J. Math. Soc. Japan
Vol. 35, No. 4, 1983

Value distribution of the Gauss maps of complete minimal surfaces in R^m

Dedicated to Professor M. Ozawa on the occasion of his 60th birthday

By Hirotaka FUJIMOTO

(Received Sept. 13, 1982)

§1. Introduction.

Concerning the value distribution of the Gauss maps of complete minimal surfaces in R^m, there have been several results obtained by R. Osserman, S.S. Chern, F. Xavier and others ([10], [2], [7], [13]). Recently, the author proved that the Gauss map of a complete minimal surface in R^m is necessarily degenerate if it omits more than m^2 hyperplanes in $P^{m-1}(C)$ located in general position ([4]). The purpose of this paper is to give several improvements of these results.

Let f be a holomorphic map of an open Riemann surface M into $P^n(C)$ and H a hyperplane in $P^n(C)$ with $f(M)\not\subset H$. For an arbitrarily fixed positive integer μ_0 we define the non-integrated defect of H for f by

$$\delta_{\mu_0}^f(H):=1-\inf\{\eta\geqq 0 : \eta \text{ satisfying condition } (*)\}.$$

Here, condition (*) means that there exists a non-negative smooth function v on M such that $\log v$ is subharmonic, $\log v\leqq\eta\log\|f\|$ and, in a neighborhood of each point $p\in f^{-1}(H)$,

$$\log v(\zeta)-\min(\nu^f(H)(p),\ \mu_0)\log|\zeta-\zeta(p)|$$

is subharmonic, where $\|f\|:=(|f_1|^2+\cdots+|f_{n+1}|^2)^{1/2}$ for a reduced representation $f=(f_1:\cdots:f_{n+1})$, ζ is a holomorphic local coordinate around p and $\nu^f(H)(p)$ denotes the intersection multiplicity of $f(M)$ and H at $f(p)$. We note that

$$\delta_{\mu_0}^f(H)=1 \tag{1.1}$$

if $f(M)\cap H=\emptyset$, or more generally, if there is a bounded holomorphic function g on M such that g has zeros of order $\nu^f(H)(p)$ at each point $p\in f^{-1}(H)$. Moreover, we can show that

This research was partially supported by Grant-in-Aids for Scientific Research (No. 00546004 and No. 57540074), Ministry of Education.

$$\delta^f_{\mu_0}(H) \geqq 1 - \frac{\mu_0}{\mu}$$

if $\nu^f(H)(p) \geqq \mu$ for every point $p \in f^{-1}(H)$.

We now consider a minimal surface M in $\boldsymbol{R}^m$, which we regard as a Riemann surface with a conformal metric. For the Gauss map G of M, the conjugate f of G is a holomorphic map of M into $P^{m-1}(\boldsymbol{C})$, and the image $f(M)$ is included in the complex quadric $Q_{m-2}(\boldsymbol{C})$ (cf., [8], p. 110). We shall prove the following

THEOREM 1.2. *If M is complete and f is non-degenerate, then for arbitrarily given hyperplanes $H_1, \cdots, H_q$ in general position we have*

$$\sum_{j=1}^{q} \delta^f_{m-1}(H_j) \leqq m^2 .$$

This is an improvement of Main Theorem of [4] by virtue of (1.1).

For the case $m=3$, there is a canonically defined biholomorphic map φ of $Q_1(\boldsymbol{C})$ onto the Riemann sphere $P^1(\boldsymbol{C})$. Instead of the Gauss map into $P^2(\boldsymbol{C})$ we shall study the holomorphic map $g = \varphi \cdot f : M \to P^1(\boldsymbol{C})$. We shall give the following improvement of a result of F. Xavier ([**13**]).

THEOREM 1.3. *Let M be a non-flat complete minimal surface in $\boldsymbol{R}^3$. Then, for arbitrarily given distinct numbers $\alpha_1, \cdots, \alpha_q$ we have*

$$\sum_{j=1}^{q} \delta^g_1(\alpha_j) \leqq 6 .$$

For the case $m=4$, there is a biholomorphic map $\psi = \psi_1 \times \psi_2$ of $Q_2(\boldsymbol{C})$ onto $P^1(\boldsymbol{C}) \times P^1(\boldsymbol{C})$. Instead of the Gauss map into $P^3(\boldsymbol{C})$, we shall study two meromorphic functions $g_i = \psi_i \cdot f$ $(i=1, 2)$. We shall give the following improvement of a result of R. Osserman ([**10**], p. 362).

THEOREM 1.4. *Let M be a non-flat complete minimal surface in $\boldsymbol{R}^4$. Then, at least one of the above-mentioned functions g_1 and g_2, say g_1, has the property that, for arbitrarily given distinct numbers $\alpha_1, \cdots, \alpha_q$,*

$$\sum_{j=1}^{q} \delta^{g_1}_1(\alpha_j) \leqq 6 .$$

To prove these results, we shall give a variant of defect relation, called non-integrated defect relation, for holomorphic maps of an open Riemann surface into the space $P^{n_1}(\boldsymbol{C}) \times \cdots \times P^{n_k}(\boldsymbol{C})$ satisfying a certain growth condition.

We shall show some preliminary properties on value distributions of meromorphic functions on the unit disc in $\boldsymbol{C}$ in § 2 and prove a basic inequality in § 3. After these preparations, we shall give the non-integrated defect relation, which will be stated in § 4 and proved in § 5. In the last section, we shall prove the above-mentioned results related to the Gauss maps as its applications.

§ 2. An estimate for logarithmic derivatives.

For later use, we give an estimate for logarithmic derivatives of meromorphic functions on the unit disc $\Delta:=\{|z|<1\}$ in C. We first recall some terminology on Nevanlinna theory.

Let φ be a nonzero meromorphic function on Δ. The counting function, the proximity function and the characteristic function of φ are defined respectively by

$$N(r, \varphi):=\int_0^r \frac{n(t, \varphi)-n(0, \varphi)}{t} dt+n(0, \varphi)\log r$$

$$m(r, \varphi):=\frac{1}{2\pi}\int_0^{2\pi} \log^+|\varphi(re^{\sqrt{-1}\theta})|\, d\theta$$

and

$$T(r, \varphi):=N(r, \varphi)+m(r, \varphi),$$

where $0<r<1$, $\log^+ x=\max(\log x, 0)$ and $n(t, \varphi)$ denotes the number of poles of φ in $\{z: |z|\leqq t\}$, each pole of order m being counted m times. The well-known First Main Theorem is stated as

$$T(r, 1/\varphi)=T(r, \varphi)+O(1). \tag{2.1}$$

The object of this section is to prove the following

PROPOSITION 2.2. *Let φ be a nonzero meromorphic function on Δ, l be a positive integer and p, p', r_0 be real numbers with $0<pl<p'<1$ and $0<r_0<1$. Then, there exists a positive constant K such that, for $r_0<r<R<1$,*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^p d\theta\leqq K\left(\frac{T(R, \varphi)}{R-r}\right)^{p'}. \tag{2.3}$$

For the proof, we give two lemmas.

LEMMA 2.4. *Let φ be a nonzero meromorphic function on Δ and l be a positive integer. We denote all zeros and poles of φ by a_μ ($\mu=1, 2, \cdots$) and b_ν ($\nu=1, 2, \cdots$) respectively, being repeated m times if they are of order m. Then, if $|z|=r<\rho<1$ and $\varphi(z)\neq 0, \infty$, we have*

$$\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(z)=\frac{l!\rho}{\pi}\int_0^{2\pi}\frac{\log|\varphi(\rho e^{\sqrt{-1}\phi})|\, e^{\sqrt{-1}\phi}}{(\rho e^{\sqrt{-1}\phi}-z)^{l+1}}d\phi$$

$$-(l-1)!\sum_{|a_\mu|<\rho}\left\{\frac{1}{(a_\mu-z)^l}-\frac{\bar{a}_\mu^l}{(\rho^2-\bar{a}_\mu z)^l}\right\}$$

$$+(l-1)!\sum_{|b_\nu|<\rho}\left\{\frac{1}{(b_\nu-z)^l}-\frac{\bar{b}_\nu^l}{(\rho^2-\bar{b}_\nu z)^l}\right\}.$$

This is easily obtained by differentiating the well-known Poisson-Jensen's formula. See [5], p. 22.

LEMMA 2.5. *Let* $r>0$ *and* $0<p<1$. *For arbitrary* $a\in \boldsymbol{C}$ *we have*

$$\int_0^{2\pi}\frac{r^p}{|re^{\sqrt{-1}\theta}-a|^p}d\theta\leqq\frac{\pi(2-p)}{1-p}.$$

PROOF. There is no loss of generality in assuming that a is real and positive. Then, if $|\theta|\leqq\pi/2$, we have

$$|re^{\sqrt{-1}\theta}-a|\geqq r|\sin\theta|\geqq\frac{2}{\pi}r|\theta|$$

and, if $\pi/2<|\theta|\leqq\pi$, we have $|re^{\sqrt{-1}\theta}-a|\geqq r$. Therefore,

$$\begin{aligned}\int_0^{2\pi}\frac{r^p}{|re^{\sqrt{-1}\theta}-a|^p}d\theta&\leqq 2\int_0^{\pi/2}\left(\frac{\pi}{2\theta}\right)^p d\theta+2\int_{\pi/2}^{\pi}d\theta\\&\leqq\frac{2^{1-p}\pi^p}{1-p}\left(\frac{\pi}{2}\right)^{1-p}+\pi\\&=\frac{\pi(2-p)}{1-p}.\end{aligned}$$

PROOF OF PROPOSITION 2.2. In the following, K_i $(i=1, 2, \cdots)$ denote some suitable constants. Since both sides of (2.3) are continuous functions of r, we may assume that φ has no zeros and no poles on $\{|z|=r\}$. Using the Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^p d\theta\\&\leqq\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^{p/p'}d\theta\right)^{p'}.\end{aligned}$$

To evaluate the right hand side of this inequality, we set $\rho=(R+r)/2$ and apply Lemma 2.4. For $|z|=r$, we have

$$\begin{aligned}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(z)\right|&\leqq\frac{l\,!\,\rho}{\pi}\int_0^{2\pi}\frac{|\log|\varphi(\rho e^{\sqrt{-1}\theta})||}{|\rho e^{\sqrt{-1}\phi}-z|^{l+1}}d\phi\\&+(l-1)!\sum_{|a_\mu|<\rho}\left\{\frac{1}{|a_\mu-z|^l}+\frac{|a_\mu|^l}{|\rho^2-\bar{a}_\mu z|^l}\right\}\\&+(l-1)!\sum_{|b_\nu|<\rho}\left\{\frac{1}{|b_\nu-z|^l}+\frac{|b_\nu|^l}{|\rho^2-\bar{b}_\nu z|^l}\right\}\end{aligned}$$

and then

$$\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^{p/p'} \leqq \left(\frac{l!\rho}{\pi}\int_0^{2\pi}\frac{|\log|\varphi(\rho e^{\sqrt{-1}\phi})||}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}d\phi\right)^{p/p'}$$

$$+\frac{(l-1)!}{r^{pl/p'}}\sum_{|a_\mu|<\rho}\left\{\left|\frac{r}{a_\mu-re^{\sqrt{-1}\theta}}\right|^{pl/p'}+\left|\frac{r}{(\rho^2/\bar{a}_\mu)-re^{\sqrt{-1}\theta}}\right|^{pl/p'}\right\}$$

$$+\frac{(l-1)!}{r^{pl/p'}}\sum_{|b_\nu|<\rho}\left\{\left|\frac{r}{b_\nu-re^{\sqrt{-1}\theta}}\right|^{pl/p'}+\left|\frac{r}{(\rho^2/\bar{b}_\nu)-re^{\sqrt{-1}\theta}}\right|^{pl/p'}\right\}.$$

Integrating each term with respect to θ and using Lemma 2.5, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^p d\theta$$

$$\leqq K_1\left(\int_0^{2\pi}d\theta\left(\int_0^{2\pi}\frac{|\log|\varphi(\rho e^{\sqrt{-1}\phi})||}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}d\phi\right)^{p/p'}\right)^{p'}$$

$$+K_2(n(\rho,\varphi)+n(\rho,1/\varphi))^{p'}$$

$$\leqq K_3\left(\int_0^{2\pi}d\theta\int_0^{2\pi}\frac{|\log|\varphi(\rho e^{\sqrt{-1}\phi})||}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}d\phi\right)^p$$

$$+K_2(n(\rho,\varphi)^{p'}+n(\rho,1/\varphi)^{p'}).$$

On the other hand, we have

$$\int_0^{2\pi}\frac{d\theta}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}\leqq\frac{1}{(\rho-r)^{l-1}}\int_0^{2\pi}\frac{d\theta}{|\rho-re^{\sqrt{-1}\theta}|^2}$$

$$=\frac{2\pi}{(\rho-r)^{l-1}(\rho^2-r^2)}$$

and, by (2.1)

$$\frac{1}{2\pi}\int_0^{2\pi}|\log|\varphi(\rho e^{\sqrt{-1}\phi})||d\phi=m(\rho,\varphi)+m(\rho,1/\varphi)$$

$$\leqq 2T(\rho,\varphi)+K_4.$$

Therefore, we can conclude

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_0^{2\pi}\frac{|\log|\varphi(\rho e^{\sqrt{-1}\phi})||}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}d\phi$$

$$=\int_0^{2\pi}|\log|\varphi(\rho e^{\sqrt{-1}\phi})||d\phi\,\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\rho e^{\sqrt{-1}\phi}-re^{\sqrt{-1}\theta}|^{l+1}}$$

$$\leqq\frac{1}{(\rho-r)^l(\rho+r)}\int_0^{2\pi}|\log|\varphi(\rho e^{\sqrt{-1}\phi})||d\phi$$

$$\leqq\frac{K_5}{(R-r)^l}T(R,\varphi),$$

because $\rho=(R+r)/2<R$, $\rho-r=(R-r)/2$ and $T(r, \varphi)$ is a non-decreasing function of r. Concerning the terms $n(\rho, \varphi)^{p'}$ and $n(\rho, 1/\varphi)^{p'}$, we can conclude easily from the definition of counting function

$$n(\rho, \varphi^{\pm 1})\leqq\frac{R}{R-\rho}(N(R, \varphi^{\pm 1})+K_6)$$

$$\leqq\frac{R}{R-\rho}(T(R, \varphi)+K_6)$$

$$\leqq\frac{2}{R-r}(T(R, \varphi)+K_6)$$

(cf., [5], p. 37). We have thus

$$\int_0^{2\pi}\left|\frac{d^{l-1}}{dz^{l-1}}\left(\frac{\varphi'}{\varphi}\right)(re^{\sqrt{-1}\theta})\right|^p d\theta$$

$$\leqq K_7\frac{T(R, \varphi)^p}{(R-r)^{pl}}+K_8\left(\frac{T(R, \varphi)}{R-r}\right)^{p'}$$

$$\leqq K_9\left(\frac{T(R, \varphi)}{R-r}\right)^{p'}.$$

§ 3. A basic inequality.

Let f be a holomorphic map of the unit disc $\varDelta$ into $P^n(\boldsymbol{C})$. Choosing homogeneous coordinates $(w_1:\cdots:w_{n+1})$ on $P^n(\boldsymbol{C})$ arbitrarily, we take a reduced representation $f=(f_1:\cdots:f_{n+1})$, where f_i $(1\leqq i\leqq n+1)$ are holomorphic functions which have no common zeros on $\varDelta$. After H. Cartan [1], we set $u(z) := \max_{1\leq i\leq n+1} \log|f_i(z)|$ and define the characteristic function of f by

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi}u(re^{\sqrt{-1}\theta})d\theta-u(0) \qquad (0\leqq r\leqq 1).$$

With each nonzero meromorphic function $\varphi=g/h$ on $\varDelta$ we can associate the holomorphic map $\tilde{\varphi}:=(g:h):\varDelta\to P^1(\boldsymbol{C})$. Note that

$$T(r, \tilde{\varphi})=N(r, \varphi)+m(r, \varphi)+O(1). \tag{3.1}$$

A bounded term in the characteristic function is not essential. We may identify $T(r, \tilde{\varphi})$ with the characteristic function for φ defined in § 2.

For a holomorphic map $f=(f_1:\cdots:f_{n+1}):\varDelta\to P^n(\boldsymbol{C})$, consider a meromorphic function $\varphi=\sum_{i=1}^{n+1}a_if_i/\sum_{i=1}^{n+1}b_if_i$, where $\sum_{i=1}^{n+1}b_if_i\not\equiv 0$. Then, we see easily

$$T(r, \varphi)\leqq T(r, f)+O(1). \tag{3.2}$$

Take hyperplanes

$$H_j: a_{j1}w_1+\cdots+a_{jn+1}w_{n+1}=0 \qquad (1\leqq j\leqq q)$$

in general position such that $f(\Delta)\not\subset H_j$. We define holomorphic functions

$$F_j=a_{j1}f_1+\cdots+a_{jn+1}f_{n+1} \qquad (1\leqq j\leqq q)$$

and denote by $W(f_1, \cdots, f_{n+1})$ the Wronskian of $f_1, \cdots, f_{n+1}$. The purpose of this section is to prove the following

PROPOSITION 3.3. *In the above situation, take positive numbers t, p', r_0 with $0<n(n+1)t/2<p'<1$ and $0<r_0<1$. Then there is a constant K such that, for $r_0<r<R<1$,*

$$\int_0^{2\pi}\left|\frac{W(f_1, \cdots, f_{n+1})}{F_1F_2\cdots F_q}\right|^t\|f\|^{t(q-n-1)}(re^{\sqrt{-1}\theta})d\theta\leqq K\left(\frac{T(r, f)}{R-r}\right)^{p'},$$

where $\|f\|=(|f_1|^2+\cdots+|f_{n+1}|^2)^{1/2}$.

For the proof, we first recall two lemmas which were shown in the previous paper [4].

LEMMA 3.4. *There is a constant K_1 such that*

$$\left|\frac{W(f_1, \cdots, f_{n+1})}{F_1F_2\cdots F_q}\right|\|f\|^{q-n-1}\leqq K_1\left(\sum_{1\leq i_1<\cdots<i_{n+1}\leq q}\left|\frac{W(F_{i_1}, \cdots, F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|\right).$$

LEMMA 3.5. *Let $F_1, \cdots, F_{n+1}$ be nonzero holomorphic functions on Δ and set $\varphi_j:=F_j/F_{n+1}$ $(1\leqq j\leqq n)$. Then, there is a polynomial $P(\cdots, u_{jl}, \cdots)$ with real positive coefficients not depending on each $F_1, \cdots, F_{n+1}$ such that*

$$\left|\frac{W(F_1, \cdots, F_{n+1})}{F_1F_2\cdots F_{n+1}}\right|\leqq P\left(\cdots, \left|\left(\frac{\varphi_j'}{\varphi_j}\right)^{(l-1)}\right|, \cdots\right).$$

More precisely, with each indeterminate u_{jl} associating weight l, we can choose P so as to be isobaric of weight $n(n+1)/2$.

We need another lemma.

LEMMA 3.6. *Let $\varphi_1, \cdots, \varphi_k$ be nonzero meromorphic functions on Δ, $l_1, \cdots, l_k$ be positive integers and $0<r_0<1$, $0<t(l_1+\cdots+l_k)<p'<1$. Then, there exists a positive constant K_2 such that for $r_0<r<R<1$*

$$\int_0^{2\pi}\left|\left(\frac{\varphi_1'}{\varphi_1}\right)^{(l_1-1)}(re^{\sqrt{-1}\theta})\cdots\left(\frac{\varphi_k'}{\varphi_k}\right)^{(l_k-1)}(re^{\sqrt{-1}\theta})\right|^t d\theta$$

$$\leqq\frac{K_2}{(R-r)^{p'}}T(R, \varphi_1)^{p's_1}\cdots T(R, \varphi_k)^{p's_k},$$

where $s_j:=l_j/(l_1+\cdots+l_k)$ $(1\leqq j\leqq k)$.

PROOF. By the generalized Hölder's inequality, we obtain

$$\int_0^{2\pi}\left|\left(\frac{\varphi_1'}{\varphi_1}\right)^{(l_1-1)}(re^{\sqrt{-1}\theta})\cdots\left(\frac{\varphi_k'}{\varphi_k}\right)^{(l_k-1)}(re^{\sqrt{-1}\theta})\right|^t d\theta$$
$$\leqq \prod_{j=1}^{k}\left(\int_0^{2\pi}\left|\left(\frac{\varphi_j'}{\varphi_j}\right)^{(l_j-1)}(re^{\sqrt{-1}\theta})\right|^{t/s_j}d\theta\right)^{s_j}.$$

Since $l_j(t/s_j)=t(l_1+\cdots+l_k)<p'<1$ by the assumption, we can apply Proposition 2.2 to show

$$\left(\int_0^{2\pi}\left|\left(\frac{\varphi_j'}{\varphi_j}\right)^{(l_j-1)}(re^{\sqrt{-1}\theta})\right|^{t/s_j}d\theta\right)^{s_j}\leqq K_3\left(\frac{T(R,\varphi_j)}{R-r}\right)^{p's_j}$$

for each $j=1, 2, \cdots, k$. This gives Lemma 3.6 because $s_1+\cdots+s_k=1$.

PROOF OF PROPOSITION 3.3. Since $t<1$, Lemma 3.4 implies that

$$\left|\frac{W(f_1,\cdots,f_{n+1})}{F_1F_2\cdots F_q}\right|^t\|f\|^{t(q-n-1)}\leqq K_4\left(\sum_{1\leq i_1<\cdots<i_{n+1}\leq q}\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|^t\right).$$

For our purpose, it suffices to show that

$$\int_0^{2\pi}\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|^t(re^{\sqrt{-1}\theta})d\theta\leqq K_5\left(\frac{T(r,f)}{R-r}\right)^{p'}$$

for arbitrary $i_1, \cdots, i_{n+1}$ with $1\leqq i_1<\cdots<i_{n+1}\leqq q$. For brevity, we set $\varphi_j:=F_{i_j}/F_{i_{n+1}}$ and $\psi_{j,l}:=(\varphi_j'/\varphi_j)^{(l-1)}$. By virtue of Lemma 3.5, we can estimate $\left|\frac{W(F_{i_1},\cdots,F_{i_{n+1}})}{F_{i_1}\cdots F_{i_{n+1}}}\right|$ from above by a positive constant multiple of the sum of some functions of type

$$\psi=|\psi_{j_1,l_1}\psi_{j_2,l_2}\cdots\psi_{j_k,l_k}|, \tag{3.7}$$

where $1\leqq j_1, j_2, \cdots, j_k\leqq n$ and $l_1+l_2+\cdots+l_k=n(n+1)/2$. We now apply Lemma 3.6 to the functions $\varphi_{j_1}, \cdots, \varphi_{j_k}$. For the function ψ given by (3.7), we have

$$\int_0^{2\pi}\left|\psi(re^{\sqrt{-1}\theta})\right|^t d\theta\leqq\frac{K_6}{(R-r)^{p'}}T(R,\varphi_1)^{p's_1}\cdots T(R,\varphi_k)^{p's_k}.$$

On the other hand, the right hand side of this inequality can be replaced by $K_7\left(\frac{T(R,f)}{R-r}\right)^{p'}$, because of (3.2) and $s_1+\cdots+s_k=1$. This completes the proof of Proposition 3.3.

§4. Non-integrated defect relation.

Let M be an open Riemann surface and f a non-constant holomorphic map of M into $P^n(\boldsymbol{C})$. For arbitrarily chosen homogeneous coordinates $(w_1:\cdots:w_{n+1})$ we take a reduced representation $f=(f_1:\cdots:f_{n+1})$. Consider a hyperplane

$$H: a_1w_1+\cdots+a_{n+1}w_{n+1}=0$$

with $f(M)\not\subset H$. Using the holomorphic function

$$F=a_1f_1+\cdots+a_{n+1}f_{n+1}, \tag{4.1}$$

we define the intersection multiplicity of $f(M)$ and H at $f(p)$ by

$$\nu^f(H)(p)=\begin{cases} 0 & \text{if } F(p)\neq 0 \\ m & \text{if } F \text{ has a zero of order } m \text{ at } p. \end{cases}$$

DEFINITION 4.2. For an arbitrarily fixed positive integer μ_0 we denote by $\alpha^f_{\mu_0}(H)$ the largest lower bound of nonnegative number η satisfying the condition that there is a subharmonic function u such that e^u is of class C^∞,

(4.3) $u\leqq\eta\log\|f\|$, and

(4.4) for each point $p\in f^{-1}(H)$, if we choose a holomorphic local coordinate ζ around p, then

$$u(\zeta)-\min(\nu^f(H)(p),\ \mu_0)\log|\zeta-\zeta(p)|$$

is subharmonic, where $\|f\|=(|f_1|^2+\cdots+|f_{n+1}|^2)^{1/2}$.

We now define the non-integrated defect of H by

$$\delta^f_{\mu_0}(H):=1-\alpha^f_{\mu_0}(H).$$

As is easily seen, $\delta^f_{\mu_0}(H)$ depends on f, H, μ_0 only. For the reason why we call $\delta^f_{\mu_0}(H)$ non-integrated defect, see the proof of Proposition 4.7 below.

PROPOSITION 4.5. *If there is a bounded nonzero holomorphic function g on M such that g has zeros of order at least* $\min(\nu^f(H)(p),\ \mu_0)$ *for each point $p\in f^{-1}(H)$, in particular, if $f(M)\cap H=\emptyset$, then $\delta^f_{\mu_0}(H)=1$.*

PROOF. Take a constant K with $|g|\leqq K$ and set $\eta:=0$, $u:=\log|g/K|$. They satisfy conditions (4.3) and (4.4), whence we have Proposition 4.5.

PROPOSITION 4.6. *If there is a positive integer $\mu>\mu_0$ such that $\nu^f(H)(p)\geqq\mu$ for each point $p\in f^{-1}(H)$, then $\delta^f_{\mu_0}(H)\geqq 1-\mu_0/\mu$.*

PROOF. Consider the function F defined by (4.1) and set $\eta:=\mu_0/\mu$, $u:=(\mu_0/\mu)\log|F/K|$, where K is a constant with $|F|\leqq K\|f\|$. They satisfy conditions (4.3) and (4.4). For, in a neighborhood of each point $p\in f^{-1}(H)$ u can be written as

$$u(\zeta)=u_0(\zeta)+\frac{\mu_0\nu^f(H)(p)}{\mu}\log|\zeta-\zeta(p)|$$

with a subharmonic function u_0 and

$$\frac{\mu_0\nu^f(H)(p)}{\mu}\geqq\mu_0\geqq\min(\nu^f(H)(p),\ \mu_0)$$

by the assumption. This implies Proposition 4.6.

For a holomorphic map f of the set $\Delta(R_0)=\{|z|<R_0\}$ $(R_0\leqq+\infty)$ into $P^n(C)$ with $\lim_{r\to R_0} T(r, f)=\infty$ and a hyperplane H in $P^n(C)$ with $f(\Delta(R_0))\not\subset H$, the defect of H is usually defined to be $\liminf_{r\to R_0}(1-N(r, H)/T(r, f))$. We consider here the modified defect

$$\delta^*_{\mu_0}(H):=\liminf_{r\to R_0}\left(1-\frac{N_{\mu_0}(r, H)}{T(r, f)}\right),$$

where $N_{\mu_0}(r, H)$ is defined by

$$N_{\mu_0}(r, H):=\int_0^r \frac{n_{\mu_0}(t)-n_{\mu_0}(0)}{t}dt+n_{\mu_0}(0)\log r$$

with the number $n_{\mu_0}(t)=\sum_{|z|\leqq t}\min(\nu^f(H)(z), \mu_0)$.

PROPOSITION 4.7. *For every positive integer* μ_0, *we have*

$$0\leqq\delta^f_{\mu_0}(H)\leqq\delta^*_{\mu_0}(H)\leqq 1\,.$$

PROOF. Let F be the function defined by (4.1). Then, there is a constant K with $|F|\leqq K\|f\|$. If we choose $\eta:=1$ and $u:=\log|F/K|$, they satisfy the conditions (4.3) and (4.4). This shows that $\delta^f_{\mu_0}(H)\geqq 0$. To show $\delta^f_{\mu_0}(H)\leqq\delta^*_{\mu_0}(H)$, take a number $\eta\geqq 0$ and a subharmonic function u satisfying the conditions (4.3) and (4.4) arbitrarily. We integrate both sides of (4.3) and get

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{\sqrt{-1}\theta})d\theta\leqq\frac{\eta}{2\pi}\int_0^{2\pi}\log\|f(re^{\sqrt{-1}\theta})\|d\theta \quad (0<r<R)\,.$$

On the other hand, we see easily

$$\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{\sqrt{-1}\theta})\|d\theta\leqq T(r, f)+O(1)$$

and, moreover

$$N_{\mu_0}(r, H)\leqq\frac{1}{2\pi}\int_0^{2\pi} u(re^{\sqrt{-1}\theta})d\theta+O(1)$$

by the use of the condition (4.4) (cf., [**6**], p. 120). Therefore,

$$\limsup_{r\to R_0}\frac{N_{\mu_0}(r, H)}{T(r, f)}=\limsup_{r\to R_0}\frac{\eta T(r, f)+O(1)}{T(r, f)}\leqq\eta\,.$$

Taking the largest lower bound of η's, we can replace the right hand side of this inequality by $\alpha^f_{\mu_0}(H)$. This gives the desired conclusion. Another relation $\delta^*_{\mu_0}(H)\leqq 1$ is obvious and so we have Proposition 4.7.

We now consider an open Riemann surface M with a conformal metric ds^2

and a holomorphic map $f=(f_1, f_2, \cdots, f_k)$ of M into $P^{n_1\cdots n_k}(C):=P^{n_1}(C)\times\cdots\times P^{n_k}(C)$.

DEFINITION 4.8. We shall call f to be *non-degenerate* if each component f_i of f is non-degenerate, namely, $f_i(M)$ is not contained in any hyperplane in $P^{n_i}(C)$.

Take a reduced representation $f_i=(f_{i1}:\cdots:f_{in_i+1})$ for each component f_i of f and set $\|f_i\|=(|f_{i1}|^2+\cdots+|f_{in_i+1}|^2)^{1/2}$.

DEFINITION 4.9. For positive numbers $\rho_1, \cdots, \rho_k$ we shall say that f satisfies *condition* $(C_{\rho_1\cdots\rho_k})$ if there exists a subharmonic function u on M such that e^u is of class C^∞ and

$$\lambda e^u \leqq \|f_1\|^{\rho_1}\cdots\|f_k\|^{\rho_k},$$

where λ is a positive real-valued function on M with $ds^2=\lambda^2|dz|^2$ for a global holomorphic function z with $dz\neq 0$ on M.

As is easily seen, this condition does not depend on particular choices of representations of $f_1, \cdots, f_k$ and ds^2.

We can prove the following non-integrated defect relation.

THEOREM 4.10. *Let M be an open Riemann surface with a complete conformal metric ds^2 which has infinite area and $f: M\to P^{n_1\cdots n_k}(C)$ be a non-degenerate holomorphic map satisfying condition $(C_{\rho_1\cdots\rho_k})$. For each i $(1\leqq i\leqq k)$ take hyperplanes $H_{i1}, \cdots, H_{iq_i}$ in $P^{n_i}(C)$ located in general position. Then, if*

$$\sum_{j=1}^{q_{i_0}} \delta_{n_{i_0}}^{f_{i_0}}(H_{i_0 j}) > n_{i_0}+1$$

for every i $(1\leqq i\leqq k)$, we have

$$\sum_{i=1}^{k} \frac{\rho_i n_i(n_i+1)}{\delta_{n_i}^{f_i}(H_{i1})+\cdots+\delta_{n_i}^{f_i}(H_{iq_i})-n_i-1} \geqq 1 .$$

The proof will be given in the next section.

REMARK 4.11. The only property of (M, ds^2) which we need for the proof of Theorem 4.10 is that $\int_M e^u d\sigma=\infty$ for any smooth subharmonic function u $(\not\equiv-\infty)$ on M. As was shown by S. T. Yau ([**14**]), this follows from the assumption that (M, ds^2) is complete and has infinite area.

§5. Proof of non-integrated defect relation.

To prove Theorem 4.10, we first recall some results in value distribution theory.

THEOREM 5.1. *Let f be a non-degenerate holomorphic map of $\Delta(R_0)=\{z: |z|<R_0\}$ $(R_0\leqq+\infty)$ into $P^n(C)$ and $H_1, \cdots, H_q$ hyperplanes in general position.* (i)

If $R_0=\infty$, *or* (ii) *if* $R_0<\infty$ *and* $\limsup_{r\to R_0} T(r, f)/-\log(R_0-r)=\infty$, *then*

$$\sum_{j=1}^{q} \delta_n^*(H_j) \leqq n+1 .$$

For the proof, see H. Cartan [**1**].

REMARK 5.2. By virtue of Proposition 4.7, the terms $\delta_n^*(H_j)$ in the above conclusion can be replaced by the non-integrated defects $\delta_n^f(H_j)$.

PROPOSITION 5.3. *Let* $f: \Delta(R_0)\to P^n(\boldsymbol{C})$ *be a non-degenerate holomorphic map with a reduced representation* $f=(f_1:\cdots:f_{n+1})$ *and consider the functions* $F_1, \cdots, F_q$ *defined by* (4.1) *for hyperplanes* $H_1, \cdots, H_q$ *in general position. Then, for every point* p *the order of poles of the meromorphic function* $\dfrac{W(f_1, \cdots, f_{n+1})}{F_1F_2\cdots F_q}$ *at* p *is not larger than* $\sum_{j=1}^{q} \min(\nu^f(H_j)(p), n)$.

For the proof, see H. Cartan [**1**].

PROPOSITION 5.4. *Let* $T(r)$ $(0\leqq r<1)$ *be a continuous increasing function with* $T(r)\geqq 1$. *Then, we can find a set* $E_0=\bigcup_{\nu=1}^{\infty}[r_\nu, r'_\nu]$ $(r'_{\nu-1}<r_\nu\leqq r'_\nu\leqq 1)$ *such that* $\int_{E_0}\dfrac{dr}{1-r}\leqq 2$ *and*

$$T\left(r+\frac{1-r}{eT(r)}\right)\leqq 2T(r)$$

for every $r\notin E_0$.

For the proof, see Hayman [**5**], pp. 38-39.

PROPOSITION 5.5. *Suppose that a positive real-valued function* $T(r)$ *on* $[0, 1)$ *satisfies the condition that*

$$T(r)\leqq \frac{1}{(1-r)^p}$$

for a set $E_0=\bigcup_{\nu=1}^{\infty}[r_\nu, r'_\nu]$ $(r'_{\nu-1}<r_\nu\leqq r'_\nu)$ *with* $\int_{E_0}\dfrac{dr}{1-r}<\infty$ *and a positive number* p. *Then there exists a positive constant* K *such that*

$$T(r)\leqq \frac{K}{(1-r)^p}$$

for every $r\in[0, 1)$.

PROOF. We may replace $T(r)$ by $T^*(r)=\sup_{r'\leqq r} T(r')$. So, it may be assumed that $T(r)$ is an increasing function. By the assumption, we see

$$\sum_\nu \int_{r_\nu}^{r'_\nu} \frac{dr}{1-r}=\sum_\nu \log\frac{1-r_\nu}{1-r'_\nu}=K_0<\infty .$$

If $r_\nu \leqq r \leqq r'_\nu$, then

$$T(r) \leqq \frac{1}{(1-r'_\nu)^p} = \frac{1}{(1-r_\nu)^p}\left(\frac{1-r_\nu}{1-r'_\nu}\right)^p$$
$$\leqq \frac{e^{pK_0}}{(1-r_\nu)^p} \leqq \frac{e^{pK_0}}{(1-r)^p}.$$

It suffices to take $K=e^{pK_0}$.

We now start to prove Theorem 4.10. Take an open Riemann surface M with a conformal metric ds^2, a holomorphic map $f=(f_1, \cdots, f_k): M \to P^{n_1 \cdots n_k}(\boldsymbol{C})$ and hyperplanes H_{ij} $(1 \leqq i \leqq k,\ 1 \leqq j \leqq q_i)$ satisfying the assumption of Theorem 4.10. Consider the universal covering surface $\varpi : \tilde{M} \to M$ with metric $d\tilde{s}^2 = \varpi^* ds^2$ and the map $\tilde{f}=(\tilde{f}_1, \cdots, \tilde{f}_k): \tilde{M} \to P^{n_1 \cdots n_k}(\boldsymbol{C})$, where $\tilde{f}_i = \varpi \cdot f_i$. Obviously, $(\tilde{M}, d\tilde{s}^2)$ and $\tilde{f}$ satisfy the assumption of Theorem 4.10 and $\delta_{n_i}^{f_i}(H_{ij}) \leqq \delta_{n_i}^{\tilde{f}_i}(H_{ij})$. According to these facts, we may assume that M is simply connected. Then, M is biholomorphic either to $\boldsymbol{C}$ or to the unit disc Δ. If $M=\boldsymbol{C}$, we have

$$\sum_{j=1}^{q} \delta_{n_i}^{f_i}(H_{ij}) \leqq n_i + 1$$

for every i and, moreover, if $M=\Delta$ and

$$\limsup_{r \to 1} \frac{T(r, f_{i_0})}{\log \frac{1}{1-r}} = \infty$$

for some i_0, then we have

$$\sum_{j=1}^{q} \delta_{n_{i_0}}^{f_{i_0}}(H_{i_0 j}) \leqq n_{i_0} + 1,$$

because of Remark 5.2.

Let us study the case $M=\Delta$ and

$$\limsup_{r \to 1} \frac{T(r, f_i)}{\log \frac{1}{1-r}} < \infty \tag{5.6}$$

for every $i=1, 2, \cdots, k$. The proof is given by reduction to absurdity. We assume that

$$\sum_{j=1}^{q_i} \delta_{n_i}^{f_i}(H_{ij}) > n_i + 1$$

for every i and

$$\sum_{i=1}^{k} \frac{\rho_i n_i (n_i+1)}{\delta_{n_i}^{f_i}(H_{i1}) + \cdots + \delta_{n_i}^{f_i}(H_{iq_i}) - n_i - 1} < 1.$$

By the definition of non-integrated defect, we can choose nonnegative numbers

η_{ij} and subharmonic functions u_{ij} such that $e^{u_{ij}}$ is of class C^{∞} and

$$\sum_{i=1}^{k} \frac{\rho_i n_i(n_i+1)}{(1-\eta_{i1})+\cdots+(1-\eta_{iq_i})-n_i-1}<1, \tag{5.7}$$

$$e^{u_{ij}} \leqq \|f_i\|^{\eta_{ij}}, \tag{5.8}$$

and in a neighborhood of each point $p \in f^{-1}(H)$

$$u_{ij}(\zeta)-\min(\nu^{f_i}(H_{ij})(p), n_i)\log|\zeta-\zeta(p)|$$

is subharmonic, where $\|f_i\|=(|f_{i1}|^2+\cdots+|f_{in_i+1}|^2)^{1/2}$ for reduced representations $f_i=(f_{i1}:\cdots:f_{in_i+1})$ and ζ is a holomorphic local coordinate around p. Set

$$v_i:=\log\left|\frac{W(f_{i1},\cdots,f_{in_i+1})}{F_{i1}\cdots F_{iq_i}}\right|+\sum_{j=1}^{q_i} u_{ij}, \tag{5.9}$$

where each F_{ij} denotes the function defined as (4.1) for a holomorphic map f_i and a hyperplane H_{ij}. As is easily seen by Proposition 5.3, each v_i is subharmonic on Δ. On the other hand, by the assumption that f satisfies condition $(C_{\rho_1\cdots\rho_k})$, there exists a subharmonic function w on Δ such that e^w is of class C^{∞} and

$$\lambda e^w \leqq \|f_1\|^{\rho_1}\cdots\|f_k\|^{\rho_k}, \tag{5.10}$$

where $ds^2=\lambda^2|dz|^2$. Set

$$t_i:=\frac{2\rho_i}{q_i-n_i-1-(\eta_{i1}+\cdots+\eta_{iq_i})}$$

$$\chi_i:=\frac{W(f_{i1},\cdots,f_{in_i+1})}{F_{i1}F_{i2}\cdots F_{iq_i}}$$

and define a subharmonic function

$$u:=2w+t_1v_1+\cdots+t_kv_k.$$

Then, by (5.10), (5.9) and (5.8) we have

$$\begin{aligned} e^u\lambda^2 &\leqq e^{t_1v_1+\cdots+t_kv_k}\|f_1\|^{2\rho_1}\cdots\|f_k\|^{2\rho_k} \\ &\leqq \prod_{i=1}^{k}|\chi_i|^{t_i}e^{t_i(u_{i1}+\cdots+u_{iq_i})}\|f_i\|^{2\rho_i} \\ &\leqq \prod_{i=1}^{k}|\chi_i|^{t_i}\|f_i\|^{t_i(\eta_{i1}+\cdots+\eta_{iq_i})+2\rho_i} \\ &\leqq \prod_{i=1}^{k}|\chi_i|^{t_i}\|f_i\|^{t_i(q_i-n_i-1)}. \end{aligned}$$

Therefore, if we set

$$s_i := \frac{t_i n_i(n_i+1)}{2} = \frac{\rho_i n_i(n_i+1)}{q_i - n_i - 1 - (\eta_{i1} + \cdots + \eta_{iq_i})}$$

$$p_i := \frac{s_i}{s_1 + \cdots + s_k}$$

and $t_i' := t_i / p_i$, we obtain by the generalized Hölder's inequality

$$\int_0^{2\pi} (e^u \lambda^2)(re^{\sqrt{-1}\theta}) d\theta \leqq \int_0^{2\pi} \Big(\prod_{i=1}^k |\chi_i|^{t_i} \|f_i\|^{t_i(q_i-n_i-1)}\Big)(re^{\sqrt{-1}\theta}) d\theta$$

$$\leqq \prod_{i=1}^k \left(\int_0^{2\pi} (|\chi_i|^{t_i'} \|f_i\|^{t_i'(q_i-n_i-1)})(re^{\sqrt{-1}\theta}) d\theta\right)^{p_i}.$$

By (5.7), $t_0 := s_1 + \cdots + s_k < 1$. Take p' with $t_0 < p' < 1$. Then, $t_i' n_i(n_i+1)/2 = t_0 < p' < 1$. We now apply Proposition 3.3 to show

$$\int_0^{2\pi} (|\chi_i|^{t_i'} \|f_i\|^{t_i'(q_i-n_i-1)}(re^{\sqrt{-1}\theta}) d\theta \leqq K_1 \left(\frac{T(R_i, f_i)}{R_i - r}\right)^{p'}$$

for all r, R_i with $r_0 < r < R_i < 1$, where $r_0 > 0$ and K_1 is a constant not depending on each r, R_i. We choose here $R_i := r + (1-r)/eT(r, f_i)$. By virtue of Proposition 5.4, we can find a set $E_0 = \bigcup_{\nu=1}^{\infty} [r_\nu, r_\nu']$ $(r_{\nu-1}' < r_\nu \leqq r_\nu' < 1)$ such that $\int_{E_0} 1/(1-r) dr < \infty$ and $T(R_i, f_i) \leqq 2T(r, f_i)$ for every $r \notin E_0$. Therefore, we get

$$\int_0^{2\pi} (e^u \lambda^2)(re^{\sqrt{-1}\theta}) d\theta \leqq K_2 \prod_{i=1}^k \left(\frac{T(r, f_i)}{1-r}\right)^{p_i p'}$$

$$\leqq K_3 \frac{1}{(1-r)^{p'}} \left(\log \frac{1}{1-r}\right)^{p'}$$

for all $r \notin E_0$, where we used (5.6). Take a number p'' with $p' < p'' < 1$. After a suitable change of a constant K_3, we can conclude

$$\int_0^{2\pi} (e^u \lambda^2)(re^{\sqrt{-1}\theta}) d\theta \leqq \frac{K_4}{(1-r)^{p''}} \qquad (r \notin E_0).$$

Moreover, we can omit here the restricted condition $r \notin E_0$ by the help of Proposition 5.5. Thus, we have

$$\iint_{\Delta} (e^u \lambda^2)(re^{\sqrt{-1}\theta}) r dr d\theta \leqq K_4 \int_0^1 \frac{r dr}{(1-r)^{p''}} < \infty.$$

On the other hand, by the result of S. T. Yau ([14]), we have necessarily

$$\iint_{\Delta} e^u d\sigma = \infty$$

(cf., Remark 4.11). This is a contradiction. So, the proof of Theorem 4.10 is completed.

§ 6. The Gauss map of a complete minimal surface in R^m.

The purpose of this section is to prove Theorems stated in § 1. Let $x=(x_1, \cdots, x_m): M \to \boldsymbol{R}^m$ be an oriented complete minimal surface in $\boldsymbol{R}^m$ $(m \geqq 3)$. For our purpose, we can replace M by the universal covering surface. So, there is no harm in assuming that M is simply connected. By associating a holomorphic local coordinate $z=u+\sqrt{-1}v$ with each positive isothermal local coordinates (u, v), M is considered as a Riemann surface with a conformal metric ds^2. Since there is no compact minimal surface in $\boldsymbol{R}^m$, M is biholomorphically isomorphic to $\boldsymbol{C}$ or the unit disc. We note here that M has infinite area because M is of non-positive curvature.

By definition, the Gauss map G of M is given by $G=\pi \cdot \partial X/\partial \bar{z}$, where z is a global holomorphic coordinate on M and π is the canonical projection of $\boldsymbol{C}^m-\{0\}$ onto $P^{m-1}(\boldsymbol{C})$. Set $n:=m-1$. The conjugate $f=\bar{G}$ of G is represented as $f=(f_1:\cdots:f_{n+1})$, where $f_i=\partial x_i/\partial z$ $(1 \leqq i \leqq n+1)$. This representation is reduced because x is an immersion. As is well known, it holds that

$$f_1^2+\cdots+f_{n+1}^2 \equiv 0\,. \tag{6.1}$$

This means that $f(M) \subseteqq Q_{m-2}(\boldsymbol{C})=\{w_1^2+\cdots+w_{n+1}^2=0\}$ $(\subset P^{m-1}(\boldsymbol{C}))$. Set $\|f\|:=(|f_1|^2+\cdots+|f_{n+1}|^2)^{1/2}$. Then, the metric ds^2 on M induced from the standard metric on $\boldsymbol{R}^m$ is given by

$$ds^2=2\|f\|^2|dz|^2\,. \tag{6.2}$$

This shows that the map $f: M \to P^n(\boldsymbol{C})$ $(n=m-1)$ satisfies condition (C_1). We can apply Theorem 4.10 to the map f and conclude

$$\frac{n(n+1)}{\sum_{j=1}^{q} \delta_n^f(H_j)-n-1} \geqq 1$$

for arbitrarily given hyperplanes $H_1, \cdots, H_q$ in general position if f is non-degenerate. This gives Theorem 1.2.

We consider next the particular case $m=3$. As is well-known, $Q_1(\boldsymbol{C})$ is biholomorphically isomorphic to the Riemann sphere $P^1(\boldsymbol{C})=\boldsymbol{C} \cup \{\infty\}$ by the map φ defined as

$$\varphi(w)=\frac{w_3}{w_1-\sqrt{-1}w_2}$$

for each $w=(w_1: w_2: w_3) \in Q_1(\boldsymbol{C})$. Instead of the map f we study the map $g=$

$\varphi\cdot f: M\rightarrow P^1(\boldsymbol{C})$. Assume that M is non-flat. Then g is not a constant. Take a nonzero holomorphic function h such that $g_1:=f_3/h$ and $g_2:=(f_1-\sqrt{-1}f_2)/h$ are both holomorphic and have no common zero. Then, g has a reduced representation $g=(g_1:g_2)$. We see easily by (6.1)

$$f_1+\sqrt{-1}f_2=-\frac{hg_1^2}{g_2}, \quad f_1-\sqrt{-1}f_2=hg_2 .$$

Therefore,

$$\begin{aligned}2\|f\|^2&=|f_1+\sqrt{-1}f_2|^2+|f_1-\sqrt{-1}f_2|^2+2|f_3|^2\\&=\frac{|h|^2}{|g_2|^2}\|g\|^4,\end{aligned}$$

where $\|g\|=(|g_1|^2+|g_2|^2)^{1/2}$. Since $\|f\|\neq 0$ everywhere, we can easily conclude that g_2/h is holomorphic on M. So, $u=\log|g_2/h|$ is subharmonic. In view of (6.2) this shows that the map $g: M\rightarrow P^1(\boldsymbol{C})$ satisfies condition (C_2). Applying Theorem 4.10 to the map g, we obtain Theorem 1.3.

Lastly, we consider the case $m=4$. The quadric $Q_2(\boldsymbol{C})$ in $P^3(\boldsymbol{C})$ is biholomorphically isomorphic to $P^1(\boldsymbol{C})\times P^1(\boldsymbol{C})$ by the map $\phi=\phi_1\times\phi_2$ defined as

$$\phi_1(w)=\frac{w_3+\sqrt{-1}w_4}{w_1-\sqrt{-1}w_2}, \quad \phi_2(w)=\frac{-w_3+\sqrt{-1}w_4}{w_1-\sqrt{-1}w_2}$$

for each $w=(w_1:\cdots:w_4)\in Q_2(\boldsymbol{C})$, where for the point $w=(w_1:\cdots:w_4)$ with $w_1=\sqrt{-1}w_2$ $\phi_i(w)$ are properly defined so as to be continuous on the totality of $Q_2(\boldsymbol{C})$ (cf., [7], p. 20). For the conjugate f of the Gauss map of a given minimal surface M, we consider the meromorphic functions $g_1=\phi_1\cdot f$ and $g_2=\phi_2\cdot f$. Instead of Theorem 1.4, we shall give the following more precise result.

THEOREM 6.3. *Let $x=(x_1, x_2, x_3, x_4): M\rightarrow \boldsymbol{R}^4$ be a non-flat complete minimal surface and $g_1, g_2: M\rightarrow P^1(\boldsymbol{C})$ be the above-mentioned meromorphic functions.*

(i) *Assume that $g_1\not\equiv$const and $g_2\not\equiv$const. Then, for arbitrary numbers $\alpha_1, \cdots, \alpha_{q_1}$ and $\beta_1, \cdots, \beta_{q_2}$ with $\alpha_i\neq\alpha_j$ if $i\neq j$ and $\beta_k\neq\beta_l$ if $k\neq l$, we have at least one of the following conclusions:*

(a) $\sum_{j=1}^{q_1}\delta_1^{g_1}(\alpha_j)\leqq 2$,

(b) $\sum_{k=1}^{q_2}\delta_1^{g_2}(\beta_k)\leqq 2$,

(c) $\dfrac{1}{\sum_{j=1}^{q_1}\delta_1^{g_1}(\alpha_j)-2}+\dfrac{1}{\sum_{k=1}^{q_2}\delta_1^{g_2}(\beta_k)-2}\geqq\dfrac{1}{2}$.

(ii) *Assume that $g_1\not\equiv$const and $g_2\equiv$const. Then, for arbitrary distinct numbers $\alpha_1, \cdots, \alpha_q$, we have*

$$\sum_{j=1}^{q} \delta_1^{g_1}(\alpha_j) \leqq 4 .$$

PROOF. As usual, we may assume that M is simply connected. Take a global coordinate z on M and set $f_i := \partial x_i/\partial z$ $(i=1, 2, 3, 4)$. We choose holomorphic functions g_{11}, g_{12}, g_{21}, g_{22} such that

$$g_1 = \frac{f_3+\sqrt{-1}f_4}{f_1-\sqrt{-1}f_2} = \frac{g_{12}}{g_{11}}, \qquad g_2 = \frac{-f_3+\sqrt{-1}f_4}{f_1-\sqrt{-1}f_2} = \frac{g_{12}}{g_{21}}$$

and $\|g_1\|^2 := |g_{11}|^2 + |g_{12}|^2 \neq 0$, $\|g_2\|^2 := |g_{21}|^2 + |g_{22}|^2 \neq 0$ everywhere. Then, since

$$(f_1, f_2, f_3, f_4) = \frac{f_1-\sqrt{-1}f_2}{2}(1+g_1g_2, \sqrt{-1}(1-g_1g_2), g_1-g_2, -\sqrt{-1}(g_1+g_2))$$

(cf., [7], p. 20), we have

$$\begin{aligned} 2\|f\|^2 &= 2(|f_1|^2+|f_2|^2+|f_3|^2+|f_4|^2) \\ &= \frac{|h|^2}{2}\left(\left|1+\frac{g_{12}g_{22}}{g_{11}g_{21}}\right|^2 + \left|1-\frac{g_{12}g_{22}}{g_{11}g_{21}}\right|^2 + \left|\frac{g_{12}}{g_{11}} - \frac{g_{22}}{g_{21}}\right|^2 + \left|\frac{g_{12}}{g_{11}} + \frac{g_{22}}{g_{21}}\right|^2\right) \\ &= \frac{|h|^2}{|g_{11}|^2|g_{21}|^2}(|g_{11}|^2+|g_{12}|^2)(|g_{21}|^2+|g_{22}|^2), \end{aligned}$$

where $h := f_1 - \sqrt{-1}f_2$. Set $u := \log(|g_{11}g_{21}|/|h|)$, which is subharmonic on M because $\|f\| \neq 0$ everywhere. The above equation can be rewritten

$$\lambda e^u = \|g_1\|\,\|g_2\|, \tag{6.4}$$

where $ds^2 = \lambda^2|dz|^2$. Consider first the case $g_1 \not\equiv$const and $g_2 \not\equiv$const. Then the map $g=(g_1, g_2): M \to P^1(\boldsymbol{C}) \times P^1(\boldsymbol{C})$ satisfies condition (C_{11}) by (6.4). Applying Theorem 4.10 to the map g, we have the conclusion (i) of Theorem 6.3. For the case $g_1 \not\equiv$const and $g_2 \equiv$const, we can take g_{21} and g_{22} as nonzero constant functions. It follows from (6.4) that the holomorphic map $g_1: M \to P^1(\boldsymbol{C})$ satisfies condition (C_1). By Theorem 4.10, we have the conclusion (ii) of Theorem 6.3.

PROOF OF THEOREM 1.4. By the assumption that M is non-flat, either g_1 or g_2 is not constant. If one of g_1 and g_2 is constant, the conclusion of Theorem 1.4 is true by the conclusion (ii) of Theorem 6.3. Assume that $g_1 \not\equiv$const and $g_2 \not\equiv$const and, for one of g_1 and g_2, say g_2, there exist distinct numbers $\beta_1, \cdots, \beta_{q_2}$ such that

$$\sum_{k=1}^{q_2} \delta_1^{g_2}(\beta_k) > 6 .$$

We have necessarily conclusion (a) or (c) of Theorem 6.3 for arbitrary distinct numbers $\alpha_1, \cdots, \alpha_q$ and the above $\beta_1, \cdots, \beta_{q_2}$. In each case, we can conclude

$$\sum_{j=1}^{q} \delta_1^{g_1}(\alpha_j) \leqq 6 .$$

This completes the proof of Theorem 1.4.

References

[1] H. Cartan, Sur les zéros des combinaisons linéaires de p fonctions holomorphes données, Mathematica, **7**(1933), 5-31.
[2] S. S. Chern and R. Osserman, Complete minimal surfaces in euclidean n-space, J. Analyse Math., **19**(1967), 15-34.
[3] H. Fujimoto, The defect relations for the derived curves of a holomorphic curve in $P^n(\boldsymbol{C})$, Tôhoku Math. J., **34**(1982), 141-160.
[4] H. Fujimoto, On the Gauss map of a complete minimal surface in $\boldsymbol{R}^m$, J. Math. Soc. Japan, **35**(1983), 279-288.
[5] W. K. Hayman, Meromorphic functions, Oxford Mathematical Monograph, Clarendon Press, Oxford, 1964.
[6] W. K. Hayman and P. B. Kennedy, Subharmonic functions, Academic Press, London, 1976.
[7] D. A. Hoffman and R. Osserman, The geometry of the generalized Gauss map, Amer. Math. Soc. Memoir, **236**, 1980.
[8] H. B. Lawson, Lectures on minimal submanifolds, Vol. 1, Publish or Perish Inc., Berkeley, 1980.
[9] R. Nevanlinna, Le théorème de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.
[10] R. Osserman, Global properties of minimal surfaces in E^3 and E^n, Ann. of Math., **80**(1964), 340-364.
[11] R. Osserman, A survey of minimal surfaces, Van Nostrand Reinhold, New York, 1969.
[12] R. Osserman, Minimal surfaces, Gauss maps, total curvature, eigenvalue estimates, and stability, The Chern Symposium 1979, Springer-Verlag.
[13] F. Xavier, The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere, Ann. of Math., **113**(1981), 211-214.
[14] S. T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Univ. Math. J., **25**(1976), 659-670.

Hirotaka FUJIMOTO
Department of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464
Japan