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An L^p theory for Schrödinger operators with nonnegative potentials

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Introduction.

This paper is concerned with some properties of the Schrödinger type operator $-\Delta+V(x)$ with nonnegative potential $V(x)\geqq 0$ in $L^p=L^p(\boldsymbol{R}^m)$ $(1<p<\infty)$. We consider the operator $-\Delta+V(x)$ as a linear *accretive* operator in L^p. The m-accretivity problem for such operators is a natural generalization of the selfadjointness problem for the special case of $p=2$.

A linear operator A with domain $D(A)$ and range $R(A)$ in L^p is said to be *accretive* if

$$\text{(A)} \qquad \mathrm{Re}(Au,\ |u|^{p-2}u)\geqq 0 \quad \text{for} \quad u\in D(A).$$

Here (f, g) denotes the pairing between $f\in L^p$ and $g\in L^q$ $(p^{-1}+q^{-1}=1)$, and (f, g) is linear in f and semilinear in g. It is well known (see e.g. Tanabe [17], Proposition 2.1.5) that condition (A) is equivalent to

$$\text{(A')} \qquad \|(A+\xi)u\|\geqq\xi\|u\| \quad \text{for all} \quad u\in D(A) \quad \text{and} \quad \xi>0.$$

If in addition $R(A+\xi)=L^p$ for some (and hence for every) $\xi>0$ then we say that A is *m-accretive*. A nonnegative selfadjoint operator is a typical example of m-accretive operators in L^2.

Now let $u\in C_0^\infty(\boldsymbol{R}^m)$. Then we have, for $p\geqq 2$,

$$\mathrm{Re}(-\Delta u,\ |u|^{p-2}u)\geqq(p-1)\int_{\boldsymbol{R}^m}|u(x)|^{p-4}\sum_{j=1}^m\left[\mathrm{Re}\frac{\partial u}{\partial x_j}\overline{u(x)}\right]^2dx.$$

If $1<p<2$ then the integral on the right-hand side should be replaced by

$$(p-1)\lim_{\delta\downarrow 0}\int_{\boldsymbol{R}^m}[|u(x)|^2+\delta]^{(p-4)/2}\sum_{j=1}^m\left[\mathrm{Re}\frac{\partial u}{\partial x_j}\overline{u(x)}\right]^2dx.$$

Let $V(x)\in L_{\mathrm{loc}}^p(\boldsymbol{R}^m)$. Then we have

$$\mathrm{Re}(V(x)u,\ |u|^{p-2}u)=\int_{\boldsymbol{R}^m}V(x)|u(x)|^p dx.$$

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Therefore, $-\Delta+V(x)+c$ (c a constant) is accretive in L^p if $V(x)$ is bounded below. So, we assume throughout this paper that $V(x)$ is nonnegative and hence $-\Delta+V(x)$ itself is accretive.

The main purpose of this paper is to present sufficient conditions for $-\Delta+V(x)$ to be m-accretive in L^p. Here the domain of $-\Delta+V(x)$ is equal to the intersection of those of $-\Delta$ and $V(x)$. The result is a generalization of those in Everitt-Giertz [**3**], Sohr [**16**] and Okazawa [**11**] to the case of $p\neq 2$. For example, $-\Delta+t|x|^{-2}$ is m-accretive in L^p if $t>p-1$. The proof is based on an abstract perturbation theorem for linear m-accretive operators in a reflexive Banach space. It should be noted that the result is also regarded as an explicit characterization of the domain of $[-\Delta+V(x)]_{\max}$ in the sense of Kato [**7**]. In this connection we note that the closure of $[-\Delta+V(x)]_{\min}$ is m-accretive in L^p because $V(x)\geqq 0$ is in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)$. This fact is pointed out by Semenov [**15**] as an application of the Kato inequality.

This paper is divided into four sections. The assertions on the m-accretivity of $-\Delta+V(x)$ are stated in §2 (see Theorems 2.1 and 2.5). §1 is the preliminaries. In §3 we consider the regularity of solutions of the Schrödinger type equations:

$$-\Delta u(x)+V(x)u(x)+\xi u(x)=v(x) \quad \text{on} \quad \boldsymbol{R}^m .$$

The result is a generalization of that in Sohr [**16**] to the case of $p\neq 2$. The proof depends on the relation of $-\Delta+V(x)$ to its adjoint operator $[-\Delta+V(x)]^*$ which will be established in §2. In particular, we shall present a criterion for the equality

$$D([-\Delta+V(x)]^\infty)=\bigcap_{n=1}^{\infty} D([-\Delta+V(x)]^n)=S(\boldsymbol{R}^m)$$

to hold, where $S(\boldsymbol{R}^m)$ is the Schwartz space of all rapidly decreasing functions on $\boldsymbol{R}^m$ (see Theorem 3.6 and Corollary 3.7). The result seems to be new even if $p=2$. The last §4 is concerned with the compactness of the resolvent

$$[-\Delta+V(x)+\zeta]^{-1}, \qquad \mathrm{Re}\,\zeta>0,$$

under an additional assumption that $V(x)\to\infty$ ($|x|\to\infty$).

§1. Preliminaries.

Let $V(x)\geqq 0$ be a function in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)$ $(1<p<\infty)$. Then $S_p=-\Delta+V(x)$ is well defined as a linear accretive operator in $L^p=L^p(\boldsymbol{R}^m)$; $D(S_p)$ contains $C_0^\infty(\boldsymbol{R}^m)$.

Let A be a linear accretive operator defined on a dense linear subspace D of a Banach space. Then A is closable (see Lumer-Phillips [**9**], Lemma 3.3) and its closure $\tilde{A}$ is also accretive. If in particular the closure $\tilde{A}$ is m-accretive, then we say that A is *essentially m-accretive* on D. In this case $\tilde{A}$ is a unique m-accretive extension of A.

The following theorem is an L^p version of the well known result of Kato [6] (see e.g. Faris [4], Kuroda [8] or Reed-Simon [12]) and is explicitly stated in Semenov [15].

THEOREM 1.1. *Let $V(x)\geqq 0$ be a function in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)$ $(1<p<\infty)$. Then $S_p=-\Delta+V(x)$ is essentially m-accretive on $C_0^\infty(\boldsymbol{R}^m)$.*

Let X be a reflexive Banach space and X^* be its adjoint. Then a linear accretive operator A with domain dense in X is essentially m-accretive on $D(A)$ if and only if its adjoint A^* is accretive in X^*. Note that in this case A^* is also m-accretive because $A^{**}=\tilde{A}$.

COROLLARY 1.2. *Let $V(x)\geqq 0$ be a function in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)\cap L^q_{\mathrm{loc}}(\boldsymbol{R}^m)$, $p^{-1}+q^{-1}=1$ $(1<p<\infty)$. Let S_p be as in Theorem 1.1. Then the adjoint of S_q is equal to $\tilde{S}_p$: $S_q^*=\tilde{S}_p$.*

In particular, $\tilde{S}_2$ is a nonnegative selfadjoint operator in L^2.

PROOF. Let ϕ, $\psi\in C_0^\infty(\boldsymbol{R}^m)$. Then we have

$$(-\Delta\phi+V(x)\phi,\ \psi)=(\phi,\ -\Delta\psi+V(x)\psi)$$

and hence $(\tilde{S}_p u, \psi)=(u, S_q\psi)$ for all $u\in D(\tilde{S}_p)$. This implies that $S_q^*\supset\tilde{S}_p$. But, $S_q^*=(\tilde{S}_q)^*$ is also m-accretive in L^p. Therefore, we obtain $S_q^*=\tilde{S}_p$. Q.E.D.

REMARK 1.3. $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)\cap L^q_{\mathrm{loc}}(\boldsymbol{R}^m)=L^r_{\mathrm{loc}}(\boldsymbol{R}^m)$ when we set $r=\max\{p, q\}$.

Let B be a linear m-accretive operator in L^p. Then $\{B_\varepsilon\}$ denotes the *Yosida approximation* of B:

$$B_\varepsilon=B(1+\varepsilon B)^{-1}=\varepsilon^{-1}[1-(1+\varepsilon B)^{-1}], \qquad \varepsilon>0.$$

B is approximated by $\{B_\varepsilon\}$ in the following sense:

$$\|Bu-B_\varepsilon u\|\to 0 \quad (\varepsilon\to+0) \qquad \text{for every} \quad u\in D(B).$$

Note that $D(B)$ is necessarily dense in L^p (see Yosida [18], VIII-§4).

LEMMA 1.4. *Let A and B be linear m-accretive operators in L^p. Let D be a core of A. Assume that there are nonnegative constants c, a and b $(b\leqq 1)$ such that for all $u\in D$,*

$$\mathrm{Re}(Au,\ F(B_\varepsilon u))\geqq -c\|u\|^2-a\|B_\varepsilon u\|\|u\|-b\|B_\varepsilon u\|^2, \tag{1.1}$$

where $F(B_\varepsilon u)=\|B_\varepsilon u\|^{2-p}|B_\varepsilon u|^{p-2}B_\varepsilon u$, $\varepsilon>0$.

If $b<1$ then $A+B$ with $D(A+B)=D(A)\cap D(B)$ is also m-accretive. If $b=1$ then $A+B$ is essentially m-accretive on $D(A+B)$.

PROOF. It suffices to show that (1.1) holds for all $u\in D(A)$ (see [11], Theorem 4.2). Let $u\in D(A)$. Then there is a sequence $\{u_n\}$ in D such that

$u_n \to u$ and $Au_n \to Au$ $(n\to\infty)$. $B_\varepsilon u_n \to B_\varepsilon u$ $(n\to\infty)$ is a consequence of the boundedness of B_ε. Therefore, $F(B_\varepsilon u_n) \to F(B_\varepsilon u)$ $(n\to\infty)$ follows from the continuity of the "duality map" F (see Kato [**5**], Lemma 1.2). Q.E.D.

REMARK 1.5. It is easy to see that $F(B_\varepsilon u_n)$ tends to $F(B_\varepsilon u)$ weakly. Let $\{F(B_\varepsilon u_{n_k})\}$ be any weakly convergent subsequence of $\{F(B_\varepsilon u_n)\}$. Then $\|f\| \leqq \liminf_{k\to\infty} \|F(B_\varepsilon u_{n_k})\| = \|B_\varepsilon u\|$ where $f = \text{w-}\lim_{k\to\infty} F(B_\varepsilon u_{n_k})$. On the other hand, we have $(B_\varepsilon u_n, F(B_\varepsilon u_n)) = \|B_\varepsilon u_n\|^2$ and hence $(B_\varepsilon u, f) = \|B_\varepsilon u\|^2$. So, we obtain $f = F(B_\varepsilon u)$.

§2. The m-accretivity of $-\Delta + V(x)$.

Let $V(x) > 0$ be a function in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m \setminus \{0\})$ and set

$$V_\varepsilon(x) = V(x)[1+\varepsilon V(x)]^{-1}, \qquad \varepsilon > 0 .$$

We denote by $B = B_p$ the maximal multiplication operator by $V(x)$:

$$Bu(x) = B_p u(x) = V(x)u(x)$$

for $u \in D(B) = \{u, V(x)u \in L^p\}$. Then B_p is m-accretive in L^p and the Yosida approximation of B_p is given by

$$B_\varepsilon u(x) = B_{p,\varepsilon} u(x) = V_\varepsilon(x)u(x) .$$

Let $A = A_p$ be the minus Laplacian in L^p:

$$Au(x) = A_p u(x) = -\Delta u(x) \qquad \text{for} \quad u \in D(A) = W^{2,p}(\boldsymbol{R}^m),$$

where $W^{2,p}(\boldsymbol{R}^m)$ is the usual Sobolev space. Then A_p is also m-accretive in L^p (cf. Tanabe [**17**], Chapter 3, §3.1).

We consider the m-accretivity of $A+B = A_p + B_p = -\Delta + V(x)$ with $D(A+B) = W^{2,p}(\boldsymbol{R}^m) \cap D(B)$ in $L^p = L^p(\boldsymbol{R}^m)$.

THEOREM 2.1. *Let A and B be as above. Assume that $V_\varepsilon(x)$ is a function of class $C^1(\boldsymbol{R}^m)$ and there are nonnegative constants c, a and b $(b \leqq 4(p-1)^{-1})$ such that on $\boldsymbol{R}^m$*

$$(2.1) \qquad |\operatorname{grad} V_\varepsilon(x)|^2 \leqq cV_\varepsilon(x) + a[V_\varepsilon(x)]^2 + b[V_\varepsilon(x)]^3, \qquad \varepsilon > 0 .$$

In the case of $1<p<2$ assume further that $c=0$.

If $b < 4(p-1)^{-1}$ then $A+B = -\Delta + V(x)$ is m-accretive in L^p. If $b = 4(p-1)^{-1}$ then $A+B$ is essentially m-accretive on $D(A+B)$.

PROOF. In order to apply Lemma 1.4, we shall show that for all $u \in C_0^\infty(\boldsymbol{R}^m)$,

$$(2.2) \qquad 4\operatorname{Re}(Au, F(B_\varepsilon u)) \geqq -(p-1)(c\|u\|^2 + a\|B_\varepsilon u\|\|u\| + b\|B_\varepsilon u\|^2) .$$

Since $|B_\varepsilon u(x)|^{p-2} B_\varepsilon u(x) = [V_\varepsilon(x)]^{p-1} |u(x)|^{p-2} u(x)$, we have

$$(Au,\ |B_\varepsilon u|^{p-2}B_\varepsilon u)=-\int_{R^m}a(x)|u(x)|^{p-2}\overline{u(x)}\Delta u(x)dx\,,$$

where we set $a(x)=[V_\varepsilon(x)]^{p-1}$. Let $p\geqq 2$. Then it follows from the same calculation as in § 5.1 of [**10**] that

$$\mathrm{Re}(Au,\ |B_\varepsilon u|^{p-2}B_\varepsilon u)\geqq\frac{1}{p}\sum_{j=1}^{m}\int_{R^m}\frac{\partial a}{\partial x_j}\frac{\partial}{\partial x_j}|u(x)|^p dx$$

$$+(p-1)\int_{R^m}a(x)|u(x)|^{p-4}\sum_{j=1}^{m}\left[\mathrm{Re}\frac{\partial u}{\partial x_j}\overline{u(x)}\right]^2 dx\,.$$

The first term on the right-hand side is larger than

$$-(p-1)\int_{R^m}a(x)|u(x)|^{p-4}\sum_{j=1}^{m}\left[\mathrm{Re}\frac{\partial u}{\partial x_j}\overline{u(x)}\right]^2 dx$$

$$-4^{-1}(p-1)^{-1}\int_{R^m}[a(x)]^{-1}|\mathrm{grad}\,a(x)|^2|u(x)|^p dx\,.$$

Therefore, we obtain

$$\mathrm{Re}(Au,\ F(B_\varepsilon u))\geqq-\frac{\|B_\varepsilon u\|^{2-p}}{4(p-1)}\int_{R^m}|\mathrm{grad}\,a(x)|^2\frac{|u(x)|^p}{a(x)}dx\,.$$

This inequality holds even if $1<p<2$. In fact, we can show that for any $\delta>0$.

$$-\mathrm{Re}\int_{R^m}a(x)[|u(x)|^2+\delta]^{(p-2)/2}\overline{u(x)}\Delta u(x)dx$$

$$\geqq-4^{-1}(p-1)^{-1}\int_U[a(x)]^{-1}|\mathrm{grad}\,a(x)|^2[|u(x)|^2+\delta]^{p/2}dx\,,$$

where U is a sufficiently large ball containing the support of u. By a simple calculation we see from (2.1) that

$$(p-1)^{-2}[a(x)]^{-1}|\mathrm{grad}\,a(x)|^2\leqq c[V_\varepsilon(x)]^{p-2}+a[V_\varepsilon(x)]^{p-1}+b[V_\varepsilon(x)]^p\,.$$

Using the Hölder inequality we obtain (2.2) for all $u\in C_0^\infty(\boldsymbol{R}^m)$. Noting that $C_0^\infty(\boldsymbol{R}^m)$ is a core of A, the conclusion follows from Lemma 1.4. Q.E.D.

Let $W(x)>0$ be another function in $L_{\mathrm{loc}}^p(\boldsymbol{R}^m\setminus\{0\})$. We denote by C the maximal multiplication operator by $W(x)$. As for the m-accretivity of $A+B+C$ with

$$D(A+B+C)=W^{2,p}(\boldsymbol{R}^m)\cap D(B)\cap D(C)\,,$$

we have

COROLLARY 2.2. *Let A, B and C be as above. Assume that both $V_\varepsilon(x)$ and $W_\varepsilon(x)$ are functions of class $C^1(\boldsymbol{R}^m)$ satisfying* (2.1) *with $b<4(p-1)^{-1}$. Then $A+B+C=-\Delta+V(x)+W(x)$ is m-accretive in L^p.*

In fact, we have (2.2) with A and B replaced by $A+B$ and C, respectively.

Next, let $V(x)>0$ be a continuous function on $\boldsymbol{R}^m\setminus\{0\}$; namely, $V(x)\in$

$L^p_{\mathrm{loc}}(\boldsymbol{R}^m \setminus \{0\})$ for every p $(1<p<\infty)$. Set

(2.3) $$b_0(p)=\min\{4(p-1),\ 4(p-1)^{-1}\} \qquad (1<p<\infty).$$

Then we have

COROLLARY 2.3. *Let A_p and B_p be as in Theorem* 2.1. *If $b<b_0(p)$ in* (2.1) *then*

(2.4) $$A_p+B_p=(A_q+B_q)^* \qquad (p^{-1}+q^{-1}=1).$$

PROOF. Noting that $p-1=(q-1)^{-1}$, we see from Theorem 2.1 (with $c=0$ except the case of $p=2$) that A_p+B_p and A_q+B_q are m-accretive in L^p and L^q, respectively. For $u\in W^{2,p}(\boldsymbol{R}^m)$ and $v\in W^{2,q}(\boldsymbol{R}^m)$ we have

$$((A_p+B_{p,\varepsilon})u,\ v)=(u,\ (A_q+B_{q,\varepsilon})v).$$

Going to the limit $\varepsilon\to+0$, we obtain

$$((A_p+B_p)u,\ v)=(u,\ (A_q+B_q)v)$$

for all $u\in D(A_p+B_p)$ and $v\in D(A_q+B_q)$. The rest part is the same as in the proof of Corollary 1.2. Q.E.D.

REMARK 2.4. The maximum of $b_0(p)$ is attained at $p=2$ (the selfadjoint case).

THEOREM 2.5. *Let A and B be as in Theorem* 2.1. *Assume instead of* (2.1) *that $V(x)\geqq 0$ is of class $C^1(\boldsymbol{R}^m)$ and*

(2.5) $$|\operatorname{grad} V(x)|^2\leqq a[V(x)+c_1]^2+b[V(x)+c_2]^3 \qquad on \quad \boldsymbol{R}^m,$$

where c_1, c_2, a and b $(b\leqq 4(p-1)^{-1})$ are nonnegative constants. Then the conclusion of Theorem 2.1 *holds. If in particular $b<4(p-1)^{-1}$ then $C_0^\infty(\boldsymbol{R}^m)$ is a core of $A+B$.*

PROOF. It suffices to show that $A+(B+1)$ (or its closure) is m-accretive. So, we may assume that $V(x)\geqq 1$. In fact, $V(x)$ in (2.5) can be replaced by $V(x)+1$. Noting this, we obtain (2.1) with $c=0$:

$$\begin{aligned}|\operatorname{grad} V_\varepsilon(x)|^2&=|\operatorname{grad} V(x)|^2[1+\varepsilon V(x)]^{-4}\\ &\leqq b[V_\varepsilon(x)]^3+[a(c_1+1)^2+b(c_2+1)^3][V_\varepsilon(x)]^2.\end{aligned}$$

It remains to show that $[(A+B)|C_0^\infty(\boldsymbol{R}^m)]^\sim=A+B$. But, since $V(x)\geqq 0$ is a function in $L^p_{\mathrm{loc}}(\boldsymbol{R}^m)$, this follows from Theorem 1.1. Q.E.D.

EXAMPLE 2.6. (i) Let $V(x)=\exp(|x|^k)$, $k\geqq 1$. Then for any $\delta>0$ we have

$$\begin{aligned}|\operatorname{grad} V(x)|^2&=k^2|x|^{2(k-1)}[V(x)]^2\\ &\leqq k\delta^{-(k-1)}[V(x)]^2+2k(k-1)\delta[V(x)]^3.\end{aligned}$$

(ii) Let $W(x)=|x|^{-l}$ $(l>2)$. Then $W_{\varepsilon}(x)=(|x|^{l}+\varepsilon)^{-1}$ and for any $\delta>0$ we have

$$|\operatorname{grad} W_{\varepsilon}(x)|^{2}\leqq l^{2}|x|^{l-2}[W_{\varepsilon}(x)]^{3}$$
$$\leqq l(l-2)\delta^{-2/(l-2)}[W_{\varepsilon}(x)]^{2}+2l\delta[W_{\varepsilon}(x)]^{3}.$$

Thus, we see from Corollary 2.2 that $-\Delta+c_{1}\exp(|x|^{k})+c_{2}|x|^{-l}$ is m-accretive in L^{p} $(k\geqq 1,\ l>2)$, where $c_{1},\ c_{2}\geqq 0$ are constants.

EXAMPLE 2.7. Let $V(x)=\beta|x|^{-2}$, where $\beta\geqq p-1$ is a constant. Then $|\operatorname{grad} V_{\varepsilon}(x)|^{2}\leqq 4\beta^{-1}[V_{\varepsilon}(x)]^{3}$ (cf. [**11**], Example 6.6). So, we have

$$\operatorname{Re}(Au,\ F(B_{\varepsilon}u))\geqq-(p-1)\beta^{-1}\|B_{\varepsilon}u\|^{2} \qquad \text{for} \quad u\in W^{2,p}(\boldsymbol{R}^{m}).$$

Therefore, $A+B=-\Delta+\beta|x|^{-2}$ $(\beta>p-1)$ is m-accretive in L^{p} and $-\Delta+(p-1)|x|^{-2}$ is essentially m-accretive on $D(A+B)$.

REMARK 2.8. Let A and B be as in Theorem 2.1 or 2.5. Then it follows from (2.2) that for all $u\in D(A)$,

$$\|B_{\varepsilon}u\|\leqq(1-b_{1})^{-1}\|(A+B_{\varepsilon})u\|+K\|u\|,$$

where $K=a_{1}(1-b_{1})^{-1}+[c_{1}(1-b_{1})^{-1}]^{1/2}$ and we have set $b_{1}=(p-1)b/4<1$ and so on (see [**11**], Lemma 1.1). Going to the limit $\varepsilon\rightarrow+0$, we have

$$\|Bu\|\leqq(1-b_{1})^{-1}\|(A+B)u\|+K\|u\|, \qquad u\in D(A+B),$$

and hence

$$\|Au\|\leqq[(1-b_{1})^{-1}+1]\|(A+B)u\|+K\|u\|, \qquad u\in D(A+B). \tag{2.6}$$

These inequalities represent the separation property of $A+B$ (see e.g. Evans-Zettl [**2**], Everitt-Giertz [**3**]).

§3. The invariant sets for the resolvents.

Let $\boldsymbol{N}$ be the set of all positive integers. In this section we shall use the multi-index notation:

$$\alpha=(\alpha_{1},\ \alpha_{2},\ \cdots,\ \alpha_{m}) \qquad \text{with} \quad |\alpha|=\sum_{j=1}^{m}\alpha_{j}, \quad \alpha_{j}\in\boldsymbol{N}\cup\{0\};$$

$D^{\alpha}u$ denotes a mixed partial derivative of u:

$$D^{\alpha}u=D_{1}^{\alpha_{1}}D_{2}^{\alpha_{2}}\cdots D_{m}^{\alpha_{m}}u, \qquad D_{j}^{\alpha_{j}}u=\partial^{\alpha_{j}}u/\partial x_{j}^{\alpha_{j}} \ (1\leqq j\leqq m).$$

Let $W^{k,p}(\boldsymbol{R}^{m})$ be the usual Sobolev space. Let A_{p} and B_{p} be as in Theorem 2.1:

$$A_{p}+B_{p}=-\Delta+V(x) \qquad \text{with} \quad D(A_{p}+B_{p})=W^{2,p}(\boldsymbol{R}^{m})\cap D(B_{p}).$$

Then, under some additional assumption, it is expected that $W^{k,p}(\boldsymbol{R}^{m})$ is mapped into $W^{k+2,p}(\boldsymbol{R}^m)$ by $(A_p+B_p+\xi)^{-1}$, $\xi>0$. More precisely, we have

PROPOSITION 3.1. *Let $k\in\boldsymbol{N}$ and $V(x)\geqq 0$ be a function of class $C^k(\boldsymbol{R}^m)$. Assume that there exist constants c_1, $c_2\geqq 0$ such that for all α with $|\alpha|\leqq k$,*

$$|D^\alpha V(x)|\leqq c_1+c_2V(x) \quad \text{on} \quad \boldsymbol{R}^m. \tag{3.1}$$

Set $u=(A_p+B_p+\xi)^{-1}v$ for $v\in W^{k,p}(\boldsymbol{R}^m)$ and $\xi>0$. Then we have

$$u\in W^{k+2,p}(\boldsymbol{R}^m), \qquad D^\alpha u\in D(B_p) \ (|\alpha|\leqq k). \tag{3.2}$$

PROOF. It follows from (3.1) with $|\alpha|=1$ that (2.5) with $b=0$ is satisfied. So, we see from Theorem 2.5 and Corollary 2.3 that A_p+B_p is m-accretive in L^p for all p $(1<p<\infty)$ and (2.4) holds.

Now we show that the assertion is true for $k=1$. To this end, it suffices to show that $\partial u/\partial x_j\in D(A_p+B_p)$ $(1\leqq j\leqq m)$ if $v\in W^{1,p}(\boldsymbol{R}^m)$. Since $u\in D(B_p)$, it follows from (3.1) with $|\alpha|=1$ that $(\partial V/\partial x_j)u\in L^p$. Consequently, we have

$$\left(\frac{\partial u}{\partial x_j}, -\Delta\phi+V(x)\phi+\xi\phi\right)=\left(\frac{\partial v}{\partial x_j}-\frac{\partial V}{\partial x_j}u, \phi\right), \qquad \phi\in C_0^\infty(\boldsymbol{R}^m).$$

Noting that $C_0^\infty(\boldsymbol{R}^m)$ is a core of A_q+B_q $(p^{-1}+q^{-1}=1)$, we see that for all $\phi\in D(A_q+B_q)$,

$$\left(\frac{\partial u}{\partial x_j}, (A_q+B_q+\xi)\phi\right)=\left(\frac{\partial v}{\partial x_j}-\frac{\partial V}{\partial x_j}u, \phi\right).$$

This implies that $\partial u/\partial x_j\in D(A_p+B_p)$ (see (2.4)).

Next, suppose that the assertion is true for all α with $|\alpha|\leqq k-1$. It then follows that

$$u\in W^{k+1,p}(\boldsymbol{R}^m), \qquad D^\beta u\in D(B_p) \ (|\beta|\leqq k-1)$$

because $v\in W^{k-1,p}(\boldsymbol{R}^m)$. Let $|\alpha|=k$. Then we have

$$(D^\alpha u, V(x)\phi)=(-1)^{|\alpha|}(V(x)u, D^\alpha\phi)-(w, \phi), \qquad \phi\in C_0^\infty(\boldsymbol{R}^m),$$

where $w(x)=\sum_{\beta<\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}V(x)\cdot D^\beta u(x)$. By virtue of (3.1) we see that $D^{\alpha-\beta}V(x)\cdot D^\beta u\in L^p$ and hence so is w, too. So, we obtain

$$(D^\alpha u, -\Delta\phi+V(x)\phi+\xi\phi)=(D^\alpha v-w, \phi), \qquad \phi\in C_0^\infty(\boldsymbol{R}^m).$$

In the same way as in the case of $k=1$ we can conclude that $D^\alpha u\in D(A_p+B_p)$ for $|\alpha|=k$. Q.E.D.

It follows from (3.2) that for $u=(A_p+B_p+\xi)^{-1}v$,

$$D^\alpha[V(x)u]=\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}V(x)\cdot D^\beta u \qquad (|\alpha|\leqq k). \tag{3.3}$$

Let $b_0(p)$ be the function which was used in Corollary 2.3. Writing

$x^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_m^{\alpha_m}$ for a multi-index α, we have

PROPOSITION 3.2. *Let $V(x)\geqq 0$ be a function of class $C^1(\boldsymbol{R}^m)$ satisfying* (2.5) *with $b<b_0(p)$. Assume that there is a constant $M>0$ such that*

$$V(x)\geqq M|x| \qquad \textit{for sufficiently large } x. \tag{3.4}$$

If $v\in L^p$ and $x^{\alpha}v(x)\in L^p$ then we have

$$x^{\alpha}(A_p+B_p+\xi)^{-1}v(x)\in D(A_p+B_p) \qquad \textit{for} \quad \xi>0\,.$$

PROOF. By assumption we see from Theorem 2.5 and Corollary 2.3 that $A_p+B_p=(A_q+B_q)^*$ is m-accretive for all p and q, $p^{-1}+q^{-1}=1$ $(1<p<\infty)$.

Set $u=(A_p+B_p+\xi)^{-1}v$. Then we have formally

$$(x^{\alpha}u,\ \Delta\phi)=(\Delta u,\ x^{\alpha}\phi)-(u,\ \phi\Delta x^{\alpha}) \tag{3.5}$$
$$-2\sum_{j=1}^{m}\left(u,\ \frac{\partial x^{\alpha}}{\partial x_j}\frac{\partial\phi}{\partial x_j}\right), \qquad \phi\in C_0^{\infty}(\boldsymbol{R}^m)\,.$$

Now let $|\alpha|=1$, i.e., $x^{\alpha}=x_i$ for some i. Then we see from (3.4) that $u\in D(B_p)$ implies $x_iu(x)\in L^p$ $(1\leqq i\leqq m)$ and hence (3.5) makes sense for $|\alpha|=1$. So, we obtain

$$(x_iu,\ -\Delta\phi+V(x)\phi+\xi\phi)=(x_iv,\ \phi)-2\left(\frac{\partial u}{\partial x_i},\ \phi\right).$$

In the same way as in the proof of Proposition 3.1 we can conclude that $x_iu(x)\in D(A_p+B_p)$ $(1\leqq i\leqq m)$.

Next, suppose that the assertion is true for all α with $|\alpha|\leqq k-1$. Since $v\in L^p$ and $x^{\alpha}v(x)\in L^p$ $(|\alpha|=k)$, it follows that $x^{\beta}v(x)\in L^p$ and hence $x^{\beta}u(x)\in D(A_p+B_p)$ for all β with $|\beta|\leqq k-1$. Consequently, $(\partial x^{\alpha}/\partial x_j)u(x)$ and $u(x)\Delta x^{\alpha}$ belong to $W^{2,p}(\boldsymbol{R}^m)$ for $|\alpha|=k$. Furthermore, by virtue of (3.4) we see that $x^{\beta}u(x)\in D(B_p)$ $(|\beta|\leqq k-1)$ implies $x^{\alpha}u(x)\in L^p$ $(|\alpha|=k)$. Therefore, (3.5) makes sense for $|\alpha|=k$ and we obtain $x^{\alpha}u(x)\in D(A_p+B_p)$. Q.E.D.

EXAMPLE 3.3. Let $m=1$ and $V(x)=\cosh x$ on $\boldsymbol{R}$. Then $|V^{(n)}(x)|\leqq V(x)$ $(n\in\boldsymbol{N})$ and $V(x)\geqq\sqrt{2}\,|x|$ on $\boldsymbol{R}$.

REMARK 3.4. Let $V(x)=|x|^2$. Then $|\mathrm{grad}\,V(x)|^2\leqq 4[V(x)+1]^2$. Set $u=(A_p+B_p+\xi)^{-1}v$ for $v\in D(B_p)$ and $\xi>0$. Then Proposition 3.2 implies that $B_pu\in D(A_p+B_p)$.

Propositions 3.1 and 3.2 are unified as follows.

PROPOSITION 3.5. *Let $k\in\boldsymbol{N}$ and $V(x)\geqq 0$ be a function of class $C^k(\boldsymbol{R}^m)$ satisfying* (3.1) *and* (3.4). *Assume that*

$$x^{\alpha}D^{\beta}v(x)\in L^p \qquad \textit{for all } \alpha,\ \beta \textit{ with } \quad |\alpha+\beta|\leqq k\,.$$

Setting $u=(A_p+B_p+\xi)^{-1}v$, we have

$$x^\alpha D^\beta u(x) \in D(A_p+B_p) \qquad \textit{for all } \alpha, \beta \textit{ with } \quad |\alpha+\beta| \leqq k.$$

PROOF. (3.1) implies that A_p+B_p is m-accretive in L^p for all p $(1<p<\infty)$. If $k=1$ then the assertion is reduced to the preceding Propositions.

Suppose that the assertion is true for $k-1$:

$$x^\alpha D^\gamma u(x) \in D(A_p+B_p) \qquad \text{for all } \alpha, \gamma \text{ with } \quad |\alpha+\gamma| \leqq k-1. \tag{3.6}$$

Since $v \in W^{k,p}(\boldsymbol{R}^m)$ and $x^\alpha v(x) \in L^p$ $(|\alpha|=k)$, it follows from Propositions 3.1 and 3.2 that $D^\beta u \in D(A_p+B_p)$ $(|\beta| \leqq k)$ and $x^\alpha u(x) \in D(A_p+B_p)$ $(|\alpha|=k)$, respectively. Furthermore, in view of (3.3) we have

$$[-\Delta+V(x)+\xi]D^\beta u(x) = D^\beta v(x) - \sum_{\gamma<\beta} \binom{\beta}{\gamma} D^{\beta-\gamma}V(x)\cdot D^\gamma u(x). \tag{3.7}$$

Here, we see from (3.1) and (3.6) that

$$D^{\beta-\gamma}V(x)\cdot[x^\alpha D^\gamma u(x)] \in L^p \qquad (|\alpha+\gamma| \leqq k-1).$$

Denoting by $w(x)$ the right-hand side of (3.7), we have $w \in L^p$ and $x^\alpha w(x) \in L^p$. Applying Proposition 3.2 to the equation $[-\Delta+V(x)+\xi]D^\beta u = w$, we obtain

$$x^\alpha D^\beta u(x) \in D(A_p+B_p) \qquad (|\alpha+\beta| \leqq k,\ |\alpha| \geqq 1,\ |\beta| \geqq 1).$$

Q.E.D.

Let $S(\boldsymbol{R}^m)$ be the Schwartz space of all rapidly decreasing functions on $\boldsymbol{R}^m$:

$$S(\boldsymbol{R}^m) = \{f \in C^\infty(\boldsymbol{R}^m)\ ;\ \sup_x [\langle x\rangle^k |D^\alpha f(x)|] < \infty \text{ for all } k, \alpha\},$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $k \in \boldsymbol{N} \cup \{0\}$.

Setting $D((A_p+B_p)^\infty) = \bigcap_{n=1}^\infty D((A_p+B_p)^n)$, we have

THEOREM 3.6. *Let $V(x) \geqq 0$ be a function of class $C^\infty(\boldsymbol{R}^m)$ satisfying* (3.4). *Assume that* (3.1) *is satisfied for all α (so that $A_p+B_p = -\Delta+V(x)$ is m-accretive in L^p). Let $n \in \boldsymbol{N}$. Then $u \in D((A_p+B_p)^n)$ implies that*

$$x^\alpha D^\beta u(x) \in L^p \qquad \textit{for all } \alpha, \beta \quad \textit{with } |\alpha+\beta| \leqq n. \tag{3.8}$$

In particular, $D((A_p+B_p)^\infty) \subset S(\boldsymbol{R}^m)$.

The proof will be given after

COROLLARY 3.7. *Let $V(x)$ be a function as in Theorem* 3.6. *Then $D((A_p+B_p)^\infty) = S(\boldsymbol{R}^m)$ if and only if $V(x)f(x) \in S(\boldsymbol{R}^m)$ for every $f \in S(\boldsymbol{R}^m)$. In this case*

$$(A_p+B_p+\zeta)^{-1}S(\boldsymbol{R}^m) = S(\boldsymbol{R}^m), \qquad \mathrm{Re}\,\zeta>0.$$

PROOF OF THEOREM 3.6. (3.8) for $n=1$ is obvious. Suppose that (3.8) is true. Let $u \in D((A_p+B_p)^{n+1})$. Then, since $(A_p+B_p+1)u = v \in D((A_p+B_p)^n)$, we have (3.8) with u replaced by v. Therefore, it follows from Proposition 3.5 that

$$x^\alpha D^\beta u(x) \in D(A_p+B_p) \qquad \text{for all } \alpha,\ \beta \text{ with } \ |\alpha+\beta| \leqq n\,.$$

Thus, we can obtain (3.8) with n replaced by $n+1$.

Next, let $u \in D((A_p+B_p)^\infty)$. Then we see that (3.8) is true for all $n \in \boldsymbol{N}$ and hence

$$x^\alpha D^\beta u(x) \in W^{k,p}(\boldsymbol{R}^m) \qquad \text{for all } \alpha,\ \beta, \ \text{ and } \ k \in \boldsymbol{N}.$$

Therefore, it follows from the Sobolev imbedding theorem (see e.g. Adams [**1**]) that $u \in C^\infty(\boldsymbol{R}^m)$ and

$$\sup\{|x^\alpha D^\beta u(x)|\,;\, x \in \boldsymbol{R}^m\} < \infty \qquad \text{for all } \ \alpha,\ \beta\,.$$

Thus, we obtain the desired inclusion. Q.E.D.

REMARK 3.8. Corollary 3.7 does not apply to $V(x)=\cosh x$ (see Example 3.3). In fact, $2(e^x+e^{-x})^{-1} \in S(\boldsymbol{R})$.

§ 4. The compactness of the resolvents.

Let $V(x) \geqq 0$ be a function of class $C^1(\boldsymbol{R}^m)$ satisfying (2.5) with $b<4(p-1)^{-1}$:

$$|\operatorname{grad} V(x)|^2 \leqq b[V(x)+c]^3 \qquad \text{on } \ \boldsymbol{R}^m\,.$$

Then $A+B=-\Delta+V(x)$ with $D(A+B)=W^{2,p}(\boldsymbol{R}^m) \cap D(B)$ is m-accretive in $L^p = L^p(\boldsymbol{R}^m)$ (see Theorem 2.5). Consequently, $A+B+\zeta$ is invertible for every ζ with $\operatorname{Re}\zeta>0$ and $(A+B+\zeta)^{-1}$ is a bounded linear operator on L^p

THEOREM 4.1. *Let $A+B=-\Delta+V(x)$ be the linear m-accretive operator obtained in Theorem* 2.5. *Assume further that*

$$V(x) \to \infty \qquad (|x| \to \infty)\,.$$

Then the resolvent $(A+B+\zeta)^{-1}$ is compact for $\operatorname{Re}\zeta>0$ *and hence $A+B$ has discrete spectrum consisting entirely of eigenvalues with finite multiplicities.*

PROOF. It suffices by the resolvent equation to show that $(A+B+1)^{-1}$ is compact. Set

$$U=\{v \in L^p\,;\, \|v\| \leqq 1\}\,.$$

We shall show that $(A+B+1)^{-1}U$ is relatively compact in L^p. Let $v \in U$ and set $u=(A+B+1)^{-1}v$. Then $u \in W^{2,p}(\boldsymbol{R}^m)$ and $\|u\| \leqq \|v\| \leqq 1$. Moreover, it follows from an estimate for the Laplacian that

$$\|u\|_{1,p} \leqq c_0(\|Au\|+\|u\|)\,,$$

where $\|u\|_{1,p}$ is the norm of $W^{1,p}(\boldsymbol{R}^m)$ (see Schechter [**14**], Theorem 3.1 of Chapter 3, Lemma 2.1 of Chapter 11). So, we see from (2.6) that

$$\|u\|_{1,p} \leqq c_1\|(A+B)u\|+(c_2+c_0)\|u\| \leqq c_0+2c_1+c_2\,.$$

Thus, $(A+B+1)^{-1}U$ is bounded in $W^{1,p}(\boldsymbol{R}^m)$. It follows from the Rellich compactness theorem (see Adams [**1**]) that for any $R>0$, $(A+B+1)^{-1}U$ is relatively compact in $L^p(\Omega_R)$, where

$$\Omega_R=\{x\in\boldsymbol{R}^m\,;\ |x|\leqq R\}.$$

Now let $\{v_n\}$ be an arbitrary sequence in U and set $u_n=(A+B+1)^{-1}v_n$. Then by a diagonal method, we can find a subsequence of $\{u_n\}$ which converges in $L^p(\Omega_R)$ for any $R>0$. We denote this subsequence again by $\{u_n\}$. By the way, we note that

$$\begin{aligned}\int_{\boldsymbol{R}^m}V(x)|u_n(x)|^p dx&\leqq \mathrm{Re}((A+B)u_n,\ |u_n|^{p-2}u_n)\\ &\leqq\|(A+B)u_n\|\,\|u_n\|^{p-1}\leqq 2\,.\end{aligned}$$

By assumption, for any $\varepsilon>0$ there is $R=R(\varepsilon)>0$ such that

$$V(x)\geqq 2(2^p+1)\varepsilon^{-1}\qquad\text{for}\quad |x|\geqq R\,.$$

So, we have

$$\begin{aligned}\int_{|x|\geqq R}|u_n(x)|^p dx&\leqq(2^p+1)^{-1}\frac{\varepsilon}{2}\int_{|x|\geqq R}V(x)|u_n(x)|^p dx\\ &<(2^p+1)^{-1}\varepsilon\,.\end{aligned}$$

Since $\{u_n\}$ is a Cauchy sequence in $L^p(\Omega_R)$, there is a positive integer $n_0=n_0(\varepsilon)$ such that for $n,\ m\geqq n_0$,

$$\int_{|x|\leqq R}|u_n(x)-u_m(x)|^p dx<(2^p+1)^{-1}\varepsilon\,.$$

Therefore, we obtain for $n,\ m\geqq n_0$,

$$\begin{aligned}\|u_n-u_m\|^p&=\Big(\int_{|x|\leqq R}+\int_{|x|\geqq R}\Big)|u_n(x)-u_m(x)|^p dx\\ &<(2^p+1)^{-1}\varepsilon+2^{p-1}\int_{|x|\geqq R}(|u_n(x)|^p+|u_m(x)|^p)dx\\ &<[(2^p+1)^{-1}+2^p(2^p+1)^{-1}]\varepsilon=\varepsilon\,,\end{aligned}$$

i.e., $\{u_n\}$ is a Cauchy sequence in L^p. Q.E.D.

In the case of $p=2$ the assertion of Theorem 4.1 holds under the simplest assumption on $V(x)$ (see Reed-Simon [**13**], Theorem XIII.67).

In view of Theorem 3.6 we obtain

COROLLARY 4.2. *Let $V(x)\geqq 0$ be a function of class $C^\infty(\boldsymbol{R}^m)$ satisfying* (3.4):

$$V(x)\geqq M|x|\qquad\textit{for sufficiently large } x\,.$$

Assume that (3.1) *is satisfied for all* α:

$$|D^\alpha V(x)|\leqq c_1+c_2V(x)\qquad\textit{on}\quad \boldsymbol{R}^m\,.$$

Then the eigenfunctions of $A_p+B_p=-\Delta+V(x)$ belong to $S(\boldsymbol{R}^m)$ and hence the spectrum of A_p+B_p is independent of p.

The following example is well known.

EXAMPLE 4.3. Let $m=1$ and $V(x)=x^2$ on $\boldsymbol{R}$. Then

$$(A_p+B_p)u(x)=-u''(x)+x^2u(x).$$

The eigenvalues of A_p+B_p and the associated eigenfunctions are given by

$$\lambda_n=2n+1, \quad \phi_n(x)=e^{-x^2/2}H_n(x) \qquad (n=0, 1, 2, \cdots),$$

where $H_n(x)$ is the Hermite polynomial.

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Added in proof. After this paper was accepted for publication, the writer noticed that an estimate in Example 2.7 is partially improved as follows. Let $A=-\Delta$ and $B=\beta|x|^{-2}$ $(\beta>0)$. Then for all $u\in W^{2,p}(\boldsymbol{R}^m)$ we have

$$\mathrm{Re}(Au,\ F(B_\varepsilon u))\geqq -2(p-1)(2p-m)p^{-1}\beta^{-1}\|B_\varepsilon u\|^2.$$

This makes sense when $p<2m/3$. If in particular $p<m/2$ then we see that $\beta^{-1}B=|x|^{-2}$ is relatively bounded with respect to $A=-\Delta$: for $u\in D(A)\subset D(B)$,

$$\beta^{-1}\|Bu\|\leqq 2^{-1}p(p-1)^{-1}(m-2p)^{-1}\|Au\|$$

(cf. [**11**], Theorem 6.8).