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The OE-property of group automorphisms

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§ 1. Introduction.

We shall discuss A. Morimoto's problem ([10]) concerned with the tolerance stability conjecture of E. C. Zeeman mentioned in F. Takens ([15]).

Let φ be a (self-) homeomorphism of a compact metric space X with a metric d. A sequence of points $\{x_i\}_{i\in Z}$ is called a δ-pseudo-orbit of φ if $d(\varphi(x_i), x_{i+1})<\delta$ for $i\in\boldsymbol{Z}$. A sequence $\{x_i\}_{i\in Z}$ is called to be ε-*traced* by $x\in X$ if $d(\varphi^i(x), x_i)<\varepsilon$ holds for $i\in\boldsymbol{Z}$. We say that (X, φ) has the *pseudo-orbit tracing property* (abbrev. P. O. T. P.) if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of φ can be ε-traced by some point $x\in X$. We know (see A. Morimoto [11] or N. Aoki [2]) that a toral automorphism has P. O. T. P. iff it is hyperbolic.

The set $\mathcal{C}(X)$ of all closed non-empty subsets of X will be a compact metric space by the Hausdorff metric $\bar{d}$ defined by

$$\bar{d}(A, B)=\max\{\max_{b\in B}\min_{a\in A} d(a, b), \max_{a\in A}\min_{b\in B} d(a, b)\}$$

for $A, B\in\mathcal{C}(X)$ (cf. C. Kuratowski [8]). We denote by $\mathrm{Orb}^{\delta}((X, \varphi))$ the set of all δ-pseudo-orbit of φ and by $\widetilde{\mathrm{Orb}}^{\delta}((X, \varphi))$ the set of all $A\in\mathcal{C}(X)$, for which there is $\{x_i\}\in\mathrm{Orb}^{\delta}((X, \varphi))$ such that $A=\mathrm{cl}\{x_i : i\in\boldsymbol{Z}\}$, cl denoting the closure. Let $E(\varphi)$ denote the set of all $A\in\mathcal{C}(X)$ such that for every $\varepsilon>0$ there is $A_\varepsilon\in\widetilde{\mathrm{Orb}}^{\varepsilon}((X, \varphi))$ with $\bar{d}(A, A_\varepsilon)<\varepsilon$. Obviously $E(\varphi)$ is closed in $\mathcal{C}(X)$. On the other hand, we define $O(\varphi)=\mathrm{cl}\{O_\varphi(x) : x\in X\}$ where $O_\varphi(x)=\mathrm{cl}\{\varphi^i(x) : i\in\boldsymbol{Z}\}$. It is clear that $O(\varphi)\subset E(\varphi)$. We call φ to have *OE-property* if $E(\varphi)=O(\varphi)$. It is easy to check that φ has *OE-property* whenever φ has P. O. T. P.

The question whether every toral automorphism with OE-property could be hyperbolic was raised by A. Morimoto ([10]). For this question we can give an answer as follows.

THEOREM. *Let X be a compact metric group and σ be an automorphism of X. If σ has OE-property, then σ has P. O. T. P.*

An easy consequence is the following

COROLLARY. *Every toral automorphism with OE-property is hyperbolic.*

For 2 and 3 dimensional toral automorphisms, the corollary was proved in

T. Sasaki ([**13**]).

We denote by $\mathscr{H}(X)$ the group of all homeomorphisms of X. Then $\mathscr{H}(X)$ becomes a complete metric group with the metric defined by $d(f, g)=\max\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)): x\in X\}$ where $f, g\in\mathscr{H}(X)$. We recall that (X, f) is topologically stable iff for every $\varepsilon>0$ there is $\delta>0$ such that for every $g\in\mathscr{H}(X)$ with $d(f, g)<\delta$ there is a continuous map $h: X\circlearrowleft$ such that $h\circ g=f\circ h$ and $d(h(x), x)<\varepsilon$ $(x\in X)$. For an automorphism σ of a compact metric abelian group X, it is well known that if (X, σ) is ergodic under the normalized Haar measure μ then it is Bernoullian under μ, and that (X, σ) is ergodic iff it is topologically mixing. In this case we remark that topological transitivity implies topological mixing.

From A. Morimoto [**10, 11, 12**], N. Aoki [**2, 3**] and the present paper, the relation among the notions of OE-property, P.O.T.P., topological stability and topological mixing for (X, σ) will be characterized as follows. In the case X is connected, OE-property is equivalent to P.O.T.P. (by Theorem), and it further implies topological mixing (by Lemma 3). However topological mixing does not imply P.O.T.P. in general (see [**11**]). If X is solenoidal, then OE-property is equivalent to topological stability (see [**2**]). When X is connected, the authors do not know whether this statement is true. In the case X is totally disconnected, every automorphism has P.O.T.P. ([**2**]) (and hence OE-property). This means that OE-property has nothing to do with topological transitivity for totally disconnected case.

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In order to show the theorem we prepare the following section.

§2. The P.O.T.P. and the OE-property of automorphisms.

Throughout this paper, we shall deal with a compact metric group X with the invariant metric d, and write the group operation by multiplicative form. Subgroups of X which we deal with will be closed. Let K be a subgroup of X and X/K denote a left coset space. The metric d of X induces the metric $d_{X/K}$ of X/K by $d_{X/K}(xK, yK)=\min_{k\in K} d(xk, y)\,(x, y\in X)$. Let σ be an automorphism of X. Its restriction and its factor will be denoted by the same symbols σ if there is no confusion.

LEMMA 1. *Let K be a completely σ-invariant subgroup of X $(\sigma(K)=K)$. Then* (i) *if (X, σ) has P.O.T.P. then $(X/K, \sigma)$ also has P.O.T.P.,* (ii) *if (X, σ) has OE-property then $(X/K, \sigma)$ also has OE-property.*

PROOF. Denote by π the natural projection from X onto X/K. If $\{x_iK\}_{i\in Z}\in\mathrm{Orb}^{\delta}((X/K, \sigma))$, then there is $\{y_i\}\in\mathrm{Orb}^{\delta}((X, \sigma))$ such that $\pi(y_i)=x_iK$ $(i\in\boldsymbol{Z})$.

To prove (i), let $\varepsilon>0$. Then there is $\delta>0$ such that $\{y_i\}\in\mathrm{Orb}^{\delta}((X, \sigma))$ implies $d(y_i, \sigma^i(y))<\varepsilon$ $(i\in\boldsymbol{Z})$ by some $y\in X$. Since $y_iK=x_iK$ $(i\in\boldsymbol{Z})$, we get $d_{X/K}(x_iK, \sigma^i(yK))<\varepsilon$ $(i\in\boldsymbol{Z})$.

Take $E\in E(\sigma_{X/K})$, then there is $E_n=\{x_i^{(n)}\}\in\mathrm{Orb}^{1/n}((X, \sigma))$ such that $\bar{d}_{X/K}(E, \mathrm{cl}\{x_i^{(n)}K\})<1/n$ where $\bar{d}_{X/K}$ is the Hausdorff metric of $\mathcal{C}(X/K)$. Since $\mathcal{C}(X)$ is compact, we can find $E'\in E(\sigma)$ such that $\bar{d}(\mathrm{cl}E_{n_j}, E')\to 0$ (as $j\to\infty$) by taking a subsequence $\{E_{n_j}\}$ suitably. Since $O(\sigma)=E(\sigma)$, we have $E=\pi(E')\in O(\sigma_{X/K})$, thus proving (ii).

Let X split into a direct product $X=\mathop{\times}\limits_{-\infty}^{\infty}\sigma^i(H)$ of normal subgroups $\sigma^i(H)$. $\tilde{X}=\mathop{\times}\limits_{-\infty}^{\infty}H$ is the space of bilateral sequence of points in H, topologized as a compact metric space in the Tychonoff topology. A metric $\tilde{d}$ is given by

$$\tilde{d}(x, y)=\max_{i\in\boldsymbol{Z}} d(x_i, y_i)/2^{|i|}.$$

The shift map $\beta\colon \tilde{X}\circlearrowleft$ is defined as usual by $\beta(x_i)=(y_i)$ where $y_i=x_{i+1}$ for all $i\in\boldsymbol{Z}$. β is a homeomorphism. It is easily checked that (X, σ) is conjugate to $(\tilde{X}, \beta)$. We call such an automorphism σ a *shift automorphism.*

LEMMA 2. *If σ is a shift automorphism, then (X, σ) has P.O.T.P.*

PROOF. Since (X, σ) is conjugate to $(\tilde{X}, \beta)$, it is enough to prove that $(\tilde{X}, \beta)$ has P.O.T.P. Take $\varepsilon>0$. For $\delta>0$ with $2\delta<\varepsilon$ and for $\{x^i\}\in\mathrm{Orb}^{\delta}((\tilde{X}, \beta))$, we have for $i\in\boldsymbol{Z}$

$$\begin{aligned}\tilde{d}(\beta(x^i), x^{i+1})&\geqq d((\beta x^i)_k, x_k^{i+1})/2^{|k|}\\ &=d(x_{k+1}^i, x_k^{i+1})/2^{|k|} \qquad (k\in\boldsymbol{Z}),\end{aligned}$$

and so $d(x_{k+1}^i, x_k^{i+1})<2^{|k|}\delta$ $(i, k\in\boldsymbol{Z})$. Put

$$x=(\cdots, x_0^{-1}, x_0^0, x_0^1, \cdots)\in\tilde{X}.$$

Obviously $(\beta^i x)_k=x_0^{i+k}$ for all $i, k\in\boldsymbol{Z}$. It follows that for $k\geqq 0$

$$d(x_k^i, x_0^{i+k})\leqq\sum_{j=0}^{k-1} d(x_{k-j}^{i+j}, x_{k-j-1}^{i+j+1})\leqq 2^{k+1}\delta$$

and similarly $d(x_k^i, x_0^{i+k})\leqq 2^{|k|+1}\delta$ for $k<0$. Hence we have for $i\in\boldsymbol{Z}$

$$\tilde{d}(x^i, \beta^i x)=\max_{k\in\boldsymbol{Z}} d(x_k^i, (\beta^i x)_k)/2^{|k|}\leqq 2\delta<\varepsilon.$$

The proof is completed.

LEMMA 3. *Assume that X is connected. If (X, σ) has OE-property, then (X, σ) is topologically transitive.*

PROOF. Let $\delta>0$ be given. Cover X by a finite family $\{U(x_i, \delta)\}_{i=1}^k$ of δ-neighborhoods such that $d(x_i, x_{i+1})<\delta$ for $1\leqq i\leqq k-1$. Since X itself is the nonwandering set of σ, for $1\leqq i\leqq k-1$ there is $n_i>0$ such that

$$\sigma^{n_i}U(x_i, \delta)\cap U(x_i, \delta)\neq\emptyset .$$

Take $z_i\in\sigma^{n_i}U(x_i, \delta)\cap U(x_{i+1}, 2\delta)$ and set

$$y_j=\begin{cases} \sigma^j(x_1) & (j<0) \\ \sigma^{j-n_1}(z_1) & (0\leqq j<n_1) \\ \vdots & \\ \sigma^{j-(n_1+\cdots+n_i)}(z_i) & (n_1+\cdots+n_{i-1}\leqq j<n_1+\cdots+n_i) \\ \vdots & \\ \sigma^{j-(n_1+\cdots+n_{k-1})}(z_{k-1}) & (n_1+\cdots+n_{k-2}\leqq j<n_1+\cdots+n_{k-1}) \\ \sigma^{j-(n_1+\cdots+n_k)}(x_k) & (j\geqq n_1+\cdots+n_k). \end{cases}$$

Then $\{y_j\}_{j\in Z}\in \mathrm{Orb}^{3\delta}((X, \sigma))$ and so $\bar{d}(X, \mathrm{cl}\{y_j\})<3\delta$. Since δ is arbitrary, we get $X\in E(\sigma)$ and by assumption $X\in O(\sigma)$. This implies that (X, σ) is topologically transitive.

Let X be a compact metric abelian group and G be the dual group of X. It is known that G is countable, discrete and torsion free. The group operation of G will be written by additive form. We define the dual automorphism γ: $G\circlearrowleft$ by $(\gamma g)(x)=g(\sigma x)$, $g\in G$ and $x\in X$.

We say that (X, σ) satisfies *condition* (A) if for every $g\in G$ there is $0\neq p(\xi)\in \boldsymbol{Z}[\xi]$ (denoting the ring of all polynomials with integer coefficients) such that $p(\gamma)g=0$, and that (X, σ) satisfies *condition* (B) if every $0\neq g\in G$ has the condition that $p(\gamma)g\neq 0$ for all $0\neq p(\xi)\in\boldsymbol{Z}[\xi]$.

LEMMA 4 ([**1**], Theorem 1). *Let X_0 be the connected component of e in X. If X is abelian, then there exists a completely σ-invariant totally disconnected subgroup X_t ($\sigma(X_t)=X_t$) such that $X=X_0X_t$, and further X_0 splits into a product $X_0=X_aX_b$ of completely σ-invariant subgroups such that*

(i) *X_a is connected and satisfies condition* (A),

(ii) *X_b is connected and satisfies condition* (B).

We call X to be *solenoidal* if X is a finite-dimensional connected abelian group. Remark that a finite-dimensional torus is solenoidal.

LEMMA 5. *Let X_a be a connected abelian group with condition* (A). *Then X_a contains a sequence $X_a\supset X_{a,1}\supset X_{a,2}\supset\cdots$ of subgroups such that $\bigcap_n X_{a,n}=\{e\}$ and for every $n\geqq 1$, $\sigma(X_{a,n})=X_{a,n}$ and $X_a/X_{a,n}$ is solenoidal.*

PROOF. This follows from the proof of Lemma 9 in N. Aoki [**1**].

LEMMA 6. *Let X_b be a connected abelian group with condition* (B). *Then (X_b, σ) has P.O.T.P.*

PROOF. This follows from the proof of (p. 196, [**1**]) and the following Lemma 7. But we shall give here a proof for completeness. Let (G, γ) be the dual of (X_b, σ) and define $K_g=\sum_{-\infty}^{\infty}\gamma^j\langle g\rangle$ for $g\in G$ as before. Since G is countable, there is a sequence $G_1 \subset G_2 \subset \cdots \subset \bigcup G_n = G$ of completely γ-invariant subgroups G_n such that $G_n = \sum_{i=1}^{n} K_{f_i}$. Let X_n be the annihilator of G_n in X_b for $n \geqq 1$, then $X_n \searrow \{e\}$ and X_b/X_n has the dual group G_n. It is known (p. 167, [**9**]) that there is the minimal divisible extension $(\bar{G}_n, \gamma)$ of (G_n, γ). Let $\boldsymbol{Q}[\xi, \xi^{-1}]$ be the ring of all polynomials of ξ and ξ^{-1} with coefficients in $\boldsymbol{Q}$. Since $\bar{G}_n$ is divisible and torsion free, we can consider $\bar{G}_n$ to be a $\boldsymbol{Q}[\xi, \xi^{-1}]$-module. Since $\boldsymbol{Q}[\xi, \xi^{-1}]$ is a principal ideal domain, there are $g_1, \cdots, g_p \in G_n$ such that $\bar{G}_n = \bigoplus_{i=1}^{p} \boldsymbol{Q}[\gamma, \gamma^{-1}] g_i$ (cf. p. 85, Theorem 2 in Chapter 7 of [**4**]). Hence $\bar{G}_n$ is expressed as $\bar{G}_n = \bigoplus_{i=1}^{p} \{\bigoplus_{-\infty}^{\infty} \gamma^j \langle g_i \rangle\}$ and so the dual of $(\bar{G}_n, \gamma)$ has P. O. T. P. by Lemma 2, so that $(X_b/X_n, \sigma)$ does so (by Lemma 1 (i)). Since n is arbitrary, we get the conclusion by using the following Lemma 7.

LEMMA 7. *If X contains a sequence $X \supset K_1 \supset \cdots$ of completely σ-invariant subgroups such that $\bigcap K_n = \{e\}$ and for every $n \geqq 1$, $(X/K_n, \sigma)$ has P. O. T. P., then (X, σ) also has P. O. T. P.*

PROOF. Let $\varepsilon > 0$ be given. Choose m so large that $\operatorname{diam}(K_m) < \varepsilon/2$. Since $(X/K_m, \sigma)$ has P. O. T. P., there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i \in \boldsymbol{Z}}$ in X there is a point $xK_m \in X/K_m$ with $d_{X/K_m}(\sigma^i(xK_m), x_iK_m) < \varepsilon/2$ ($i \in \boldsymbol{Z}$). Since $\operatorname{diam}(K_m) < \varepsilon/2$, it follows that $d(\sigma^i(x), x_i) < \varepsilon$ for $i \in \boldsymbol{Z}$.

LEMMA 8 ([**3**]). *Let K be as in Lemma 1. If $(X/K, \sigma)$ and (K, σ) have P. O. T. P., then (X, σ) also has P. O. T. P.*

LEMMA 9 ([**3**]). *Assume that X is totally disconnected. Then every automorphism has P. O. T. P.*

LEMMA 10. *Let K be a completely σ-invariant open subgroup of X. Then (X, σ) has P. O. T. P. iff (K, σ) has P. O. T. P. If (X, σ) has OE-property, then so does (K, σ).*

PROOF. Since K is open and closed, it is easily seen that (K, σ) has P. O. T. P. [OE-property] whenever (X, σ) has P. O. T. P. [OE-property]. If (K, σ) has P. O. T. P., then (X, σ) has the same property since X/K is discrete by Lemmas 8 and 9.

§ 3. Proof of Theorem.

The proof will be divided into five parts.

[I] *Solenoidal case.*

Throughout this part, X will be an r-dimensional solenoidal group with the invariant metric d and σ will be an automorphism of X. As before let (G, γ) be the dual of (X, σ). Since $\operatorname{rank}(G) = r < \infty$ and G is torsion free, an into isomorphism $\varphi: G \to \boldsymbol{Q}^r$ exists ($\boldsymbol{Q}^r$ denotes the vector space over $\boldsymbol{Q}$), so that $\bar{\gamma}=\varphi\circ\gamma\circ\varphi^{-1}$ is an automorphism of $\varphi(G)$. Since rank$((G))=$rank$(\boldsymbol{Q}^r)=r$, $\bar{\gamma}$ is extended on $\boldsymbol{Q}^r$ and further on $\boldsymbol{R}^r$. We shall denote again by γ the extension on $\boldsymbol{R}^r$.

The following Lemmas 11 and 12 are known (see § 1, [2]).

LEMMA 11. *Uuder the above notations, there exist a homomorphism* $\psi: \boldsymbol{R}^r \to X$ *and a totally disconnected subgroup* F *of* X *such that*

(i) $\psi\circ\gamma=\sigma\circ\psi$,

(ii) $X=\psi(\boldsymbol{R}^r)F$,

(iii) *there is a closed neighborhood* U *of* 0 *in* $\boldsymbol{R}^r$ *so that* $\psi: U \to X$ *is an into homeomorphism,* $\psi(U)\cap F=\{e\}$ *and* $\psi(U)F$ *is a closed neighborhood of* e *in* X *(we shall write* $\psi(U)\times F$ *such a neighborhood* $\psi(U)F$*).*

LEMMA 12. *Let* F *be as in Lemma* 11. *Then* F *contains subgroups* F^+, F^- *and* H *such that*

(i) $\sigma(H)=H$,

(ii) $F^+\supset\sigma(F^+)\supset\cdots\supset\bigcap_0^\infty\sigma^n(F)=\{e\}$,

(iii) $F^-\supset\sigma^{-1}(F^-)\supset\cdots\supset\bigcap_0^\infty\sigma^{-n}(F)=\{e\}$,

(iv) $F=F^+\times F^-\times H$.

If in particular G *is finitely generated under* γ *(i.e.* G *is the group generated by* $\bigcup_{-\infty}^{\infty}\gamma^i(\Lambda)$ *for some finite subset* Λ*), then one has* $H=\{e\}$.

MAIN LEMMA 13. *Assume that* X *is solenoidal. If* (X, σ) *has OE-property, then it has P.O.T.P.*

PROOF. If $(\boldsymbol{R}^r, \gamma)$ is hyperbolic, then (X, σ) has P.O.T.P. (see Theorem 2, [2]). Assuming that $(\boldsymbol{R}^r, \gamma)$ is not hyperbolic, we shall derive a contradiction.

By the assumption there are $0\neq g_0\in G$ $(\subset\boldsymbol{R}^r)$ and an irreducible polynomial $p(\xi)$ over $\boldsymbol{Q}$ such that $p(\gamma)g_0=0$ and $p(\xi)$ has some roots of modulus one. Let G_0 denote the subgroup generated by $\{\gamma^j(g_0): j\in\boldsymbol{Z}\}$, and denote by K the annihilator of G_0 in X. Obviously $\sigma(K)=K$ and G_0 is the dual of X/K. By Lemma 1 (ii), $(X/K, \sigma)$ has OE-property. We shall prove that this can not happen because (G_0, γ) is not hyperbolic.

For convenience we replace X/K by X and so G_0 by G (remark that $G=G_0$ is finitely generated under γ). Then $F=F^+\times F^-$ by Lemma 12. As usual $\boldsymbol{R}^r=E^s\oplus E^c\oplus E^u$ where E^s, E^c and E^u are the subspaces corresponding to the eigenvalues of γ with modulus less than one, equal to one and greater than one, respectively. Now γ_{E^s} is essentially a contraction. So we shall use a norm on E^s relative to which γ_{E^s} is actually a contraction. Similarly, we shall use a norm on E^u relative to which γ_{E^u} is an expansion. With these norms, there is $\lambda\in(0, 1)$ such that

$$|\gamma(v^s)|\leqq\lambda|v^s| \quad (v^s\in E^s) \quad \text{and} \quad |\gamma(v^u)|\geqq\lambda^{-1}|v^u| \quad (v^u\in E^u).$$

Since $p(\xi)$ is irreducible over $\boldsymbol{Q}$, γ_{E^c} is an isometry: i.e. with some norm

$$|\gamma(v^c)|=|v^c| \quad (v^c\in E^c).$$

This follows from the fact that by Jordan normal form in the real field, γ_{E^c} is expressed as the matrix

$$\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_m \end{bmatrix}$$

where $\gamma_i=[\pm 1]$ or $\gamma_i=\begin{bmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{bmatrix}$ for some θ_i with $0<\theta_i<2\pi$.

Clearly $\|v\|=\max\{|v^s|, |v^c|, |v^u|\}$ is equivalent to the usual norm of $\boldsymbol{R}^r$. If $B(\alpha)=\{v\in\boldsymbol{R}^r: \|v\|<\alpha\}$ for $\alpha>0$, then there is $\alpha_1>0$ such that $\phi(B(\alpha_1))\times F$ is a closed neighborhood of e in X (by Lemma 11 (iii)). For $x=x_1x_2$ with $x_1\in\phi(B(\alpha_1))$ and $x_2\in F$, put $\rho(x)=\min\{\alpha_1, \max\{\|\phi^{-1}(x_1)\|, d(x_2, 0)\}\}$ and define a metric d_1 of X by

$$d_1(x, y)=\begin{cases} \rho(x-y) & \text{if } x-y\in\phi(B(\alpha_1))\times F \\ \alpha_1 & \text{otherwise}. \end{cases}$$

The metric d_1 is compatible with the original topology of X, and in particular $d_1(\phi(v), O)=\|v\|$ for $v\in B(\alpha_1)$. Denote

$$B^s(\alpha_1)=B(\alpha_1)\cap E^s,\ B^c(\alpha_1)=B(\alpha_1)\cap E^c \quad\text{and}\quad B^u(\alpha_1)=B(\alpha_1)\cap E^u.$$

Then the choice of the norm yields

$$\begin{aligned}\phi(B(\alpha_1))\times F&=\phi(B(\alpha_1))\times F^+\times F^- \\ &=\phi(B^s(\alpha_1))\times\phi(B^c(\alpha_1))\times\phi(B^u(\alpha_1))\times F^+\times F^-.\end{aligned}$$

For $\alpha\in(0, \alpha_1]$, we define $F^\pm(\alpha)=\{x\in F^\pm: d_1(x, 0)\leqq\alpha\}$. Clearly $F^\pm(\alpha)$ is a closed neighborhood of the identity in $F^\pm$. Choose and fix $\alpha\in(0, \alpha_1)$ such that

$$(*)\qquad \begin{aligned} &\gamma^{-1}(B^s(\alpha))\subset B^s(\alpha_1), && \gamma(B^u(\alpha))\subset B^u(\alpha_1), \\ &\sigma^{-1}(F^+(\alpha))\subset F^+, && \sigma(F^-(\alpha))\subset F^-. \end{aligned}$$

Take $v_0\in B^c(\alpha/2)\setminus B^c(\alpha/4)$. For every $n\geqq 1$ we set a sequence $\{v_{n,i}\}_{i\in\boldsymbol{Z}}\in B^c(\alpha/2)$ by

$$v_{n,i}=\begin{cases} 0 & (i\leqq 0) \\ i\gamma^i(v_0)/n & (0<i<n) \\ \gamma^i(v_0) & (n\leqq i). \end{cases}$$

It follows easily that $\{v_{n,i}\}_{i\in\boldsymbol{Z}}\in\mathrm{Orb}^{1/n}((B^c(\alpha_1), \gamma))$ for $n\geqq 1$. Put $E_n=\mathrm{cl}\{v_{n,i}: i\in\boldsymbol{Z}\}$

for $n\geqq 1$. Since $(B^c(\alpha_1), d_1)$ is a compact metric space, as before the Hausdorff metric $\bar{d}_1$ is defined on $\mathcal{C}(B^c(\alpha_1))$. Then $\mathcal{C}(B^c(\alpha_1))$ is compact under $\bar{d}_1$. Hence $\bar{d}_1(E_n, E)\to 0$ (as $n\to\infty$) for some $E\in\mathcal{C}(B^c(\alpha_1))$ (choosing a subsequence if necessary), so that $E\in E(\gamma_{B^c(\alpha_1)})$. On the other hand, E contains the zero element 0 of $B^c(\alpha_1)$ and $E\cap\{B^c(\alpha/2)\setminus B^c(\alpha/4)\}\neq\emptyset$ holds. Since $\gamma_{B^c(\alpha_1)}$ is an isometry, we have $E\notin O(\gamma_{B^c(\alpha_1)})$.

Since ψ: $B(\alpha_1)\to X$ is an into homeomorphism, we get easily $\bar{d}(\psi(E_n), \psi(E))\to 0$ as $n\to\infty$ where $\bar{d}$ is the Hausdorff metric of $\mathcal{C}(X)$. Therefore $\psi(E)\in E(\sigma)$. However it is checked that $\psi(E)\notin O(\sigma)$. Indeed, if $\psi(E)\in O(\sigma)$ then for $n\geqq 1$ there is $y_n\in X$ such that

$$(**)\qquad \bar{d}(\psi(E), O_\sigma(y_n))<1/n\,.$$

Since $\bar{d}(\psi(E_n), \psi(E))\to 0$ as $n\to\infty$, we have $\bar{d}(\psi(E_m), O_\sigma(y_m))<\alpha/2$ for m sufficiently large. By the definition of $\bar{d}$, for every $j\in\boldsymbol{Z}$ there is $i\in\boldsymbol{Z}$ such that

$$d(\psi(v_{m,i}), \sigma^j(y_m))<\alpha/2\,.$$

Hence for every $j\in\boldsymbol{Z}$

$$d(0, \sigma^j(y_m))\leqq d(0, \psi(v_{m,i}))+d(\psi(v_{m,i}), \sigma^j(y_m))<\alpha\,.$$

Using $(*)$, we have for every $J>0$

$$y_m\in\{\bigcap_{j=-J}^{J}\sigma^j\psi(B^s(\alpha_1))\}\times\psi(B^c(\alpha_1))\times\{\bigcap_{j=-J}^{J}\sigma^j\psi(B^u(\alpha_1))\}$$
$$\times\{\bigcap_{j=-J}^{J}\sigma^j(F^+)\}\times\{\bigcap_{j=-J}^{J}\sigma^j(F^-)\},$$

which implies $O_\sigma(y_m)\subset\psi(B^c(\alpha_1))$ (by the definition of the metric d_1 and Lemma 12 (ii), (iii)). It is clear that $\psi^{-1}(O_\sigma(y_m))=O_\gamma(\psi^{-1}(y_m))$. From $(**)$, we have $\bar{d}_1(O_\gamma(\psi^{-1}(y_n)), E)\to 0$ as $n\to\infty$, thus contradicting $E\notin O(\gamma_{B^c(\alpha_1)})$.

[II] *Connected abelian case.*

MAIN LEMMA 14. *Assume that X is connected and abelian. If (X, σ) has OE-property, then (X, σ) has P.O.T.P.*

PROOF. Note that X splits into a direct product $X=X_aX_b$ of subgroups as in Lemma 4. Let $\{X_{a,n}\}$ be a sequence of subgroups of X_a as in Lemma 5. Since $X_a/X_{a,n}$ $(n\geqq 1)$ is solenoidal and $X/X_bX_{a,n}$ is a factor of $X_a/X_{a,n}$, $X/X_bX_{a,n}$ is clearly solenoidal. By Main Lemma 13, $(X/X_bX_{a,n}, \sigma)$ has P.O.T.P. and hence $(X/X_b, \sigma)$ also has P.O.T.P. by Lemma 7. Therefore we get that (X, σ) has P.O.T.P. using Lemmas 6 and 8.

[III] *Abelian case.*

MAIN LEMMA 15. *Assume that X is abelian. If (X, σ) has OE-property, then (X, σ) has P.O.T.P.*

PROOF. Let X_t be as in Lemma 4. Since X_t is totally disconnected, (X_t, σ) has P.O.T.P. by Lemma 9. Since X/X_t is connected, $(X/X_t, \sigma)$ has P.O.T.P. by Main Lemma 14, and therefore the conclusion is obtained by Lemma 8.

[IV] *Connected non-abelian case.*

MAIN LEMMA 16. *Assume that X is connected and non-abelian. If (X, σ) has OE-property, then (X, σ) has P.O.T.P.*

First we shall prepare some useful lemmas.

LEMMA 17 (3.4, [**18**]). *Let X be as in Main Lemma 16. If X splits into a direct product $\times_{i\in I} D_i$ of algebraically simple non-abelian groups D_i, then this splitting is unique, and every normal subgroup of X is equal to the product of some collection of the groups D_i.*

LEMMA 18 (pp. 88-93, [**16**]). *Let X be as in Main Lemma 16. Then there exist in X normal subgroups A and B such that*

(i) *A is the connected component of e in the center Z_X of X,*

(ii) *B is isomorphic to $B'/Z=(\times_{i\in I} L_i')/Z$ where L_i' $(i\in I)$ is simply connected compact simple Lie groups and Z is a subgroup of the center $Z_{B'}$ of B', and*

(iii) *$X=AB$.*

The following is an easy consequence of Lemma 18.

LEMMA 19. *Under the assumption and the notations of Lemma 18, if Z_B is the center of B, then*

(i) *B/Z_B is isomorphic to $B'/Z_{B'}=\times_{i\in I}(L_i'Z_{B'}/Z_{B'})$,*

(ii) *B/Z_B splits into a direct product $B/Z_B=\times_{i\in I} L^{(i)}$ of $L^{(i)}=L_iZ_B/Z_B$ where L_i $(i\in I)$ is a simply connected compact simple Lie subgroup of B, and B/Z_B is a group without center,*

(iii) *Z_B is totally disconnected and normal in X,*

(iv) *Z_B can be expressed as $Z_B=\prod_{i\in I} Z_i$ where Z_i $(i\in I)$ is the center of L_i, and it is central in X,*

(v) *X/AZ_B is isomorphic to B/Z_B, and $AZ_B=Z_X$.*

We remark that $\sigma(A)=A$ for every automorphism σ (by Lemma 18 (i)).

LEMMA 20. *Under the notations of Lemma 18, let φ be an isomorphism from B/Z_B onto X/Z_X defined by $\varphi(xZ_B)=xZ_X$, $x\in B$. Then for every automorphism σ, $\sigma(B)=B$ and $(X/Z_X, \sigma)$ is isomorphic to $(B/Z_B, \sigma)$ under φ.*

PROOF. Define $\psi(xZ_B)=\sigma(x)\sigma(Z_B)$ for $x\in B$, then ψ: $B/Z_B\to\sigma(B)/\sigma(Z_B)$ is an isomorphism. Since B is normal in X, $\sigma(Z_B)(\sigma(B)\cap B)/\sigma(Z_B)$ is a normal subgroup of $\sigma(B)/\sigma(Z_B)$. Since $B/Z_B=\times_{i\in I} L^{(i)}$ by Lemma 19 (ii), we have

$$\sigma(B)/\sigma(Z_B)=\times_{i\in I}\psi(L^{(i)})$$

and hence by Lemma 17

$$\sigma(Z_B)(\sigma(B)\cap B)/\sigma(Z_B)=\underset{i\in I_0}{\times}\phi(L^{(i)})$$

where I_0 is some subset of I. Since

$$\sigma(B)/\sigma(Z_B)=\{\underset{i\in I_0}{\times}\phi(L^{(i)})\}\times\{\underset{i\notin I_0}{\times}\phi(L^{(i)})\},$$

we have

$$\begin{aligned}\sigma(B)B/\sigma(Z_B)B&\cong\sigma(B)/\sigma(Z_B)(\sigma(B)\cap B)\\&\cong\{\sigma(B)/\sigma(Z_B)\}/\{\sigma(Z_B)(\sigma(B)\cap B)/\sigma(Z_B)\}\\&\cong\underset{i\notin I_0}{\times}\phi(L^{(i)})\end{aligned}$$

(the notation "$\cong$" means that two topological groups are isomorphic).

To complete the proof, we denote by D the kernel of the projection from $A\times B$ onto X. Then there is an isomorphism φ_1: $(A\times B)/D\to X$. Let π_0, π_1 and π_2 be the projections in the following diagram

$$\begin{tikzcd} A\times B \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & A \arrow[d, "\pi_2"] \\ (A\times B)/D \arrow[r, "F"'] & A/\pi_0(D) \end{tikzcd}$$

where F is defined by $F\circ\pi_1(a, b)=\pi_2\circ\pi_0(a, b)$, $a\in A$ and $b\in B$. It is clear that F is a continuous homomorphism. Now define by $\sigma'=\varphi_1^{-1}\circ\sigma\circ\varphi$ an automorphism of $(A\times B)/D$. Since $F(\sigma'(\{e\}\times B)D/D)$ is abelian and the kernel of F is a subgroup $(\{e\}\times B)D/D$,

$$\sigma'[(\{e\}\times B)D/D][(\{e\}\times B)D/D]/[(\{e\}\times B)D/D]$$

is abelian. Hence $\sigma(B)B/B$ is abelian and $\sigma(B)B/\sigma(Z_B)B$ must be trivial, and so $\sigma(B)\subset\sigma(Z_B)B$. Taking the connected component of the identity of $\sigma(Z_B)B$, we get $\sigma(B)\subset B$ since $\sigma(B)$ is connected and $\sigma(Z_B)$ is totally disconnected. By symmetry we have $\sigma(B)=B$. The second statement is obtained easily from the definition of the map φ.

PROOF OF MAIN LEMMA 16.

Since $\sigma(B)=B$ (by Lemma 20), $\sigma(Z_B)=Z_B$ and Z_B is totally disconnected. To get the conclusion, it will be enough to prove that $(X/Z_B, \sigma)$ has P.O.T.P.

By Lemmas 18 (iii) and 19 (v), we have $X/Z_B=AZ_B/Z_B\times B/Z_B$. Since AZ_B/Z_B is connected and abelian, Main Lemma 14 ensures that $(AZ_B/Z_B, \sigma)$ has P.O.T.P. On the other hand, by Lemma 19 (ii), B/Z_B is expressed as $B/Z_B=\underset{i\in I}{\times}L^{(i)}$ where $L^{(i)}$ $(i\in I)$ is algebraically simple. Let us put

$$(***)\qquad \begin{aligned} M_1 &= \times\{L^{(i)} : \sigma^n(L^{(i)}) \neq L^{(i)} \quad \text{for all } n \neq 0\} \quad \text{and} \\ M_2 &= \times\{L^{(i)} : \sigma^n(L^{(i)}) = L^{(i)} \quad \text{for some } n \neq 0\}. \end{aligned}$$

By Lemma 17, B/Z_B is expressed as the direct product splitting

$$B/Z_B = M_1 \times M_2 .$$

Since, for $i \in I$ there is $i' \in I$ such that $\sigma(L^{(i)}) = L^{(i')}$ (by Lemma 17), we have

$$M_1 = \mathop{\times}_{n=-\infty}^{\infty} \sigma^n \{ \mathop{\times}_{i \in I_1} L^{(i)} \}$$

where I_1 is a suitable subset of I. Hence (M_1, σ) has P.O.T.P. by Lemma 2. M_2 is expressed as

$$M_2 = \mathop{\times}_{i} U_i$$

where each U_i is a σ-invariant semi-simple Lie group. Since σ_{U_i} is an automorphism of U_i, σ_{U_i} leaves invariant the Killing form $\boldsymbol{B}$ of U_i, which is negative definite. Hence σ_{U_i} is an isometry under the invariant Riemannian metric on U_i induced by $-\boldsymbol{B}$, so that σ_{M_2} is an isometry under some metric. Since (M_2, σ) has OE-property, it is topologically transitive by Lemma 3. Hence (M_2, σ) is minimal (cf. see p. 121, [**17**]), so that $M_2 = \{e\}$. Hence $(B/Z_B, \sigma) = (M_1, \sigma)$ has P.O.T.P.

[V] *General case.*

MAIN LEMMA 21. *Let X be a compact metric group. If (X, σ) has OE-property, then (X, σ) has P.O.T.P.*

For the proof we need the following

LEMMA 22. *Let X_0 denote the connected component of e in X. Assume that the dimension of X_0 is finite. Then there exists in X a totally disconnected normal subgroup H such that X_0H is open in X and $\sigma(X_0H) = X_0H$ holds.*

PROOF. We denote by X^* the set of equivalence classes of irreducible unitary representations of X. If $X_0 \neq \{e\}$, then we can take $g \in X^*$ such that $g(X_0) \neq \{e\}$ (the existence of such a representation g is a consequence of Peter-Weyl's theorem). Let $H^{(1)}$ denote the kernel of g, then it is normal in X and $X_0H^{(1)} = g^{-1}(g(X_0))$ holds. Denote by $g(X)_0$ the connected component of e in $g(X)$. Then $g(X_0) \subset g(X)_0$ and $g(X)_0/g(X_0)$ is connected. It is clear that $g(X)_0/g(X_0)$ is a factor group of $g^{-1}g(X)_0/X_0$. Hence $g(X)_0/g(X_0)$ is totally disconnected: i.e. $g(X_0) = g(X)_0$. Since $g(X)$ is a Lie group, $g(X_0)$ is open in $g(X)$.

Therefore $X_0H^{(1)}$ is also open in X. Let $H_0^{(1)}$ be the connected component of the identity e in $H^{(1)}$, then we get $H_0^{(1)} \subsetneqq X_0$ and hence $\dim(H_0^{(1)}) < \dim(X_0)$. Again, take $f \in X^*$ such that $f(H_0^{(1)}) \neq \{e\}$ and denote by f' the restriction on $H^{(1)}$ of f. Then the kernel $H^{(2)}$ of f' is a normal subgroup of X. Indeed, it is obvious that $H^{(2)}$ is a subgroup. The normality of it follows from the fact that for every $x\in X$, $xH^{(1)}x^{-1}=H^{(1)}$ and $f'(xhx^{-1})=f(x)f(h)f(x^{-1})=e$ for every $h\in H^{(2)}$. Since $f'(H_0^{(1)})$ is open in $f'(H^{(1)})$, $H_0^{(1)}H^{(2)}=f'^{-1}(f'(H_0^{(1)}))$ is also open in $H^{(1)}$, so that $H^{(1)}/H_0^{(1)}H^{(2)}$ is finite. It is easy to see that $X_0H^{(1)}/X_0H^{(2)}$ is a factor group of $H^{(1)}/H_0^{(1)}H^{(2)}$. Hence $X_0H^{(2)}$ is open in $X_0H^{(1)}$ and so in X. Let $H_0^{(2)}$ be the connected component of e in $H^{(2)}$, then $\dim(H_0^{(2)})<\dim(H_0^{(1)})$.

Repeating the above argument, we see that X contains a sequence $\{H_0^{(k)}\}$ of normal subgroups such that

$$\dim(X_0)>\dim(H_0^{(1)})>\dim(H_0^{(2)})>\cdots$$

and for every k, $X_0H^{(k)}$ is open in X. Since $\dim(X_0)<\infty$, there is $n\geqq 1$ such that $H^{(n)}$ is totally disconnected. We write

$$D=H^{(n)} \quad \text{and} \quad A_m=D\sigma(D)\cdots\sigma^m(D) \qquad (m\geqq 1).$$

Let $\pi: X\to X/X_0$ be the natural projection, then π is an open map and so $\{\pi(A_m)\}_{m\geqq 1}$ is an increasing sequence of open subgroups of X/X_0 (because each A_mX_0 is open in X). Put $\dot{K}=\bigcup_{m\geqq 1}\pi(A_m)$. Then $\dot{K}$ is compact. Hence there is $M>0$ such that $\dot{K}=\pi(A_M)$. Since D is totally disconnected, so is A_M. We get that $\sigma(\dot{K})=\dot{K}$: i.e. $\sigma(X_0A_M)=X_0A_M$. For, let μ be the normalized Haar measure of X/X_0. Then $\mu(\bigcup_{j\geqq 1}\sigma^j(\dot{K}\setminus\sigma(\dot{K})))=\sum_{j\geqq 1}\mu(\dot{K}\setminus\sigma(\dot{K}))=\infty$ unless $\sigma(\dot{K})=\dot{K}$ since $\dot{K}\setminus\sigma(\dot{K})$ is open and compact. This can not happen and the proof is completed.

Lemma 23. *Let X_0 and H be as in Lemma 22. If X_0 is abelian, then H is chosen such that $\sigma(H)=H$ holds.*

Proof. Let X_1 and X_2 be subgroups of X. Denote by $[X_1, X_2]$ the subgroup generated by points of the forms $[x_1, x_2]=x_1^{-1}x_2^{-1}x_1x_2$, $x_1\in X_1$ and $x_2\in X_2$. Since H is normal in X, $[X_0, H]\subset X_0\cap H=\{e\}$ and so

$$[X_0H, X_0H]=[X_0, X_0][H, H]=[H, H].$$

Since $X_0H/[H, H]=(X_0[H, H]/[H, H])(H/[H, H])$ is abelian, by Lemma 4 there is a completely σ-invariant subgroup $H_t/[H, H]$ such that $H_t/[H, H]$ is totally disconnected and $X_0H/[H, H]=(X_0[H, H]/[H, H])(H_t/[H, H])$. It is easy to see that $\sigma(H_t)=H_t$ and H_t is totally disconnected. This H_t is our requirement.

Let X_0 be as in Lemma 22 and assume that X_0 is abelian. We denote by (G, γ) the dual of (X_0, σ) as before. As usual, $C(X_0)$ denotes the Banach space of all complex valued continuous functions imposed with the uniform norm. Denoting by $\langle\cdot, g\rangle$ a character g of X_0, we get $\hat{G}=\{\langle\cdot, g\rangle : g\in G\}\subset C(X_0)$. It follows easily that $\hat{G}$ is discrete in $C(X_0)$. Let $\psi_y : \hat{G}\circlearrowleft$ be an automorphism defined by

$$\langle x, \psi_y g\rangle=\langle yxy^{-1}, g\rangle \qquad (g\in G \quad \text{and} \quad y\in X).$$

LEMMA 24. (i) *ϕ_y is continuous in y and* (ii) *for $g \in G$, $\{\langle \cdot, \phi_y g\rangle : y \in X\}$ is a finite set.*

PROOF. It is easy to see that for $g \in G$

$$\sup_{x \in X_0} |\langle x, \phi_y g\rangle - \langle x, g\rangle| \to 0 \quad \text{as} \quad y \to e .$$

Define a map $\varphi_g : X \to \hat{G}$ by $\varphi_g(y) = \langle \cdot, \phi_y g\rangle$ for $g \in G$ and $y \in X$. Then φ_g is continuous since ϕ_y is continuous in y. Hence $\varphi_g(X) \subset \hat{G}$, and $\varphi_g(X)$ is finite.

LEMMA 25. *For $g \in G$, there exists an open normal subgroup U_g such that $X_0 \subset U_g$ and $\phi_y(g) = g$ for all $y \in U_g$.*

PROOF. Since $\{\phi_y(g) : y \in X\}$ is finite by Lemma 24 (ii), there is in X an open subgroup U'_g such that $X_0 \subset U'_g$ and $\phi_y(g) = g$ for all $y \in U'_g$ (by using Lemma 24 (i)). Note that X/X_0 is totally disconnected and compact. Then there is an open normal subgroup U_g, so that $X_0 \subset U_g \subset U'_g$. This U_g is the desired subgroup.

Let G_A be the maximal subgroup of G whose dual satisfies condition (A).

LEMMA 26. *There exists a completely σ-invariant open normal subgroup X_1 such that $X_0 \subset X_1$ and $\phi_y(G_A) = G_A$ for all $y \in X_1$.*

PROOF. If $0 \neq g \in G_A$, then there is $0 \neq p(\xi) \in \boldsymbol{Z}[\xi]$ with degree $(p(\xi)) = k$ such that $p(\gamma)g = 0$. By Lemma 25 there is an open normal subgroup V so that $\phi_v(\gamma^i g) = \gamma^i g$ $(0 \leqq i \leqq k)$ for all $v \in V$. Note that G is torsion free. It follows that $\phi_v(\gamma^i g) = \gamma^i g$ for all $i \in \boldsymbol{Z}$ and all $v \in V$. By compactness there is $m > 0$ such that $X_1 = V\sigma(V) \cdots \sigma^m(V)$ is completely σ-invariant. Therefore $\phi_y(g) = g$ for all $y \in X_1$. Since g is arbitrary in G_A, we get the conclusion of the lemma.

LEMMA 27. *Let G_A and X_1 be as in Lemma* 26. *Then there exists a completely σ-invariant subgroup K of X_0 such that*

(i) *K is normal in X_1,*

(ii) *K has the dual group G/G_A and satisfies condition* (B),

(iii) *X_0/K has the dual group G_A.*

PROOF. Since $\phi_y(G_A) = G_A$ for all $y \in X_1$ by Lemma 26, the annihilator K of G_A in X_0 is normal in X_1. Since K and X_0/K have the dual groups G/G_A and G_A respectively, the assertions (ii) and (iii) hold.

LEMMA 28. *Let X_1, K and G_A be as in Lemma* 27. *Then there exists a sequence $X_0 \supset X^{(1)} \supset \cdots$ of completely σ-invariant subgroups such that $\bigcap X^{(i)} = K$ and for every $i \geqq 1$*

(i) *$X^{(i)}$ is normal in X_1,*

(ii) *$X_0/X^{(i)}$ is solenoidal.*

PROOF. By Lemma 27 (iii), the dual group of X_0/K satisfies condition (A). Let g be a character of X_0/K: i.e. $g \in G_A$. Then $\{\phi_y(g) : y \in X_1\}$ is finite by Lemma 24 (ii). Hence the rank of the subgroup generated by

$$\{\gamma^i\phi_y(g)\ :\ -\infty<i<\infty,\ y\in X_1\}$$

is finite. By using this, we get easily the conclusion of the lemma.

LEMMA 29. *Let X_0 be the connected component of e in X as before. Assume that X_0 has no center. If U is a completely σ-invariant Lie group in X_0 and U is normal in X_0, then there is a completely σ-invariant open subgroup X_1 of X such that $X_1\supset X_0$ and U is normal in X_1. If in particular (X, σ) has OE-property, then X_0 does not contain such subgroups U.*

PROOF. Let L be a subgroup of X_0. We may assume that L is algebraically simple and normal in X_0. Choose a representation $g\in X^*$ such that $g(L)\neq\{e\}$ and let F be the kernel of g. Then F is a normal subgroup of X such that X/F is a Lie group and $F\cap L=\{e\}$ holds. Note that $x^{-1}Lx\subset X_0$ and $g(x^{-1}Lx)\neq\{e\}$ for $x\in X$. Then $\mathcal{O}=\{x^{-1}Lx\ :\ x\in X\}$ is a finite sequence of subgroups that are normal in X_0. For, if $\mathcal{O}$ is infinite, then $R=\prod(x^{-1}Lx)$ splits into the infinite direct product $R=\times(x^{-1}Lx)$ and $F\cap R=\{e\}$ by Lemma 17. But FR/F is a Lie group and $FR/F\cong R$. This can not happen. Therefore $\{x\in X: x^{-1}Lx=L\}$ is an open subgroup of X. By assumption X_0 is represented as $X_0=\times L^{(i)}$ with the notations of Lemma 19 (ii). Since $U\subset X_0$, U splits into a direct product of a finite family of $\{L^{(i)}\}$ (by Lemma 17). Hence $X_1=\{x\in X\ :\ x^{-1}Ux=U\}$ is a completely σ-invariant open subgroup of X (since $\sigma(U)=U$).

Let V be a direct factor such that $X_0=V\times U$. Then V is normal in X_1 and $\sigma(V)=V$ holds (this is obtained using Lemma 17). If (X, σ) has OE-property then (X_1, σ) and $(X_1/V, \sigma)$ both have OE-property. Since $X_0/V\cong U$, by (5.1, [18]) we can find a completely σ-invariant normal subgroup $\dot{C}$ of X_1/V such that $\dot{C}\cap X_0/V$ is trivial and $\dot{C}\times X_0/V$ is open in X_1/V. Since $(X_1/V, \sigma)$ has OE-property, as in the proof of Main Lemma 16 we get $X_1=V$: i.e. $U=\{e\}$.

PROOF OF MAIN LEMMA 21.

As before let X_0 be the connected component of e in X. With the notations of Lemma 18 (iii), X_0 splits into a product $X_0=AB$ of subgroups that are normal in X (Lemma 20). Since $\sigma(B)=B$, we have $\sigma(Z_B)=Z_B$ where Z_B is the center of B. Note that Z_B is normal in X. Let us put

$$\dot{X}=X/Z_B,\quad \dot{A}=AZ_B/Z_B,\quad \dot{B}=B/Z_B \quad\text{and}\quad \dot{X}_0=X_0/Z_B\,.$$

Then $\dot{A}$, $\dot{B}$ and $\dot{X}_0$ are normal in $\dot{X}$ and completely σ-invariant. Note that $\dot{X}_0=\dot{A}\times\dot{B}$ holds. By Lemma 1 (ii), $(\dot{X}, \sigma)$ has OE-property.

To get the conclusion of Main Lemma 21, we need only to prove that $(\dot{X}, \sigma)$ has P.O.T.P. (because (Z_B, σ) has P.O.T.P. by Lemma 9 and hence by Lemma 8, (X, σ) has P.O.T.P.). Note that $(\dot{X}/\dot{A}, \sigma)$ has OE-property. Since $\dot{X}_0/\dot{A}\cong\dot{B}$, $\dot{X}_0/\dot{A}$ has no center. By Lemma 29, $\dot{B}$ does not contain non-trivial σ-invariant Lie groups that are normal in $\dot{X}_0$: i.e. $\dot{B}=M_1$ where M_1 is the group in (***). Therefore $(\dot{B}, \sigma)$ has P.O.T.P. (by Lemma 2).

Thus it is enough to show that $(\dot{X}/\dot{B}, \sigma)$ has P.O.T.P. For convenience put $Y=\dot{X}/\dot{B}$ and $Y_0=\dot{X}_0/\dot{B}$. Clearly Y_0 is the connected component of the identity of Y and Y_0 is abelian. Let (G, γ) be the dual of (Y_0, σ) as before. Since Y_0 is connected, G is torsion free. Let G_A be the maximal subgroup of G whose dual satisfies condition (A). Then there is in Y a completely σ-invariant open normal subgroup Y_1 (by Lemma 26), and by Lemma 27 there is a subgroup K such that $\sigma(K)=K$, K is normal in Y_1 and Y_0/K has the dual group G_A. By using Lemma 28, we have that Y_1 contains a sequence $Y_0 \supset Y^{(1)} \supset \cdots$ of completely σ-invariant subgroups such that $\bigcap Y^{(i)}=K$ and for every $i \geqq 1$, $Y^{(i)}$ is normal in Y_1 and $Y_0/Y^{(i)}$ is solenoidal. Since $Y_0/Y^{(i)}$ is the connected component of the identity in $Y_1/Y^{(i)}$, $Y_1/Y^{(i)}$ contains a totally disconnected normal subgroup $H_i/Y^{(i)}$ such that $Y_0H_i/Y^{(i)}$ is open in $Y_1/Y^{(i)}$ and $\sigma(Y_0H_i/Y^{(i)})=Y_0H_i/Y^{(i)}$ holds (by Lemma 22).

Since Y_1 is open in Y and (Y, σ) has OE-property (by Lemma 1 (ii)), (Y_1, σ) has OE-property (see Lemma 10), and hence $(Y_0H_i/Y^{(i)}, \sigma)$ also has OE-property. By Lemma 23 we remark that $H_i/Y^{(i)}$ is chosen such that $\sigma(H_i/Y^{(i)})=H_i/Y^{(i)}$ holds. Hence $(H_i/Y^{(i)}, \sigma)$ has P.O.T.P. On the other hand, since $(Y_0H_i/Y^{(i)})/(H_i/Y^{(i)})$ is connected, by Lemma 1 (i) and Main Lemma 16 the system has P.O.T.P., and so does $(Y_0H_i/Y^{(i)}, \sigma)$ by Lemma 8. By using Lemma 10 we get that $(Y_1/Y^{(i)}, \sigma)$, and hence $(Y/Y^{(i)}, \sigma)$, has P.O.T.P. Since $Y^{(i)} \searrow K$, $(Y/K, \sigma)$ must have P.O.T.P. (by Lemma 7). Since K has the dual group G/G_A, (K, σ) satisfies condition (B) by Lemma 27 (ii), and so (K, σ) has P.O.T.P. (by Lemmas 6 and 7). Therefore $(Y, \sigma)=(\dot{X}/\dot{B}, \sigma)$ has P.O.T.P. The proof of Main Lemma 21 is completed.

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