{\sqrt{|\xi|^2+m^2}}\sin t\sqrt{|\xi|^2+m^2}\,\mathcal{F}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote Fourier transform and inverse Fourier transform, respectively.

Our main result is the following

THEOREM. *Let* $3<p<5$.

(a) *If* u_- *is any solution of* (KG) *of finite energy, there exists a unique solution* u *of* (NLKG) *in some time interval* $(-\infty, T]$ *such that*

$$u\in L^\infty((-\infty, T]: H^1(\boldsymbol{R}^3))\cap C((-\infty, T];\ L^2(\boldsymbol{R}^3))$$
$$\cap L^{2(p-1)}((-\infty, T]\times\boldsymbol{R}^3)\cap L^4((-\infty, T]\times\boldsymbol{R}^3)$$

with $u_t\in L^\infty((-\infty, T]: L^2(\boldsymbol{R}^3))$, *and* $\|u(t)-u_-(t)\|_e\to 0$ *as a.e.* $t\to-\infty$.

(b) *If* $\|u_-\|_e$ *is sufficiently small, then* u *exists for all time,* $u\in L^{2(p-1)}(\boldsymbol{R}_t^1\times\boldsymbol{R}_x^3)\cap L^4(\boldsymbol{R}_t^1\times\boldsymbol{R}_x^3)$, *and there exists a unique solution* u_+ *of* (KG) *such that* $\|u(t)-u_+(t)\|_e\to 0$ *as a.e.* $t\to+\infty$, *and*

$$\|u_+\|_e^2=\|u_-\|_e^2=\|u\|_e^2+2\int H(u)dx.$$

2. Preliminaries.

Let $s\in\boldsymbol{R}$ and $1\leqq p\leqq\infty$. $H^{s,p}(\boldsymbol{R}^3)$ is the usual Sobolev space of fractional order s of all L^p-functions u such that

$$\|u\|_{s,p}=\|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}u\|_p<\infty\,.$$

We denote $H^{s,2}(\boldsymbol{R}^3)$ by $H^s(\boldsymbol{R}^3)$. Let $1<p,\ q<\infty$ and write for $s>0$, $s=[s]+\sigma$

where $[s]$ denotes the largest integer less than s and $0<\sigma<1$. The Besov space $B_p^{s,q}(\boldsymbol{R}^3)$ is the completion of $\mathcal{S}(\boldsymbol{R}^3)$ in the norm

$$\|u\|_{s,p,q}=\|u\|_p+\left(\int_0^1 t^{-\sigma q}\sup_{|k|\leqq t}\sum_{|\alpha|\leqq[s]}\|D^\alpha(u_k-u)\|_p^q\frac{dt}{t}\right)^{1/q}$$

where $u_k(x)=u(x+k)$ (see [1]).

In the following constants will be denoted by C, and will change from line to line. If necessary, we indicate dependence of constants on the quantities α, β, $\cdots$ by $C(\alpha, \beta, \cdots)$.

We now consider the free equation (KG) with Cauchy data $v(x, 0)=f(x)$; $\frac{\partial}{\partial t}v(x, 0)=g(x)$. The solution v of (KG) with the above Cauchy data is given by $v=\frac{d}{dt}E(t)f+E(t)g$. The following two lemmas concerning L^p-estimates of solutions of (KG) are essential tools for the study.

LEMMA 1 ([3], [5], [6]). *Let* s, $s'\geqq 0$, $1\leqq q\leqq\infty$, $1\leqq p'\leqq 2$, $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\delta=\frac{1}{2}-\frac{1}{p}$. *Then*

$$\|E(t)g\|_{s',p,q}\leqq K(t)\|g\|_{s,p',q}\qquad t\geqq 0 \tag{1}$$

where

$$K(t)=\begin{cases} Ct^{-2\delta}, & t\geqq 1 \\ Ct^{1+s-s'-6\delta}, & 0<t<1, \end{cases}$$

provided $4\delta\leqq 1+s-s'$.

We may replace $\|\cdot\|_{s',p,q}$ *and* $\|\cdot\|_{s,p',q}$ *by* $\|\cdot\|_{s',p}$ *and* $\|\cdot\|_{s,p'}$, *respectively.*

LEMMA 2 ([14]). *Assume* $(f, g)\in H^1(\boldsymbol{R}^3)\times L^2(\boldsymbol{R}^3)$. *Then*

$$\|v\|_{L^r([0,\infty)\times\boldsymbol{R}^3)}\leqq C(\|f\|_{1,2}+\|g\|_2)=C\|(f, g)\|_e \tag{2}$$

provided $\frac{10}{3}\leqq r\leqq 8$.

LEMMA 3. *Let* $3<p<5$, $p_1=2(p-1)$, $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $\sigma=\frac{(p-3)(3p-7)}{2(p-1)(p-2)}$ *and* $\rho=\frac{p-3}{p-2}$. *Then for every* $u\in H^1(\boldsymbol{R}^3)\cap L^{p_1}(\boldsymbol{R}^3)$ *we have*

$$\|h(u)\|_{\sigma+\varepsilon,p_1',q}\leqq C\|u\|_{1,2}^\rho\|u\|_{p_1}^{p-\rho} \tag{3}$$

where ε *is a positive number so small that* $\sigma+\varepsilon<\rho$, *provided* $q\geqq\frac{1}{\rho}=\frac{p-2}{p-3}$.

PROOF. Note that $\sigma<\rho$ for $3<p<5$.

We have

$$\|h(u)\|_{\sigma+\varepsilon,p_1',q}$$
$$=\|h(u)\|_{p_1'}+\left(\int_0^1 t^{-(\sigma+\varepsilon)q}\sup_{|k|\leqq t}\|h(u_k)-h(u)\|_{p_1'}^q\frac{dt}{t}\right)^{1/q}.$$

The first term of the right hand side is dominated by

$$C\||u|^{p-1}u\|_{p_1'}\leqq C\|u\|_2^\rho\|u\|_{p_1}^{p-\rho}\leqq C\|u\|_{1,2}^\rho\|u\|_{p_1}^{p-\rho}$$

since $\dfrac{p-\rho}{p_1}+\dfrac{\rho}{2}=\dfrac{1}{p_1'}$. Analogously we have

$$\begin{aligned}\|h(u_k)-h(u)\|_{p_1'}&\leqq C\|(|u_k|^{p-1}+|u|^{p-1})|u_k-u|\|_{p_1'}\\&\leqq C\|(|u_k|^{p-\rho}+|u|^{p-\rho})|u_k-u|^\rho\|_{p_1'}\\&\leqq C\|u\|_{p_1}^{p-\rho}\|u_k-u\|_2^\rho .\end{aligned}$$

Hence the second term is estimated above by

$$C\|u\|_{p_1}^{p-\rho}\|u\|_{(\sigma+\varepsilon)/\rho,2,\rho q}^\rho\leqq C\|u\|_{p_1}^{p-\rho}\|u\|_{1,2}^\rho$$

since $0<(\sigma+\varepsilon)/\rho<1$ and $H^{1,2}(\boldsymbol{R}^3)\subset B_2^{(\sigma+\varepsilon)/\rho,\rho q}(\boldsymbol{R}^3)$ provided $\rho q\geqq 1$.

Therefore we obtain

$$\|h(u)\|_{\sigma+\varepsilon,p_1',q}\leqq C\|u\|_{1,2}^\rho\|u\|_{p_1}^{p-\rho}.$$

Q. E. D.

For $s\in\boldsymbol{R}$ define

$$(\mathcal{H}_s u)(t)=\int_s^t E(t-\tau)h(u(\tau))d\tau .$$

LEMMA 4. *Suppose that all the hypotheses of Lemma 3 hold. Then for any* $u\in L^\infty(\boldsymbol{R};\ H^1(\boldsymbol{R}^3))\cap L^{p_1}((-\infty,\ T]\times\boldsymbol{R}^3)\,(s\leqq T)$ *we have*

$$\left(\int_{T_1}^T \|\mathcal{H}_s u(t)\|_{p_1}^{p_1}dt\right)^{1/p_1}\leqq C\sup_{t\in\boldsymbol{R}}\|u\|_{1,2}^\rho\left(\int_{T_1}^T \|u(t)\|_{p_1}^{p_1}dt\right)^{(p-\rho)/p_1} \tag{4}$$

for any T_1 *with* $T_1\leqq s\leqq T$.

PROOF. By virtue of Lemma 1, we have

$$\|\mathcal{H}_s u(t)\|_{p_1}\leqq C\left[\int_{I_1}|t-\tau|^{1+\sigma-6\delta}\|h(u(\tau))\|_{\sigma,p_1'}d\tau\right.\\ \left.+\int_{I_2}|t-\tau|^{-2\delta}\|h(u(\tau))\|_{\sigma,p_1'}d\tau\right]$$

where $I_1=[s,\ t]\cap[t-1,\ t+1]$ and $I_2=[s,\ t]\setminus[t-1,\ t+1]$. Since $2\delta\geqq-(1+\sigma-6\delta)>0$ and $|t-\tau|\geqq 1$ on I_2, we have

$$\|\mathcal{H}_s u(t)\|_{p_1}\leqq C\int_s^t|t-\tau|^{1+\sigma-6\delta}\|h(u(\tau))\|_{\sigma,p_1'}d\tau .$$

Since $B_{p_1'}^{\sigma+\varepsilon,q}\subset H^{\sigma,p_1'}$ for any $\varepsilon>0$ and $1\leqq q\leqq\infty$, we get

$$\|\mathcal{H}_s u(t)\|_{p_1}\leqq C\int_s^t|t-\tau|^{1+\sigma-6\delta}\|h(u(\tau))\|_{\sigma+\varepsilon,p_1',q}d\tau .$$

By the singular integral inequality and Lemma 3, we obtain

$$\left(\int_{T_1}^{T}\|\mathcal{H}_s u(t)\|_{p_1}^{p_1}dt\right)^{1/p_1}\leqq C\left(\int_{T_1}^{T}\|h(u(\tau))\|_{\sigma+\varepsilon,p_1',q}^{r}dt\right)^{1/r}$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^{\rho}\left(\int_{T_1}^{T}\|u(t)\|_{p_1'}^{r(p-\rho)}dt\right)^{1/r}$$

where

$$\frac{1}{r}=\frac{1}{p_1}+2+\sigma-6\delta=\frac{p^2-3p+3}{2(p-1)(p-2)}.$$

Since $r(p-\rho)=p_1$ we have

$$\left(\int_{T_1}^{T}\|\mathcal{H}_s u(t)\|_{p_1}^{p_1}dt\right)^{1/p_1}\leqq C\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^{\rho}\left(\int_{T_1}^{T}\|u(t)\|_{p_1}^{p_1}dt\right)^{(p-\rho)/p_1}.\quad \text{Q.E.D.}$$

LEMMA 5. *Let* $s\leqq T$, $p_1=2(p-1)$ *and* $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $(p>2)$. *Then for every* $u, v\in L^{p_1}((-\infty, T]\times\boldsymbol{R}^3)\cap L^4((-\infty, T]\times\boldsymbol{R}^3)$, *we have*

(5) $$\left(\int_{-\infty}^{T}\|\mathcal{H}_s u(t)-\mathcal{H}_s v(t)\|_4^4dt\right)^{1/4}$$

$$\leqq C\left(\int_{-\infty}^{T}\|u(t)\|_{p_1}^{p_1}dt+\int_{-\infty}^{T}\|v(t)\|_{p_1}^{p_1}dt\right)^{1/2}\left(\int_{-\infty}^{T}\|u(t)-v(t)\|_4^4dt\right)^{1/4}.$$

PROOF. By virtue of Lemma 1, we have

$$\|\mathcal{H}_s u(t)-\mathcal{H}_s v(t)\|_4\leqq C\int_s^t|t-\tau|^{-1/2}\|h(u(\tau))-h(v(\tau))\|_{4/3}d\tau$$

$$\leqq C\int_s^t|t-\tau|^{-1/2}\|(|u|^{p-1}+|v|^{p-1})|u-v|\|_{4/3}d\tau$$

$$\leqq C\int_s^t|t-\tau|^{-1/2}(\|u\|_{p_1}^{p-1}+\|v\|_{p_1}^{p-1})\|u-v\|_4d\tau$$

since $\frac{p-1}{p_1}+\frac{1}{4}=\frac{3}{4}$.

By the singular integral inequality and Hölder's inequality, we obtain

$$\left(\int_{-\infty}^{T}\|\mathcal{H}_s u(t)-\mathcal{H}_s v(t)\|_4^4dt\right)^{1/4}$$

$$\leqq C\left(\int_{-\infty}^{T}(\|u(t)\|_{p_1}^{p-1}+\|v(t)\|_{p_1}^{p-1})^{4/3}\|u(t)-v(t)\|_4^{4/3}dt\right)^{3/4}$$

$$\leqq C\left[\int_{-\infty}^{T}(\|u(t)\|_{p_1}^{p_1}+\|v(t)\|_{p_1}^{p_1})dt\right]^{1/2}\left(\int_{-\infty}^{\infty}\|u(t)-v(t)\|_4^4dt\right)^{1/4}.\qquad \text{Q.E.D.}$$

3. Proof of Theorem.

Consider the integral equation

$$(\mathrm{CP})_s \qquad u(t)=\frac{d}{dt}E(t)f_-+E(t)g_-+\int_s^t E(t-\tau)h(u(\tau))d\tau$$
$$=u_-(t)+\mathcal{H}_s(u(t)).$$

Formally $u(t)$ is the solution of (NLKG) with the initial values at $t=s$,

$$u(s)=u_-(s)\,;\ \frac{du}{dt}(s)=\frac{du_-}{dt}(s).$$

For the moment we assume that $u(t)$ is a smooth solution of $(\mathrm{CP})_s$. Then u satisfies the standard energy equality:

$$\|u(t)\|_e^2+2\int_{R^3}H(u(x,t))dx=\|u(s)\|_e^2+2\int_{R^3}H(u(x,s))dx\,. \tag{6}$$

Since $H(u)\geqq 0\ \forall u\in\boldsymbol{R}$, $\int_{R^3}H(u(x,s))dx\leqq C\|u(s)\|_{1,2}^{p+1}$ and $\|u(s)\|_e=\|(f_-,g_-)\|_e$, we have

$$\|u(t)\|_e\leqq M_1(\|(f_-,g_-)\|_e) \tag{7}$$

where $M_1=M_1(r)$ is a continuous nondecreasing function on $[0,\infty)$ such that $\lim_{r\searrow 0}M_1(r)=M_1(0)=0$.

Taking the L^{p_1}-norm of both sides of $(\mathrm{CP})_s$ and using Lemma 4, we obtain for any T_1, T with $T_1\leqq s\leqq T$

$$\begin{aligned}\Big(\int_{T_1}^T\|u(t)\|_{p_1}^{p_1}dt\Big)^{1/p_1}&\leqq\Big(\int_{T_1}^T\|u_-(t)\|_{p_1}^{p_1}dt\Big)^{1/p_1}\\&\quad+C\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^{\rho}\Big(\int_{T_1}^T\|u(t)\|_{p_1}^{p_1}dt\Big)^{(p-\rho)/p_1}\\&\leqq\Big(\int_{T_1}^T\|u_-(t)\|_{p_1}^{p_1}dt\Big)^{1/p_1}\\&\quad+CM_1(\|(f_-,g_-)\|_e)^{\rho}\Big(\int_{T_1}^T\|u(t)\|_{p_1}^{p_1}dt\Big)^{(p-\rho)/p_1}\end{aligned} \tag{8}$$

where $p-\rho=(p^2-3p+3)/(p-2)>1$ for $3<p<5$.

By virtue of Lemma 2, we have

$$\Big(\int_{-\infty}^{\infty}\|u_-(t)\|_{p_1}^{p_1}dt\Big)^{1/p_1}\leqq C\|(f_-,g_-)\|_e\,.$$

Therefore if $\|(f_-,g_-)\|_e$ is sufficiently small or if $T<0$ and $|T|$ is sufficiently large, i.e., $\int_{-\infty}^T\|u_-(t)\|_{p_1}^{p_1}dt$ is sufficiently small, from (8) we deduce

$$\Big(\int_{-\infty}^T\|u(t)\|_{p_1}^{p_1}dt\Big)^{1/p_1}\leqq M_2(\|(f_-,g_-)\|_e,T) \tag{9}$$

where $M_2=M_2(r, T)$ is a continuous nondecreasing function on $[0, r_0)\times(-\infty, T_0)$ for some $r_0\leqq+\infty$ and $T_0\leqq+\infty$, such that

$$\lim_{r\to r_0} M_2(r, T)=+\infty,\ \lim_{r\to 0} M_2(r, T)=0 \qquad \text{for every } T$$

and

$$\lim_{T\to T_0} M_2(r, T)\leqq+\infty,\ \lim_{T\to-\infty} M_2(r, T)=0 \qquad \text{for every } r.$$

In the former case $T\to M_2(r, T)$ is bounded and we can take $T=+\infty$ in (9). In the latter case we may take $r_0=+\infty$.

Taking the L^4-norm of both sides of $(\mathrm{CP})_s$ and using Lemma 5, we get for $s\leqq T$

$$\begin{aligned}(10)\qquad \Big(\int_{-\infty}^{T}\|u(t)\|_4^4dt\Big)^{1/4}&\leqq\Big(\int_{-\infty}^{T}\|u_-(t)\|_4^4dt\Big)^{1/4}\\ &\quad+\Big(\int_{-\infty}^{T}\|u(t)\|_{p_1}^{p_1}dt\Big)^{1/2}\Big(\int_{-\infty}^{T}\|u(t)\|_4^4dt\Big)^{1/4}\\ &\leqq C\|(f_-, g_-)\|_e+M_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\\ &\quad\times\Big(\int_{-\infty}^{T}\|u(t)\|_4^4dt\Big)^{1/4}.\end{aligned}$$

Hence, if we choose $\|(f_-, g_-)\|_e$ so small that $M_2(\|(f_-, g_-)\|_e, T)^{p_1/2}<1$ or if we choose T such that $T<0$ and $|T|$ is sufficiently large so that $M_2(\|(f_-, g_-)\|_e, T)^{p_1/2}<1$, we obtain

$$(11)\qquad \Big(\int_{-\infty}^{T}\|u(t)\|_4^4dt\Big)^{1/4}\leqq M_3(\|(f_-, g_-)\|_e, T),$$

where $M_3=M_3(r, T)$ is a continuous nondecreasing function on $[0, r_0']\times(-\infty, T_0')$ for some $r_0'\leqq+\infty$ and $T_0'\leqq+\infty$, satisfying the same property as M_2.

We now prove the existence of global solutions of $(\mathrm{CP})_s$. If f_- and g_- are sufficiently smooth, then it is well-known that $(\mathrm{CP})_s$ has the unique global classical solution u satisfying (NLKG) in the classical sense. Therefore let $\{(f_n, g_n)\}$ be a sequence of functions such that $(f_n, g_n)\in C_0^\infty(\boldsymbol{R}^3)\times C_0^\infty(\boldsymbol{R}^3)$ and

$$(f_n, g_n)\to(f_-, g_-) \quad \text{in} \quad H^1(\boldsymbol{R}^3)\times L^2(\boldsymbol{R}^3)$$

and let u_n be the corresponding solution of $(\mathrm{CP})_s$ with

$$u_n(s)=\frac{d}{dt}E(s)f_n;\ \frac{d}{dt}u_n(s)=E(s)g_n.$$

Assume that $\|(f_-, g_-)\|_e<\min(r_0, r_0')$ and without loss of generality assume that for some $\delta>0$

$$\|(f_n, g_n)\|_e\leqq\|(f_-, g_-)\|_e+\delta<\min(r_0, r_0')$$

for all n. From (7), (9) and (11) we obtain a priori estimates

(12) $$\sup_{t\in\mathbf{R}} \|u_n(t)\|_{1,2} \leqq C$$

(13) $$\sup_{t\in\mathbf{R}} \left\|\frac{d}{dt}u_n(t)\right\|_2 \leqq C$$

(14) $$\left(\int_{-\infty}^{T} \|u_n(t)\|_{p_1}^{p_1} dt\right)^{1/p_1} \leqq C$$

and

(15) $$\left(\int_{-\infty}^{T} \|u_n(t)\|_4^4 dt\right)^{1/4} \leqq C$$

for $-\infty < T < \min(T_0, T_0')$, where C is a positive constant independent of n.

Hence there exist a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) and a function $u \in L^\infty(\mathbf{R}: H^1(\mathbf{R}^3)) \cap L^{p_1}((-\infty, T] \times \mathbf{R}^3) \cap L^4((-\infty, T] \times \mathbf{R}^3)$ such that $u_n \to u$ weakly star in $L^\infty(\mathbf{R}; H^1(\mathbf{R}^3))$ and weakly in $L^p((-\infty, T] \times \mathbf{R}^3) \cap L^4((-\infty, T] \times \mathbf{R}^3)$; $\frac{du_n}{dt} \to \frac{d}{dt}u$ weakly star in $L^\infty(\mathbf{R}: L^2(\mathbf{R}^3))$. Moreover u satisfies the estimates (7), (9) and (11).

To show that u solves (NLKG) in the sense of distribution is accomplished by the standard manner. It is shown in [4] and [15] that u is uniquely determined by its Cauchy data. We have

$$\langle u(\cdot), \phi \rangle = \langle u_-(\cdot), \phi \rangle + \langle \mathcal{H}_s(u(\cdot)), \phi \rangle \qquad \forall \phi \in C_0^\infty(\mathbf{R}^4).$$

By virtue of Lemma 5 we see that $\mathcal{H}_s(u) \in L^4((-\infty, T] \times \mathbf{R}^3)$. Hence u satisfies $(\mathrm{CP})_s$ in $L^4((-\infty, T] \times \mathbf{R}^3)$.

Thus we have

PROPOSITION 1. *Let* $3 < p < 5$. *For any* $(f_-, g_-) \in H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$ *there exists a unique solution* $u_s(t)$ *of* $(\mathrm{CP})_s$ *satisfying*

$$u_s \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^3)); \quad \frac{d}{dt}u_s \in L^\infty(\mathbf{R}; L^2(\mathbf{R}^3)).$$

Furthermore if s *is sufficiently near* $-\infty$,

$$u_s \in L^{2(p-1)}((-\infty, T] \times \mathbf{R}^3) \cap L^4((-\infty, T] \times \mathbf{R}^3)$$

for some T *with* $s \leqq T$.

We now prove (a) in Theorem. Let $\{s_n\}$ be a sequence in $\mathbf{R}$ such that $s_n \to -\infty$. Then $\{u_{s_n}\}$ satisfies a priori estimates (7), (9), (11) with replacing u by u_{s_n}. Moreover $\{u_{s_n}\}$ is a Cauchy sequence in $L^4((-\infty, T] \times \mathbf{R}^3)$ for some $T \in \mathbf{R}$. Note that from (10) it follows that

(16) $$\left(\int_{-\infty}^{s_n} \|u_{s_n}(t)\|_4^4 dt\right)^{1/4} \leqq 2\left(\int_{-\infty}^{s_n} \|u_-(t)\|_4^4 dt\right)^{1/4}$$

for sufficiently large n such that $M_2(\|(f_-, g_-)\|_e, s_n)^{p_1/2} \leqq \frac{1}{2}$. We have for $s_m < s_n$

$$\begin{aligned}\|u_{s_n}(t)-u_{s_m}(t)\|_4 &= \|\mathcal{H}_{s_n}(u_{s_n}(t))-\mathcal{H}_{s_m}(u_{s_m}(t))\|_4 \\ &\leqq \int_{s_n}^t \|E(t-\tau)(h(u_{s_n}(\tau))-h(u_{s_m}(\tau)))\|_4 d\tau \\ &\quad + \int_{s_m}^{s_n} \|E(t-\tau)h(u_{s_m}(\tau))\|_4 d\tau .\end{aligned}$$

In much the same way as in the proof of Lemma 5, we obtain for $s_n \leqq T$

$$\begin{aligned}&\left(\int_{-\infty}^T \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\leqq C\left(\int_{-\infty}^T [\|u_{s_n}(t)\|_{p_1}^{p_1}+\|u_{s_m}(t)\|_{p_1}^{p_1}]\, dt\right)^{1/2} \\ &\qquad \times \left(\int_{-\infty}^T \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\quad + C\left(\int_{s_m}^{s_n} \|u_{s_m}(t)\|_{p_1}^{p_1} dt\right)^{1/2}\left(\int_{s_m}^{s_n} \|u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\leqq 2CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\left(\int_{-\infty}^T \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\quad + CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\left(\int_{s_m}^{s_n} \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right. \\ &\qquad\qquad \left. + \int_{s_m}^{s_n} \|u_{s_n}(t)\|_4^4\, dt\right)^{1/4} \\ &\leqq 3CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\left(\int_{-\infty}^T \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\quad + CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\left(\int_{s_m}^{s_n} \|u_{s_m}(t)\|_4^4 dt\right)^{1/4} .\end{aligned}$$

Choose T so that $3CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2} \leqq \frac{1}{2}$. We have

$$\begin{aligned}&\left(\int_{-\infty}^T \|u_{s_n}(t)-u_{s_m}(t)\|_4^4 dt\right)^{1/4} \\ &\leqq 2CM_2(\|(f_-, g_-)\|_e, T)^{p_1/2}\left(\int_{-\infty}^{s_n} \|u_{s_n}(t)\|_4^4\, dt\right)^{1/4} \\ &\leqq \text{const.}\left(\int_{-\infty}^{s_n} \|u_-(t)\|_4^4 dt\right)^{1/4} \qquad \text{(by (16))}\end{aligned} \tag{17}$$

which tends to zero as $s_n, s_m \to -\infty$. Hence there exists a $u \in L^\infty((-\infty, T];$

$H^1(\boldsymbol{R}^3))\cap L^{p_1}((-\infty, T]\times\boldsymbol{R}^3)\cap L^4((-\infty, T]\times\boldsymbol{R}^3)$ with $\frac{d}{dt}u\in L^\infty((-\infty, T]: L^2(\boldsymbol{R}^3))$ such that as $s_n\to-\infty$, $u_{s_n}\to u$ weakly star in $L^\infty((-\infty, T]; H^1(\boldsymbol{R}^3))$, weakly in $L^{p_1}((-\infty, T]\times\boldsymbol{R}^3)$ and strongly in $L^4((-\infty, T]\times\boldsymbol{R}^3)$; $\frac{d}{dt}u_{s_n}\to\frac{du}{dt}$ weakly star in $L^\infty((-\infty, T]; L^2(\boldsymbol{R}^3))$. Furthermore u satisfies

$$u(t)=u_-(t)+\int_{-\infty}^t E(t-\tau)h(u(\tau))d\tau. \tag{18}$$

From (17) it follows that

$$\left(\int_{-\infty}^T\|u_s(t)-u(t)\|_4^4dt\right)^{1/4}\leqq\text{const.}\left(\int_{-\infty}^s\|u_-(t)\|_4^4dt\right)^{1/4}\to 0\quad\text{as}\quad s\to-\infty. \tag{19}$$

We next show that $\|u(t)-u_-(t)\|_e\to 0$ as $t\to-\infty$. First we have

$$\|u_s(t)\|_e^2+2\int_{\boldsymbol{R}^3}H(u_s(x, t))dx\leqq\|u_-(s)\|_e^2+2\int_{\boldsymbol{R}^3}H(u_-(x, s))dx\qquad\text{a.e. } t. \tag{20}$$

The first term on the right equals $\|(f_-, g_-)\|_e^2$. The second term is in $L^1(-\infty, +\infty)$. Indeed we have

$$\begin{aligned}\int_{-\infty}^\infty\int H(u_-(x, s))dxds&\leqq C\int_{-\infty}^\infty\int|u_-(x, s)|^{p+1}dxds\\&\leqq C\left(\int_{-\infty}^\infty\int|u_-(x, s)|^{2(p-1)}dxds\right)^{1/2}\\&\qquad\times\left(\int_{-\infty}^\infty\int|u_-(x, s)|^4dxds\right)^{1/2}.\end{aligned} \tag{21}$$

From (19) we get

$$\begin{aligned}&\int_{-\infty}^T\left|\int(H(u_s(x, t))-H(u(x, t)))dx\right|dt\\&\leqq C\int_{-\infty}^T\int(|u_s|^p+|u|^p)|u_s-u|dxdt\\&\leqq C\left\{\left(\int_{-\infty}^T\|u_s(t)\|_{2(p-1)}^{2(p-1)}dt\right)^{1/2}\left(\int_{-\infty}^T\|u_s(t)\|_4^4dt\right)^{1/4}\right.\\&\qquad\left.+\left(\int_{-\infty}^T\|u(t)\|_{2(p-1)}^{2(p-1)}dt\right)^{1/2}\left(\int_{-\infty}^T\|u(t)\|_4^4dt\right)^{1/4}\right\}\\&\quad\times\left(\int_{-\infty}^T\|u_s(t)-u(t)\|_4^4dt\right)^{1/4}\\&\leqq C(T)\left(\int_{-\infty}^T\|u_s(t)-u(t)\|_4^4dt\right)^{1/4}\to 0\quad\text{as}\quad s\to-\infty.\end{aligned} \tag{22}$$

Let ξ be a nonnegative real L^∞ function with compact support and integral 1.

Since $u_{s_n}\to u$ weakly star in $L^\infty((-\infty, T]: H^1(\boldsymbol{R}^3))$ and $\frac{d}{dt}u_{s_n}\to\frac{d}{dt}u$ weakly star in $L^\infty((-\infty, T]: L^2(\boldsymbol{R}^3))$, we see that

(23)
$$\int\|u(t)\|_e^2\xi(t)dt\leqq\liminf_{s_n\to-\infty}\int\|u_{s_n}(t)\|_e^2\xi(t)dt\,.$$

From (19), (21), and (22) it follows that there exists a sequence $s_n\to-\infty$ such that

$$\int_{\boldsymbol{R}^3}H(u_-(x, s_n))dx\to 0\,,$$

$$\|u_{s_n}(t)-u(t)\|_4\to 0 \qquad \text{for almost every } t\,,$$

and

$$\int_{\boldsymbol{R}^3}H(u_{s_n}(x, t))dx\to\int_{\boldsymbol{R}^3}H(u(x, t))dx \qquad \text{for almost every } t\,.$$

Hence

$$\int\Big[\|u(t)\|_e^2+2\int H(u(x, t))\,dx\Big]\xi(t)dt\leqq\|(f_-, g_-)\|_e^2\,,$$

which implies that

(24)
$$\|u(t)\|_e^2+2\int H(u(x, t))dx\leqq\|(f_-, g_-)\|_e^2 \qquad \text{a.e. } t\,.$$

Put $w_1(t)=E'(-t)u(t)+E(-t)u'(t)$. Then $w_1(t)$ satisfies

(25)
$$w_1(t)=f_-+\int_{-\infty}^t E(-\tau)h(u(\tau))d\tau\,.$$

Lemma 1 and Hölder's inequality give

(26)
$$\begin{aligned}\|w_1(t)-f_1\|_4&\leqq C\int_{-\infty}^t|\tau|^{-1/2}\|\,|u|^{p-1}u\|_{4/3}d\tau\\ &\leqq C\int_{-\infty}^t|\tau|^{-1/2}\|u\|_{2(p-1)}^{p-1}\|u\|_4 d\tau\\ &\leqq C\Big(\int_{-\infty}^t|\tau|^{-2}d\tau\Big)^{1/4}\Big(\int_{-\infty}^t\|u\|_{2(p-1)}^{2(p-1)}d\tau\Big)^{1/2}\\ &\quad\times\Big(\int_{-\infty}^t\|u\|_4^4 d\tau\Big)^{1/4}\end{aligned}$$

which tends to zero as $t\to-\infty$.

Put $w_2(t)=E''(-t)u(t)+E'(-t)u'(t)$. Then $w_2(t)$ satisfies

(27)
$$\langle w_2(t), \phi\rangle=\langle g_-, \phi\rangle+\int_{-\infty}^t\Big\langle\Big(\frac{d}{dt}E(\tau)\Big)h(u(\tau)), \phi\Big\rangle d\tau$$

$$=\langle g_-, \phi\rangle+\int_{-\infty}^{t}\frac{d}{dt}\langle E(\tau)h(u(\tau)), \phi\rangle d\tau$$

$$-\int_{-\infty}^{t}\left\langle E(\tau)h'(u(\tau))\frac{du}{dt}(\tau), \phi\right\rangle d\tau$$

$$=\langle g_-, \phi\rangle+\langle E(t)h(u(t)), \phi\rangle$$

$$-\int_{-\infty}^{t}\left\langle E(\tau)h'(u(\tau))\frac{du}{dt}(\tau), \phi\right\rangle d\tau$$

for any $\phi\in C_0^\infty(\boldsymbol{R}^3)$. Indeed we have used the fact that

$$\left|\left\langle\frac{d}{dt}E(\tau)h(u(\tau)), \phi\right\rangle\right|=\left|\left\langle h(u(\tau)), \frac{d}{dt}E(\tau)\phi\right\rangle\right|$$

$$\leqq C\|u\|_p^p\left\|\frac{d}{dt}E(\tau)\phi\right\|_\infty$$

$$\leqq C|\tau|^{-3/2}\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^p\|\phi\|_{3,1}$$

which means that $\tau\to\left\langle\frac{d}{dt}E(\tau)h(u(\tau)), \phi\right\rangle$ is in $L^1(-\infty, t)$ for $t<0$,

(28)
$$|\langle E(t)h(u(t)), \phi\rangle|=|\langle h(u(t)), E(t)\phi\rangle|$$

$$\leqq C\|u\|_p^p\|E(t)\phi\|_\infty$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^p\|E(t)\phi\|_{1,4}$$

$$\leqq C|t|^{-1/2}\sup_{t\in\boldsymbol{R}}\|u(t)\|_{1,2}^p\|\phi\|_{1,4/3}$$

and

(29)
$$\left|\int_{-\infty}^{t}\left\langle E(\tau)h'(u(\tau))\frac{du}{d\tau}(\tau), \phi\right\rangle d\tau\right|$$

$$=\left|\int_{-\infty}^{t}\left\langle h'(u(\tau))\frac{du}{d\tau}(\tau), E(\tau)\phi\right\rangle d\tau\right|$$

$$\leqq C\int_{-\infty}^{t}\left\||u(\tau)|^{p-1}\frac{du}{d\tau}(\tau)\right\|_1\|E(\tau)\phi\|_\infty d\tau$$

$$\leqq C\int_{-\infty}^{t}\|u(\tau)\|_{2(p-1)}^{p-1}\left\|\frac{du}{d\tau}(\tau)\right\|_2\|E(\tau)\phi\|_\infty d\tau$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\left\|\frac{du}{dt}(t)\right\|_2\left(\int_{-\infty}^{t}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{1/2}$$

$$\times\left(\int_{-\infty}^{t}\|E(t)\phi\|_\infty^2 d\tau\right)^{1/2}$$

$$\leqq C|t|^{-1/3}\sup_{t\in\boldsymbol{R}}\left\|\frac{du}{dt}(t)\right\|_2\left(\int_{-\infty}^{t}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{1/2}\|\phi\|_{4/3,6/5}$$

since

$$\|E(\tau)\phi\|_\infty\leqq C\|E(\tau)\phi\|_{1,6}\leqq C|\tau|^{-2/3}\|\phi\|_{4/3,6/5}.$$

From (28) and (29), we see that

$$\|w_2(t)-g_-\|_{H^{-2,6}}\to 0 \quad\text{as}\quad t\to-\infty$$

since $H^{2,6/5}\subset H^{1,4/3}\cap H^{4/3,6/5}$.

On the other hand

$$\|(w_1(t),\ w_2(t))\|_e=\left\|\left(u(t),\ \frac{d}{dt}u(t)\right)\right\|_e=\|u(t)\|_e \qquad \text{a.e. } t$$

since $\dfrac{d}{ds}E(s-t)u(t)+E(s-t)\dfrac{d}{dt}u(t)=\dfrac{d}{ds}E(s)w_1(t)+E(s)w_2(t)$. From (24), $\|u(t)\|_e$ is bounded for almost every t. Hence $w_1(t)$ converges to f_- weakly in $H^1(\boldsymbol{R}^3)$ and $w_2(t)$ converges to g_- weakly in $L^2(\boldsymbol{R}^3)$ as a.e. $t\to-\infty$.

Since

$$\int_{-\infty}^{T}\int H(u(x,\ t))dxdt\leqq C\left(\int_{-\infty}^{T}\|u(t)\|_{2(p-1)}^{2(p-1)}dt\right)^{1/2}\left(\int_{-\infty}^{T}\|u(t)\|_4^4dt\right)^{1/2},$$

there exists a sequence $t_n\to-\infty$ such that

$$\int H(u(x,\ t_n))dx\to 0.$$

Hence

$$\|(f_-,\ g_-)\|_e^2\leqq\liminf_{t_n\to-\infty}\|(w_1(t_n),\ w_2(t_n))\|_e^2$$

$$=\liminf_{t_n\to-\infty}\left(\|u(t_n)\|_e^2+2\int H(u(x,\ t_n))dx\right)\leqq\|(f_-,\ g_-)\|_e^2.$$

Therefore

$$\|u(t)\|_e^2+2\int H(u(x,\ t))\,dx=\|(f_-,\ g_-)\|_e^2 \qquad \text{a.e. } t,$$

and

$$\left(u(t),\ \frac{du}{dt}(t)\right)\to(f_-,\ g_-)\quad\text{strongly in } H^1(\boldsymbol{R}^3)\times L^2(\boldsymbol{R}^3)$$

as a.e. $t\to-\infty$. Hence

$$\|u(t)-u_-(t)\|_e\to 0 \quad\text{as}\quad \text{a.e. } t\to-\infty.$$

This completes the proof of the assertion (a) in Theorem.

We next prove (b) in Theorem.

Exactly in the same manner as in Proposition 1, we have

PROPOSITION 2. *Let* $3<p<5$. *If* $\|(f_-, g_-)\|_e$ *is sufficiently small, there exists a unique solution* $u_s(t)$ *of* $(\mathrm{CP})_s$ *satisfying*

$$u_s\in L^\infty(\boldsymbol{R}:\ H^1(\boldsymbol{R}^3))\cap L^{2(p-1)}(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)\cap L^4(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)$$

with

$$\frac{d}{dt}u_s\in L^\infty(\boldsymbol{R}:\ L^2(\boldsymbol{R}^3)).$$

From Proposition 2 we have a unique solution u of (18) such that

$$u\in L^\infty(\boldsymbol{R}:\ H^1(\boldsymbol{R}^3))\cap L^{2(p-1)}(\boldsymbol{R}^4)\cap L^4(\boldsymbol{R}^4)$$

with

$$\frac{d}{dt}u\in L^\infty(\boldsymbol{R}:\ L^2(\boldsymbol{R}^3)).$$

Define

$$f_+=f_- -\int_{-\infty}^{\infty}E(\tau)h(u(\tau))d\tau. \tag{30}$$

Then $f_+\in L^4(\boldsymbol{R}^3)$. Indeed we have

$$\left\|\int_{-\infty}^{\infty}E(\tau)h(u(\tau))d\tau\right\|_4$$

$$\leqq\int_{I_1}\tau^{-1/2}\|h(u(\tau))\|_{4/3}d\tau+\int_{I_2}\tau^{-1/2}\|h(u(\tau))\|_{4/3}d\tau,$$

where $I_1=[-1,+1]$ and $I_2=\boldsymbol{R}\setminus[-1,+1]$. The first term on the right is dominated by

$$C\sup_{t\in\boldsymbol{R}}\|u\|_{1,2}^{3/2}\int_{I_1}\tau^{-1/2}\|u\|_{2(p-1)}^{p-(3/2)}d\tau$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\|u\|_{1,2}^{3/2}\left(\int_{I_1}|\tau|^{-1/2\times 4(p-1)/(2p-1)}d\tau\right)^{(2p-1)/4(p-1)}$$

$$\times\left(\int_{-\infty}^{\infty}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{(2p-3)/4(p-1)}$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\|u\|_{1,2}^{3/2}\left(\int_{-\infty}^{\infty}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{(2p-3)/4(p-1)},$$

since

$$\|h(u)\|_{4/3}\leqq C\|u\|_{2(p-1)}^{p-3/2}\|u\|_r^{3/2}$$

where $r=6-\dfrac{6}{p}<6$.

The second term on the right is estimated by

$$C\left(\int_{I_2}\tau^{-2}d\tau\right)^{1/4}\left(\int_{I_2}\||u|^p\|_{4/3}^{4/3}d\tau\right)^{3/4}$$

$$\leqq C\left(\int_{-\infty}^{\infty}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{2/3}\left(\int_{-\infty}^{\infty}\|u(\tau)\|_4^4 d\tau\right)^{1/3}.$$

Define

$$g_+=g_- -\int_{-\infty}^{\infty}E(\tau)h'(u(\tau))\frac{du}{dt}(\tau)d\tau. \tag{31}$$

Then $g_+\in H^{-2,6}(\boldsymbol{R}^3)$. Indeed we have for any $\phi\in C_0^\infty(\boldsymbol{R}^3)$

$$\left|\left\langle\int_{-\infty}^{\infty}E(\tau)h'(u(\tau))\frac{du}{d\tau}(\tau)d\tau,\ \phi\right\rangle\right|$$

$$=\left|\int_{-\infty}^{\infty}\left\langle h'(u(\tau))\frac{du}{d\tau}(\tau),\ E(\tau)\phi\right\rangle d\tau\right|$$

$$\leqq C\int_{I_1}\|u(\tau)\|_{2(p-1)}^{p-1}\left\|\frac{du}{dt}(\tau)\right\|_2\|E(\tau)\phi\|_\infty d\tau$$

$$+\int_{I_2}\|u(\tau)\|_{2(p-1)}^{p-1}\left\|\frac{du}{d\tau}(\tau)\right\|_2\|E(\tau)\phi\|_\infty d\tau.$$

In much the same way as in (29), the second term on the right is dominated by

$$C\sup_{t\in\boldsymbol{R}}\left\|\frac{du}{dt}(t)\right\|_2\left(\int_{-\infty}^{\infty}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{1/2}\|\phi\|_{4/3,6/5}.$$

The first term on the right is estimated above by

$$C\sup_{t\in\boldsymbol{R}}\left\|\frac{du}{dt}(t)\right\|_2\int_{I_1}\|u(\tau)\|_{2(p-1)}^{p-1}|\tau|^{-1/2+(3/2)\varepsilon}d\tau\|\phi\|_{1,4/(3-\varepsilon)}$$

$$\leqq C\sup_{t\in\boldsymbol{R}}\left\|\frac{du}{dt}(t)\right\|_2\left(\int_{-\infty}^{\infty}\|u(\tau)\|_{2(p-1)}^{2(p-1)}d\tau\right)^{1/2}\|\phi\|_{1,4/(3-\varepsilon)}$$

where $0<\varepsilon<1$, since

$$\|E(\tau)\phi\|_\infty\leqq C\|E(\tau)\phi\|_{1,4/(1+\varepsilon)}$$

$$\leqq|\tau|^{-1/2+(3/2)\varepsilon}\|\phi(\tau)\|_{1,4/(3-\varepsilon)}.$$

Since $H^{2,6/5}\subset H^{1,4/(3-\varepsilon)}\cap H^{4/3,6/5}$, we conclude that $g_+\in H^{-2,6}(\boldsymbol{R}^3)$. From (25) and (30) we have

$$w_1(t)-f_+=\int_t^\infty E(\tau)h(u(\tau))d\tau.$$

Exactly as in the proof of (26),

$$\|w_1(t)-f_+\|_4\leqq C\int_t^\infty|\tau|^{-1/2}\|u\|_{2(p-1)}^{p-1}\|u\|_4 d\tau$$

$$\leqq C\left(\int_t^\infty|\tau|^{-2}d\tau\right)^{1/4}\left(\int_{-\infty}^{\infty}\|u\|_{2(p-1)}^{2(p-1)}d\tau\right)^{1/2}\left(\int_{-\infty}^{\infty}\|u\|_4^4 d\tau\right)^{1/2}$$

which tends to zero as $t\to+\infty$. From (27) and (31), we have

$$w_2(t)-g_+=E(t)h(u(t))+\int_t^\infty E(\tau)h'(u(\tau))\frac{du}{dt}(\tau)d\tau\,.$$

Then $\|w_2(t)-g_+\|_{-2,6}\to 0$ as $t\to+\infty$ exactly as in the proof of (28) and (29).

On the other hand

$$\|(w_1(t),\ w_2(t))\|_e=\left\|u(t),\ \frac{d}{dt}u(t)\right\|_e=\|u(t)\|_e \qquad \text{a.e. } t\,.$$

Hence, $\|(w_1(t),\ w_2(t))\|_e$ is bounded for almost every t. So $f_+\in H^1(\boldsymbol{R}^3)$, $g_+\in L^2(\boldsymbol{R}^3)$, and $w_1(t)$ converges weakly to f_+ in $H^1(\boldsymbol{R}^3)$, $w_2(t)$ converges weakly to g_+ in $L^2(\boldsymbol{R}^3)$ as a.e. $t\to-\infty$.

Since

$$\int_{-\infty}^\infty\int H(u(x,\ t))dx\,dt\leqq C\Big(\int_{-\infty}^\infty\|u(t)\|_{2(p-1)}^{2(p-1)}dt\Big)^{1/2}\Big(\int_{-\infty}^\infty\|u(t)\|_4^4dt\Big)^{1/2},$$

there exists a sequence $t_n\to+\infty$ such that

$$\int H(u(x,\ t_n))dx\to 0\,.$$

Hence

$$\begin{aligned}(32)\qquad \|(f_+,\ g_+)\|_e^2&\leqq\liminf_{t_n\to+\infty}\|(w_1(t_n),\ w_2(t_n))\|_e^2\\ &=\liminf_{t_n\to+\infty}\Big(\|u(t_n)\|_e^2+2\int H(u(x,\ t_n))dx\Big)\\ &=\|(f_-,\ g_-)\|_e^2\,.\end{aligned}$$

Consider the equation

$$(33)\qquad w_s(t)=u_+(t)+\int_s^t E(t-\tau)h(w_s(\tau))d\tau$$

with

$$u_+(t)=\frac{d}{dt}E(t)f_++E(t)g_+\,.$$

Exactly in the same way as in the proofs of Proposition 1 and Proposition 2, using (32), we see that there exists a unique solution $w_s(t)$ of (33) satisfying

$$w_s\in L^\infty(\boldsymbol{R}:\ H^1(\boldsymbol{R}^3))\cap L^{2(p-1)}(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)\cap L^4(\boldsymbol{R}_t\times\boldsymbol{R}_x^3)$$

with

$$\frac{d}{dt}w_s\in L^\infty(\boldsymbol{R}:\ L^2(\boldsymbol{R}^3))\,.$$

Furthermore

$$\left(\int_{-\infty}^{\infty}\|w_s(t)-u(t)\|_4^4dt\right)^{1/4}\to 0 \quad \text{as} \quad s\to+\infty\,. \tag{34}$$

Indeed we have

$$\begin{aligned}u_+(t)&=\frac{d}{dt}E(t)f_-+E(t)g_-\\&\quad-\int_{-\infty}^{\infty}\frac{d}{dt}E(t)E(\tau)h(u(\tau))d\tau-\int_{-\infty}^{\infty}E(t)E(\tau)h'(u(\tau))\frac{du}{dt}(\tau)d\tau\\&=u_-(t)-\int_{-\infty}^{\infty}\left[\frac{d}{dt}E(t)E(\tau)-E(t)\frac{dE}{dt}(\tau)\right]h(u(\tau))d\tau\\&=u_-(t)+\int_{-\infty}^{\infty}E(t-\tau)h(u(\tau))d\tau\end{aligned}$$

in $H^{-2,6}(\boldsymbol{R}^3)$. Hence

$$\begin{aligned}w_s(t)-u(t)&=\int_s^tE(t-\tau)h(w_s(\tau))d\tau+\int_t^{\infty}E(t-\tau)h(u(\tau))d\tau\\&=\int_t^{\infty}E(t-\tau)[h(w_s(\tau))-h(u(\tau))]d\tau\\&\quad-\int_s^{\infty}E(t-\tau)h(w_s(\tau))d\tau\,.\end{aligned}$$

Exactly in the same manner as in the proof of (17) we have

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\|w_s(t)-u(t)\|_4^4dt\right)^{1/4}\\&\quad\leqq \text{const.}\left(\int_s^{\infty}\|w_s(t)\|_4^4dt\right)^{1/4}\leqq \text{const.}\left(\int_s^{\infty}\|u_+(t)\|_4^4dt\right)^{1/4}\\&\quad\to 0 \quad \text{as} \quad s\to+\infty\,.\end{aligned}$$

Here we used the fact that $\|(f_+, g_+)\|_e\leqq\|(f_-, g_-)\|_e$. Therefore there exists a sequence $s_j\to+\infty$ such that

$$\|w_{s_j}(t)-u(t)\|_4\to 0 \qquad \text{almost everywhere}.$$

We have

$$\|w_s(t)\|_e^2+2\int_{\boldsymbol{R}^3}H(w_s(x, t))dx\leqq\|u_+(s)\|_e^2+2\int_{\boldsymbol{R}^3}H(u_+(x, s))dx \qquad \text{a.e.}$$

In much the same way as in the proof of (24), we obtain

$$\|u(t)\|_e^2+2\int H(u(x, t))dx\leqq\|(f_+, g_+)\|_e^2 \qquad \text{a.e.} \tag{35}$$

From (32) and (35) it follows that

$$\|u(t)\|_e^2+2\int H(u(x, t))dx=\|(f_+, g_+)\|_e^2=\|(f_-, g_-)\|_e^2 .$$

Hence

$$\|u(t)-u_+(t)\|_e \to 0 \qquad \text{as a.e. } t\to +\infty .$$

This completes the proof of b) in Theorem.

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