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A construction of geometric structures on Seifert fibered spaces

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§ 1. Introduction.

Since Thurston described his geometric views on 3-manifold topology, we have had quite many opportunities of recognizing its incredible importance. According to this recognition, a couple of expository descriptions of geometric structures on 3-manifolds, say in [4] [6], have been presented. In particular in the article by Peter Scott [6] there is a complete description of geometric structures on Seifert fibered spaces. A geometric structure on a Seifert fibered space there is given by finding a faithful discrete representation of its fundamental group to the isometric transformation group. In this paper we shall give an alternative construction of geometric structures by using the idea of Thurston's hyperbolic Dehn surgery. We shall describe this only for Seifert fibered spaces over S^2 with precisely 3 exceptional fibers since it is the typical case and simultaneously the method is easily generalized to the other case.

An immediate application of the construction is computation of volumes. The normalized volume for a closed Lorentz manifold is known to be a topological invariant (see [2], [4]). We shall take the trefoil knot as an example for computing volumes of the resultant manifolds of Dehn surgery along it. Though taking the trefoil seems to be special for general discussion, it is enough to see how the volumes fill up $\boldsymbol{R}^+$, which is in contrast with the case of hyperbolic volumes.

In the next three sections, we discuss geometric Dehn surgery in general situation and review 3-dimensional geometries in the sense of Thurston [8] and Seifert fibrations. The main construction and Dehn surgery along the trefoil knot are dealt in the last two sections.

§ 2. Geometric Dehn surgery.

In this section, we generalize Thurston's hyperbolic Dehn surgery [7]. Roughly speaking, it is the study of completion of an incomplete geometric toral

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end from the Dehn surgical viewpoint. We need several conditions in the process that the completion turns out to be the result of Dehn surgery, and our purpose is to establish those by Lemma 2.1.

We first have to describe our setting. Let (X, G) be a pair of a simply connected 3-dimensional riemannian homogeneous space X with its topological transformation group G which acts transitively and effectively as isometries. It is called a geometry. An (X, G)-structure on a manifold is a riemannian structure locally modeled on (X, G). We are only interested in the completion of a toral end, so let U be a closed neighborhood of a toral end, which is homeomorphic to $T^2\times(0, 1]$. When U has a specific (X, G)-structure, denote it by Σ. This notation means both the underlying space U and its specific geometric structure hereafter. Then by analytic continuation, we get a developing map $d : \tilde{\Sigma}\to X$ of the universal covering $\tilde{\Sigma}$ of Σ to X (see [7]). The developed image is unique up to multiplication by elements of G. Also using the developed image $d(\tilde{\Sigma})\subset X$, we get a holonomy mapping $h : \pi_1(\Sigma)=H\to G$, which represents how Σ is distorted (see also [7]). The holonomy group $h(H)$ is uniquely determined by Σ up to conjugation in G. Denote the developed image $d(\tilde{\Sigma})\subset X$ by V and the holonomy group $h(H)\subset G$ by Γ. Then our first condition is

(1) the completion $C(V)$ of V is the union of V with either a geodesic circle or a geodesic line.

Notice that a neighborhood of this circle or line, which will be denoted by L, admits a geometric structure.

The developing map $d : \tilde{\Sigma}\to X$ extends to a surjective map: $C(\tilde{\Sigma})\to C(V)$ between completions because d does not increase distances. Let $\tilde{L}$ be the pre-image of L by this mapping. Since d is also an isometric immersion, $\tilde{L}$ is homeomorphic to a line by the condition (1). Then the action of H on $\tilde{\Sigma}$ extends to one on $C(\tilde{\Sigma})=\tilde{\Sigma}\cup\tilde{L}$. Here we have two cases for the manner of extensions. One is the case when the orbit of any point on $\tilde{L}$ by H is dense in $\tilde{L}$. The other case is our second condition,

(2) the orbit of $x\in\tilde{L}$ by H is discrete in $\tilde{L}$.

Since the projection : $\tilde{\Sigma}\to\Sigma$ extends to a surjective map: $C(\tilde{\Sigma})\to C(\Sigma)$ between completions by the same reason as before, $C(\Sigma)$ turns out to be topologically a manifold and is homeomorphic to the union of U and a circle.

We now specify the longitude l and the meridian m of U. These represent the loops in U and simultaneously the elements of $\pi_1(U)$. Let $J\subset H$ be the stabilizer of $\tilde{L}$, $\{g\in H; g\cdot x=x \text{ for all } x\in\tilde{L}\}$. Since H/J acts non trivially, discontinuously and effectively on the line $\tilde{L}$, it is isomorphic to an infinite cyclic group $\boldsymbol{Z}$ and hence J is also an infinite cyclic group generated by $m^p l^q$ where p and q are coprime integers. The third condition is

(3) J is contained in the kernel of h.

This means that Γ acts effectively on L and J acts trivially on $C(V)$.

The third condition implies that the mapping: $C(\tilde{\Sigma})\to C(V)$ induces a surjective map: $C(\tilde{\Sigma})/J=C(\tilde{\Sigma}/J)\to C(V)$. Our last condition is

(4) it is an immersion.

This is to avoid overlaps around L.

On the other hand, if (1)$\sim$(4) hold, then $C(\tilde{\Sigma}/J)$ admits a geometric structure induced by this immersion and also it follows from the construction that the surjective map: $C(\tilde{\Sigma}/J)\to C(V)$ becomes a covering so that the group of covering transformations, $H/J\cong \boldsymbol{Z}$, acts on $C(\tilde{\Sigma}/J)$ as isometries with respect to its geometric structure. Thus its quotient $C(\Sigma)$ also admits a geometric structure which is an extension of the original one Σ. Notice that a loop representing $m^p l^q$ bounds a disk in $C(\Sigma)$. We have then established

LEMMA 2.1. *If Σ satisfies the conditions* (1)$\sim$(4), *then by completion, Σ extends to a geometric structure on some solid torus bounded by ∂U. A simple closed curve representing $m^p l^q$ bounds a disk in the resultant solid torus.*

Here are two viewpoints for the resultant solid torus. If we originally had a solid torus with a specified meridian m instead of the toral end, then the resultant solid torus would be the result of (p, q) Dehn surgery along the core of the original one. If we consider the parallel loops to the longitude as forming a Seifert fibration on U, then the core of the resultant solid torus becomes an exceptional fiber with Dehn surgical index (p, q).

§ 3. Geometries.

Following Thurston [8], we briefly describe 3-dimensional geometries. However, before going into details, let us note that we require our geometries (X, G) to have the common properties below:

(i) There is an equivariant projection $p:(X, G)\to(P, K)$, where (P, K) is a 2-dimensional geometry that is either the spherical geometry $(S^2, SO(3))$, the euclidean geometry $(E^2, \mathrm{Isom}^+E^2)$ or the hyperbolic geometry $(H^2, PSL_2\boldsymbol{R})$.

(ii) The kernel of $p: G\to K$ is a 1-parameter subgroup S of G which acts principally on X. S is isomorphic to S^1 or $\boldsymbol{R}$ according to whether G is compact or not.

The product of any of the 2-dimensional geometry with the 1-dimensional euclidean geometry has of course these properties and will be in our interest. Also there is a twisted geometry for each 2-dimensional geometry and hence we will get six geometries.

Since we required the properties above for simplicity, our geometries are not quite equal to Thurston's, however they are actually subgeometries which have smaller transformation groups. Let us start with (X, G) which covers the

2-dimensional spherical geometry. The product geometry in this case is $S^2\times E=(S^2\times E, SO(3)\times \boldsymbol{R})$. The topological type of closed manifolds in this geometry is just $S^2\times S^1$ and hence there is almost nothing interesting for us. The twisted geometry is a subgeometry $S^3=(S^3, G)$ of the 3-dimensional spherical geometry $(S^3, SO(4))$. To see what G is, let us think of the Hopf fibration: $S^3\to CP^1$; $(z_1, z_2)\mapsto[z_1 : z_2]$ where $(z_1, z_2)\in \boldsymbol{C}^2$ such that $|z_1|^2+|z_2|^2=1$. Then the subgroup $\tilde{K}$ of $SO(4)$ which consists of the left rotations,

$$\left\{U\begin{pmatrix} \cos\phi & -\sin\phi & & \\ \sin\phi & \cos\phi & & 0 \\ & 0 & \cos\phi & \sin\phi \\ & & -\sin\phi & \cos\phi \end{pmatrix}U^{-1}\ ;\ U\in SO(4)\right\}$$

is isomorphic to $SU(2)\cong S^3$, and induces the orthogonal action of $\tilde{K}/\{\pm I\}\cong SO(3)$ on downstairs $CP^1=S^2$. Then G is a subgroup of $SO(4)$ generated by $\tilde{K}$ and the 1-parameter subgroup S parametrized by

$$\begin{array}{ccc} \boldsymbol{R}/2\pi\boldsymbol{Z} & \longrightarrow & S\subset SO(4) \\ \Downarrow & & \Downarrow \\ \theta & \longmapsto & \begin{pmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & 0 \\ & 0 & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{pmatrix}. \end{array}$$

Since $\tilde{K}$ commutes with S and $\tilde{K}\cap S=\{\pm I\}$, G is isomorphic to $S^3\times S^1/\boldsymbol{Z}_2$ as a group.

We next describe geometries which cover the 2-dimensional euclidean geometry. The product geometry in this case is $E^3=(E^2\times E^1, \mathrm{Isom}^+E^2\times \boldsymbol{R})$, which is a subgeometry of the 3-dimensional euclidean geometry. The twisted geometry is the nilpotent geometry $N=(X, G)$. Here X is a twisted product of E with E^2, and G is an extension of the group of isometries of E^2 by $\boldsymbol{R}$,

$$\boldsymbol{R}\longrightarrow G\longrightarrow \mathrm{Isom}^+E^2\,,$$

which is embedded in the affine transformation group A^3 by

$$\begin{array}{l} A^3\cong GL(3)\ltimes \boldsymbol{R}^3 \\ \quad\cup \\ \quad G\ \ni \left(\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ \dfrac{-b\cos\phi+a\sin\phi}{2} & \dfrac{b\sin\phi+a\cos\phi}{2} & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ \tilde{\theta} \end{pmatrix}\right) \\ \quad\downarrow \\ \mathrm{Isom}^+E^2\cong SO(2)\ltimes \boldsymbol{R}^2\ni\left(\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right) \end{array}$$

where $\tilde{\theta}$ runs over all $\boldsymbol{R}$. The fiber of the identity of Isom^+E^2 forms S in this geometry, which is clearly contained in the center of G. We parametrize S by

$\theta=2\pi\tilde{\theta}$ for uniformity in future.

The last two geometries cover the 2-dimensional hyperbolic geometry. The product geometry is $H^2\times E=(H^2\times E, PSL_2\boldsymbol{R}\times\boldsymbol{R})$. The twisted geometry in this case is the Lorentz geometry $S\tilde{L}_2=(X, G)$. Here X can be seen as the universal covering of the unit tangent sphere bundle $T(H^2)$ over H^2. Since $PSL_2\boldsymbol{R}$ acts on H^2 as isometries, its derivative acts on $T(H^2)$ and then we get the action of the universal covering group $S\tilde{L}_2\boldsymbol{R}$ of $PSL_2\boldsymbol{R}$ on X. Also $S^1\cong\boldsymbol{R}/2\pi\boldsymbol{Z}$ acts on $T(H^2)$ as simultaneous rotations of all vectors keeping their base point fixed, so does $\boldsymbol{R}$ on X. Let $\tilde{I}\in S\tilde{L}_2\boldsymbol{R}$ be the lift of the identity element of $PSL_2\boldsymbol{R}$ which acts on X as the unit simultaneous translation of fibers in the positive direction and let N be an infinite cyclic subgroup of the direct product $S\tilde{L}_2\boldsymbol{R}\times\boldsymbol{R}$ generated by $\tilde{I}\times\{-1\}$. Then $G=S\tilde{L}_2\boldsymbol{R}\times\boldsymbol{R}/N$.

REMARK. The universal covering group $\tilde{K}$ of K is embedded in G in the spherical and Lorentz geometries.

§ 4. Seifert fibered spaces.

The normalized Dehn surgical invariant to describe a Seifert fibered space consists of the topological type of the orbit manifold, the obstruction class and the normalized Dehn surgical indices for singular fibers. See for example [5]. To avoid insignificant confusion, just think of orientable Seifert fibered spaces over orientable surfaces. Then its bundle structure is determined by genus g of the orbit manifold, an integral obstruction class n and indices of singular fibers $(p_1, q_1), \cdots, (p_r, q_r)$ where p_j and q_j are coprime integers such that $0<q_j<p_j$. For example, a manifold with these data has the fundamental group isomorphic to

$$\langle a_1, b_1, \cdots, a_g, b_g, s_1, \cdots, s_r, h \mid [a_i, h]=[b_i, h]=[s_j, h]=e,$$
$$s_j^{p_j}h^{q_j}=e,\ s_1\cdots s_r[a_1, b_1]\cdots[a_g, b_g]=h^n\rangle.$$

There are two important invariants when we regard it as a circle fibration over a 2-dimensional orbifold as in § 13 of [7]. Those are euler number of the fibration and euler characteristic of the orbit orbifold and they can be computed by

$$e^{\mathrm{orb}}=q_1/p_1+\cdots+q_r/p_r+n \qquad \text{and}$$
$$\chi^{\mathrm{orb}}=(2-2g)-\sum_{j=1}^{r}(1-1/p_j)$$

respectively.

This description is useful to construct a geometric structure on a given Seifert fibered space, however it has some disadvantage to construct a continuous family of structures as in the case of hyperbolic Dehn surgery and we had better use the unnormalized Dehn surgical invariant which has no obstruction class. It is defined again by pairs of exponents of s_j and h in the relation of the fundamental group but by choosing meridional elements $s_1, \cdots, s_r$ so that

$$s_1 \cdots s_r[a_1, b_1] \cdots [a_g, b_g]=e,$$

and hence we cannot expect normalizing inequalities $0<q_j<p_j$. In case there is no singular fiber, a regular fiber should be considered temporarily as a singular fiber with index $(1, n)$. The integer n in this case represents the euler class of the bundle.

In the next section, we deal only with Seifert fibered spaces over the 2-sphere with precisely 3 exceptional fibers. It can be seen as a result of Dehn surgery on (3 punctured 2-sphere)$\times S^1$. When its unnormalized indices are (p, p'), (q, q') and (r, r'), the orbifold invariants are $e^{\mathrm{orb}}=p'/p+q'/q+r'/r$ and $\chi^{\mathrm{orb}}=1/p+1/q+1/r$ respectively.

§5. Construction of geometric structures.

Let $\Delta^{\cdot}$ be a triangle and let Δ be $\Delta^{\cdot}-\{\text{vertices}\}$. By specifying $\Delta_{ABC}^{\cdot}$, we mean the geodesic triangle on P having the vertices A, B, C with corner angles σ_A, σ_B and σ_C. Here P is either S^2, E^2 or H^2 according to whether $\sigma_A+\sigma_B+\sigma_C$ is greater than, equal to or less than π. Also Δ_{ABC} stands for $\Delta_{ABC}^{\cdot}-\{A, B, C\}$.

Choose a geometric structure Σ_1 on $\Delta\times S^1$ such that

(1) $d(\Sigma_1)$ is isometric to $p^{-1}(\Delta_{ABC})$, and

(2) h(a generator of $\pi_1(\Sigma_1))=\theta\in S\subset G$.

This is unique up to isometry. Choose also a geometric structure Σ_2 on $\Delta\times S^1$ satisfying (2) and (1)* instead of (1).

(1)* $d(\Sigma_2)$ is isometric to $p^{-1}(\bar{\Delta}_{ABC})$, where $\bar{\Delta}_{ABC}$ is the reflection image of Δ_{ABC} along the geodesic $\overline{AB}$ on P.

Let us now construct an incomplete geometric structure on $Q\times S^1$ where Q is a 3 punctured 2-sphere. We first study the case of product geometries. By the definition of Σ_1 and Σ_2, we have an orientation reversing isometry $\phi: \Sigma_1\to\Sigma_2$ induced by an isometry of X which covers the reflection of P along $\overline{AB}$. We now use the symbol $\frown$ to indicate the face of Σ whose developed image projects to the geodesic specified under it. Then identify the face $\widehat{AB}$ of Σ_1 with $\widehat{AB}$ of Σ_2 by $\phi|\widehat{AB}$, $\widehat{AC}$ of Σ_1 with $\widehat{AC'}$ of Σ_2 by $-\theta_A{}^*\cdot\phi|\widehat{AC}$ and $\widehat{BC}$ of Σ_1 with $\widehat{BC'}$ of Σ_2 by $\theta_B{}^*\cdot\phi|\widehat{BC'}$ where C' is the new vertex of $\bar{\Delta}_{ABC}$. Here $-\theta_A{}^*(\theta_B{}^*)$ means an isometry of $\widehat{AC'}$ ($\widehat{BC'}$) to itself induced by $-\theta_A\in S$ ($\theta_B\in S$). This determines an incomplete geometric structure on $Q\times S^1$. Denote it by Ω.

We then compute the holonomy of Ω. To see this, we first need to specify the meridional loops α and β in Ω. Choose a base point b in Σ_1 and take three arcs which start from b and which terminate at three faces respectively. The union of the arcs with its image by ϕ is a tree in Ω. The loop α is defined by travelling from b to $\widehat{AC'}$ along the tree through $\widehat{AB}$, going down along the fiber on $\widehat{AC'}$ by θ_A (it then meets the tree again) and coming back to b again along the tree. The loop β is defined by travelling from b to $\widehat{BC}$ along the tree, going down along the fiber on $\widehat{BC'}$ by θ_B and coming back to b again along the tree through $\widehat{AB}$. Then by the construction of Ω, we have

LEMMA 5.1. *The holonomy of Ω is determined by*

$$\begin{cases} h(\alpha)=r_A(2\sigma_A)\theta_A \\ h(\beta)=r_B(2\sigma_B)\theta_B \\ h(f)=\theta\,, \end{cases}$$

where $r_*(2\sigma_*)$ stands for the element of $K\cong K\times\{0\}\subset K\times S=G$ which rotates P round the specified point $*$ with given angle $2\sigma_*$ and f is a fiber which goes through b.

Let us now look at the end of Ω around A. A pair of meridian and longitude is then given by α and f. To find when the completion of this end is a result of geometric Dehn surgery, let us check the conditions of Lemma 2.1. The condition (1) is obviously satisfied by the construction. To satisfy (2), θ and θ_A must be rationally related, say by $\theta_A{}^p=\theta^{-p'}$ for some pair of coprime integers p and p'. Then we need to require by the condition (3) that $(h(\alpha))^p=(h(f))^{-p'}$. Since $h(f)$ is contained in $S\subset G$, so must be $(h(\alpha))^p$. This means that $p(2\sigma_A)\equiv 0\ (2\pi)$. Finally the condition (4) requires that $2\sigma_A$ is actually equal to $2\pi/p$. Conversely since $(h(\alpha))^p=(r_A(2\pi/p)\theta_A)^p=\theta_A{}^p$, a non trivial solution of the equation, $\theta_A{}^p=\theta^{-p'}$ with respect to θ_A and θ for some coprime integers p and p', defines a geometric structure Ω of which the completion of the end around A becomes the result of (p, p') Dehn surgery with respect to α and f. The analogous condition for geometric completion of the end around B is obtained by the same way.

Since our geometry here is product, we have the identity.

$$r_A(2\sigma_A)r_B(2\sigma_B)r_C(2\sigma_C)=\mathrm{id}\times\{0\}$$

in G. Then $\gamma=(\alpha\beta)^{-1}$ and f constitute a pair of meridian and longitude for the end of Ω around C, and we get an equation in S for geometric completion with respect to θ_A, θ_B and θ for some pairs (p, p'), (q, q') and (r, r') of coprime integers,

$$\text{(I)}\qquad \begin{cases} p\theta_A+p'\theta=0 \\ q\theta_B+q'\theta=0 \\ -r(\theta_A+\theta_B)+r'\theta=0. \end{cases}$$

Its non trivial solution defines a geometric structure Ω which extends by completion to a geometric structure on the result of (p, p'), (q, q') and (r, r') Dehn surgeries on $Q\times S^1$. Here we wrote the equation linearly since S is abelian. Now there is a non trivial solution whenever $p'/p+q'/q+r'/r=0$ and by completion we get a Seifert fibered space over S^2 with 3 exceptional fibers unnormalizingly indexed by (p, p'), (q, q') and (r, r'). The condition is equivalent to the vanishing of euler numbers. The euler characteristic of the orbit orbifold, $1/p+1/q+1/r-1$, reflects on the geometry where Δ_{ABC} is in. Thus we have

PROPOSITION 5.2. *A Seifert fibered space over S^2 with 3 exceptional fibers with vanishing euler number admits a geometric structure of type either $S^2\times E$, E^3 or $H^2\times E$ according to whether the euler characteristic of the orbit orbifold is positive, zero or negative.*

REMARK. In fact there is no $S^2\times E$-manifold in this class but I state this for the generalization.

We next discuss the twisted case. The first different point is how to identify Σ_1 with Σ_2. Since we have no natural orientation reversing isometry between these as ψ, we need to find its substitution. To see this, remember that the identification of Σ_1 with Σ_2 can be understood by looking at the developed images. The developed images of Σ_1 and Σ_2 are $p^{-1}(\Delta_{ABC})$ and $p^{-1}(\bar{\Delta}_{ABC})$ respectively, and the identity map of X induces the identification of $\widehat{AB}$ of Σ_1 with $\widehat{AB}$ of Σ_2. Let $r_*(2\sigma_*)^{\cdot}$ be an isometry in G such that its projected image in K is $r_*(2\sigma_*)$ and such that it moves each point of the fiber on $*$ by $2\sigma_*$ in the positive direction. Then $r_A(2\sigma_A)^{\cdot}$ induces the isometry ψ_A of $\widehat{AC}$ to $\widehat{AC'}$ and $r_B(2\sigma_B)^{\cdot}$ induces the isometry ψ_B of $\widehat{BC}$ to $\widehat{BC'}$. Notice that $r_*(2\sigma_*)^{\cdot}$ is contained in the universal covering group $\tilde{K}\subset G$ of K in the case of spherical and Lorentz geometries. Then identify $\widehat{AC}$ with $\widehat{AC'}$ by $-\theta_A{}^*\cdot\psi_A$ and identify also $\widehat{BC}$ with $\widehat{BC'}$ by $\theta_B{}^*\cdot\psi_B$. Here $-\theta_A{}^*$ and $\theta_B{}^*$ mean the same as before. This clearly determines an incomplete geometric structure on $Q\times S^1$ and let us again denote it by Ω.

To see the holonomy of Ω, we specify the loops α and β with the following holonomical properties basically by the same method as before.

LEMMA 5.3. *The holonomy of Ω is determined by*

$$\begin{cases} h(\alpha)=r_A(2\sigma_A)^{\cdot}\theta_A \\ h(\beta)=r_B(2\sigma_B)^{\cdot}\theta_B \\ h(f)=\theta\,. \end{cases}$$

Then again by Lemma 2.1, we can find the condition for the end of Ω around A to be completed geometrically. It says that $2\sigma_A$ is actually equal to $2\pi/p$ for some integer p, and since $(h(\alpha))^p=(r_A(2\pi/p)^{\cdot}\theta_A)^p=2\pi+p\theta_A$, a solution of the equation; $2\pi+p\theta_A+p'\theta=0$ in S with respect to θ_A and θ for some integer p' coprime to p, defines a geometric structure Ω of which the completion of the end around A turns out to be the result of (p, p') Dehn surgery with respect to α and f. Here 2π means the unit simultaneous translation of fibers. There is an analogous condition for the geometric completion of the end around B.

We now come to the turning point. In the case of spherical and Lorentz geometries, $\tilde{K}$ was embedded in G, and $r_*(2\sigma_*)^{\cdot}$ was in $\tilde{K}$. The structure Ω is modeled on these geometries when $\sigma_A+\sigma_B+\sigma_C\neq\pi$. Then we have the identity;

$$(*) \qquad r_A(2\sigma_A)^{\cdot}r_B(2\sigma_B)^{\cdot}r_C(2\sigma_C)^{\cdot}=2\pi\in S\subset G\,.$$

The proof of this identity can be found in [1]. Again since $\gamma=(\alpha\beta)^{-1}$ and f constitute a pair of meridian and longitude for the end of Ω around C, to get a geometric structure Ω of which three ends can be geometrically completed simultaneously, we only need a non trivial solution of the linear equation in S:

$$\text{(II)} \qquad \begin{cases} 2\pi+p\theta_A+p'\theta=0 \\ 2\pi+q\theta_B+q'\theta=0 \\ 2\pi-r(2\pi+\theta_A+\theta_B)+r'\theta=0\,, \end{cases}$$

with respect to θ_A, θ_B and θ for some pair of coprime integers (p, p'), (q, q') and (r, r'). Again there is a solution iff $p'/p+q'/q+r'/r\neq0$ and $1/p+1/q+1/r-1\neq0$, and by replacing these conditions by euler numbers and euler characteristics, we have

PROPOSITION 5.4. *A Seifert fibered space over S^2 with 3 exceptional fibers with non vanishing euler number and non vanishing euler characteristic admits a spherical or Lorentz structure according to whether the euler characteristic of its orbit orbifold is positive or negative.*

The remaining is the nilpotent case. Turning point was the identity (*). In this case, $\sigma_A+\sigma_B+\sigma_C=\pi$ and we have the corresponding identity by elementary trigonometric calculus:

$$r_A(2\sigma_A)^{\cdot}r_B(2\sigma_B)^{\cdot}r_C(2\sigma_C)^{\cdot}=2\pi(1-\text{area}\,\Delta_{ABC})\in S\subset G\,.$$

Then the corresponding linear equation which follows from the same argument is

$$\text{(III)} \qquad \begin{cases} 2\pi+p\theta_A+p'\theta=0 \\ 2\pi+q\theta_B+q'\theta=0 \\ 2\pi-r(2\pi(1-\text{area}\,\Delta_{ABC})+\theta_A+\theta_B)+r'\theta=0\,, \end{cases}$$

and we have

PROPOSITION 5.5. *A Seifert fibered space over S^2 with 3 exceptional fibers with non vanishing euler number but vanishing euler characteristic admits a* *nilpotent structure.*

To construct a geometric structure for a given Seifert fibered space in general, the same argument works out almost equally well by starting with the fundamental domain of the orbit orbifold, which in our case is an a priori fixed quadrangle $ACBC'$. The only point we further need is a generalization of (*). In the case of Lorentz geometry, which is the most general case, if the orbit manifold has genus g with r elliptic point singularities of order $p_1, \cdots, p_r$, then the formula is

$$r_{A_1}(2\pi/p_1)^{\cdot} \cdots r_{A_r}(2\pi/p_r)^{\cdot}[a_1^{\cdot}b_1^{\cdot}] \cdots [a_g^{\cdot}b_g^{\cdot}]=2\pi(r-(2-2g)),$$

where A_j is the vertices of the fundamental domain corresponding to the singularities and $a_i^{\cdot}$ and $b_i^{\cdot}$ are any lifts of the projective images of a_i and b_i into the Fuchsian group $\pi_1(M)/\langle h\rangle\subset PSL_2\boldsymbol{R}$ to the universal covering group $S\widetilde{L}_2\boldsymbol{R}$. There is also a corresponding generalization of the formula for each other geometry. Eventually we can verify existence theorems in [4] and [6].

THEOREM. *A Seifert fibered space admits a geometric structure according to the table;*

euler characteristic / euler number	>0	$=0$	<0
$=0$	$S^2\times E$	E^3	$H^2\times E$
$\neq 0$	S^3	N	$S\widetilde{L}_2$

§ 6. Dehn surgery along the trefoil knot.

The complement of a torus knot admits a Seifert fibration over D^2 with 2 exceptional fibers. Thus every Dehn surgery along it produces a Seifert fibered space over S^2 with at most 3 exceptional fibers except when the Dehn surgical slope matches up with a regular fiber. This is the result of [3]. Taking the trefoil knot as an example, we review this fact from the geometric viewpoint.

The circle action on $S^3=\{|z_1|^2+|z_2|^2=1\}$ defined by $(z_1, z_2)\to(e^{3i\theta}z_1, e^{2i\theta}z_2)$ for $\theta\in S^1\cong\boldsymbol{R}/2\pi\boldsymbol{Z}$ gives a Seifert fibration of S^3 over S^2 with 2 exceptional fibers, $\{z_1=0\}\cap S^3$ and $\{z_2=0\}\cap S^3$. Any regular fiber is a trefoil knot in S^3 and therefore the fibration of S^3 also gives one on the complement of a trefoil knot by deleting one regular fiber K.

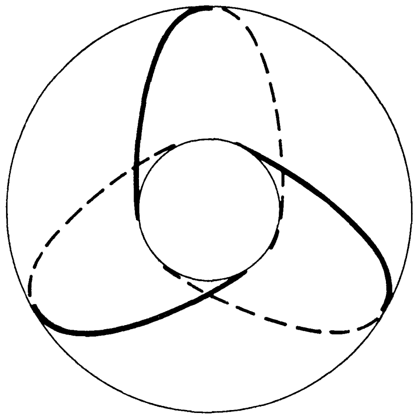

Figure 1.

We can choose meridional loops α and β for the singular fibers so that $\gamma=(\alpha\beta)^{-1}$ represents the ordinal meridian of K and so that the Dehn surgical invariants of singular fibers in terms of α, β and a regular fiber f are $(2, -1)$ and $(3, 1)$. Notice that the ordinal longitude is represented by $f\gamma^6$, and hence the (p, q) Dehn surgery along K in terms of γ and f is the $(p-6q, q)$ Dehn surgery along K in the usual sense.

Start with the geodesic triangle Δ_{ABC} on P such that $\sigma_A=\pi/2$, $\sigma_B=\pi/3$ and $\sigma_C=\pi/r$. P is again either S^2, E^2 or H^2 according to whether $r>6$, $=6$ or <6. Here r moves continuously in $6/5<r<\infty$. Introducing new continuous variable x and y, let us think of the linear equations (I)*, (II)* and (III)* obtained by letting $(p, p')=(2, -1)$, $(q, q')=(3, 1)$ and $(r, r')=(x+6y, y)$ in (I), (II), (III). Constructing a geometric structure on $Q\times S^1$ as in §5 and then taking completions around A and B, we eventually get an incomplete geometric structure on S^3-K. Then form a family of geometric structures on S^3-K continuously parametrized by x and y with some catastrophe. The structures corresponding to the solution of (I)* lie on the line: $\{x=0\}$ since it is the condition to get a non trivial solution. Similarly, the structures corresponding to the solutions of (II)* and (III)* lie on the open domain: $\{x\neq 0\}\cap\{x+6y\neq 6, >6/5\}$ and the line: $\{x\neq 0\}\cap\{x+6y=6\}$ respectively. Look at the x-y plane which is a coordinate of generalized Dehn surgery invariants in the usual sense. We describe it only for $x+6y\geqq 0$ because of its symmetry.

The geometric structure on S^3-K at (x, y), which is primitive in the integral lattice on the x-y plane, can be geometrically completed and its completion is diffeomorphic to the resultant manifold of (p, q) Dehn surgery along K in the usual sense. Thus we have eventually described geometric structures of such manifolds except when $(p, q)=(6, -1)$ and $(1-6q, q)$. Now think of the latter case. Since in this surgery the surgery slope intersects a regular fiber once,

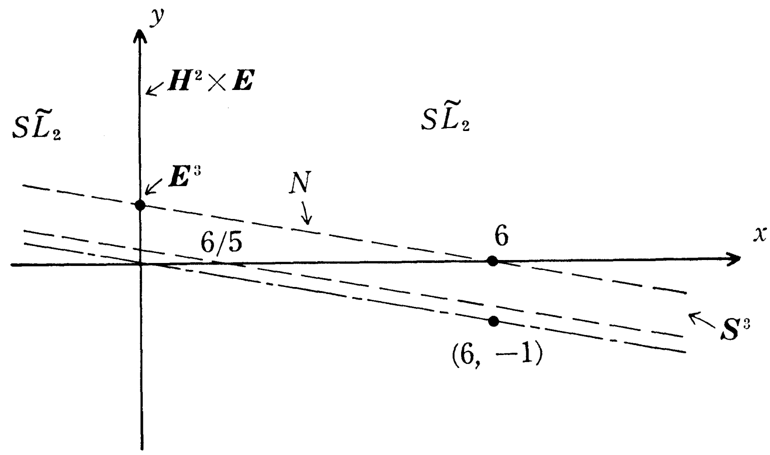

Figure 2.

the resultant manifold inherits a Seifert fibration from S^3-K without producing a new exceptional fiber, and hence is a lens space. The resultant fibration is a circle fibration over a 2-orbifold underlying on S^2 with two elliptic point singularities of order 2 and 3. Since it is bad, in particular it is not a 2-dimensional spherical orbifold. This is why our construction does not work out for this case. When $(p, q)=(6, -1)$, it lies completely outside of our argument. In fact the resultant manifold of the $(6, -1)$ Dehn surgery along K is a connected sum of lens spaces $L(3, 2)\sharp L(2, 3)$, and hence is not irreducible (see [**3**]).

Now we want to discuss the limit of geometric structures on S^3-K when (x, y) goes to (∞, ∞) along the line, $y=kx$, for some $k\in \boldsymbol{R}\cup\{\infty\}$. Then the geodesic triangle Δ_{ABC} on H^2 tends to a generalized one by making C approach the circle at ∞. At the limit, we get a complete end around C and hence a complete geometric structure on S^3-K itself. Notice that S^3-K admits both $H^2\times E$ and Lorentz structures according to $k=\infty$ or not. Let $M_{(x,y)}$ be a geometric structure on S^3-K at the point (x, y). Then since area $\Delta_{ABC}=(x+6y-6)\pi/6(x+6y)$ by the Gauss-Bonnet theorem, we have

$$\text{volume } M_{(x,y)}=\frac{2(x+6y-6)^2\pi^2}{3|x(x+6y)|}=\frac{2(1+6y/x-6/x)^2\pi^2}{3|1+6y/x|}$$

except for the case $x=0$. Thus if $y=kx$ $(k\neq\infty)$, then

$$\lim_{x\to\infty} \text{volume } M_{(x,y)}=\frac{2|1+6k|\pi^2}{3}.$$

In particular, the limiting complete geometric structure on S^3-K depends on the slope k. When $x=0$, the structure of $M_{(x,y)}$ itself is not uniquely determined and also there is no primitive pair on this line except $(0, 1)$. Hence we do not examine this case any more.

Now, the completion of $M_{(p,q)}$ was a Lorentz manifold for $p+6q>6$ and its Lorentz volume was a topological invariant. For a given $k\in\boldsymbol{R}$, there is a sequence of a pair of coprime integers $\{(p_i, q_i)\}_{i\geq 1}$ so that $p_i\to\infty$ and $q_i/p_i\to k$ when $i\to\infty$. Since the volume of $M_{(p_i,q_i)}$ is equal to the volume of its completion, we have

THEOREM. *The set of Lorentz volumes of the resultant manifolds of (p, q) Dehn surgery along the trefoil knot is dense in $\boldsymbol{R}^+$, where (p, q) runs over all primitive pairs with $p+6q>6$.*

REMARK. The distribution of Lorentz volumes in $\boldsymbol{R}^+$ contrasts with the results of Jørgensen and Thurston [7].

REMARK. Lorentz volume is a topological invariant for closed manifolds but not for non-compact manifolds. Actually S^3-K takes a Lorentz structure with arbitrary given volume.

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