J. Math. Soc. Japan
Vol. 35, No. 4, 1983

# Sylow 2-intersections, 2-fusion, and 2-factorizations in finite groups of characteristic 2 type

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(Received Feb. 19, 1982)

## Introduction.

There is a close relationship between Sylow intersections, fusion, and factorizations in finite groups. This is probably best illustrated by the following examples. Let $p$ be a prime and $G$ be a group of order divisible by $p$. Define $\mathcal{H}_0$ to be the set of all nonidentity $p$-subgroups $H$ of $G$ such that $N_G(H)/H$ is $p$-isolated in the sense of Goldschmidt [10]. Let $\mathcal{N}_0$ be the set of the normalizers of the elements of $\mathcal{H}_0$. Then the following holds.

(1) *$\mathcal{N}_0$ controls Sylow $p$-intersections in $G$.*

(2) *$\mathcal{N}_0$ controls $p$-fusion in $G$.*

(3) *If $G$ is not $p$-isolated and $S \in \mathrm{Syl}_p(G)$, then*

$$G=\langle N\in\mathcal{N}_0\,;\,S\cap N\in\mathrm{Syl}_p(N)\rangle N_G(S)\,.$$

In the above, (1) is essentially a lemma in [11, (2.3)], and the reader is referred to Kondo [16, Lemma 2] for a generalization of (1) and the precise meaning of 'control' in (1) (the definition of the control in the most general form will be given in the first section of the present paper). The proposition (2) is a theorem of Goldschmidt [10, Theorem 3.4] improving Alperin's fusion theorem [1]. The proposition (3) is considered to be a sort of $p$-factorization theorem, and is an easy consequence of (1). It has already been pointed out that (2) can easily be derived from (1) also [12, Proposition 2.4], [16, Theorem 1].

Still more interesting than (1), (2), and (3) are the following theorems of Aschbacher [3] and P. McBride.

(4) *If $G$ is a group of characteristic 2 type, $S\in\mathrm{Syl}_2(G)$, and $G$ is not generated by the normalizers of nontrivial characteristic subgroups of $S$, then either $G$ is 2-isolated or some maximal 2-local subgroup of $G$ has a block in $\mathcal{X}$.*

(5) *If $G$ is a group of characteristic 2 type in which each simple section of each 2-local subgroup is of known type and if 2-fusion in $G$ is not controlled by the normalizers of nontrivial characteristic subgroups of a Sylow 2-subgroup of $G$, then some maximal 2-local subgroup of $G$ has a block in $\mathcal{X}$.*

In the above, (5) was announced at the A.M.S. Summer Institute held at the

University of California, Santa Cruz, in 1979. Now since (4) and (5) are analogous to (3) and (2), respectively, and since (2) and (3) are easy consequences of (1), it seems natural to ask whether there is a theorem on the control of Sylow 2-intersections from which (4) and (5) are easily derived. In this paper, we show that such a theorem exists. Our result may be phrased as follows.

(6) *We can assign to each nonidentity* 2*-group* $S$ *a pair of nonidentity characteristic subgroups,* $A_S$ *and* $B_S$*, with the following properties:* $A_S \leq \Omega_1(Z(S))$*, and whenever* $G$ *is a group of characteristic* 2 *type in which each simple section of each* 2*-local subgroup is of known type, Sylow* 2*-intersections in* $G$ *are controlled by* $C_G(A_S)$, $N_G(B_S)$ *(as* $S$ *ranges over* $\mathrm{Syl}_2(G)$*), and maximal* 2*-local subgroups of* $G$ *having a block in* $\mathcal{X}$.[(1)]

This result is obtained by combining the theorems 4.2 and 4.10 of this paper, a variant of a theorem of Glauberman and Niles [**9**] proved in the thesis [**7**] of N. R. Campbell, and a theorem of Aschbacher [**4**] on GF(2)-representations. Combining (6) with the theorems 1.4 and 1.5 of this paper, we can make improvements on (4) and (5) under the assumption that each simple section of each 2-local subgroup of $G$ is of known type (this assumption is actually superfluous, since it is reported that the program to classify the finite simple groups has been finished).

It should be mentioned that Foote [**8**] has developed a theory of blocks in groups of characteristic 2 type, and that R. Solomon and S. K. Wong have studied the so-called 'standard blocks' in groups of characteristic 2 type [**17**], [**18**]. If some maximal 2-local subgroup of a simple group $G$ of characteristic 2 type has a block in $\mathcal{X}$, then we can identify $G$ by their work.

The organization of the paper is as follows. In the first section, we give a definition of control of Sylow $p$-intersections and control of $p$-fusion by a normal set of subgroups modelled after (1) and (2), and then we study the relationship between control of Sylow $p$-intersections, control of $p$-fusion, and $p$-factorizations. Furthermore, we prove three fundamental theorems 1.7, 1.11, and 1.12. In the second and third sections, we give a brief summary of two basic tools to be used in the proof of the main theorem of this paper: control of Sylow 2-intersections in groups of Chev(2) type and groups of alternating type, and GF(2)-representations of finite groups. In the fourth section, we prove the main theorem of this paper, Theorem 4.11. In the concluding remarks, we precisely restate the proposition (6) and its consequences using the terminology and notation to be introduced in the first and fourth sections.

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(1) When I wrote this manuscript, I was unable to explicitly define $A_S$ and $B_S$. Some progress has been made since then, and we are now able to explicitly define $A_S$ and $B_S$. For instance, we may define $A_S = \Omega_1(Z(S))$. For details, the reader is referred to my paper "Characteristic pairs for 2-groups" which will be published elsewhere.

## 1. Sylow intersections, fusion, and factorizations.

Let $p$ be a prime, $G$ be a group of order divisible by $p$, and $\mathfrak{F}$ be a normal set of subgroups of $G$. For $S\in\mathrm{Syl}_p(G)$, let $\mathfrak{F}(S)=\{X\in\mathfrak{F}\,;\,S\cap X\in\mathrm{Syl}_p(X)\}$.

1.1 DEFINITION. The normal set $\mathfrak{F}$ of subgroups of $G$ is said to *control Sylow $p$-intersections* in $G$ if for each pair $S$, $T$ of *distinct* Sylow $p$-subgroups of $G$ with $S\cap T\neq 1$, there exist Sylow $p$-subgroups $S_0, S_1, \cdots, S_n$ of $G$, elements $X_1, \cdots, X_n$ of $\mathfrak{F}$, and an element $x_i\in X_i$ for each $i$ satisfying the following conditions:

(1) $S_0=S$ and $S_n=T$;

(2) $X_i\in\mathfrak{F}(S_{i-1})\cap\mathfrak{F}(S_i)$ for each $i$;

(3) $S_i^{x_i}=S_{i-1}$ for each $i$;

(4) $S\cap T\leq S_i\cap X_i$ for each $i$.

When $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$ are as above, we say that $S$ is conjugate to $T$ via $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$, or via $\mathfrak{F}$, even if $S=T$ or $S\cap T=1$.

1.2 DEFINITION. The normal set $\mathfrak{F}$ of subgroups of $G$ is said to *control $p$-fusion* in $G$ if $\mathfrak{F}$ satisfies the following condition: whenever $A$ is a subset $\neq 1$ of $S\in\mathrm{Syl}_p(G)$ and $g$ is an element of $G$ with $A^g\leq S$, there exist elements $Y_1, \cdots, Y_n$ of $\mathfrak{F}(S)$, an element $y_i\in Y_i$ for each $i$, and an element $y\in\mathrm{N}_G(S)$ such that

(1) $A^g=A^{y_1\cdots y_n y}$, and

(2) $A^{y_1\cdots y_i}\leq S\cap Y_i$ for each $i$.

If we replace the condition (1) above by the stronger condition

(1′) $g=y_1\cdots y_n y$,

we obtain the definition of *strong control of $p$-fusion* in $G$.

1.3 LEMMA. *Let $S$, $T\in\mathrm{Syl}_p(G)$ and assume that $S$ is conjugate to $T$ via $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$. Then there exist elements $Y_1, Y_2, \cdots, Y_n$ of $\mathfrak{F}(S)$ and an element $y_i\in Y_i$ for each $i$ such that $y_1y_2\cdots y_i=x_i\cdots x_2x_1$ and $(S\cap T)^{y_1y_2\cdots y_i}\leq S\cap Y_i$ for each $i$.*

PROOF. Define $y_i=x_i^{x_{i-1}\cdots x_2x_1}$ and $Y_i=X_i^{x_{i-1}\cdots x_2x_1}$. It readily follows by induction on $i$ that $y_1y_2\cdots y_i=x_i\cdots x_2x_1$. As $x_i\in X_i\in\mathfrak{F}(S_{i-1})$ and $S_{i-1}^{x_{i-1}\cdots x_2x_1}=S$, $y_i\in Y_i\in\mathfrak{F}(S)$. We may deduce as follows:

$$\begin{aligned}(S\cap T)^{y_1y_2\cdots y_i}&=(S\cap T)^{x_i\cdots x_2x_1}\\ &\leq S_i^{x_i\cdots x_2x_1}\cap X_i^{x_{i-1}\cdots x_2x_1}\\ &=S\cap Y_i\,.\end{aligned}$$

The proof is complete.

1.4 THEOREM. *The normal set $\mathfrak{F}$ of subgroups of $G$ controls Sylow $p$-intersections in $G$ if and only if $\mathfrak{F}$ strongly controls $p$-fusion in $G$.*

PROOF. Assume that $\mathcal{F}$ controls Sylow $p$-intersections in $G$. Suppose $S\in \mathrm{Syl}_p(G)$, $1\neq A\leq S$, $g\in G$, and $A^g\leq S$. Let $T=S^{g^{-1}}$. Then $A\leq S\cap T$ and so $S\cap T\neq 1$. If $S=T$, then $g\in \mathrm{N}_G(S)$. Assume $S\neq T$. Then $S$ is conjugate to $T$ via, say, $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$. Choose $Y_i$ and $y_i$ as in 1.3. Then $A^{y_1y_2\cdots y_i}\leq (S\cap T)^{y_1y_2\cdots y_i}\leq S\cap Y_i$ and $S=T^{x_n\cdots x_2x_1}=S^{g^{-1}y_1y_2\cdots y_n}$, and so $g\in y_1y_2\cdots y_n\mathrm{N}_G(S)$. This shows that $\mathcal{F}$ strongly controls $p$-fusion in $G$.

Assume that $\mathcal{F}$ strongly controls $p$-fusion in $G$. Suppose $S, T\in \mathrm{Syl}_p(G)$, $S\neq T$, and $S\cap T\neq 1$. Let $A=S\cap T$ and choose $g\in G$ so that $T^g=S$. Then $A^g\leq S$ and so there exist elements $Y_1, \cdots, Y_n$ of $\mathcal{F}(S)$, an element $y_i\in Y_i$ for each $i$, and an element $y\in \mathrm{N}_G(S)$ such that $g=y_1\cdots y_ny$ and $A^{y_1\cdots y_i}\leq S\cap Y_i$ for each $i$. Let $z_i=(y_1\cdots y_i)^{-1}$, $S_i=S^{z_i}$, $X_i=Y_i^{z_i}$, and $x_i=y_i^{z_i}$. Then $S$ is conjugate to $T$ via $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$. Therefore, $\mathcal{F}$ controls Sylow $p$-intersections in $G$.

1.5 THEOREM. *If the normal set $\mathcal{F}$ of subgroups of $G$ controls Sylow $p$-intersections in $G$ and if $G$ is not $p$-isolated, then $G=\langle\mathcal{F}(S)\rangle \mathrm{N}_G(S)$ for each* $S\in \mathrm{Syl}_p(G)$.

PROOF. Let $g\in G-\mathrm{N}_G(S)$ and set $T=S^{g^{-1}}$. As $G$ is not $p$-isolated, $S$ is joined to $T$ by a chain of Sylow $p$-subgroups of $G$ containing $S\cap T$ such that the adjacent Sylow $p$-subgroups are distinct and intersect nontrivially. Therefore, $S$ is conjugate to $T$ via, say, $\{S_i\}$, $\{X_i\}$, and $\{x_i\}$. As in the proof of 1.4, we have $g\in y_1y_2\cdots y_n\mathrm{N}_G(S)$, where $y_i\in Y_i\in\mathcal{F}(S)$ for each $i$. Therefore, $g\in \langle\mathcal{F}(S)\rangle\mathrm{N}_G(S)$.

1.6 DEFINITION. Let $\mathcal{H}_0=\mathcal{H}_{0,p,G}$ be the set of all nonidentity $p$-subgroups $H$ of $G$ such that $\mathrm{N}_G(H)/H$ is $p$-isolated. For each $H\in\mathcal{H}_0$, let $\mathrm{N}_G^*(H)$ be the subgroup of $\mathrm{N}_G(H)$ containing $H$ such that $\mathrm{N}_G^*(H)/H$ is the unique minimal subnormal subgroup of $\mathrm{N}_G(H)/H$ of order divisible by $p$. That $\mathrm{N}_G^*(H)$ exists follows from the following fact (see [**16**, §1]): if $X$ is a $p$-isolated group of order divisible by $p$, then any normal subgroup of $X$ of order divisible by $p$ is also $p$-isolated, and the intersection of any two normal subgroups of $X$ of order divisible by $p$ is also of order divisible by $p$.

1.7 THEOREM. *If for each $H\in\mathcal{H}_{0,p,G}$, $\mathrm{N}_G^*(H)$ is contained in some member of the normal set $\mathcal{F}$ of subgroups of $G$, then $\mathcal{F}$ controls Sylow $p$-intersections in $G$.*

PROOF. Suppose the theorem is false, and choose $S, T\in\mathrm{Syl}_p(G)$ so that

(1) $S\neq T$ and $H=S\cap T\neq 1$,

(2) $S$ is not conjugate to $T$ via $\mathcal{F}$, and

(3) $|H|$ is maximal subject to (1) and (2).

Choose $Q, R\in\mathrm{Syl}_p(G)$ so that $\mathrm{N}_S(H)\leq\mathrm{N}_Q(H)\in\mathrm{Syl}_p(\mathrm{N}_G(H))$ and $\mathrm{N}_T(H)\leq\mathrm{N}_R(H)\in\mathrm{Syl}_p(\mathrm{N}_G(H))$. Then $H<S\cap Q$ and $H<R\cap T$. If $Q=R$, then $S\neq Q\neq T$ and $S$ is conjugate to $T$ via $\mathcal{F}$ by (3), a contradiction. Therefore, $Q\neq R$, and $Q$ is not conjugate to $R$ via $\mathcal{F}$ by (2). So $Q\cap R=H$ by (3), and replacing $S$, $T$ by $Q$, $R$, we may assume that $\mathrm{N}_S(H)$ and $\mathrm{N}_T(H)$ are Sylow $p$-subgroups of $\mathrm{N}_G(H)$. It

then follows from (3) that $N_G(H)/H$ is $p$-isolated, and so $N=N_G^*(H)$ is contained in some member $X$ of $\mathcal{F}$. We may choose $U, V\in \mathrm{Syl}_p(G)$ so that $S\cap N\leq U\cap X\in \mathrm{Syl}_p(X)$, $T\cap N\leq V\cap X\in \mathrm{Syl}_p(X)$, and $V^x=U$ for some $x\in X$. As $H<S\cap U$ and $H<V\cap T$, $S$ is conjugate to $T$ via $\mathcal{F}$ by (3). This is a contradiction proving 1.7.

1.8 LEMMA. *Let $S\in \mathrm{Syl}_p(G)$ and let $G_i$ $(i=1, 2)$ be subgroups of $G$ containing $S$ with $G=G_1G_2$. Then for each $g\in G$ there exists $U\in \mathrm{Syl}_p(G)$ such that $S\cap S^g\leq U\leq G_1\cap G_2{}^g$.*

PROOF. Let $g=g_2g_1$ with $g_i\in G_i$ $(i=1, 2)$. Then $S^{g_1}\leq G_1\cap G_2{}^{g_1}=G_1\cap G_2{}^g$ and $S\cap S^g\leq G_1\cap G_2{}^g$. Therefore, the assertion follows from Sylow's theorem.

1.9 LEMMA. *Suppose $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{D}$ are normal sets of subgroups of $G$ and the index of each member of $\mathcal{E}\cup\mathcal{F}$ in $G$ is not divisible by $p$. Assume that for each $S\in \mathrm{Syl}_p(G)$ and each $E\in\mathcal{E}(S)$ there exist $F_1, F_2\in\mathcal{F}(S)$ such that $E=(F_1\cap E)(F_2\cap E)$. Then if $\mathcal{E}\cup\mathcal{D}$ controls Sylow $p$-intersections in $G$, so does $\mathcal{F}\cup\mathcal{D}$.*

PROOF. Suppose $S, T\in \mathrm{Syl}_p(G)$ and $E\in\mathcal{E}(S)\cap\mathcal{E}(T)$. Choose $F_1, F_2\in\mathcal{F}(S)$ so that $E=(F_1\cap E)(F_2\cap E)$. As $S, T\in \mathrm{Syl}_p(E)$, there is an element $g\in E$ such that $T=S^g$. As $S\leq(F_1\cap E)\cap(F_2\cap E)$, there exists $U\in \mathrm{Syl}_p(G)$ such that $S\cap T\leq U\leq (F_1\cap E)\cap(F_2\cap E)^g$ by 1.8. As $\langle S, U\rangle\leq F_1$ and $\langle U, T\rangle\leq F_2{}^g$, $S$ is conjugate to $T$ via $\mathcal{F}$. This proves 1.9.

1.10 DEFINITION. Let $f$ be a mapping which associates with each $p$-subgroup $P$ of $G$ a set $f(P)$ of subgroups of $P$ such that

(1) $N_G(P)\leq N_G(F)$ for each $F\in f(P)$, and

(2) $f(P)^g=f(P^g)$ for each $g\in G$.

For each subgroup $M$ of $G$ of order divisible by $p$, define

$$\mathcal{F}_M=\{N_M(F)\,;\ F\in f(P),\ P\in \mathrm{Syl}_p(M)\}.$$

Then $\mathcal{F}_M$ becomes a normal set of subgroups of $M$, and if $P\in \mathrm{Syl}_p(M)$, then $\mathcal{F}_M(P)=\{N_M(F)\,;\ F\in f(P)\}$. Let $\mathcal{F}=\mathcal{F}_G$. We say that $M$ is *$\mathcal{F}$-regular* if $\mathcal{F}_M$ controls Sylow $p$-intersections in $M$. If $M$ is not $\mathcal{F}$-regular, we say that $M$ is *$\mathcal{F}$-singular*. Let $\mathcal{F}'=\mathcal{F}'_G$ be the set of all $\mathcal{F}$-singular maximal $p$-local subgroups of $G$. Notice that $\mathcal{F}'$ is also a normal set of subgroups of $G$.

1.11 THEOREM. *Let the notation be as in* 1.10 *and assume that for each nonidentity $p$-subgroup $P$ of $G$, $f(P)$ consists of nonidentity subgroups of $P$. Then $\mathcal{F}\cup\mathcal{F}'$ controls Sylow $p$-intersections in $G$.*

PROOF. Let $\mathcal{D}=\mathcal{F}\cup\mathcal{F}'$ and assume that $\mathcal{D}$ does not control Sylow $p$-intersections in $G$. Choose $S, T\in \mathrm{Syl}_p(G)$ so that

(1) $S\neq T$ and $H=S\cap T\neq 1$,

(2) $S$ is not conjugate to $T$ via $\mathcal{D}$, and

(3) $|H|$ is maximal subject to (1) and (2).

For each pair $S, T$ as above, we may choose a maximal $p$-local subgroup $M$ of $G$

so that

(4) $H<S\cap M$ and $H<T\cap M$.

(Any maximal $p$-local subgroup containing $N_G(H)$ satisfies this condition.) Let $\mathfrak{T}$ be the set of triples $(S, T, M)$ satisfying (1)-(4), and choose $(S, T, M)\in\mathfrak{T}$ so that $|S\cap M||T\cap M|$ is maximal. Choose $U, V\in\mathrm{Syl}_p(G)$ so that $S\cap M\leq U\cap M\in\mathrm{Syl}_p(M)$, $T\cap M\leq V\cap M\in\mathrm{Syl}_p(M)$, and $U=V^m$ for some $m\in M$. Then $U\cap V=H$ and $U$ is not conjugate to $V$ via $\mathfrak{D}$ by (2) and (3). So $S\cap M$ and $T\cap M$ are Sylow $p$-subgroups of $M$ by the maximality of $|S\cap M||T\cap M|$. Replacing $S, T$ by $U, V$, we may assume that $S=T^m$ for some $m\in M$. If $M$ is $\mathfrak{F}$-singular, then $M\in\mathfrak{F}'\leq\mathfrak{D}$ and $S$ is conjugate to $T$ via $\mathfrak{D}$, contrary to (2). So $M$ is $\mathfrak{F}$-regular, and there exist Sylow $p$-subgroups $U_0, U_1, \cdots, U_n$ of $M$ containing $H$ with $U_0=S\cap M$, $U_n=T\cap M$, and $U_{i-1}\neq U_i$ for each $i$, and there exist elements $X_1, \cdots, X_n$ of $\mathfrak{F}_M$ with $\langle U_{i-1}, U_i\rangle\leq X_i$ for each $i$. Let $U_i\leq S_i\in\mathrm{Syl}_p(G)$ for each $i$ with $S_0=S$ and $S_n=T$. Then, for some $i$, $S_{i-1}$ is not conjugate to $S_i$ via $\mathfrak{D}$ by (2), and so $S_{i-1}\cap S_i=H$ by (3). Replacing $S, T$ by $S_{i-1}, S_i$, we may assume that $\langle S\cap M, T\cap M\rangle$ is contained in some member $X$ of $\mathfrak{F}_M$. As $X=N_M(F)$ for some $F\in f(S\cap M)\cap f(T\cap M)$, $S\neq S\cap M$ and $T\neq T\cap M$ by (2). But then $S\cap M<N_S(S\cap M)\leq N_G(F)$ and $T\cap M<N_T(T\cap M)\leq N_G(F)$, so if $N$ is a maximal $p$-local subgroup of $G$ containing $N_G(F)$, $(S, T, N)\in\mathfrak{T}$ and $|S\cap M||T\cap M|<|S\cap N|\cdot|T\cap N|$. This is a contradiction completing the proof.

1.12 Theorem. *Let $S\in\mathrm{Syl}_p(G)$ and let $G_i$ $(i=1, 2)$ be subgroups of $G$ containing $S$ with $G=G_1G_2$. If $G_i$ $(i=1, 2)$ is $\mathfrak{F}$-regular, where $\mathfrak{F}$ is as in* 1.10, *then so is $G$.*

Proof. Applying 1.9 to the sets $\{G\}$, $\mathfrak{D}=\{G_1{}^G\}\cup\{G_2{}^G\}$, and $\emptyset$, we have that $\mathfrak{D}$ controls Sylow $p$-intersections in $G$. As each member of $\mathfrak{D}$ is $\mathfrak{F}$-regular, so is $G$.

1.13 Corollary. *Let $S\in\mathrm{Syl}_p(G)$ and assume $G=N_G(F_1)N_G(F_2)$ for some pair $F_1, F_2$ of elements of $f(S)$, where $f$ is as in* 1.10. *Then $G$ is $\mathfrak{F}$-regular.*

## 2. Control of Sylow 2-intersections in groups of Chev(2) type and groups of alternating type.

In this section, we consider the following situation.

2.1 Hypothesis. *$G$ is a finite group, $N$ is a normal subgroup of $G$, $G/N$ is a 2-group, and $N$ is a central product of quasisimple groups $L=L_1, L_2, \cdots, L_k$, which are all conjugate in $G$.*

Let Chev(2) denote the collection of all quasisimple groups $L$ with $O_2(L)=1$ such that $L/Z(L)$ is isomorphic to a simple group of Lie type and of characteristic 2. Here we consider $A_6\cong Sp_4(2)'$, $SU_3(3)\cong G_2(2)'$, and ${}^2F_4(2)'$ to be of Lie type and of characteristic 2. Thus the 3-fold covering group $\hat{A}_6$ of $A_6$ is a member of

Chev(2). For $L\in\mathrm{Chev}(2)$, a *Borel subgroup* of $L$ is a Sylow 2-normalizer of $L$, and a *parabolic subgroup* of $L$ is a subgroup containing a Borel subgroup. Borel subgroups and $L$ itself are called the *trivial* parabolic subgroups.

We list the main results of [**13**].

2.2. *Under Hypothesis* 2.1 *with* $L\in\mathrm{Chev}(2)$, *if* $H\in\mathcal{H}_{0,2,G}$, *then there exists a proper subgroup* $M$ *of* $G$ *containing* $\mathrm{N}_G^*(H)$ *such that* $|G:M|$ *is odd and* $\mathrm{O}^2(\mathrm{C}_M(\mathrm{O}_2(M)))=\mathrm{Z}(N)$, *except when one of the following holds:*

(1) $L\cong\mathrm{SL}_2(2^m)$, $\mathrm{(P)SU}_3(2^m)$, *or* $\mathrm{Sz}(2^{2m-1})$, $m\geqq 2$;

(2) $L\cong\mathrm{(P)SL}_3(2^m)$, $\mathrm{Sp}_4(2^m)'$, *or* $\hat{\mathrm{A}}_6$, *and if* $S\in\mathrm{Syl}_2(G)$, *then* $\mathrm{N}_S(L)$ *contains an element which interchanges the two nontrivial parabolic subgroups of* $L$ *containing* $S\cap L$.

2.3. *Under Hypothesis* 2.1 *with* $L\cong\mathrm{A}_n$, $n\geqq 7$, *if* $H\in\mathcal{H}_{0,2,G}$, *then the following holds:*

(1) *if* $n\neq 2^m+1$ *for any integer* $m$, *then there exists a proper subgroup* $M$ *of* $G$ *containing* $\mathrm{N}_G^*(H)$ *such that* $|G:M|$ *is odd;*

(2) *if* $n$ *is even, then there exists a proper subgroup* $M$ *of* $G$ *containing* $\mathrm{N}_G^*(H)$ *such that* $|G:M|$ *is odd and* $\mathrm{C}_M(\mathrm{O}_2(M))\leq\mathrm{O}_2(M)$;

(3) *if* $n\equiv 3 \pmod 4$, *then there exists a proper subgroup* $M$ *of* $G$ *containing* $\mathrm{N}_G^*(H)$ *such that* $|G:M|$ *is odd and* $\mathrm{O}^2(\mathrm{C}_M(\mathrm{O}_2(M)))=1$ *or* $\langle x_1, x_2, \cdots, x_k\rangle$, *where* $x_i$ *is a* 3-*cycle in* $L_i\cong\mathrm{A}_n$ *for each* $i$.

2.4. *If* $G=\Sigma_n$, $n$ *odd, and* $H\in\mathcal{H}_{0,2,G}$, *then either there exists a subgroup* $M$ *of* $G$ *containing* $\mathrm{N}_G(H)$ *such that* $M\cong\Sigma_{n-1}\times\Sigma_1$ *or* $\mathrm{N}_G(H)\cong\Sigma_3\times S$, *where* $S$ *is a Sylow* 2-*subgroup of* $\Sigma_{n-3}$. *Here the symbols* $\cong$ *denote the isomorphism of permutation groups.*

### 3. GF(2)-representations of finite groups.

Throughout this section, let $G$ be a finite group and $V$ be a faithful $\mathrm{GF}(2)G$-module. Define $\mathcal{O}=\mathcal{O}(G, V)$ to be the set of all nonidentity elementary abelian 2-subgroups $A$ of $G$ satisfying $|A|\geqq|V:\mathrm{C}_V(A)|$, and define $\mathcal{P}=\mathcal{P}(G, V)$ to be the set of all nonidentity elementary abelian 2-subgroups $A$ of $G$ satisfying $|A||\mathrm{C}_V(A)|\geqq|B||\mathrm{C}_V(B)|$ for each subgroup $B$ of $A$. Let $\mathcal{P}^*=\mathcal{P}^*(G, V)$ be the set of all minimal elements of $\mathcal{P}$ under the partial order $\leq_{(V)}$ defined by: $A\leq_{(V)}B$ if and only if $A\leq B$ and $|A||\mathrm{C}_V(A)|=|B||\mathrm{C}_V(B)|$. Let $\mathcal{P}_0^*=\mathcal{P}_0^*(G, V)$ (resp. $\mathcal{P}_0=\mathcal{P}_0(G, V)$) be the set of all elements of $\mathcal{P}^*$ (resp. $\mathcal{P}$) contained in $\mathrm{O}_{2',2}(G)$, and let $\mathcal{P}_1^*=\mathcal{P}_1^*(G, V)=\mathcal{P}^*-\mathcal{P}_0^*$. Let $G_i^*=\langle\mathcal{P}_i^*\rangle$ for $i\in\{0, 1\}$ and $G_0=\langle\mathcal{P}_0\rangle$. When $X$ is a group and $W$ is a $\mathrm{GF}(2)X$-module, let $W(X)=[W, X]/C_{[W,X]}(X)$.

One of the objects of the theory of GF(2)-representations is to determine the structure of $\langle\mathcal{P}\rangle$ and its action on $V$. Here we list those theorems on GF(2)-representations which are needed in this paper. Although most of them are essentially proved by Aschbacher [**2**, **3**, **4**, **5**], we will give their complete proofs in [**14**].

3.1. *Suppose* $\mathcal{P}_0^* \neq \emptyset$ *and* $O_2(G_0^*)=1$. *Let* $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_n$ *be the* $G_0^*$*-orbits on* $\mathcal{P}_0^*$, *and set* $N_i = \langle \mathcal{O}_i \rangle$ *and* $V_i = [V, N_i]$ *for each* $i$. *Then the following holds:*

(1) $N_i \cong \mathrm{SL}_2(2)$ *for each* $i$ *and* $G_0^* = N_1 \times N_2 \times \cdots \times N_n$;

(2) $(V_i)_{N_i}$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2)$*-module of dimension* 2 *for each* $i$ *and* $[V, G_0^*] = V_1 \oplus V_2 \oplus \cdots \oplus V_n$;

(3) $V = [V, G_0^*] \oplus C_V(G_0^*)$.

3.2. *If* $O_2(G_0^*)=1$, *then* $G_0 = G_0^*$.

3.3. *If* $\mathcal{P}_1^* \neq \emptyset$ *and* $O_2(G_1^*)=1$, *then* $\mathrm{E}(G_1^*) \neq 1$ *and* $C_{G_1^*}(\mathrm{E}(G_1^*)) = O(G_1^*) = Z(G_1^*)$. *Here* $\mathrm{E}(G_1^*)$ *is the maximal semisimple normal subgroup of* $G_1^*$.

Let $Q$ be the collection of all quadruples $(X, W, A, K)$, where $X$ is a finite group, $W$ is a faithful $\mathrm{GF}(2)X$-module, $A \in \mathcal{O}(X, W)$, and $K$ is a quasisimple normal subgroup of $X$ such that $O_2(K)=1$, $C_X(K)=Z(K)$, and $X=KA$.

3.4. *If* $L$ *is a quasisimple component of* $G$ *with* $O_2(L)=1$ *and* $\langle \mathcal{P} \rangle \not\leq C_G(L)$, *then there exists* $(X, W, A, K) \in Q$ *such that* $K$ *is a homomorphic image of* $L$.

By definition, the *natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$*-module* is the set of all two dimensional row vectors with coefficients in $\mathrm{GF}(2^m)$ considered a $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$-module, and the *natural* $\mathrm{GF}(2)\Sigma_n$*-module* or $\mathrm{GF}(2)\mathrm{A}_n$*-module* is the unique nontrivial composition factor of the natural permutation module for $\Sigma_n$ or $\mathrm{A}_n$ over $\mathrm{GF}(2)$.

3.5. *If* $G \cong \mathrm{SL}_2(2^m)$ *and* $A \in \mathcal{O}$, *then* $V/C_V(G)$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$*-module and* $|A| = |V : C_V(A)| = 2^m$.

3.6. *Suppose* $S \in \mathrm{Syl}_2(G)$, $L$ *is a quasisimple component of* $G$, $[L, \langle \mathcal{P}^*(S, V) \rangle] \neq 1$, $[C_V(S), L] \neq 0$, *and* $L \cong (\mathrm{P})\mathrm{SL}_3(2^m)$, $\mathrm{Sp}_4(2^m)'$, *or* $\hat{\mathrm{A}}_6$. *Then* $N_S(L)$ *normalizes the two nontrivial parabolic subgroups of* $L$ *containing* $S \cap L$.

3.7. *Suppose* $\mathcal{P} \neq \emptyset$, $S \in \mathrm{Syl}_2(G)$, $L$ *is a subnormal subgroup of* $G$, *and* $L \cong \mathrm{SL}_2(2^m)$, $\mathrm{A}_{2m-1}$, $m \geqq 2$, *or* $L \cong \hat{\mathrm{A}}_7$, *the* 3*-fold covering group of* $\mathrm{A}_7$. *When* $L \cong \mathrm{A}_3$, *assume* $|[V, L]| = 4$. *Then the following holds:*

(1) *if* $[L, \langle \mathcal{P} \rangle] \neq 1$, *then* $L \not\cong \hat{\mathrm{A}}_7$ *and* $C_S(C_{[V,L]}(\langle \mathcal{P}(S, V) \rangle)) \leq N_S(L)$;

(2) *if* $[L, \langle \mathcal{P}^* \rangle] \neq 1$, *then* $V(L)$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$*-module or by the natural* $\mathrm{GF}(2)\mathrm{A}_{2m-1}$*-module, or else* $L \cong \mathrm{A}_7$ *and* $|[V, L]| = 16$;

(3) *if* $L \cong \mathrm{SL}_2(2^m)$, $C_G(L)=1$, *and* $V(L)$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$*-module, then* $C_S(C_{[V,L]}(\langle \mathcal{P}(S, V) \rangle)) = S \cap L$;

(4) *if* $L \cong \mathrm{A}_{2m-1}$, $m \geqq 3$, $C_G(L)=1$, *and* $V(L)$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{A}_{2m-1}$*-module, then* $G \cong \Sigma_{2m-1}$ *and* $C_S(C_{[V,L]}(\langle \mathcal{P}(S, V) \rangle))$ *is generated by all transpositions in* $S$.

## 4. Control of Sylow 2-intersections by a characteristic pair.

4.1 DEFINITION. For a 2-group $S$, let $\mathcal{A}(S)$ be the set of all elementary abelian subgroups of $S$ of maximal order. Set $J(S)=\langle\mathcal{A}(S)\rangle$, the *Thompson subgroup* of $S$, and $K(S)=C_S(\Omega_1(Z(J(S))))$. ($K(S)$ is sometimes denoted by $\tilde{J}(S)$ and is called the *Baumann subgroup of* $S$ after Baumann [6]). For any finite group $G$ with $S\in \mathrm{Syl}_2(G)$, let $J(G)=\langle J(S)^G\rangle$ and $K(G)=\langle K(S)^G\rangle$. Let $\mathcal{Q}(S)$ be the collection of all finite groups $G$ satisfying the following set of conditions:

(1) $S\in \mathrm{Syl}_2(G)$;

(2) $C_G(O_2(G))\leq O_2(G)$;

(3) $\bar{G}=G/C_G(\Omega_1(Z(O_2(G))))$ is isomorphic to $SL_2(2^m)$ for some $m$;

(4) when $V=\Omega_1(Z(O_2(G)))$ is regarded as a $GF(2)\bar{G}$-module, $V(\bar{G})$ is induced by the natural $GF(2)SL_2(2^m)$-module;

(5) $O_2(G)\in \mathrm{Syl}_2(C_G(V))$;

(6) $[O_2(G), O^2(G)]\not\leq V$;

(7) $S$ is contained in a unique maximal subgroup of $G$;

(8) $G=K(G)$.

A *characteristic pair* for the 2-group $S$ is a pair $S_1$, $S_2$ of characteristic subgroups of $S$ such that whenever $G\in\mathcal{Q}(S)$, either $S_1\lhd G$ or $S_2\lhd G$. The characteristic pair is said to be *nontrivial* if $S_1\neq 1\neq S_2$. A work of N. R. Campbell shows that for each nonidentity 2-group $S$ there exists a nontrivial characteristic pair $S_1$, $S_2$ satisfying $S_1\leq\Omega_1(Z(S))$ [7]. We say that such a pair is of *Glauberman-Niles type* after [9].(2)

Now fix a characteristic pair $T_1$, $T_2$ for each 2-group $T$ satisfying $T=K(T)$, and for an arbitrary 2-group $S$ define $C_i(S)=(K(S))_i$ for $i\in\{1, 2\}$.(3) For each group $G$ of even order, let

$$\mathcal{C}_G=\{N_G(C_1(S)), N_G(C_2(S)), N_G(\Omega_1(Z(S)));\ S\in \mathrm{Syl}_2(G)\}.$$

Then $\mathcal{C}_G$ is a normal set of subgroups of $G$, and we may use the terminology of the first section (especially 1.10) for $\mathcal{C}_G$.

4.2 THEOREM. *Suppose the* $C_i(S)$ *are defined as above by the fixed characteristic pairs* $T_1$, $T_2$ *of Glauberman-Niles type for all 2-groups* $T$ *satisfying* $T=K(T)$. *For each group* $G$ *of even order, let* $\mathcal{D}_G=\{C_G(C_1(S)\cap\Omega_1(Z(S))), N_G(C_2(S));\ S\in \mathrm{Syl}_2(G)\}$. *Then* $\mathcal{D}_G\cup\mathcal{C}'_G$ *controls Sylow 2-intersections in* $G$.

(2) In the paper mentioned in Footnote (1), I have defined a nonidentity characteristic subgroup $Q(S)$ for each nonidentity 2-group $S$ and shown that $\Omega_1(Z(S))$ and $Q(S)$ form a characteristic pair of Glauberman-Niles type for $S$.

(3) Of course, we identify isomorphic 2-groups. More precisely, if $\alpha$ is an isomorphism of a 2-group $S$ onto a 2-group $R$, then we define $C_i(R)=C_i(S)^{\alpha}$. Hence, if $G$ is a group of even order, the mapping $f$ which associates with each 2-subgroup $P$ of $G$ the set $\{C_1(P), C_2(P), \Omega_1(Z(P))\}$ satisfies the conditions (1) and (2) in 1.10.

PROOF. Let $S\in\mathrm{Syl}_2(G)$, $T=\mathrm{C}_S(\mathrm{C}_1(S))$, and $N=\mathrm{N}_G(\mathrm{C}_1(S))$. Then $T\in\mathrm{Syl}_2(\mathrm{C}_G(\mathrm{C}_1(S)))$ and $\mathrm{K}(T)=\mathrm{K}(S)$ as $\mathrm{C}_1(S)\le\mathrm{Z}(\mathrm{K}(S))$. Therefore,

$$N=\mathrm{C}_G(\mathrm{C}_1(S))\mathrm{N}_G(T)=\mathrm{C}_N(\mathrm{C}_1(S)\cap\Omega_1(\mathrm{Z}(S)))\mathrm{N}_N(\mathrm{C}_2(S)).$$

Similarly, if $M=\mathrm{N}_G(\Omega_1(\mathrm{Z}(S)))$, then

$$M=\mathrm{C}_M(\mathrm{C}_1(S)\cap\Omega_1(\mathrm{Z}(S)))\mathrm{N}_M(\mathrm{C}_2(S)).$$

So $\mathcal{D}_G\cup\mathcal{C}'_G$ controls Sylow 2-intersections in $G$ by 1.9 and 1.11.

4.3 HYPOTHESIS. *$G$ is a group of even order satisfying $\mathrm{C}_G(\mathrm{O}_2(G))\le\mathrm{O}_2(G)$. If $K$ is a quasisimple section of $G$ with $(X, W, A, K)\in\mathcal{Q}$ for some $X$, $W$, and $A$, then $K\in\mathrm{Chev}(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1}); m\geqq 2\}$ or $K\cong\mathrm{A}_n$, $n\geqq 7$.*

By a theorem of Aschbacher [4], the group $G$ of even order with $\mathrm{C}_G(\mathrm{O}_2(G))\le\mathrm{O}_2(G)$ satisfies Hypothesis 4.3 if each simple section of $G$ is of known type.

4.4 LEMMA. *Under Hypothesis 4.3, if $H$ is a section of $G$, $U$ is a faithful $\mathrm{GF}(2)H$-module, and $L$ is a quasisimple component of $H$ such that $\mathrm{O}_2(L)=1$ and $[L, \langle\mathcal{P}(H, U)\rangle]\ne 1$, then $L\in\mathrm{Chev}(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1}); m\geqq 2\}$ or $L\cong\mathrm{A}_n$, $n\geqq 7$.*

PROOF. There exists an element $(X, W, A, K)\in\mathcal{Q}$ such that $K$ is a homomorphic image of $L$ by 3.4. The assertion, therefore, follows from 4.3 as $L\not\cong\hat{\mathrm{A}}_7$ by 3.7.

4.5 THEOREM. *Under Hypothesis 4.3, if $G$ is $\mathcal{C}$-singular, then for each $S\in\mathrm{Syl}_2(G)$ there exists a subgroup $X$ of $G$ satisfying the following conditions:*

(1) $X=[X, \mathrm{J}(S)]$;

(2) $X=\mathrm{O}^2(X)$;

(3) $[\mathrm{O}_2(G), X]\le V=\Omega_1(\mathrm{Z}(\mathrm{O}_2(G)))$;

(4) $\bar{X}=X/\mathrm{C}_X(V(X))$ *is isomorphic to* $\mathrm{A}_3$ *or* $\mathrm{SL}_2(2^m)$, $m\geqq 2$;

(5) *when regarded as a* $\mathrm{GF}(2)\bar{X}$*-module,* $V(X)$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{A}_3$*-module or by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$*-module.*

PROOF. We call a group $X$ as above a *$\mathcal{C}$-singular subgroup* of $G$ with respect to $S$. Until 4.5 is proved, let $G$ be a minimal counterexample to 4.5. Furthermore, let $S\in\mathrm{Syl}_2(G)$, $Q=\mathrm{O}_2(G)$, $V=\Omega_1(\mathrm{Z}(Q))$, $C=\mathrm{C}_G(V)$, and $\bar{G}=G/C$. We show in a series of reductions, (a)$\sim$(l), that $G\in\mathcal{G}(S)$ and $S=\mathrm{K}(S)$. It would then follow that $\mathrm{C}_1(S)$ or $\mathrm{C}_2(S)$ is normal in $G$, which is a contradiction as $G$ is $\mathcal{C}$-singular.

(a) *If $H$ is a proper subgroup of $G$ containing $\mathrm{J}(S)Q$, then $H$ is $\mathcal{C}$-regular.*

PROOF. Suppose $H$ is $\mathcal{C}$-singular, and let $\mathrm{J}(S)Q\le T\in\mathrm{Syl}_2(H)$. Then $H$ has a $\mathcal{C}$-singular subgroup $X$ with respect to $T$ by the minimality of $G$. Let $W=\Omega_1(\mathrm{Z}(\mathrm{O}_2(H)))$. Then $[V, X]\le[Q, X]\le[\mathrm{O}_2(H), X]\le W\le V$ and so, as $X=\mathrm{O}^2(X)$, $[V, X]=[Q, X]=[W, X]$. As $\mathrm{J}(T)=\mathrm{J}(S)$, $X$ is a $\mathcal{C}$-singular subgroup of $G$ with respect to $S$.

(b) *If $H$ and $K$ are subgroups of $G$ containing $S$ such that $G=HK$, then $H=G$ or $K=G$.*

PROOF. Suppose $H\neq G\neq K$. As $H$ and $K$ are $\mathcal{C}$-regular by (a), so is $G$ by 1.12, a contradiction.

(c) *The following holds:*

(1) *if $M$ is a normal subgroup of $G$ with $MS\neq G$, then $M$ is 2-closed;*

(2) $Q\in \mathrm{Syl}_2(C)$;

(3) $\mathrm{O}_2(\bar{G})=1$;

(4) *any maximal subgroup of $G$ containing $S$ also contains $C$.*

PROOF. Let $T=S\cap M$. Then $G=MS\mathrm{N}_G(T)$. As $MS\neq G$, $\mathrm{N}_G(T)=G$ by (b) and so $M$ is 2-closed. As $CS\leq \mathrm{N}_G(\Omega_1(\mathrm{Z}(S)))$ and $G$ is $\mathcal{C}$-singular, $CS\neq G$. So $C$ is 2-closed by (1), and $Q\in\mathrm{Syl}_2(C)$. Let $N/C=\mathrm{O}_2(G/C)$. Then $NS=CS\neq G$, so $N$ is 2-closed by (1), and $Q\in \mathrm{Syl}_2(N)$, proving (3). Suppose $H$ is a maximal subgroup of $G$ containing $S$ and $C\not\leq H$. Then $G=CSH$ and $CS\neq G\neq H$, contrary to (b).

(d) $\mathrm{J}(S)\not\leq C$.

PROOF. Suppose $\mathrm{J}(S)\leq C$. Then $\mathrm{J}(S)\leq \mathrm{C}_S(V)=Q$ by (c.2), $\mathrm{J}(S)=\mathrm{J}(Q)$, and $V\leq\Omega_1(\mathrm{Z}(\mathrm{J}(Q)))=\Omega_1(\mathrm{Z}(\mathrm{J}(S)))$. By the definition of $\mathrm{K}(S)$, $\mathrm{K}(S)\leq \mathrm{C}_S(V)=Q$. But then $\mathrm{K}(S)=\mathrm{K}(Q)\lhd G$ and $\mathrm{C}_i(S)\lhd G$ for $i\in\{1, 2\}$. This is a contradiction as $G$ is $\mathcal{C}$-singular.

(e) *$\mathcal{P}(\bar{G}, V)\neq\emptyset$, and $\bar{G}$ has a normal subgroup $\bar{N}=N/C$ such that $G/N$ is a 2-group and such that $\bar{N}$ is a central product of conjugate subgroups $\bar{L}=\bar{L}_1, \bar{L}_2, \cdots, \bar{L}_k$ of $\bar{G}$ with $\bar{L}\cong \mathrm{SL}_2(2)$ or $\bar{L}\in \mathrm{Chev}(2)-\{(\mathrm{P})\mathrm{SU}_3(2^m), \mathrm{Sz}(2^{2m-1}); m\geqq 2\}$ or $\bar{L}\cong \mathrm{A}_n$, $n\geqq 7$.*

PROOF. If $A\in\mathcal{A}(S)$ and $A\not\leq C$, then $\bar{A}\in\mathcal{P}(\bar{G}, V)$,(4) so $\mathcal{P}(\bar{G}, V)\neq\emptyset$ by (d). Choose $i\in\{0, 1\}$ so that $\mathcal{P}_i^*(\bar{G}, V)\neq\emptyset$. If $i=0$, let $N/C=\langle\mathcal{P}_0^*(\bar{G}, V)\rangle$, while if $i=1$, let $N/C=\mathrm{E}(\langle\mathcal{P}_1^*(\bar{G}, V)\rangle)$. Then $N\lhd G$, and $\bar{N}$ is a central product, $\bar{N}=\bar{L}_1*\bar{L}_2*\cdots*\bar{L}_k$, where $\bar{L}_i\cong\mathrm{SL}_2(2)$ for all $i$ or $\bar{L}_i$ is quasisimple for all $i$ by 3.1, 3.3, and (c.3). Notice that $\{\bar{L}_1, \bar{L}_2, \cdots, \bar{L}_k\}$ is a normal set of subgroups of $\bar{G}$ by the Krull-Remak-Schmidt theorem. If $\bar{M}=M/C$ is a normal subgroup of $\bar{G}$ of even order, then $G=MS$ by (c.1). So $G=NS$, and $\bar{L}_1, \bar{L}_2, \cdots, \bar{L}_k$ are all conjugate in $\bar{G}$. If $\bar{L}_1$ is quasisimple, then $\bar{L}_1$ has the structure as described in (e) by 4.3 and 4.4.

(f) *One of the following holds:*

(1) $\bar{L}\cong\mathrm{SL}_2(2^m)$, $m\geqq 1$;

---

(4) Let $V$ be a normal elementary abelian 2-subgroup of a group $G$, $\bar{G}=G/\mathrm{C}_G(V)$, and $A$ an elementary abelian 2-subgroup of maximal order. Then $|\bar{A}||\mathrm{C}_V(\bar{A})|\geqq|\bar{B}||\mathrm{C}_V(\bar{B})|$ for each subgroup $\bar{B}$ of $\bar{A}$. Although this fact appears well known, I supply a proof. Let $\mathrm{C}_A(V)\leq B\leq A$. Then $|A|\geqq|B\mathrm{C}_V(B)|$ and $\mathrm{C}_V(A)=A\cap V$ by the maximality of $|A|$, so $|A:B|\geqq|\mathrm{C}_V(B):\mathrm{C}_V(A)|$. This completes the proof.

(2) $\bar{L}\cong(\mathrm{P})\mathrm{SL}_3(2^m)$, $\mathrm{Sp}_4(2^m)'$, $m\geqq 1$, *or* $\hat{\mathrm{A}}_6$, *and* $\mathrm{N}_{\bar{S}}(\bar{L})$ *contains an element which interchanges the two nontrivial parabolic subgroups of* $\bar{L}$ *containing* $\bar{S}\cap\bar{L}$;

(3) $\bar{L}\cong\mathrm{A}_{2m+1}$, $m\geqq 2$.

PROOF. If $\bar{G}$ is 2-isolated, then $\bar{G}=\bar{L}\cong\mathrm{SL}_2(2^m)$. Therefore, assume that $\bar{G}$ is not 2-isolated. We may also assume $\bar{L}\not\cong\mathrm{SL}_2(2)$. If none of (1), (2), and (3) holds, then the set $\bar{\mathcal{M}}$ of all maximal subgroups of $\bar{G}$ of odd index controls Sylow 2-intersections in $\bar{G}$ by 2.2, 2.3, and 1.7. Let $\mathcal{M}$ be the set of the inverse images of all elements of $\bar{\mathcal{M}}$. Since $\bar{G}$ is not 2-isolated and $\bar{\mathcal{M}}$ controls Sylow 2-intersections in $\bar{G}$, it follows that $\mathcal{M}$ controls Sylow 2-intersections in $G$. As each member of $\mathcal{M}$ is $\mathcal{C}$-regular by (a), $G$ is $\mathcal{C}$-regular, a contradiction.

(g) *One of the following holds:*

(1) $\bar{L}\cong\mathrm{SL}_2(2^m)$, $m\geqq 1$, *and* $V(\bar{L})$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$-*module;*

(2) $\bar{L}\cong\mathrm{A}_{2m+1}$, $m\geqq 2$, *and* $V(\bar{L})$ *is induced by the natural* $\mathrm{GF}(2)\mathrm{A}_{2m+1}$-*module.*

PROOF. As $G$ is $\mathcal{C}$-singular, $\mathrm{C}_V(\bar{S})=\Omega_1(\mathrm{Z}(S))\neq\mathrm{C}_V(\bar{G})$. So $[\mathrm{C}_V(\bar{S}),\bar{L}]\neq 0$ and Case (2) of (f) does not occur by 3.6. The definition of $\bar{L}$ in the proof of (e), 3.1, and 3.7.2 show that (1) or (2) holds.

(h) *The following holds:*

(1) *$S$ is contained in a unique maximal subgroup of $G$;*

(2) $\langle\mathcal{C}_G(S)\rangle\neq G$.

PROOF. (1) follows from (c.4) and the structure of $\bar{G}$ described in (e) and (g), as a Sylow 2-subgroup of $\mathrm{SL}_2(2^m)$ (resp. $\mathrm{A}_{2m+1}$) is contained in a unique maximal subgroup of $\mathrm{SL}_2(2^m)$ (resp. $\mathrm{A}_{2m+1}$). As $G$ is $\mathcal{C}$-singular, each member of $\mathcal{C}_G(S)$ is a proper subgroup of $G$ containing $S$. So (2) follows from (1).

(i) $G=\mathrm{K}(G)$.

PROOF. Let $M=\mathrm{K}(G)$, $T=S\cap M$, $R=\mathrm{O}_2(M)$, $W=\Omega_1(\mathrm{Z}(R))$, and $D=\mathrm{C}_M(W)$. As $\mathrm{K}(T)=\mathrm{K}(S)$ and $\mathrm{N}_M(\Omega_1(\mathrm{Z}(T)))\leq\mathrm{N}_M(T)\mathrm{C}_M(\Omega_1(\mathrm{Z}(S)))$, $\langle\mathcal{C}_M(T)\rangle\leq\langle\mathcal{C}_G(S)\rangle$. As $\mathrm{J}(T)=\mathrm{J}(S)\not\leq C$ by (d), $M$ is not 2-closed, and so $G=MS$ by (c.1). So $M\neq\langle\mathcal{C}_M(T)\rangle$ by (h.2), and it follows from 1.5 that $M$ is $\mathcal{C}$-singular. Therefore, if $G\neq M$, then $M$ has a $\mathcal{C}$-singular subgroup $X$ with respect to $T$ by the minimality of $G$. Now, $[Q,M]\leq Q\cap M=R$ and $[V,M]\leq V\cap R=V\cap W$. So

$$\mathrm{O}^2(\mathrm{C}_M(V\cap W))\leq C\cap M,$$

and $[V,X]\leq[Q,X]\leq W$. If $[W,X]\leq V$, then $[V,X]=[Q,X]=[W,X]$, and $X$ is a $\mathcal{C}$-singular subgroup of $G$ with respect to $S$. So $[W,X]\not\leq V$ and, as $X$ acts irreducibly on $W(X)$, $[V\cap W,X]=1$. We conclude that

$$X\leq C\cap M.$$

As $\mathrm{O}^2(D)\leq C\cap M$ and as $\mathrm{O}_2(M/C\cap M)=1$ by (c.3), $D\leq C\cap M$. Also, $\mathrm{O}_2(M/D)\leq C\cap M/D$ by (c.3). As $R\in\mathrm{Syl}_2(C\cap M)$ by (c.2), we conclude that

$$C\cap M/D\leq \mathrm{O}(M/D)$$

and that

$$\mathrm{O}_2(M/D)=1.$$

As $\mathrm{J}(T)=\mathrm{J}(S)\not\leq D$ by (d), $\mathcal{P}(M/D, W)\neq\emptyset$. Choose $i\in\{0, 1\}$ so that $\mathcal{P}_i^*(M/D, W)\neq\emptyset$. If $i=0$, let $K/D=\langle\mathcal{P}_0^*(M/D, W)\rangle$, and if $i=1$, let $K/D=\mathrm{E}(\langle\mathcal{P}_1^*(M/D, W)\rangle)$. As in the proof of (e), we have that $G=KS$ and so

$$XD/D\leq C\cap M/D\leq \mathrm{O}(K/D).$$

If $i=1$, then $XD/D\leq \mathrm{Z}(K/D)$, which is a contradiction as $X$ acts irreducibly on $W(X)$. So $i=0$ and $K/D=J_1/D\times\cdots\times J_n/D$, where $J_1/D\cong \mathrm{SL}_2(2)$ and $J_1,\cdots,J_n$ are all conjugate in $G$ (as in the proof of (e)). In particular, $M/D=\mathrm{O}_{3,2}(M/D)$ and so $\mathcal{P}(M/D, W)=\mathcal{P}_0(M/D, W)$. Therefore, $\mathrm{J}(M)D/D\leq\langle\mathcal{P}(M/D, W)\rangle=K/D$ by 3.2. As $\mathrm{J}(M)D/D$ contains an element of $\mathcal{P}_0^*(M/D, W)$, $J_i/D\leq \mathrm{J}(M)D/D$ for some $i$ by 3.1. We conclude that $K/D=\mathrm{J}(M)D/D$. Similarly, we have $N/C=\mathrm{J}(G)C/C$ and so, as $\mathrm{J}(M)=\mathrm{J}(G)$, $N/C\cong(K/D)/(C\cap M/D)$. As both $N/C$ and $K/D$ are direct products of $\mathrm{SL}_2(2)$'s and as $C\cap M/D\leq \mathrm{O}(K/D)$, we must have $C\cap M=D$. But then $[W, X]=1$, a contradiction.

(j) *$\bar{G}\cong \mathrm{SL}_2(2^m)$, $m\geq 1$, and $V(\bar{G})$ is induced by the natural* $\mathrm{GF}(2)\mathrm{SL}_2(2^m)$-*module.*

PROOF. As $\overline{\mathrm{J}(S)}\leq\langle\mathcal{P}(\bar{S}, V)\rangle$, $\mathrm{C}_V(\langle\mathcal{P}(\bar{S}, V)\rangle)\leq \mathrm{C}_V(\overline{\mathrm{J}(S)})\leq\Omega_1(\mathrm{Z}(\mathrm{J}(S)))$. So $\overline{\mathrm{K}(S)}\leq \mathrm{C}_{\bar{S}}(\mathrm{C}_V(\langle\mathcal{P}(\bar{S}, V)\rangle))$, and it follows from 3.7.1 that $\mathrm{K}(S)\leq \mathrm{N}_G(L)$.[5] Therefore, $L\lhd G$ by (i). If Case (1) of (g) occurs, then $\mathrm{K}(S)\leq L$ by 3.7.3; so $G=L$ and we are done. Therefore, assume that (2) of (g) holds. Then $\bar{G}\cong\Sigma_{2m+1}$ and $\overline{\mathrm{K}(S)}$ is contained in the subgroup $\bar{T}$ of $\bar{S}$ generated by the transpositions in $\bar{S}$ by 3.7.4. Let $\mathcal{M}$ be the set of all subgroups $X$ of $G$ containing $C$ such that either $\bar{X}\cong\Sigma_{2m}\times\Sigma_1$ or $\bar{X}\cong\Sigma_3\times P$, where $P\in \mathrm{Syl}_2(\Sigma_{2m-2})$. Then the set $\{\bar{X}; X\in\mathcal{M}\}$ controls Sylow 2-intersections in $\bar{G}$ by 2.4 and so $G=\langle\mathcal{M}(S)\rangle$ by 1.5. If $X\in\mathcal{M}(S)$, then $\bar{T}\leq\bar{S}\cap\bar{X}$ by the structure of $\bar{X}$. So $\mathrm{K}(S)\leq S\cap X$, and $X$ is $\mathcal{C}$-regular by (a). In particular, $X=\langle\mathcal{C}_X(S\cap X)\rangle$ by 1.5. As $\mathrm{K}(S)=\mathrm{K}(S\cap X)$ and $\mathrm{N}_X(\Omega_1(\mathrm{Z}(S\cap X)))\leq \mathrm{N}_X(S\cap X)\mathrm{C}_X(\Omega_1(\mathrm{Z}(S)))$, we conclude that $X\leq\langle\mathcal{C}_G(S)\rangle$. But then $G=\langle\mathcal{C}_G(S)\rangle$, contrary to (h.2).

(k) $[Q, \mathrm{O}^2(G)]\not\leq V$.

PROOF. Assume $[Q, \mathrm{O}^2(G)]\leq V$. Let $X$ be a $\mathrm{J}(S)$-invariant subgroup of $G$ minimal subject to the condition $\mathrm{O}^2(\bar{G})=\bar{X}$. Then $X$ is a $\mathcal{C}$-singular subgroup of $G$ with respect to $S$ by (j).

(l) $S=\mathrm{K}(S)$.

PROOF. Let $Z=\Omega_1(\mathrm{Z}(\mathrm{J}(S)))$ and $W=VZ$. Choose $g\in G$ and $A\in\mathcal{A}(S)$ so that $G=\langle S, S^g\rangle$ and $A\not\leq C$ (this is possible by (d), (h), and (j)). Then $|A:Q\cap A|=$

(5) $L$ is the complete inverse image of $\bar{L}$.

$|V:V\cap A|=2^m$ by 3.5; so $B=V(Q\cap A)\in\mathcal{A}(S)$ and $S=QA$. Therefore, $V\cap A\le Z\le Q\cap A$. Now since $B^g\in\mathcal{A}(S)$ and $B^g\le Q$, it follows that $W\le B^g$ and so $[W, A^g]\le[B^g, A^g]\le V^g=V$. As $G=\langle S, A^g\rangle$, we have $W\lhd G$ and so $W=VZ^g$. As $V=(V\cap A)(V\cap A^g)$ by 3.5, $W=(V\cap A)Z^g$ and we conclude that $Z=(V\cap A)\cdot(Z\cap Z^g)\le\Omega_1(\mathrm{Z}(S))(Z\cap Z^g)$. Now since $G=\langle\mathrm{K}(S), \mathrm{K}(S^g), Q\rangle$, it follows that $Z\cap Z^g\lhd G$ and so $Z\cap Z^g\le\mathrm{Z}(G)$ by (i). Therefore, $Z=\Omega_1(\mathrm{Z}(S))$ and $S=\mathrm{K}(S)$.

We have shown that $G\in\mathcal{G}(S)$ and $S=\mathrm{K}(S)$. The proof of 4.5 is, therefore, complete.

4.6 COROLLARY. *Under Hypothesis* 4.3, *if* $S\in\mathrm{Syl}_2(G)$, *then the following holds:*

(1) $\mathrm{N}_G(\mathrm{J}(S))$ *is* $\mathcal{C}$-*regular;*

(2) *if* $G$ *is* $\mathcal{C}$-*singular, then so is* $\mathrm{J}(G)S$.

PROOF. As $\mathrm{N}_G(\mathrm{J}(S))$ can not have a $\mathcal{C}$-singular subgroup, $\mathrm{N}_G(\mathrm{J}(S))$ is $\mathcal{C}$-regular by 4.5. As $G=\mathrm{J}(G)S\mathrm{N}_G(\mathrm{J}(S))$, (2) follows from (1) and 1.12.

4.7 DEFINITION. A 2-*component* of a finite group $G$ is a subnormal subgroup $B$ of $G$ such that $B=\mathrm{O}^2(B)$ and $B/\mathrm{O}_2(B)$ is quasisimple or of odd prime order (Gorenstein and Walter [**15**] used the term '2-component' in a different sense). A 2-component $B$ of $G$ is of *Aschbacher type* if

(1) there exists a unique noncentral chief factor $U$ of $B$ within $\mathrm{O}_2(B)$,

(2) $\bar{B}=B/\mathrm{O}_2(B)\cong\mathrm{SL}_2(2^m)$ or $\mathrm{A}_{2m-1}$, $m\geqq 2$, and

(3) when considered a $\mathrm{GF}(2)\bar{B}$-module, $U$ is induced by the natural $\mathrm{GF}(2)\cdot\mathrm{SL}_2(2^m)$-module or by the natural $\mathrm{GF}(2)\mathrm{A}_{2m-1}$-module.

4.8 LEMMA. *If* $B$ *is a* 2-*component of a finite group* $G$, *then* $[\mathrm{O}_2(G), B]=[\mathrm{O}_2(B), B]$.

PROOF. As $B\lhd\lhd\mathrm{O}_2(G)B$, $B=\mathrm{O}^2(\mathrm{O}_2(G)B)$ and so $\mathrm{O}_2(G)\le\mathrm{N}_G(B)$. Therefore, $[\mathrm{O}_2(G), B]\le\mathrm{O}_2(B)\le\mathrm{O}_2(G)$. As $B=\mathrm{O}^2(B)$, $[\mathrm{O}_2(G), B]=[\mathrm{O}_2(B), B]$.

4.9 LEMMA. *Let* $G$ *be a simple group such that* $G\in\mathrm{Chev}(2)$ *or* $G\cong\mathrm{A}_n$, $n\geqq 5$. *Then there is no nontrivial* $2'$-*automorphism of* $G$ *centralizing a Sylow* 2-*subgroup of* $G$.

PROOF. When $G\cong\mathrm{A}_n$, $n\geqq 6$, the assertion follows from the fact that a Sylow 2-subgroup of $\mathrm{A}_n$ $(n\geqq 6)$ is self-normalizing in $\mathrm{A}_n$ (a proof of this will be given in [**13**]). Therefore, assume $G\in\mathrm{Chev}(2)$. Let $S\in\mathrm{Syl}_2(G)$ and choose representatives $M_1, \cdots, M_l$ of all conjugacy classes of maximal 2-local subgroups of $G$ so that $\mathrm{O}_2(M_i)\le S$ for each $i$. If a nontrivial $2'$-automorphism $\alpha$ of $G$ centralizes $S$, then as $\mathrm{C}_{M_i}(\mathrm{O}_2(M_i))\le\mathrm{O}_2(M_i)$, $\alpha$ centralizes $M_i$. So $H=\langle M_1, \cdots, M_l\rangle$ is a proper subgroup of $G$, and as is well known, $H$ is strongly embedded in $G$. Therefore, $G\cong\mathrm{SL}_2(2^m)$, $\mathrm{PSU}_3(2^m)$, or $\mathrm{Sz}(2^{2m-1})$, $m\geqq 2$. We can now verify 4.9 using the well known structure of the automorphism groups of these simple groups. (We remark that the above argument applies to all simple groups of characteristic 2 type.)

4.10 THEOREM. *Under Hypothesis* 4.3, *if $G$ is $\mathcal{C}$-singular, then $G$ has a 2-component $B$ of Aschbacher type such that* $[O_2(G), B]\leq \Omega_1(Z(O_2(G)))$.

PROOF. Let $G$ be a minimal counterexample. Furthermore, let $S\in \mathrm{Syl}_2(G)$, $Q=O_2(G)$, $V=\Omega_1(Z(Q))$, $C=C_G(V)$, and $\bar{G}=G/C$. We shall derive a contradiction in a series of reductions.

(a) $O_2(\bar{G})=1$ *and* $Q\in \mathrm{Syl}_2(C)$.

PROOF. Let $N/C$ be a normal 2-subgroup of $G/C$. Assume $Q\notin \mathrm{Syl}_2(N)$, and let $T=S\cap N$ and $H=N_G(T)$. Then $G=HCS$ and $H\neq G$. As $CS$ is $\mathcal{C}$-regular, $H$ is $\mathcal{C}$-singular by 1.12, and so $H$ has a 2-component $B$ of Aschbacher type such that $[O_2(H), B]\leq \Omega_1(Z(O_2(H)))$ by the minimality of $G$. As $T\leq O_2(H)$, $[Q, B]\leq [T, B]\leq V$. So $[C, B]\leq C_C(Q/V)\leq Q$ and, as $B=O^2(BQ)$, $N=CT\leq N_G(B)$. As $G=NH$, $B$ is subnormal in $G$ and so $B$ is a 2-component of $G$ of Aschbacher type with $[Q, B]\leq V$. Therefore, we must have $Q\in \mathrm{Syl}_2(N)$, proving (a).

(b) *$G$ has a 2-component $B$ such that $[Q, B]\leq V$ and such that* $B/O_2(B)\in \mathrm{Chev}(2)-\{SL_2(2^m), (P)SU_3(2^m), Sz(2^{2m-1})\,;\, m\geqq 2\}$ *or* $B/O_2(B)\cong A_{2m}$, $m\geqq 3$, *or else* $B/O_2(B)\cong A_7$ *and* $|[V, B]|=16$.

PROOF. Let $L_0=C_G(Q/V)$ and $H_0=L_0SC$. Choose a $\mathcal{C}$-singular subgroup $X$ of $G$ with respect to $S$, whose existence was proved in 4.5. For $n=1, 2, \cdots$, define inductively $L_n=\langle X^{H_{n-1}}\rangle$ and $H_n=L_nSC$. Then $H_n$ is a subgroup, $L_{n+1}\lhd H_n$, and $L_{n+1}\leq L_n$. Therefore, $L_n \lhd\lhd G$. Choose $n$ so that $L_n=L_{n+1}$. Let $H=H_n$, $L=L_n$, $P/C=O_2(H/C)$, and $W=C_V(P)$. As $O_2(LC/C)=1$ by (a), $[P, LC]\leq P\cap LC=C$ and so $C_{LC}(W)=C$ by the $A\times B$-lemma. Therefore, $C_H(W)=P$ as $H/LC$ is a 2-group. Now $[W, X]\neq 1$ and $X$ acts irreducibly on $V(X)$. So $[Q, X]\leq [V, X]\leq W$. Since $L=\langle X^H\rangle$, we conclude that $[Q, L]\leq W$.

As $[X, J(S)]=X\not\leq P$, $J(S)\not\leq P$. Therefore, $W$ is a faithful $GF(2)(H/P)$-module and $\mathcal{P}(H/P, W)\neq\emptyset$. Choose $i\in\{0, 1\}$ so that $\mathcal{P}_i^*(H/P, W)\neq\emptyset$. If $i=0$, define $J/P=\langle\mathcal{P}_0^*(H/P, W)\rangle$ while if $i=1$, define $J/P=E(\langle\mathcal{P}_1^*(H/P, W)\rangle)$. Then $J/P$ is a central product of subgroups $K_1/P, \cdots, K_n/P$, and either $K_i/P\cong SL_2(2)$ for each $i$ or $K_i/P$ is quasisimple for each $i$ by 3.1 and 3.3. Let $B=O^2(K_1\cap L)$. Then, as $C_C(Q/V)\leq Q$, $B$ is a 2-component of $G$ such that $[Q, B]\leq W$, and either $B/O_2(B)\cong A_3$ or $B/O_2(B)\cong K_1/P$ is quasisimple. If $B/O_2(B)$ is quasisimple, then $B/O_2(B)\in \mathrm{Chev}(2)-\{(P)SU_3(2^m), Sz(2^{2m-1})\,;\, m\geqq 2\}$ or $B/O_2(B)\cong A_n$, $n\geqq 7$, by 4.3 and 4.4.

Assume $B/O_2(B)\cong SL_2(2^m)$ or $A_{2m-1}$, $m\geqq 2$. Then $W(B/O_2(B))=W(K_1/P)$ is induced by the natural $GF(2)SL_2(2^m)$-module or by the natural $GF(2)A_{2m-1}$-module, or else $B/O_2(B)\cong A_7$ and $[W, B]=[W, K_1/P]$ has order 16 by 3.1 and 3.7. As $[W, B]=[V, B]=[Q, B]=[O_2(B), B]$ by 4.8, either $B$ is of Aschbacher type or $B/O_2(B)\cong A_7$ and $|[V, B]|=16$.

(c) *Let $B$ be a 2-component of $G$ as described in* (b). *Then* $G=\langle B^G\rangle S$.

PROOF. Let $N=\langle B^G\rangle$, $T=S\cap NQ$, and $H=N_G(T)$. Assume $G\neq NS$. If $NS$

is $\mathcal{C}$-singular, then $NS$ has a 2-component $B$ of Aschbacher type with $[O_2(NS), B] \leq \Omega_1(Z(O_2(NS)))$. As $B \leq N$, $B$ is a 2-component of $G$, and $[Q, B] \leq V$. Therefore, $NS$ is $\mathcal{C}$-regular and, as $G=HNS$, $H$ is $\mathcal{C}$-singular by 1.12. As $H \neq G$, $H$ has a 2-component $D$ of Aschbacher type such that $[T, D] \leq V$. In particular, $[T, D] \leq Q$ and so $[N, D] \leq Q$ by 4.9. As $G=NH$, $DQ \lhd\lhd G$. Therefore, $D$ is a 2-component of $G$ of Aschbacher type and $[Q, D] \leq V$.

(d) $C=Q$.

PROOF. Let $N=\langle B^G \rangle$ where $B$ is as in (b). Then $C \leq NQ$ by (a) and (c). As $C \cap N \leq C_C(Q/V) \leq Q$, $C=Q(C \cap N)=Q$.

(e) *$G$ is $\mathcal{C}$-regular.*

PROOF. Let $B$ be as in (b) and set $N=\langle B^G \rangle$. Assume $B/O_2(B) \in \mathrm{Chev}(2)-\{SL_2(2^m), (P)SU_3(2^m), Sz(2^{2m-1}); m \geqq 2\}$ or $B/O_2(B) \cong A_{2m}$, $m \geqq 3$. $\bar{G}$ satisfies Hypothesis 2.1 by (c). Therefore, Sylow 2-intersections in $\bar{G}$ are controlled by the set $\bar{\mathcal{M}}$ of all proper subgroups $\bar{M}$ such that $|\bar{G}:\bar{M}|$ is odd and $O^2(C_{\bar{M}}(O_2(\bar{M}))) \leq Z(\bar{N})$ by 1.7, 2.2, 2.3, and 3.6. Let $\mathcal{M}$ be the set of the inverse images of the elements of $\bar{\mathcal{M}}$. Then $\mathcal{M}$ controls Sylow 2-intersections in $G$ as $\bar{G}$ is not 2-isolated. (The above argument was used in the proof of 4.5, the steps (f) and (g).) Suppose $M \in \mathcal{M}$ and $M$ is $\mathcal{C}$-singular. Then $M$ has a 2-component $D$ of Aschbacher type such that $[O_2(M), D] \leq \Omega_1(Z(O_2(M)))$. As $[O_2(\bar{M}), \bar{D}]=1$ by (d), $\bar{D} \leq Z(\bar{N})$. This is a contradiction as $\bar{D}$ acts irreducibly on $V(D)$ by 4.8. So each member of $\mathcal{M}$ is $\mathcal{C}$-regular, and it follows that $G$ is $\mathcal{C}$-regular.

Assume $B/O_2(B) \cong A_7$. In this case, $\bar{N}=\bar{B}_1 \times \cdots \times \bar{B}_k$ with $\bar{B}_i \cong A_7$ for each $i$, and Sylow 2-intersections in $\bar{G}$ are controlled by the set $\bar{\mathcal{M}}$ of all proper subgroups $\bar{M}$ such that $|\bar{G}:\bar{M}|$ is odd and $O^2(C_{\bar{M}}(O_2(\bar{M})))=1$ or $\langle \bar{x}_1, \cdots, \bar{x}_k \rangle$, where $\bar{x}_i$ is a 3-cycle in $\bar{B}_i$ by 2.3. Sylow 2-intersections in $G$ are again controlled by the set $\mathcal{M}$ of the inverse images of elements of $\bar{\mathcal{M}}$. Suppose $M \in \mathcal{M}$ and $M$ is $\mathcal{C}$-singular. Then $M$ has a 2-component $D$ of Aschbacher type such that $\bar{D} \leq \langle \bar{x}_1, \cdots, \bar{x}_k \rangle$, where $\bar{x}_i$ is a 3-cycle in $\bar{B}_i$. Since $\bar{B}_i$ acts irreducibly on $[V, \bar{B}_i]$, it follows that $[V, \bar{N}]=[V, \bar{B}_1] \oplus \cdots \oplus [V, \bar{B}_k]$. Also, $\bar{x}_i$ acts fixed-point-freely on $[V, \bar{B}_i]$. Therefore, $|[V, \bar{D}]| \geqq |[V, \bar{B}_1]|=16$. This is a contradiction as $|[V, \bar{D}]|=4$. Therefore, each member of $\mathcal{M}$ is $\mathcal{C}$-regular, and so is $G$.

We have derived a contradiction, proving 4.10.

4.11 THEOREM. *Under Hypothesis* 4.3, *if $G$ is $\mathcal{C}$-singular, then $G$ has a 2-component $B$ of Aschbacher type such that* $[O_2(G), B] \leq \Omega_1(Z(O_2(G)))$ *and* $B=[B, K(S)]=[B, J(S)]$ *for any* $S \in \mathrm{Syl}_2(G)$.

PROOF. Let $S \in \mathrm{Syl}_2(G)$, $Q=O_2(G)$, $V=\Omega_1(Z(Q))$, and $C=C_G(V)$. It follows from 4.6 and 4.10 that $G$ has a 2-component $B$ of Aschbacher type such that $[Q, B] \leq V$ and $B \leq J(G)$. As $[V, B]=[Q, B]=[O_2(B), B]$ by 4.8, 3.7.1 shows $J(S) \leq N_G(BC)$. As $B=O^2(C_{BC}(Q/V))$, $J(S) \leq N_G(B)$. Therefore, $B \lhd J(G)$.

Suppose $[B, J(S)] < B$. Then $[B, J(S)] \leq O_2(B)$. As $B$ acts irreducibly on

$V(B)$, $[V(B), \mathrm{J}(S)]=1$. But then, as $\mathrm{J}(G)=\langle \mathrm{J}(S)^{\mathrm{J}(G)}\rangle$, $[V(B), \mathrm{J}(G)]=1$, a contradiction. Therefore, $[B, \mathrm{J}(S)]=B$. It then follows from 3.7.1 that $\mathrm{K}(S)\leq \mathrm{N}_G(B)$. Therefore, $[B, \mathrm{K}(S)]=B$.

**Concluding remarks.**

Suppose the $\mathrm{C}_i(S)$ in Definition 4.1 are defined by using fixed characteristic pairs $T_1$, $T_2$ of Glauberman-Niles type for all 2-groups $T$ satisfying $T=\mathrm{K}(T)$. Then Theorem 4.2 shows that Sylow 2-intersections in a group $G$ of even order are controlled by the set consisting of

$$\mathrm{C}_G(\mathrm{C}_1(S)\cap\Omega_1(\mathrm{Z}(S))), \qquad (S\in \mathrm{Syl}_2(G)),$$

$$\mathrm{N}_G(\mathrm{C}_2(S)), \qquad (S\in \mathrm{Syl}_2(G)), \qquad \text{and}$$

the $\mathcal{C}$-singular maximal 2-local subgroups.

Suppose further that $G$ is of characteristic 2 type and that each simple section of each 2-local subgroup of $G$ is of known type. Then each maximal 2-local subgroup $M$ of $G$ satisfies Hypothesis 4.3 (with $G$ replaced by $M$). Hence if $M$ is $\mathcal{C}$-singular, Theorem 4.11 shows that $M$ has a 2-component $B$ of Aschbacher type such that $[\mathrm{O}_2(M), B]\leq\Omega_1(\mathrm{Z}(\mathrm{O}_2(M)))$ and $B=[B, \mathrm{K}(R)]=[B, \mathrm{J}(R)]$ for each $R\in \mathrm{Syl}_2(M)$. Let us call such a 2-component $B$ an *Aschbacher block* of $M$. By the theorems 1.4 and 1.5, control of Sylow 2-intersections implies control of 2-fusion and 2-factorizations. Thus, we obtain the following result.

THEOREM. *Let $S$ be a nonidentity 2-group, $T=\mathrm{K}(S)$, and $(T_1, T_2)$ a characteristic pair of Glauberman-Niles type for $T$. Let $G$ be a group of characteristic 2 type such that $S\in \mathrm{Syl}_2(G)$ and suppose each simple section of each 2-local subgroup of $G$ is of known type. Then Sylow 2-intersections and 2-fusion in $G$ are controlled by the set consisting of*

*the conjugates of* $\mathrm{C}_G(T_1\cap\Omega_1(\mathrm{Z}(S)))$,

*the conjugates of* $\mathrm{N}_G(T_2)$, *and*

*the maximal 2-local subgroups of $G$ having an Aschbacher block.*

*If furthermore $G$ is not 2-isolated, then $G$ is generated by* $\mathrm{C}_G(T_1\cap\Omega_1(\mathrm{Z}(S)))$, $\mathrm{N}_G(T_2)$, *and the maximal 2-local subgroups $M$ of $G$ with an Aschbacher block such that* $S\cap M\in \mathrm{Syl}_2(M)$.

The theorems of Aschbacher and McBride mentioned in the introduction are immediate corollaries of the above theorem.

## References

[1] J. L. Alperin, Sylow intersections and fusion, J. Algebra, **6** (1967), 222-241.
[2] M. Aschbacher, On the failure of the Thompson factorization in 2-constrained groups, Proc. London Math. Soc., **43** (1981), 425-449.
[3] ———, A factorization theorem for 2-constrained groups, Proc. London Math. Soc., **43** (1981), 450-477.
[4] ———, GF(2)-representations of finite groups, Amer. J. Math., **104** (1982), 683-771.
[5] M. Aschbacher, D. Gorenstein and R. Lyons, The embedding of 2-locals in finite groups of characteristic 2-type, Ann. of Math., **114** (1981), 335-456.
[6] B. Baumann, Über endliche Gruppen mit einer zu $L_2(2^n)$ isomorphen Factorgruppe, Proc. Amer. Math. Soc., **74** (1979), 215-222.
[7] N. R. Campbell, Pushing up in finite groups, Ph. D. Thesis, California Institute of Technology, 1979.
[8] R. Foote, Component type theorems for finite groups in characteristic 2, Illinois J. Math., **26** (1982), 62-111.
[9] G. Glauberman and R. Niles, A pair of characteristic subgroups for pushing-up in finite groups, Proc. London Math. Soc. (to appear).
[10] D. M. Goldschmidt, A conjugation family for finite groups, J. Algebra, **16** (1970), 138-142.
[11] K. Gomi, A characterization of the groups $PSL(3, 2^n)$ and $PSp(4, 2^n)$, J. Math. Soc. Japan, **26** (1974), 549-574.
[12] ———, Sylow 2-intersections and split $BN$-pairs of rank two, J. Fac. Sci. Univ. Tokyo Sect. IA, **23** (1976), 1-22.
[13] ———, Control of Sylow 2-intersections in groups of Chev(2) type and groups of alternating type, Sci. Papers College Gen. Ed. Univ. Tokyo, **32** (1982), 15-31.
[14] ———, On GF(2)-representations of finite groups, "Group Theory and Its Applications", Reports on Researches Supported by Grant-in-Aid for Scientific Research No. 56460001 (ed. T. Kondo), College of General Education, Univ. Tokyo, 1983.
[15] D. Gorenstein and J. H. Walter, Balance and generation in finite groups, J. Algebra, **33** (1975), 224-287.
[16] T. Kondo, On Alperin-Goldschmidt's fusion theorem, Sci. Papers College Gen. Ed. Univ. Tokyo, **28** (1978), 159-166.
[17] R. Solomon, On certain 2-local blocks (to appear).
[18] R. Solomon and S. K. Wong, On $L_2(2^n)$-blocks (to appear).

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