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On minimal immersions of $\boldsymbol{R}^2$ into $\boldsymbol{P}^n(\boldsymbol{C})$

By Katsuei KENMOTSU*)

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Let $\boldsymbol{P}^n(\boldsymbol{C})$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4ρ and M^n be a compact (real) n-dimensional totally real, minimal submanifold of $\boldsymbol{P}^n(\boldsymbol{C})$. It is known by Chen and Ogiue [**1**] and Ludden, Okumura and Yano [**7**] that if the square length of the second fundamental form of the submanifold is smaller than or equal to $n(n+1)\rho/2n-1$, then M^n is totally geodesic or $n=2$ and M^2 is a flat Clifford torus in $\boldsymbol{P}^2(\boldsymbol{C})$. This gives the characterization of the flat torus in the set of totally real minimal immersions of M^2 into $\boldsymbol{P}^2(\boldsymbol{C})$.

The purpose of this paper is to generalize their theorem to the case of minimal immersions of real two dimensional surfaces into $\boldsymbol{P}^n(\boldsymbol{C})$ of any dimension n. The total realness of the immersion is not assumed previously. The first theorem of this paper proves that if the square length of the second fundamental form of a compact minimal surface M^2 in $\boldsymbol{P}^n(\boldsymbol{C})$ is smaller than or equal to 2ρ, then M^2 is superminimal or M^2 is flat and totally real in $\boldsymbol{P}^n(\boldsymbol{C})$.

Superminimal surfaces in $\boldsymbol{P}^n(\boldsymbol{C})$ have been studied extensively by Chern and Wolfson [**2**] and Wolfson [**9**]. Therefore we shall determine all isometric totally real, minimal immersions of the Euclidean 2-plane $\boldsymbol{R}^2$ into $\boldsymbol{P}^n(\boldsymbol{C})$. Main results of this paper insist that they constitute an $(n-2)$-parameter family and each of them must be an orbit of an abelian Lie subgroup of $U(n+1)$. These results are proved by applying the work by Chern and Wolfson [**2**].

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1. Preliminaries.

We introduce some results of Chern and Wolfson's theory [**2**] used for the proof of equations of Gauss and Codazzi of minimal immersions of a real surface into a Kaehlerian manifold.

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Let Y be a Kaehlerian manifold of complex dimension n of constant holomorphic sectional curvature 4ρ such that the metric is represented by $ds^2=\sum w_\alpha \bar{w}_\alpha$, where $\{w_\alpha\}$ is a local field of unitary coframes on Y and $\alpha, \beta, \gamma, \cdots$ run through 1 to n. We denote by $w_{\alpha\beta}$ the unitary connection forms with respect to the w_α. They satisfy structure equations:

$$dw_\alpha=\sum w_{\alpha\beta}\wedge w_\beta, \qquad w_{\alpha\beta}+\bar{w}_{\beta\alpha}=0, \tag{1.1}$$

$$dw_{\alpha\beta}=\sum w_{\alpha\gamma}\wedge w_{\gamma\beta}+\Omega_{\alpha\beta}, \tag{1.2}$$

$$\Omega_{\alpha\beta}=-\rho(w_\alpha\wedge\bar{w}_\beta+\delta_{\alpha\beta}\sum w_\gamma\wedge\bar{w}_\gamma). \tag{1.3}$$

Let M be an oriented real two dimensional Riemannian manifold and $x: M\to Y$ be an isometric minimal immersion. By using isothermal parameters, the Riemannian metric of M is written by $ds_M^2=\phi\bar{\phi}$, where ϕ is a complex valued one form and it is defined up to a complex factor of norm one.

It is proved in [2] that if x is neither holomorphic nor anti-holomorphic, then, except at isolated points, there exist a smooth real valued function α and fields of unitary coframes such that

$$w_1=\cos\frac{\alpha}{2}\cdot\phi, \quad w_2=\sin\frac{\alpha}{2}\cdot\bar{\phi}, \quad w_3=\cdots=w_n=0 \qquad \text{along } M, \tag{1.4}$$

and they satisfy

$$\frac{1}{2}[d\alpha+\sin\alpha\cdot(w_{11}+w_{22})]=a\phi. \tag{1.5}$$

$$w_{12}=c\bar{\phi}, \tag{1.6}$$

$$\begin{cases} \cos\dfrac{\alpha}{2}\cdot w_{\lambda 1}=a_\lambda\phi, \\ \sin\dfrac{\alpha}{2}\cdot w_{\lambda 2}=c_\lambda\bar{\phi} \qquad (\lambda\geqq 3), \end{cases} \tag{1.7}$$

for some complex valued smooth functions a, c, a_λ and c_λ.

x is called *superminimal* if $c\equiv 0$. We put

$$\sigma=|a|^2+|c|^2+\sum|a_\lambda|^2+\sum|c_\lambda|^2, \tag{1.8}$$

and get

PROPOSITION 1. *Let B be the second fundamental form of x. Then the Gauss equation of x is*

$$|B|^2=4\sigma=4\left\{\left(2-\frac{3}{2}\sin^2\alpha\right)\rho-\frac{1}{2}K\right\}, \tag{1.9}$$

where K is the Gaussian curvature of M.

PROOF. We set

$$(1.10)\qquad \begin{cases} \cos\dfrac{\alpha}{2}\cdot w_1+\sin\dfrac{\alpha}{2}\cdot \bar{w}_2=\phi=\theta_1+i\theta_2\,, \\ \sin\dfrac{\alpha}{2}\cdot w_1-\cos\dfrac{\alpha}{2}\cdot \bar{w}_2=\theta_3+i\theta_4\,, \\ w_\lambda=\theta_{2\lambda-1}+i\theta_{2\lambda} \qquad (\lambda\geqq 3)\,. \end{cases}$$

Then $\{\theta_A\}$, $1\leqq A, B, \cdots \leqq 2n$, is a Darboux coframe of the underlying Riemannian structure of Y (cf. [**2**]). Let θ_{AB} be the Riemannian connection forms for θ_A. We can put $\theta_{i\lambda}=\sum h_{\lambda ij}\theta_j$ $(\lambda\geqq 3)$ along $x(M)$, where $h_{\lambda ij}$, $1\leqq i, j\leqq 2$, is the second fundamental tensor and we know

$$(1.11)\qquad |B|^2=\sum h_{\lambda ij}^2=2\sum(h_{\lambda 11}^2+h_{\lambda 12}^2)\,,$$

because of $h_{\lambda 11}+h_{\lambda 22}=0$. The structure equations of M are

$$(1.12)\qquad \begin{cases} d\phi=-i\theta_{12}\wedge\phi\,, \\ d\theta_{12}=-\dfrac{i}{2}K\,\phi\wedge\bar{\phi}\,. \end{cases}$$

The exterior differentiation of the first formula in (1.10) gives, by means of (1.1) and (1.10),

$$\begin{aligned} d(\theta_1+i\theta_2)=&\Big(\cos^2\frac{\alpha}{2}\cdot w_{11}-\sin^2\frac{\alpha}{2}\cdot w_{22}\Big)\wedge\theta_1 \\ &+i\Big(\cos^2\frac{\alpha}{2}\cdot w_{11}-\sin^2\frac{\alpha}{2}\cdot w_{22}\Big)\wedge\theta_2 \\ &-\Big\{w_{12}+\frac{1}{2}(d\alpha-\sin\alpha\cdot(w_{11}+w_{22}))\Big\}\wedge\theta_3 \\ &+i\Big\{w_{12}-\frac{1}{2}(d\alpha-\sin\alpha\cdot(w_{11}+w_{22}))\Big\}\wedge\theta_4 \\ &+\sum\Big\{\cos\frac{\alpha}{2}\cdot w_{1\lambda}+\sin\frac{\alpha}{2}\cdot\bar{w}_{2\lambda}\Big\}\wedge\theta_{2\lambda-1} \\ &+\sum i\Big\{\cos\frac{\alpha}{2}\cdot w_{1\lambda}-\sin\frac{\alpha}{2}\cdot\bar{w}_{2\lambda}\Big\}\wedge\theta_{2\lambda}\,. \end{aligned}$$

We can get also similar formulas for $d(\theta_3+i\theta_4)$ and $d(\theta_{2\lambda-1}+i\theta_{2\lambda})$ represented as linear combinations of θ_A on Y. Since these are valid on Y, by the uniqueness of the connection forms θ_{AB} on Y, we get

$$(1.13)\qquad \begin{cases} \theta_{12}=i\Big(\cos^2\dfrac{\alpha}{2}\cdot w_{11}-\sin^2\dfrac{\alpha}{2}\cdot w_{22}\Big)\,, \\ \theta_{13}+i\theta_{23}=-w_{12}-\dfrac{1}{2}(d\alpha-\sin\alpha\cdot(w_{11}+w_{22}))\,, \\ \theta_{14}+i\theta_{24}=i\Big\{w_{12}-\dfrac{1}{2}(d\alpha-\sin\alpha\cdot(w_{11}+w_{22}))\Big\}\,, \end{cases}$$

$$\begin{cases} \theta_{1,2\lambda-1}+i\theta_{2,2\lambda-1}=\cos\frac{\alpha}{2}\cdot w_{1\lambda}+\sin\frac{\alpha}{2}\cdot \bar{w}_{2\lambda}\,, \\ \theta_{1,2\lambda}+i\theta_{2,2\lambda}=i\left(\cos\frac{\alpha}{2}\cdot w_{1\lambda}-\sin\frac{\alpha}{2}\cdot \bar{w}_{2\lambda}\right), \end{cases}$$

which prove the first formula of (1.9) by (1.5), the complex conjugate of (1.5), (1.6) and (1.7). The latter part of (1.9) has been proved in [**2**]. In fact, it follows from the exterior differentiation of the first formula of (1.13). We shall remark that Proposition 1 can also be proved by a result in [**9**]. Q.E.D.

REMARK 1. The normal bundle of x has a splitting by (1.10). Under this decomposition, $|a|^2+|c|^2$ and hence $\sum|a_\lambda|^2+\sum|c_\lambda|^2$ are scalar invariants.

We hope to get Codazzi equations of x:

PROPOSITION 2. *Let $x: M\to Y$ be an isometric minimal immersion of M into a Kaehlerian manifold of constant holomorphic sectional curvature. Then we have*

$$\begin{aligned} da&=ia\theta_{12}-a\cos\alpha\cdot(w_{11}+w_{22})+\frac{3}{4}\rho\sin 2\alpha\cdot\bar{\phi} \\ &\quad -\tan\frac{\alpha}{2}\cdot\sum|a_\lambda|^2\bar{\phi}+\cot\frac{\alpha}{2}\cdot\sum|c_\lambda|^2\bar{\phi}+a_{,1}\phi\,, \end{aligned} \tag{1.14}$$

$$dc=-ic\theta_{12}+c(w_{11}-w_{22})+c_{,1}\bar{\phi}\,, \tag{1.15}$$

$$\begin{aligned} da_\lambda&=ia_\lambda\theta_{12}+\sin^2\frac{\alpha}{2}\cdot a_\lambda(w_{11}+w_{22})-a_\lambda w_{11} \\ &\quad -\cot\frac{\alpha}{2}\cdot\bar{c}c_\lambda\cdot\bar{\phi}+\sum a_\mu w_{\lambda\mu}+a_{\lambda,1}\phi\,, \end{aligned} \tag{1.16}$$

$$\begin{aligned} dc_\lambda&=-ic_\lambda\theta_{12}+\cos^2\frac{\alpha}{2}\cdot c_\lambda(w_{11}+w_{22})-c_\lambda w_{22} \\ &\quad +\tan\frac{\alpha}{2}\cdot ca_\lambda\cdot\phi+\sum c_\mu w_{\lambda\mu}+c_{\lambda,1}\bar{\phi}\,, \end{aligned} \tag{1.17}$$

for some locally defined functions $a_{,1}$, $c_{,1}$, $a_{\lambda,1}$ and $c_{\lambda,1}$.

PROOF. Only a proof of (1.14) shall be given in this paper because other formulas can be shown in the same way.

By taking the exterior differentiation of (1.5) and using structure equations of M and Y, we get

$$\begin{aligned} \Big\{da-ia\theta_{12}+a\cos\alpha\cdot(w_{11}+w_{22})-\frac{3}{4}\rho\sin 2\alpha\cdot\bar{\phi} \\ +\tan\frac{\alpha}{2}\cdot\sum|a_\lambda|^2\cdot\bar{\phi}-\cot\frac{\alpha}{2}\cdot\sum|c_\lambda|^2\cdot\bar{\phi}\Big\}\wedge\phi=0\,, \end{aligned}$$

which implies (1.14).

We remark that (1.5) is used in the proof of (1.16), but in the case of (1.17) its complex conjugate is used. Q.E.D.

For a holomorphic (resp. anti-holomorphic) curve, we have $\sin(\alpha/2)=0$, (resp. $\cos(\alpha/2)=0$).

On the other hand, x is totally real if and only if $\cos\alpha=0$. This is equivalent to the usual definition of the totally real immersion by virtue of [3].

2. Minimal immersions with $|B|^2\leqq 2\rho$.

We wish to study minimal immersions $x: M^2\to Y$ with $|B|^2\leqq 2\rho$. By [2, p. 69], $|c|^2$ is a scalar invariant of x. If x is superminimal, then c is identically zero on M^2, [9]. The following result proves generalization of [1] and [7].

THEOREM 3. *Let M^2 be a complete connected two dimensional Riemannian manifold and $x: M^2\to Y$ be an isometric minimal immersion such that $|B|^2\leqq 2\rho$. Then either x is superminimal, or M^2 is flat and x is totally real.*

PROOF. ρ is non-negative and $\rho=0$ if and only if $B=0$. Now we assume $\rho>0$. By (1.9), we have $2\rho-|B|^2=2(K-3\cos^2\alpha\cdot\rho)$, which implies, by the assumption,

$$K\geqq 3\cos^2\alpha\cdot\rho\geqq 0\,. \tag{2.1}$$

By Theorem 2 in [2], $P=\cos(\alpha/2)\cdot\sin(\alpha/2)\cdot\bar{c}\cdot\phi^3$ is a holomorphic 3-form on M^2. It follows from the standard method (for instance, see S. T. Yau [9]) that we have

$$\frac{1}{4}\Delta\log(|\sin\alpha\cdot c|)=3K\,, \tag{2.2}$$

at points of $\sin\alpha\cdot c\neq 0$, where Δ is the Laplacian of the metric of M^2.

Assume that $c\neq 0$ and x is not holomorphic, nor anti-holomorphic, which is equivalent to $\sin\alpha\cdot c\neq 0$. By (2.1) and (2.2) $\log(|\sin\alpha|\cdot|c|)$ is a subharmonic function on M^2 and $|\sin\alpha|\cdot|c|$ is bounded above, because of $|\sin\alpha|^2\cdot|c|^2\leqq|c|^2\leqq|B|^2\leqq 2\rho$. By the theorem of Blanc, Fiala and Huber [4], $|\sin\alpha|\cdot|c|$ is a positive constant. Hence we get $K=0$ and $\cos\alpha=0$, so that x is totally real and $|B|^2=2\rho$.

If x is holomorphic (or anti-holomorphic) with $|B|^2\leqq 2\rho$, then x is totally geodesic by Lawson [6] and in particular c is identically zero. Q.E.D.

COROLLARY 4. *Let $x: M^2\to \boldsymbol{P}^n(\boldsymbol{C})$ be an isometric minimal immersion with $|B|^2\leqq 2\rho$ of a compact Riemannian manifold M^2 into $\boldsymbol{P}^n(\boldsymbol{C})$. Then* (1) *$M^2$ is homeomorphic to a two dimensional sphere and x is superminimal, or* (2) *M^2 is a flat torus and x is totally real.*

PROOF. By (2.1), K is non-negative. The Gauss-Bonnet's Theorem implies that the genus of M^2 is zero or one. When M^2 is of genus zero, x is superminimal by [9]. If M^2 is homeomorphic to a torus, then K must be identically zero and x is totally real. Q.E.D.

REMARK 2. In Theorem 3, if we assume $|B|^2<2\rho$, then x is superminimal and $K>0$.

In his Ph. D. Thesis [9], Wolfson studied extensively the case (1) in Corollary 4. Therefore in the rest of this paper we shall classify all minimal immersions in the case (2) of this Corollary.

3. Totally real minimal immersions of $\boldsymbol{R}^2$ into Y.

Let $\boldsymbol{R}^2$ be the 2-dimensional Euclidean plane with standard coordinates (u, v) and the flat metric be given by $ds^2=du^2+dv^2$. In this section we shall study an isometric totally real minimal immersion $x:\boldsymbol{R}^2\to Y$. The standard Euclidean metric of $\boldsymbol{R}^2$ implies

$$\phi=du+idv \quad \text{and} \quad \theta_{12}\equiv 0. \tag{3.1}$$

By the definition of the total realness of x, we have

$$\alpha\equiv\frac{\pi}{2}, \tag{3.2}$$

which implies $|B|^2=2\rho$ by (1.9). Hence ρ is non-negative. By adding (1.5) and its complex conjugate, we have $d\alpha=a\phi+\overline{a\phi}$. Considering (3.2), we get

$$a\equiv 0. \tag{3.3}$$

By the first formula of (1.13), (3.1) and (3.2), we have $w_{11}=w_{22}$. On the other hand, we know $w_{11}+w_{22}=0$ by (1.5) in this case, which implies

$$w_{11}=w_{22}=0. \tag{3.4}$$

By (1.15), (3.1) and (3.4), we get $dc=c_{,1}\bar{\phi}$. In the proof of Theorem 3, we obtained $|c|^2=\text{constant}\geqq 0$ because of (3.2). From these two facts, c must be a complex constant.

From (1.7) and (3.2), we get

$$w_{\lambda 1}=\sqrt{2}\,a_\lambda\phi, \quad w_{\lambda 2}=\sqrt{2}\,c_\lambda\bar{\phi} \qquad (\lambda\geqq 3). \tag{3.5}$$

By virtue of (1.14), (3.1), (3.2), (3.3) and (3.4), we get

$$\sum|a_\lambda|^2=\sum|c_\lambda|^2, \qquad a_{,1}\equiv 0. \tag{3.6}$$

By (1.8), (1.9) and (3.3), we have

$$\sum|a_\lambda|^2=\frac{1}{2}\Big(\frac{1}{2}\rho-|c|^2\Big)=\text{constant}\geqq 0. \tag{3.7}$$

Let e_α be the dual frames of w_α. Then e_λ, $\lambda\geqq 3$, are defined up to the transformation

$$e_\lambda \longrightarrow \sum a_{\lambda\nu} e_\nu, \qquad (a_{\lambda\nu}): \text{unitary matrix}$$

under which $w_{\lambda i}$ transform as

$$w_{\lambda i} \longrightarrow \sum a_{\lambda\nu} w_{\nu i}.$$

Hence $\sum a_\lambda \bar{c}_\lambda$ is also independent of the choice of e_λ.

LEMMA 5. *For any fixed standard coordinates of $\boldsymbol{R}^2$, we have*

$$\sum a_\lambda \bar{c}_\lambda = (\textit{complex}) \textit{ constant}. \tag{3.8}$$

PROOF. Formulas (1.16) and (1.17) have simple expressions in this case:

$$da_\lambda = -\bar{c} c_\lambda \bar{\phi} + \sum a_\mu w_{\lambda\mu} + a_{\lambda,1}\phi; \tag{3.9}$$

$$dc_\lambda = c a_\lambda \phi + \sum c_\mu w_{\lambda\mu} + c_{\lambda,1}\bar{\phi}. \tag{3.10}$$

By making use of (3.6), (3.9) and (3.10), we get $d(\sum a_\lambda \bar{c}_\lambda) \wedge \phi = 0$, which shows that $\sum a_\lambda \bar{c}_\lambda$ is holomorphic. Since we have

$$|\sum a_\lambda \bar{c}_\lambda|^2 \leqq (\sum |a_\lambda|^2)^2 = \frac{1}{4}\left(\frac{1}{2}\rho - |c|^2\right)^2,$$

it must be constant by the Liouville's theorem. Q.E.D.

We define $a_{2,\lambda}$ and $c_{2,\lambda}$ by

$$w_{1\lambda} = \overline{a_{2,\lambda}\phi}, \qquad w_{2\lambda} = c_{2,\lambda}\phi \qquad (\lambda \geqq 3). \tag{3.5$'$}$$

Then we have

$$\sum |a_{2,\lambda}|^2 = \sum |c_{2,\lambda}|^2 = \frac{1}{2}\rho - |c|^2 \geqq 0, \tag{3.6$'$}$$

$$\sum a_{2,\lambda} c_{2,\lambda} = \text{constant}, \tag{3.8$'$}$$

$$da_{2,\lambda} = -\overline{c c_{2,\lambda}\phi} + \sum a_{2,\mu} w_{\lambda\mu} + a_{2,\lambda,1}\phi, \tag{3.9$'$}$$

$$dc_{2,\lambda} = \overline{c a_{2,\lambda}\phi} - \sum c_{2,\mu} w_{\mu\lambda} + c_{2,\lambda,1}\phi. \tag{3.10$'$}$$

We set

$$k_1 = (\sum |a_{2,\lambda}|^2)^{1/2},$$

which is a non-negative constant.

LEMMA 6. *If $k_1 = 0$, then $x(\boldsymbol{R}^2)$ is contained in some real 4-dimensional totally geodesic submanifold of Y.*

PROOF. By (3.5)$'$ and (3.6)$'$, we have

$$\begin{cases} w_{13} = w_{14} = \cdots = w_{1n} = 0, \\ w_{23} = w_{24} = \cdots = w_{2n} = 0, \qquad \text{along} \quad x(\boldsymbol{R}^2) \end{cases} \tag{3.11}$$

and $|c|^2 = \rho/2$. By (1.10), we have

$$(3.12)\qquad w_3=\cdots=w_n=0 \quad \text{along} \quad x(\boldsymbol{R}^2).$$

Equations (3.11) and (3.12) show that there exists a real 4-dimensional totally geodesic C^∞-submanifold in the underlying Riemannian manifold of Y containing $x(\boldsymbol{R}^2)$. Q.E.D.

In the case of $k_1>0$, we shall prove the following two lemmas. We set

$$(3.13)\qquad n_1=|\sum \bar{a}_{2,\lambda}e_\lambda \wedge \sum c_{2,\lambda}e_\lambda|,$$

which is independent of the choice of e_λ and constant on $\boldsymbol{R}^2$ by (3.6)′ and (3.8)′.

LEMMA 7. *Assume that $k_1>0$ and $n_1=0$. Then $x(\boldsymbol{R}^2)$ is fully contained in a real 6-dimensional totally geodesic submanifold of Y.*

PROOF. Since normal vectors $\sum \bar{a}_{2,\lambda}e_\lambda$ and $\sum c_{2,\lambda}e_\lambda$ are linearly dependent and have the same length, we can set $\sum c_{2,\lambda}e_\lambda=\exp(i\theta)\sum \bar{a}_{2,\lambda}e_\lambda$, where θ is a real constant by (3.8)′. Put

$$e_3^*=\frac{1}{k_1}\sum \bar{a}_{2,\lambda}e_\lambda,$$

and $\{e_\alpha^*\}$, $3\leqq\alpha\leqq n$, denotes a new unitary frame along $x(\boldsymbol{R}^2)$. Then we have

$$(3.14)\qquad \begin{cases} w_{13}^*=k_1\bar{\phi}, & w_{1\lambda}^*=0, \quad \lambda\geqq 4, \\ w_{23}^*=e^{i\theta}k_1\phi, & w_{2\lambda}^*=0, \quad \lambda\geqq 4. \end{cases}$$

For the proof of these formulas, we compute,

$$\begin{aligned} w_{1\lambda}^* &= \langle De_1, e_\lambda^*\rangle=\langle\sum w_{1\alpha}e_\alpha, e_\lambda^*\rangle \\ &= \langle\sum \overline{a_{2,\alpha}\phi}\, e_\alpha, e_\lambda^*\rangle \qquad \text{(by (3.5)′)} \\ &= \langle k_1e_3^*, e_\lambda^*\rangle\bar{\phi}=k_1\bar{\phi} \qquad (\lambda=3), \\ &\quad \text{or } =0 \qquad (\lambda\geqq 4), \end{aligned}$$

where D denotes the covariant differentiation of Y and $\langle\,,\rangle$ is the hermitian inner product of Y defined by $\langle U, V\rangle=\sum \omega_\alpha(U)\overline{\omega_\alpha(V)}$ for any tangent vectors U and V of Y.

Other formulas are also proved by the similar way. Taking the exterior differentiation of (3.14), we have $w_{3\lambda}^*=0$ $(\lambda\geqq 4)$, along $x(\boldsymbol{R}^2)$. We drop the asterisks. By considering the differential system $\{w_4=\cdots=w_n=0,\ w_{i\lambda}=0,\ w_{3\lambda}=0,\ i=1, 2,\ \lambda\geqq 4\}$ on the underlying Riemannian manifold of Y, there exists a real 6-dimensional totally geodesic C^∞-submanifold of Y containing $x(\boldsymbol{R}^2)$. Q.E.D.

In case of $k_1>0$ and $n_1>0$, we can put

$$(3.15)\qquad \begin{cases} e_3^*=\dfrac{1}{k_1}\sum \bar{a}_{2,\lambda}e_\lambda, \\ e_4^*=A_2/|A_2|, \qquad A_2=\sum c_{2,\lambda}e_\lambda-\langle\sum c_{2,\lambda}e_\lambda, e_3^*\rangle e_3^*, \end{cases}$$

and $\{e_3^*, e_4^*, e_\alpha^*, \alpha\geqq 5\}$ denotes a new unitary frame along $x(\boldsymbol{R}^2)$. For the fixed coordinates of $\boldsymbol{R}^2$, e_3^* and e_4^* are globally defined vector fields along $x(\boldsymbol{R}^2)$.

LEMMA 8. *In case of $k_1>0$ and $n_1>0$, we have*

$$(3.16)\qquad \begin{cases} w_{11}=0\,, \quad w_{12}=c_1\bar{\phi}\,, \quad w_{13}=k_1\bar{\phi}\,, \quad w_{14}=\cdots=w_{1n}=0\,, \\ w_{22}=0\,, \quad w_{23}=k_1c_{2,3}\phi\,, \quad w_{24}=k_1c_{2,4}\phi\,, \quad w_{25}=\cdots=w_{2n}=0\,, \\ w_{33}=-c_1c_{2,3}\phi+\overline{c_1c_{2,3}\phi}\,, \quad w_{34}=c_{2,4}(-c_1\phi+c_2\bar{\phi})\,, \\ w_{44}=\overline{c_2c_{2,3}\phi}-c_2c_{2,3}\bar{\phi}\,, \end{cases}$$

where k_1 and $c_{2,4}$ are real positive constants and all other coefficients in (3.16) *are complex constants and satisfy*

$$(3.17)\qquad k_1^2=\frac{\rho}{2}-|c_1|^2>0\,, \qquad |c_{2,3}|^2+c_{2,4}{}^2=1\,, \qquad \frac{\rho}{2}-|c_2|^2\geqq 0\,.$$

PROOF. The proof for formulas of $w_{i\lambda}$, $\lambda\geqq 3$, is similar to that of Lemma 7. Since we have $n_1=k_1^2c_{2,4}$ for the new frame, $c_{2,4}$ is a positive number and an invariant of x. By using (3.9)′ for the $\{e_\alpha^*\}$, we get $w_{33}^*=-c_1c_{2,3}\phi+\overline{c_1c_{2,3}\phi}$, because of $w_{33}^*+\bar{w}_{33}^*=0$. From $dw_{14}^*=0$ and $dw_{24}^*=0$, we get $(w_{34}^*+c_1c_{2,4}\phi)\wedge\bar{\phi}=0$ and $(c_{2,3}w_{34}^*+c_{2,4}w_{44}^*)\wedge\phi=0$, from which we can put $w_{34}^*=-c_1c_{2,4}\phi+c_2c_{2,4}\bar{\phi}$, and $w_{44}^*=\overline{c_2c_{2,3}\phi}-c_2c_{2,3}\bar{\phi}$. It is left to prove the constancy of c_2: By the definition of e_3^*, e_4^* and (3.9)′, we know

$$\begin{aligned} w_{34}^*\wedge\phi&=\langle De_3^*, e_4^*\rangle\wedge\phi \\ &=\frac{1}{k_1|A_2|}\Big\{\sum\overline{a_{2,\beta,1}c_{2,\beta}}-\frac{\sum\overline{a_{2,\alpha}c_{2,\alpha}}}{k_1^2}\sum a_{2,\beta}\bar{a}_{2,\beta,1}\Big\}\bar{\phi}\wedge\phi\,. \end{aligned}$$

From the exterior differentiation of $\sum a_{2,\beta}\bar{a}_{2,\beta}$, we get, by (3.6)′ and (3.9)′,

$$(3.18)\qquad \sum\bar{a}_{2,\beta}a_{2,\beta,1}=c_1\sum a_{2,\beta}c_{2,\beta}=\text{complex constant.}$$

The exterior differentiation of (3.9)′ gives

$$da_{2,\lambda,1}\wedge\phi=\sum a_{2,\mu,1}w_{\lambda\mu}\wedge\phi+\Big\{\frac{\rho}{2}a_{2,\lambda}-\bar{c}_{2,\lambda}\sum a_{2,\mu}c_{2,\mu}\Big\}\phi\wedge\bar{\phi}\,,$$

from which we can compute

$$d(\sum\bar{a}_{2,\beta}a_{2,\beta,1})\wedge\phi=\Big\{-\sum|a_{2,\beta,1}|^2+\frac{1}{2}\rho k_1^2-|\sum a_{2,\beta}c_{2,\beta}|^2\Big\}\phi\wedge\bar{\phi}\,.$$

By coupling it with (3.18), we get

$$(3.19)\qquad \sum|a_{2,\beta,1}|^2=\frac{1}{2}\rho k_1^2-|\sum a_{2,\beta}c_{2,\beta}|^2=\text{constant.}$$

Similar calculation proves $d(\sum a_{2,\beta,1}c_{2,\beta})\wedge\phi=0$. Therefore $\sum a_{2,\beta,1}c_{2,\beta}$ is a globally defined holomorphic function on $\boldsymbol{R}^2$ for the fixed ϕ. By (3.6)′ and (3.19), it is bounded above on $\boldsymbol{R}^2$, hence it must be a constant function, which implies the constancy of c_2.

By $dw^*_{1\lambda}=dw^*_{2\lambda}=0$, $\lambda\geqq 5$, we have $w^*_{3\lambda}\wedge\bar{\phi}=0$ and $(c_{2,3}w^*_{3\lambda}+c_{2,4}w^*_{4\lambda})\wedge\phi=0$. Since we know $c_{2,4}\neq 0$, we can put

$$(3.20)\qquad \begin{cases} w^*_{3\lambda_2}=c_{2,4}\overline{a_{3,\lambda_2}\phi}\,, \\ w^*_{4\lambda_2}=c_{3,\lambda_2}\phi-c_{2,3}\overline{a_{3,\lambda_2}\phi}\,, \end{cases}$$

where λ_2, $\cdots$ run through 5 to n. The exterior differentiation of w^*_{33} gives

$$\sum w^*_{3\lambda_2}\wedge\bar{w}^*_{3\lambda_2}=-\Big(\frac{\rho}{2}-|c_2|^2\Big)c^2_{2,4}\phi\wedge\bar{\phi}$$

and hence, by (3.20), we have

$$(3.21)\qquad \sum|a_{3,\lambda_2}|^2=\Big(\frac{\rho}{2}-|c_2|^2\Big)\geqq 0\,.$$

We drop the asterisks again. Q. E. D.

By using $dw_{44}=0$, we know

$$\sum w_{4\lambda_2}\wedge\bar{w}_{4\lambda_2}=|c_{2,4}|^2\Big(\frac{1}{2}\rho-|c_2|^2\Big)\phi\wedge\bar{\phi}\,.$$

Coupling this formula with the latter formula of (3.20), we have

$$(3.22)\qquad \sum|c_{3,\lambda_2}|^2=\sum|a_{3,\lambda_2}|^2=\frac{\rho}{2}-|c_2|^2.$$

Put

$$(3.23)\qquad \begin{cases} k_2=\{\sum|a_{3,\lambda_2}|^2\}^{1/2}, \\ n_2=|\sum\bar{a}_{3,\lambda_2}e_{\lambda_2}\wedge\sum c_{3,\lambda_2}e_{\lambda_2}|, \end{cases}$$

which are independent of the choice of frame e_i and e_{λ_2}.

LEMMA 9. *n_2 is a constant.*

PROOF. It is sufficient to prove that

$$(3.24)\qquad \sum a_{3,\lambda_2}c_{3,\lambda_2}\quad\text{is a constant.}$$

The method of a proof of this formula is similar to that of Lemma 5: At first we shall obtain the third order Codazzi equations:

$$(3.25)\qquad \begin{cases} (da_{3,\lambda_2}-\sum a_{3,\mu_2}w_{\lambda_2\mu_2}+\overline{(c_2c_{3,\lambda_2})\phi})\wedge\phi=0\,; \\ (dc_{3,\lambda_2}+\sum c_{3,\mu_2}w_{\mu_2\lambda_2}-\overline{(c_2a_{3,\lambda_2})\phi})\wedge\phi=0\,, \end{cases}$$

which follow from the exterior differentiation of (3.20).

By virtue of (3.22) and (3.25), we get $d(\sum a_{3,\lambda_2}c_{3,\lambda_2})\wedge\phi=0$, which proves that the function $\sum a_{3,\lambda_2}c_{3,\lambda_2}$ is holomorphic. Moreover it is bounded above by (3.22). By the Liouville's theorem, it must be constant. Q. E. D.

k_2 is a non-negative constant.

LEMMA 10. *If* $k_1n_1>0$ *and* $k_2=0$, *then there exists a real* 8 *dimensional totally geodesic* C^∞*-submanifold of* Y *containing* $x(\boldsymbol{R}^2)$ *fully.*

PROOF. By (3.16), we have $w_{i\lambda_2}=0$. By (3.20) and (3.22), we have $w_{3\lambda_2}=w_{4\lambda_2}=0$ along $x(\boldsymbol{R}^2)$. By (1.10), we have $w_5=\cdots=w_n=0$. It follows from these facts that there exists a real 8 dimensional totally geodesic C^∞-submanifold of Y containing $x(\boldsymbol{R}^2)$. Q.E.D.

n_2 is a non-negative constant.

LEMMA 11. *In the case of* $k_1k_2>0$, $n_1>0$ *and* $n_2=0$, *there exists a real* 10 *dimensional totally geodesic submanifold of* Y *containing* $x(\boldsymbol{R}^2)$ *fully.*

PROOF. By (3.23) and Lemma 9, normal vectors $\sum \bar{a}_{3,\lambda_2}e_{\lambda_2}$ and $\sum c_{3,\lambda_2}e_{\lambda_2}$ are linearly dependent and have the same length. Hence we can set $\sum c_{3,\lambda_2}e_{\lambda_2}=\exp(i\theta)\sum a_{3,\lambda_2}e_{\lambda_2}$, where θ is a real constant by (3.24). Let

$$e_5^*=\frac{1}{k_2}\sum \bar{a}_{3,\lambda_2}e_{\lambda_2},$$

and $\{e_3, e_4; e_5^*; e_\alpha^*, \alpha\geqq 6\}$ be a new unitary frame.

With respect to this frame, we have

$$w_{35}^*=c_{2,4}k_2\bar{\phi}, \qquad w_{3\lambda}^*=0, \quad \lambda\geqq 6,$$

$$w_{45}^*=\frac{\sum a_{3,\mu_2}c_{3,\mu_2}}{k_2}\phi-c_{2,3}k_2\bar{\phi}, \qquad w_{4\lambda}^*=0, \quad \lambda\geqq 6.$$

By assumptions, $\sum a_{3,\mu_2}c_{3,\mu_2}$ is non-zero. From $dw_{\beta\lambda}^*=0$, $\beta=3, 4$, $\lambda\geqq 6$, we have $w_{35}^*\wedge w_{5\lambda}^*=w_{45}^*\wedge w_{5\lambda}^*=0$, which implies $w_{5\lambda}^*=0$, $\lambda\geqq 6$. Since we obtained the system $w_{\alpha\beta}^*=0$, $1\leqq\alpha\leqq 5$, $\beta\geqq 6$, there exists a real 10 dimensional totally geodesic submanifold of Y containing $x(\boldsymbol{R}^2)$. Q.E.D.

In case of $k_1k_2>0$ and $n_1n_2>0$, we can put

$$\text{(3.26)} \qquad \begin{cases} e_5^*=\dfrac{1}{k_2}\sum \bar{a}_{3,\lambda_2}e_{\lambda_2}, \\ e_6^*=A_3/|A_3|, \qquad A_3=\sum c_{3,\lambda_2}e_{\lambda_2}-\langle\sum c_{3,\lambda_2}e_{\lambda_2}, e_5^*\rangle e_5^*, \end{cases}$$

and $\{e_3, e_4; e_5^*, e_6^*; e_\alpha^*, \alpha\geqq 7\}$ is the new unitary frame along $x(\boldsymbol{R}^2)$.

With respect to this frame, a direct computation shows, where we dropped the asterisks again,

$$\text{(3.27)} \qquad \begin{cases} w_{35}=k_2c_{2,4}\bar{\phi}, \qquad w_{36}=\cdots=w_{3n}=0; \\ w_{45}=k_2(c_{3,5}\phi-c_{2,3}\bar{\phi}), \qquad c_{3,5}=\sum a_{3,\lambda_2}c_{3,\lambda_2}/k_2^2, \\ w_{46}=k_2c_{3,6}\phi, \qquad w_{47}=\cdots=w_{4n}=0, \qquad c_{3,6}=|A_3|/k_2>0, \\ w_{55}\wedge\bar{\phi}=-c_{3,5}c_2\phi\wedge\bar{\phi}, \\ w_{56}=c_{3,6}(-c_2\phi+c_3\bar{\phi}), \\ c_3=\{k_2^2\sum \overline{a_{3,\lambda_2,1}c_{3,\lambda_2}}-(\sum \overline{a_{3,\lambda_2}c_{3,\lambda_2}})\sum a_{3,\lambda_2}\overline{a_{3,\lambda_2,1}}\}/|A_3|^2k_2^2, \end{cases}$$

where $a_{3,\lambda_2,1}$ is defined by

$$da_{3,\lambda_2}-\sum a_{3,\mu_2}w_{\lambda_2\mu_2}+\overline{c_2c_{3,\lambda_2}\phi}=a_{3,\lambda_2,1}\phi\,. \tag{3.28}$$

LEMMA 12. *c_3 is a constant.*

PROOF. For the proof, it is sufficient to get the following facts:

$$\sum \bar{a}_{3,\lambda_2}a_{3,\lambda_2,1}=c_2\sum a_{3,\lambda_2}c_{3,\lambda_2} \quad \text{is constant,} \tag{3.29}$$

$$\sum a_{3,\lambda_2,1}c_{3,\lambda_2} \quad \text{is constant.} \tag{3.30}$$

(3.29) follows from the exterior differentiation of $\sum a_{3\lambda_2}\bar{a}_{3,\lambda_2}=$constant. For a proof of (3.30), we prepare

$$da_{3,\lambda_2,1}\wedge\phi=\sum a_{3,\mu_2,1}w_{\lambda_2\mu_2}\wedge\phi+\Big(\frac{\rho}{2}a_{3,\lambda_2}-\bar{c}_{3,\lambda_2}\sum a_{3,\lambda_2}c_{3,\lambda_2}\Big)\phi\wedge\bar{\phi}\,, \tag{3.31}$$

which follows from (3.28). By this formula, (3.25) and (3.29), we shall compute

$$\begin{aligned}0&=d(\sum \bar{a}_{3,\lambda_2}a_{3,\lambda_2,1})\wedge\phi\\&=\Big\{-\sum|a_{3,\lambda_2,1}|^2+\frac{\rho}{2}k_2^2-|\sum a_{3,\lambda_2}c_{3,\lambda_2}|^2\Big\}\phi\wedge\bar{\phi}\,.\end{aligned}$$

$\sum a_{3,\lambda_2,1}c_{3,\lambda_2}$ is independent of the choice of e_{λ_2}. By making use of (3.31), (3.25), (3.29) and (3.21), we get $d(\sum a_{3,\lambda_2,1}c_{3,\lambda_2})\wedge\phi=0$. We have proved that the function $\sum a_{3,\lambda_2,1}c_{3,\lambda_2}$ is holomorphic and bounded above on $\boldsymbol{R}^2$, hence it must be constant. Q.E.D.

From $dw_{46}=0$, we get $w_{66}=\overline{c_{3,5}c_3\phi}-c_{3,5}c_3\bar{\phi}$. By $dw_{3\lambda}=dw_{4\lambda}=0$, $\lambda\geqq 7$, we can put

$$\begin{cases} w_{5\lambda_3}=c_{3,6}\overline{a_{4,\lambda_3}\phi}\,,\\ w_{6\lambda_3}=c_{4,\lambda_3}\phi-c_{3,5}\overline{a_{4,\lambda_3}\phi}\,,\end{cases} \tag{3.32}$$

where $\lambda_3, \cdots$ run through 7 to n.

By (3.27) and the last formula of (1.1), we get $w_{55}=-c_{3,5}c_2\phi+\overline{c_{3,5}c_2\phi}$. From $dw_{55}=dw_{66}=0$, we get

$$\sum|c_{4,\lambda_3}|^2=\sum|a_{4,\lambda_3}|^2=\frac{\rho}{2}-|c_3|^2\geqq 0\,. \tag{3.33}$$

Summarizing these computations, we have proved the following.

LEMMA 13. *In the case of $k_1k_2>0$ and $n_1n_2>0$, there exists a unitary coframe such that*

$$\begin{cases} w_{11}=0\,, \quad w_{12}=c_1\bar{\phi}\,, \quad w_{13}=k_1\bar{\phi}\,, \quad w_{14}=\cdots=w_{1n}=0\,,\\ w_{22}=0\,, \quad w_{23}=k_1c_{2,3}\phi\,, \quad w_{24}=k_1c_{2,4}\phi\,, \quad w_{25}=\cdots=w_{2n}=0\,,\\ w_{33}=-c_{2,3}c_1\phi+\overline{c_{2,3}c_1\phi}\,, \quad w_{34}=c_{2,4}(-c_1\phi+c_2\bar{\phi})\,, \quad w_{35}=k_2c_{2,4}\bar{\phi}\,,\\ \qquad w_{36}=\cdots=w_{3n}=0\,,\end{cases}$$

$$(3.34)\quad \begin{cases} w_{44}=\overline{c_{2,3}c_2\phi}-c_{2,3}c_2\bar{\phi}\,, \quad w_{45}=k_2(c_{3,5}\phi-c_{2,3}\bar{\phi})\,, \quad w_{46}=k_2c_{3,6}\phi\,, \\ \qquad w_{47}=\cdots=w_{4n}=0\,, \\ w_{55}=-c_{3,5}c_2\phi+\overline{c_{3,5}c_2\phi}\,, \quad w_{56}=c_{3,6}(-c_2\phi+c_3\bar{\phi})\,, \quad w_{5\lambda_3}=c_{3,6}\overline{a_{4,\lambda_3}\phi}\,, \\ w_{66}=\overline{c_{3,5}c_3\phi}-c_{3,5}c_3\bar{\phi}\,, \quad w_{6\lambda_3}=c_{4,\lambda_3}\phi-c_{3,5}\overline{a_{4,\lambda_3}\phi}\,, \end{cases}$$

where all coefficients in (3.34) are constants and satisfy

$$(3.35)\quad \begin{cases} k_1^2+|c_1|^2=k_2^2+|c_2|^2=\frac{\rho}{2}, \quad |c_{2,3}|^2+c_{2,4}^2=|c_{3,5}|^2+{c_{3,6}}^2=1\,; \\ \sum|a_{4,\lambda_3}|^2+|c_3|^2=\sum|c_{4,\lambda_3}|^2+|c_3|^2=\frac{\rho}{2}, \quad c_{2,4}c_{3,6}>0\,. \end{cases}$$

REMARK 3. By $n_2=k_2^2c_{3,6}$, $c_{3,6}$ is also an invariant of x.

Put

$$k_3=\{\sum|a_{4,\lambda_3}|^2\}^{1/2},$$

$$n_3=|\sum \bar{a}_{4,\lambda_3}e_{\lambda_3}\wedge\sum c_{4,\lambda_3}e_{\lambda_3}|,$$

which are independent of the choice of e_i and e_{λ_3}. They are non-negative constants. If $k_3=0$, then there exists a real 12-dimensional totally geodesic C^∞-submanifold of Y containing $x(\boldsymbol{R}^2)$ fully. If k_3 is positive and $n_3=0$, then there exists a real 14-dimensional totally geodesic C^∞-submanifold of Y containing $x(\boldsymbol{R}^2)$ fully. Their proofs are similar to those of Lemmas 10 and 11.

Continuing this way, we can define a global frame field $\{e_A\}$ such that if $k_1k_2\cdots k_s>0$ and $n_1n_2\cdots n_s>0$, we have

$$(3.36)\quad \begin{cases} w_1=\frac{1}{\sqrt{2}}\phi\,, \quad w_2=\frac{1}{\sqrt{2}}\bar{\phi}\,, \quad w_3=\cdots=w_n=0\,, \\ w_{2t+1,2t+1}=-c_tc_{t+1,2t+1}\phi+\overline{c_tc_{t+1,2t+1}\phi}\,, \\ w_{2t+1,2t+2}=c_{t+1,2t+2}(-c_t\phi+c_{t+1}\bar{\phi})\,, \\ w_{2t+1,2t+3}=k_{t+1}c_{t+1,2t+2}\bar{\phi}\,, \quad w_{2t+1,\lambda}=0\,, \quad \lambda\geqq 2t+4\,, \\ w_{2t+2,2t+2}=\overline{c_{t+1}c_{t+1,2t+1}\phi}-c_{t+1}c_{t+1,2t+1}\bar{\phi}\,, \\ w_{2t+2,2t+3}=k_{t+1}(c_{t+2,2t+3}\phi-c_{t+1,2t+1}\bar{\phi})\,, \\ w_{2t+2,2t+4}=k_{t+1}c_{t+2,2t+4}\phi\,, \quad w_{2t+2,\lambda}=0\,, \quad \lambda\geqq 2t+5\,, \quad t=0,1,\cdots,s-1\,, \\ w_{2s+1,2s+1}=-c_sc_{s+1,2s+1}\phi+\overline{c_sc_{s+1,2s+1}\phi}\,, \\ w_{2s+1,2s+2}=c_{s+1,2s+2}(-c_s\phi+c_{s+1}\bar{\phi})\,, \\ w_{2s+1,\lambda_{s+1}}=c_{s+1,2s+2}\overline{a_{s+2,\lambda_{s+1}}\phi}\,, \quad 2s+3\leqq\lambda_{s+1}\leqq n\,, \\ w_{2s+2,2s+2}=\overline{c_{s+1}c_{s+1,2s+1}\phi}-c_{s+1}c_{s+1,2s+1}\bar{\phi}\,, \\ w_{2s+2,\lambda_{s+1}}=c_{s+2,\lambda_{s+1}}\phi-c_{s+1,2s+1}\overline{a_{s+2,\lambda_{s+1}}\phi}\,, \end{cases}$$

where coefficients of the system (3.36) satisfy $c_0=0$, $c_{1,2}=1$, $c_{1,1}=0$,

$$(3.37)\qquad \begin{cases} k_t^2+|c_t|^2=\dfrac{\rho}{2}, \qquad |c_{t+1,2t+1}|^2+c_{t+1,2t+2}{}^2=1\,, \\ \qquad c_{t+1,2t+2}>0\,, \quad k_t>0\,, \quad t=1, 2, \cdots, s\,; \\ \sum_{\lambda_{s+1}} |a_{s+2,\lambda_{s+1}}|^2=\sum_{\lambda_{s+1}} |c_{s+2,\lambda_{s+1}}|^2=\dfrac{\rho}{2}-|c_{s+1}|^2. \end{cases}$$

It follows from $n_t=k_t^2 c_{t+1,2t+2}$ and the above formulas that invariants k_t and n_t satisfy

$$(3.38)\qquad n_t \leqq k_t^2 \leqq \frac{\rho}{2}.$$

We can define

$$k_{s+1}=\{\sum_{\lambda_{s+1}} |a_{s+2,\lambda_{s+1}}|^2\}^{1/2},$$

$$n_{s+1}=|\sum_{\lambda_{s+1}} a_{s+2,\lambda_{s+1}} e_{\lambda_{s+1}} \wedge \sum \bar{c}_{s+2,\lambda_{s+1}} e_{\lambda_{s+1}}|.$$

It can be proved that k_{s+1} and n_{s+1} are non-negative constants. If $k_{s+1}=0$, then $x(\boldsymbol{R}^2)$ is contained in some real $4(s+1)$-dimensional totally geodesic submanifold of Y and if $k_{s+1}>0$ and $n_{s+1}=0$, then $x(\boldsymbol{R}^2)$ is contained in some real $2(2s+3)$-dimensional totally geodesic submanifold of Y. Therefore, if the immersion is full, that is, the image is not contained in a totally geodesic submanifold of Y and $n=2m$, then we have $k_1 \cdots k_{m-1}>0$, $n_1 \cdots n_{m-1}>0$, $k_m=0$ and hence $|c_m|^2=\rho/2$. We may assume $c_m=(\rho/2)^{1/2}$ by applying a suitable orthogonal transformation of the Euclidean coordinates of $\boldsymbol{R}^2$ if necessary.

When the immersion is full and $n=2m+1$, we have $k_1 k_2 \cdots k_m>0$, $n_1 n_2 \cdots n_{m-1}>0$, $n_m=0$ and hence $|c_{m+1,2m+1}|^2=1$, $c_{m+1,2m+2}=0$. Then taking suitable ϕ, we may assume $c_{m+1,2m+1}=1$.

We summarize our results in the following theorem:

THEOREM 14. *Let $x:\boldsymbol{R}^2\to Y$ be an isometric totally real minimal immersion of the real Euclidean two plane into a complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature 4ρ. Then $\rho\geqq 0$. If $\rho=0$, then x is totally geodesic. When ρ is positive and the immersion x is full, we have $n-2$ invariants: $k_1, \cdots, k_{m-1}$; $n_1, \cdots, n_{m-1}$ in case of $n=2m$ ($k_1, \cdots, k_m$; $n_1, \cdots, n_{m-1}$ in case of $n=2m+1$) which are positive constants and satisfy the condition* (3.38).

Moreover, the Frenet-Borůvka equations for such an x are given by (3.36).

4. The case when the ambient space is the complex projective space.

We shall apply the results of Section 3 to the case that Y is the complex projective space $\boldsymbol{P}^n(\boldsymbol{C})$ with the Fubini-Study metric.

Let's introduce notations for geometry of $\boldsymbol{P}^n(\boldsymbol{C})$. For $W=(w_0, \cdots, w_n)$, $Z=(z_0, \cdots, z_n)\in\boldsymbol{C}^{n+1}$, the usual hermitian inner product is defined by $(W, Z)=$

$\sum w_a\bar{z}_a$, where we use the index range $0\leqq a, b, c, \cdots \leqq n$. The n-dimensional complex projective space $\boldsymbol{P}^n(\boldsymbol{C})$ is the orbit space of $\boldsymbol{C}^{n+1}-\{0\}$ under the action of the group $\{Z\to\alpha Z: \alpha\in\boldsymbol{C}-\{0\}\}$. $\pi: \boldsymbol{C}^{n+1}-\{0\}\to\boldsymbol{P}^n(\boldsymbol{C})$ denotes the projection.

For a point $p\in\boldsymbol{P}^n(\boldsymbol{C})$, we take a vector $Z\in\pi^{-1}(p)$, which is called a homogeneous coordinate vector of p. Tangent space of $\boldsymbol{P}^n(\boldsymbol{C})$ at p is identified with $\{W\in\boldsymbol{C}^{n+1}: (Z, W)=0\}$, (cf. [5, p. 273]).

$\boldsymbol{P}^n(\boldsymbol{C})$ is diffeomorphic to the coset space of the unitary group $U(n+1)$:

$$(4.1)\qquad U(n+1)\xrightarrow{\lambda} U(n+1)/U(n)\xrightarrow{h} U(n+1)/U(1)\times U(n)=\boldsymbol{P}^n(\boldsymbol{C}).$$

We identify the unitary group $U(n+1)$ with the space of all unitary frames $\{Z_a\}$, $Z_a\in\boldsymbol{C}^{n+1}-\{0\}$, satisfying $(Z_a, Z_b)=\delta_{ab}$. Under this identification the first projection in (4.1) is defined by assigning to the frame $\{Z_a\}$ its first vector Z_0. h is the Hopf fibering.

Maurer-Cartan forms θ_{ab} of the unitary group $U(n+1)$ are defined by

$$(4.2)\qquad dZ_a=\sum \theta_{ab}Z_b, \qquad \theta_{ab}+\bar{\theta}_{ba}=0.$$

They satisfy the Maurer-Cartan equations: $d\theta_{ab}=\sum \theta_{ac}\wedge\theta_{cb}$. The Fubini-Study metric on $\boldsymbol{P}^n(\boldsymbol{C})$ is given by

$$ds^2=\sum \theta_{0\alpha}\bar{\theta}_{0\alpha}.$$

If we set

$$(4.3)\qquad \omega_\alpha=\theta_{0\alpha}, \qquad \omega_{\alpha\beta}=-(\theta_{\beta\alpha}-\delta_{\alpha\beta}\theta_{00}),$$

then these forms satisfy conditions (1.1) and (1.2). It follows that they are the connection forms of the Fubini-Study metric. Its curvature forms are

$$\Omega_{\alpha\beta}=-\omega_\alpha\wedge\omega_\beta-\delta_{\alpha\beta}\sum \omega_\delta\wedge\bar{\omega}_\delta$$

which prove that the space has constant holomorphic sectional curvature 4, i.e., $\rho=1$.

With these preparations on the geometry of $\boldsymbol{P}^n(\boldsymbol{C})$, let $x: \boldsymbol{R}^2\to\boldsymbol{P}^n(\boldsymbol{C})$ be a totally real, isometric minimal immersion. We assume that the image is not contained in a totally geodesic $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ of $\boldsymbol{P}^n(\boldsymbol{C})$. We wish to define a unitary frame field $\{Z_a\}$ over a neighborhood $U\subseteqq\boldsymbol{R}^2$ along x as C^∞-maps

$$Z_a: U\subseteqq\boldsymbol{R}^2\longrightarrow \boldsymbol{C}^{n+1}-\{0\}$$

such that:

1) $\pi\cdot Z_0: U\to\boldsymbol{P}^n(\boldsymbol{C})$ is the restriction of the immersion x;

2) $\{Z_0, Z_1, \cdots, Z_n\}$ is a unitary frame in $\boldsymbol{C}^{n+1}$ for each point $x\in\boldsymbol{R}^2$.

Each Z_a is defined up to the multiplication by a complex number of norm one: $Z_a\to Z_a^*=\exp(i\tau_a)Z_a$, τ_a real. Such a frame will be called the *Frenet-Borůvka frame* of the immersion x.

Let x be any point of $\boldsymbol{R}^2$ and choose $Z: U \subseteqq \boldsymbol{R}^2 \rightarrow \boldsymbol{C}^{n+1} - \{0\}$ to be a homogeneous coordinate vector for x and put

$$Z_0 = Z/(Z, Z)^{1/2}.$$

From the result given in Section 3, there is a unitary frame $\{e_\alpha\}$ along $x(\boldsymbol{R}^2)$ such that they satisfy (3.36) and (3.37). $Z_\alpha \in \boldsymbol{C}^{n+1}$ corresponds to e_α by the identification $T_x(\boldsymbol{P}^n(\boldsymbol{C})) \cong \{W \in \boldsymbol{C}^{n+1} : (Z, W) = 0\}$. Then $\{Z_0, Z_1, \cdots, Z_n\}$ is a unitary frame field over $U \subseteqq \boldsymbol{R}^2$ along x. We have, by (3.36), (4.2) and (4.3),

$$
\begin{aligned}
dZ_0 &= \theta_{00} Z_0 + \frac{1}{\sqrt{2}} \phi Z_1 + \frac{1}{\sqrt{2}} \bar{\phi} Z_2, \\
dZ_1 &= -\frac{1}{\sqrt{2}} \bar{\phi} Z_0 + \theta_{00} Z_1 + \bar{c}_1 \phi Z_2 + k_1 \bar{\phi} Z_3, \\
dZ_2 &= -\frac{1}{\sqrt{2}} \phi Z_0 - c_1 \bar{\phi} Z_1 + \theta_{00} Z_2 + k_1 \overline{c_{2,3}\phi} Z_3 + k_1 c_{2,4} \bar{\phi} Z_4, \\
dZ_3 &= -k_1 \bar{\phi} Z_1 - k_1 c_{2,3} \phi Z_2 + (\theta_{00} + c_1 c_{2,3} \phi - \overline{c_1 c_{2,3} \phi}) Z_3 \\
&\quad + \bar{c}_{2,4} (\bar{c}_2 \phi - \overline{c_1 \phi}) Z_4 + \overline{k_2 c_{2,4}} \phi Z_5, \\
dZ_4 &= -k_1 c_{2,4} \phi Z_2 + c_{2,4} (c_1 \phi - c_2 \bar{\phi}) Z_3 + (\theta_{00} - \overline{c_2 c_{2,3}} \phi + c_2 c_{2,3} \bar{\phi}) Z_4 \\
&\quad + k_2 (\overline{c_{3,5} \phi} - \bar{c}_{2,3} \phi) Z_5 + k_2 c_{3,6} \bar{\phi} Z_6,
\end{aligned}
$$

(4.4)

$$
\begin{aligned}
dZ_{2k+1} &= -k_k c_{k,2k} \bar{\phi} Z_{2k-1} + k_k (-c_{k+1,2k+1} \phi + c_{k,2k-1} \bar{\phi}) Z_{2k} \\
&\quad + (\theta_{00} + c_k c_{k+1,2k+1} \phi - \overline{c_k c_{k+1,2k+1} \phi}) Z_{2k+1} \\
&\quad + c_{k+1,2k+2} (-\overline{c_k \phi} + \bar{c}_{k+1} \phi) Z_{2k+2} + k_{k+1} c_{k+1,2k+2} \phi Z_{2k+3}, \\
dZ_{2k+2} &= -k_k c_{k+1,2k+2} \phi Z_{2k} + c_{k+1,2k+2} (c_k \phi - c_{k+1} \bar{\phi}) Z_{2k+1} \\
&\quad + (\theta_{00} - \overline{c_{k+1} c_{k+1,2k+1}} \phi + c_{k+1} c_{k+1,2k+1} \bar{\phi}) Z_{2k+2} \\
&\quad + k_{k+1} (\overline{c_{k+2,2k+3} \phi} - \bar{c}_{k+1,2k+1} \phi) Z_{2k+3} + k_{k+1} c_{k+2,2k+4} \bar{\phi} Z_{2k+4}, \\
&\qquad\qquad k = 1, \cdots, m-1,
\end{aligned}
$$

$$
dZ_n \begin{cases}
= -k_{m-1} c_{m,2m} \phi Z_{2m-2} + c_{m,2m} (c_{m-1} \phi - c_m \bar{\phi}) Z_{2m-1} \\
\quad + \left(\theta_{00} - \frac{1}{\sqrt{2}} (\bar{c}_{m,2m+1} \phi - c_{m,2m+1} \bar{\phi})\right) Z_n, \quad (n = 2m), \qquad \text{or} \\
= -k_m c_{m,2m} \bar{\phi} Z_{2m-1} + k_m (-c_{m+1,2m+1} \phi + c_{m,2m-1} \bar{\phi}) Z_{2m} \\
\quad + (\theta_{00} + (c_m \phi - \overline{c_m \phi})) Z_n, \qquad (n = 2m+1).
\end{cases}
$$

A rotation $Z_0 \rightarrow Z_0^* = \exp(i\psi) Z_0$, ψ real, induces the change

$$\theta_{00} \longrightarrow \theta_{00}^* = i d\psi + \theta_{00}$$

on the Maurer-Cartan forms. Since θ_{00} is closed and purely imaginary, by the first formula of (3.36) and (4.3), we can assume $\theta_{00}^* = 0$ by taking suitable function ψ on U. Dropping the asterisks, we have

(4.5) $$d\,{}^t[Z_0, Z_1, \cdots, Z_n] = [A dz + B d\bar{z}]\,{}^t[Z_0, Z_1, \cdots, Z_n],$$

where A and B are constant complex matrices which are algebraically determined by $n-2$ real numbers k_t and n_t. We wish to solve (4.5) as follows: Since ${}^t[Z_0, \cdots, Z_n] \in U(n+1)$, they satisfy $A\,{}^t\bar{A} = {}^t\bar{A}A$, $B\,{}^t\bar{B} = {}^t\bar{B}B$ and $B + {}^t\bar{A} = 0$. Therefore there exists some unitary matrix T such that

$$
{}^t\bar{T}AT = \begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}, \qquad {}^t\bar{T}BT = \begin{bmatrix} -\bar{\lambda}_0 & & & 0 \\ & -\bar{\lambda}_1 & & \\ & & \ddots & \\ 0 & & & -\bar{\lambda}_n \end{bmatrix},
$$

where λ_a are eigenvalues of A. Hence we have

$$
d\left({}^t\bar{T}\begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_n \end{bmatrix}\right) = \begin{bmatrix} \lambda_0 dz - \overline{\lambda_0 dz} & & 0 \\ & \ddots & \\ 0 & & \lambda_n dz - \overline{\lambda_n dz} \end{bmatrix} \cdot {}^t\bar{T}\begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_n \end{bmatrix}.
$$

This is easily solved and hence the general solution of the total differential equation (4.5) is given by, for a fixed $(u_0, v_0) \in U$,

$$
(4.6) \qquad {}^t[Z_0(u, v), \cdots, Z_n(u, v)] = TG\,{}^t\bar{T}\,{}^t[Z_0(u_0, v_0), \cdots, Z_n(u_0, v_0)],
$$

where we get

$$
(4.7) \qquad G = \{(\exp(\lambda_a z - \overline{\lambda_a z})\delta_{ab}) : z = u + iv,\ (u, v) \in \boldsymbol{R}^2\}.
$$

As we can assume ${}^t[Z_0(u_0, v_0), \cdots, Z_n(u_0, v_0)] = T$, we obtained $Z_0(u, v) = GZ_0(u_0, v_0)$. This solution $Z_0(u, v)$ has a unique extension over $\boldsymbol{R}^2$, hence we proved

THEOREM 15. *Let $x : \boldsymbol{R}^2 \to \boldsymbol{P}^n(\boldsymbol{C})$ be a totally real, isometric minimal immersion such that the image is not contained in a totally geodesic submanifold of $\boldsymbol{P}^n(\boldsymbol{C})$. Then x is homogeneous in the sense that $x(\boldsymbol{R}^2)$ is an orbit of an abelian Lie subgroup of $U(n+1)$.*

There exists an $(n-2)$-parameter family Σ_n of isometric minimal and full immersions of $\boldsymbol{R}^2$ into $\boldsymbol{P}^n(\boldsymbol{C})$.

REMARK 4. In case of $n=2$, the above theorem proves that there exists a unique totally real flat minimal surfaces in $\boldsymbol{P}^2(\boldsymbol{C})$, which has been proved earlier by [7] and later by [8], independently.

When $n>2$, Σ_n includes *at least* $n-2$ different minimal immersions because, for given $n-2$ positive numbers k_t and $c_{t+1,2t+2}$ satisfying $k_t^2 - \rho/2 \leqq 0$ and $c_{t+1,2t+2}{}^2 - 1 \leqq 0$, we can define complex numbers c_t and $c_{t+1,2t+1}$ by (3.37) with some ambiguity.

Main result of this paper is also stated as follows:

Let $x: M^2 \to \boldsymbol{P}^n(\boldsymbol{C})$ be an isometric minimal immersion of a complete simply connected two dimensional Riemannian manifold into the complex projective space of constant holomorphic sectional curvature 4. If $|B|^2 \leqq 2$ on M^2 and x is not superminimal, then M^2 is isometric to $\boldsymbol{R}^2$ and $x(\boldsymbol{R}^2) \in \Sigma_m$ for some $m \leqq n$.

References

[1] B. Y. Chen and K. Ogiue, On totally real submanifolds, Trans. Amer. Math. Soc., **193** (1974), 257-266.

[2] S. S. Chern and J. G. Wolfson, Minimal surfaces by moving frames, Amer. J. Math., **105** (1983), 59-83.

[3] J. H. Eschenburg, R. A. Tribuzy and I. V. Guadalupe, The fundamental equations of minimal surfaces in $\boldsymbol{CP}^2$, Math. Ann., **270** (1985), 571-598.

[4] A. Huber, On subharmonic functions and differential geometry in the large, Comm. Math. Helv., **32** (1957), 13-72.

[5] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 2, Interscience Publishers, New York, 1969.

[6] B. Lawson, The Riemannian geometry of holomorphic curves, Carolina Conference Proceedings: Holomorphic mappings and minimal surfaces 1970, 86-107.

[7] G. D. Ludden, M. Okumura and K. Yano, A totally real surface in $\boldsymbol{CP}^2$ that is not totally geodesic, Proc. Amer. Math. Soc., **53** (1975), 186-190.

[8] H. Naitoh, Totally real parallel submanifolds in $\boldsymbol{P}^n(\boldsymbol{C})$, Tokyo J. Math., **4** (1981), 279-306.

[9] J. G. Wolfson, Minimal surfaces in complex manifolds, Ph. D. Thesis, University of California, Berkeley, 1982.

[10] S. T. Yau, Submanifolds with constant mean curvature I, Amer. J. Math., **96** (1974), 346-366.

Katsuei KENMOTSU
Department of Mathematics
College of General Education
Tôhoku University
Kawauchi, Sendai 980
Japan