J. Math. Soc. Japan
Vol. 38, No. 2, 1986

Structure of the scattering operator for time-periodic Schrödinger equations

By Shu NAKAMURA

(Received Oct. 31, 1984)

§1. Introduction.

In this paper we study the structure of the scattering operator for time-periodic Schrödinger equations with period ω:

$$i\frac{\partial}{\partial t}\phi(t, x)=(-\Delta+V(t, x))\phi(t, x), \qquad \phi(t, \cdot)\in\mathcal{H}=L^2(\boldsymbol{R}^n), \tag{1.1}$$

$$V(t+\omega, x)=V(t, x)\in\boldsymbol{R}, \qquad (t\in\boldsymbol{R},\ x\in\boldsymbol{R}^n). \tag{1.2}$$

Under suitable conditions on $V(t, x)$ to be specified below, (1.1) generates a unitary evolution operator $U(t, s)$, $-\infty<t, s<\infty$, and for each $s\in\boldsymbol{R}$, the wave operators defined by

$$W_{\pm}(s)=\operatorname*{s-lim}_{t\to\pm\infty}U(s, t)e^{-i(t-s)H_0}, \qquad H_0=-\Delta, \tag{1.3}$$

exist and are complete: $\operatorname{Ran}W_{\pm}(s)=\mathcal{H}^{\mathrm{ac}}(U(s+\omega, s))$ (see Yajima [**14**], Howland [**6**], Kitada-Yajima [**8**], Nakamura [**11**]). Then the scattering operator defined by

$$S(s)=W_+(s)^*W_-(s) \tag{1.4}$$

is unitary, and by virtue of the time-periodicity, it satisfies

$$S(s)e^{-i\omega H_0}=e^{-i\omega H_0}S(s). \tag{1.5}$$

It follows that, if we denote a spectral representation of H_0 by $(\tilde{F}(\lambda), \mathfrak{X}(\lambda), d\lambda)$, $S(s)$ is decomposed as

$$\begin{cases} S(s)=\sum_{\mu\in\boldsymbol{Z}}S_{\mu}, \\ \tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)S_{\mu}\phi=\tilde{S}_{\mu}(\lambda)\tilde{F}(\lambda)\phi \quad (\text{a.e. }\lambda), \\ \tilde{S}_{\mu}(\lambda)\in\mathrm{B}\Big(\mathfrak{X}(\lambda),\ \mathfrak{X}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)\Big), \end{cases} \tag{1.6}$$

for any $\phi\in\mathcal{H}$. We call $\{\tilde{S}_{\mu}(\lambda)\}$ S-matrices (see §2 for details).

In this paper we are concerned with the structure of $S(s)$ or $\{\tilde{S}_{\mu}(\lambda)\}$, and show that the decay of $\tilde{S}_{\mu}(\lambda)$ as μ tends to infinity is completely determined by the smoothness property of $V(t, x)$ in t. We assume that the potential $V(t, x)$ satisfies

ASSUMPTION $(A)_\beta$. For some $p>n$, $\alpha>1/2$, and $\beta>0$, $t\rightarrow(1+|x|^2)^\alpha\cdot V(t, x)$ is an $(L^p(\boldsymbol{R}^n)+L^\infty(\boldsymbol{R}^n))$-valued $C^{1+\beta}$-function.

We denote the eigenvalues of $U(s+\omega, s)$ by $\{e^{-i\omega\lambda_j}\}_{j=1,2,\ldots}$, and set the exceptional set $\mathcal{E}$ as

$$\mathcal{E}=\left\{\frac{2\pi}{\omega}\mu+\lambda_j : \mu\in\boldsymbol{Z},\ j=1, 2, \cdots\right\}\cup\frac{2\pi}{\omega}\boldsymbol{Z}. \tag{1.7}$$

Under Assumption $(A)_\beta$, it is known that $\mathcal{E}$ is a closed set with no accumulation points except $(2\pi/\omega)\boldsymbol{Z}$ (see Nakamura [**11**] Theorem 2.18). Our main result is formulated as follows.

THEOREM 1. *Let* $(A)_\beta$ *be satisfied. Suppose that* J *is a compact subset of* $\boldsymbol{R}$ *such that* $J\cap\mathcal{E}=\emptyset$, *and that* $\varepsilon<\beta$ *is a positive constant. Then*

$$\|P_{\{\lambda:\lambda>E\}}(H_0)S(s)P_J(H_0)\|<CE^{-(1/4+\varepsilon)} \qquad (E>0), \tag{1.8}$$

where $\{P_\Omega(H_0)\}$ *is the spectral measure of* H_0.

Scattering theory for time-periodic Schrödinger equations has been studied by Schmidt [**13**], Yajima [**14**], Howland [**6**], Kitada-Yajima [**8**], and others ([**1**], [**3**], [**11**]), and the existence and the completeness of the wave operators have been proved by them. To prove these properties, Schmidt used the trace class method of Birman-Kato; Yajima and Howland employed a time-periodic version of the Howland stationary theory for time-dependent Hamiltonians ([**5**]); and Kitada-Yajima used a variation of Enss time-dependent method ([**2**]). See Yajima [**15**] for further references. On the other hand, representation of the scattering operator for time-independent Schrödinger operators had been known in the physical literature since 1950's, and proved rigorously by stationary scattering theory (see the note for XII-§6 of Reed-Simon [**12**], and Kuroda [**10**] for example). Here we shall combine an abstract representation formula given by Kuroda [**9**] with the method of Yajima-Howland to obtain a representation of $S(s)$ (see Theorem 2).

In §2 we review the method of Yajima-Howland, and construct an explicit representation of $S(s)$. In §3 we estimate $\{\tilde{S}_\mu(\lambda)\}$ and prove Theorem 1.

NOTATIONS. We shall use the following notations throughout the paper.

We denote the set of natural numbers by $\boldsymbol{N}$, integers by $\boldsymbol{Z}$, and reals by $\boldsymbol{R}$. We write $\boldsymbol{R}^n$ for the Euclidian n-space.

For a Hilbert space $\mathcal{H}$ and a measure space M, we write $L^p(M, \mathcal{H})$ for the $\mathcal{H}$-valued L^p-space on M, and write $l^p(\mathcal{H})=L^p(\boldsymbol{Z}, \mathcal{H})$. For a pair of Banach spaces (X, Y), $\mathrm{B}(X, Y)$ denotes the Banach space of all bounded operators from

X to Y, and we write $\mathrm{B}(X)=\mathrm{B}(X, X)$.

$H^{\gamma}(\boldsymbol{R}^n)$ is the Sobolev space of order γ on $\boldsymbol{R}^n$, and $H_{\alpha}^{\gamma}(\boldsymbol{R}^n)$ denotes the weighted Sobolev space:

$$H_{\alpha}^{\gamma}(\boldsymbol{R}^n)=\{\phi\in\mathcal{S}'(\boldsymbol{R}^n) : (1+|x|^2)^{\alpha/2}\phi(x)\in H^{\gamma}(\boldsymbol{R}^n)\}. \tag{1.9}$$

We write $L_{\alpha}^2(\boldsymbol{R}^n)=H_{\alpha}^0(\boldsymbol{R}^n)$ for the weighted L^2-space. For $m\in\boldsymbol{N}$ and $0<\beta<1$, $C^{m+\beta}(\boldsymbol{R})$ denotes the class of C^m-functions whose m-th derivative is Hölder continuous of order β.

For a function $F=F(x)$, we denote the multiplication operator by $F(x)$ by the same symbol F. We write $\langle x\rangle=\sqrt{1+|x|^2}$ for $x\in\boldsymbol{R}^n$.

$\mathcal{F}_{x\to\xi}$ denotes the Fourier transform from $\boldsymbol{R}_x^n$-space to $\boldsymbol{R}_{\xi}^n$-space and is defined by

$$(\mathcal{F}_{x\to\xi}\phi)(\xi)=\hat{\phi}(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n}e^{-ix\cdot\xi}\phi(x)dx. \tag{1.10}$$

$\mathcal{F}_{t\to\mu}\phi$ denotes the Fourier series expansion of ϕ on $[0, \omega)$ and is defined by

$$(\mathcal{F}_{t\to\mu}\phi)_{\mu}=\omega^{-1/2}\int_0^{\omega}e^{-i2\pi\mu t/\omega}\phi(t)dt. \tag{1.11}$$

We define the energy support of $\phi\in L^2(\boldsymbol{R}^n)$ by

$$\text{E-supp}\,\phi=\{|\xi|^2 : \xi\in\operatorname{supp}(\mathcal{F}_{x\to\xi}\phi)\}. \tag{1.12}$$

§ 2. Representation of the scattering operator.

In this section we assume $(\mathrm{A})_{\beta}$. Then it is known that (1.1) generates a unitary evolution operators.

PROPOSITION 2.1. *There exists a set of unitary operators* $\{U(t, s): t, s\in\boldsymbol{R}\}$ *such that*

$$(t, s)\to U(t, s): \quad \textit{strongly continuous.} \tag{2.1}$$

$$U(t, s)=U(t, r)U(r, s) \qquad (t, r, s\in\boldsymbol{R}). \tag{2.2}$$

$$U(t+\omega, s+\omega)=U(t, s) \qquad (t, s\in\boldsymbol{R}). \tag{2.3}$$

$$U(t, s)H^2(\boldsymbol{R}^n)=H^2(\boldsymbol{R}^n) \qquad (t, s\in\boldsymbol{R}). \tag{2.4}$$

$$\frac{d}{dt}U(t, s)\phi=i(H_0+V(t))U(t, s)\phi \qquad (t, s\in\boldsymbol{R},\ \phi\in H^2(\boldsymbol{R}^n)), \tag{2.5}$$

$$\frac{d}{ds}U(t, s)\phi=-iU(t, s)(H_0+V(s))\phi \qquad (t, s\in\boldsymbol{R},\ \phi\in H^2(\boldsymbol{R}^n)),$$

where the derivatives are taken in the strong sense.

For the proof see Kato [7], Yajima [14].

It is known also that the wave operators exist and are complete (Kitada-

Yajima [8], see also Yajima [14], Howland [6], Nakamura [11]).

PROPOSITION 2.2. *The wave operators defined by*

$$W_{\pm}(s)=\operatorname*{s-lim}_{t\to\pm\infty} U(s, t)e^{-i(t-s)H_0} \tag{2.6}$$

exist and are complete:

$$\operatorname{Ran} W_{\pm}(s)=\mathcal{H}^{ac}(U(s, s+\omega)). \tag{2.7}$$

Now, following Yajima [14] and Howland [6], we introduce

$$\mathcal{K}=L^2(\boldsymbol{T}, \mathcal{H})\cong L^2(\boldsymbol{T})\otimes\mathcal{H}, \qquad \mathcal{H}=L^2(\boldsymbol{R}^n), \quad \boldsymbol{T}=\boldsymbol{R}/\omega, \tag{2.8}$$

and we define the propagators $\mathcal{U}_0(\sigma)$, $\mathcal{U}(\sigma)$ by

$$(\mathcal{U}_0(\sigma)\Psi)(t)=e^{-i\sigma H_0}\Psi(t-\sigma), \tag{2.9}$$

$$(\mathcal{U}(\sigma)\Psi)(t)=U(t, t-\sigma)\Psi(t-\sigma), \tag{2.10}$$

for $\Psi=\{\Psi(t): t\in\boldsymbol{T}, \Psi(t)\in\mathcal{H}\}\in\mathcal{K}$. It follows easily from Proposition 2.1 that $\{\mathcal{U}_0(\sigma): \sigma\in\boldsymbol{R}\}$ and $\{\mathcal{U}(\sigma): \sigma\in\boldsymbol{R}\}$ are one parameter unitary groups on $\mathcal{K}$. Then by Stone's theorem, there exist self-adjoint operators K_0 and K such that $\mathcal{U}_0(\sigma)=e^{-i\sigma K_0}$ and $\mathcal{U}(\sigma)=e^{-i\sigma K}$.

We define $\mathcal{U}_{0s}$ and $\mathcal{U}_s\in\mathrm{B}(\mathcal{H}, \mathcal{K})$ by

$$(\mathcal{U}_{0s}\phi)(t)=e^{-i(t-s)H_0}\phi \qquad (0\leqq t<\omega), \tag{2.11}$$

$$(\mathcal{U}_s\phi)(t)=U(t, s)\phi \qquad (0\leqq t<\omega), \tag{2.12}$$

for $\phi\in\mathcal{H}$, where we identify $\boldsymbol{T}$ with $[0, \omega)$.

LEMMA 2.3 (Yajima [14], Howland [6]). *Wave operators defined by*

$$\mathcal{W}_{\pm}=\operatorname*{s-lim}_{t\to\pm\infty}\mathcal{U}(-\sigma)\mathcal{U}_0(\sigma) \tag{2.13}$$

exist and are complete: $\operatorname{Ran}\mathcal{W}_{\pm}=\mathcal{K}^{ac}(K)$. *Thus the scattering operator defined by* $\mathcal{S}=\mathcal{W}_+^*\mathcal{W}_-$ *is unitary. Moreover,*

$$\mathcal{W}_{\pm}\mathcal{U}_{0s}=\mathcal{U}_sW_{\pm}(s), \tag{2.14}$$

$$\mathcal{U}_{0s}^*\mathcal{S}\mathcal{U}_{0s}=\omega S(s). \tag{2.15}$$

PROOF. From the definitions (2.9)~(2.12), we see for $\Psi\in\mathcal{K}$,

$$\begin{aligned}(\mathcal{U}(\sigma)^*\mathcal{U}_0(\sigma)\Psi)(t) &\qquad (0\leqq t<\omega)\\ =U(t, t+\sigma)e^{-i\sigma H_0}\Psi(t)&\\ =U(t, s)\{U(s, t+\sigma)e^{-i(\sigma+t-s)H_0}\}e^{i(t-s)H_0}\Psi(t).&\end{aligned} \tag{2.16}$$

By this formula, we obtain the existence of $\mathcal{W}_{\pm}$, (2.14) and (2.15). If we set $(\tilde{\mathcal{U}}_s\Psi)(t)=U(t, s)\Psi(t)$ $(0\leqq t<\omega)$, we see also

$$(2.17)\qquad (\mathcal{U}(\omega)\Psi)(t)=U(t+\omega,\, t)\Psi(t)$$
$$=U(t,\, s)U(s+\omega,\, s)U(s,\, t)\Psi(t)$$
$$=(\tilde{\mathcal{U}}_s(I\otimes U(s+\omega,\, s))\tilde{\mathcal{U}}_s^*\Psi)(t).$$

Thus we have

$$(2.18)\qquad \mathcal{K}^{ac}(K)=\mathcal{K}^{ac}(\mathcal{U}(\omega))$$
$$=\mathcal{K}^{ac}(\tilde{\mathcal{U}}_s(I\otimes U(s+\omega,\, s))\tilde{\mathcal{U}}_s^*)$$
$$=\tilde{\mathcal{U}}_s\mathcal{K}^{ac}(I\otimes U(s+\omega,\, s))$$
$$=\tilde{\mathcal{U}}_s(L^2(\boldsymbol{T})\otimes \operatorname{Ran} W_\pm(s))\qquad \text{(by Proposition 2.2)}.$$

The completeness follows from (2.16) and (2.18). □

Let us consider K_0. Denote by $\gamma(\rho)$ the trace operator: $\gamma(\rho)\psi(x)=\psi(x)$ $(x\in\rho S^{n-1})$, $H^\alpha(\boldsymbol{R}^n)\to L^2(\rho S^{n-1})$, and define $\tilde{F}(\lambda): L_\alpha^2(\boldsymbol{R}^n)\to L^2(\lambda^{1/2}S^{n-1})\equiv\mathcal{X}(\lambda)$ by $\tilde{F}(\lambda)=2^{-1/2}\lambda^{-1/4}\gamma(\lambda^{1/2})\mathcal{F}_{x\to\xi}$ if $\lambda\geqq 0$, and $\tilde{F}(\lambda)=0$ if $\lambda<0$. Then it is well-known that $(\tilde{F}(\lambda),\, \mathcal{X}(\lambda),\, d\lambda)$ provides a spectral representation of H_0 i.e.

$$(2.19)\qquad P_\Omega(H_0)=\int_{\lambda\in\Omega}\tilde{F}(\lambda)^*\tilde{F}(\lambda)d\lambda\qquad (\Omega:\text{ a Borel set of } \boldsymbol{R})$$

where the integral is a Riemann integral of $\mathrm{B}(L_\alpha^2(\boldsymbol{R}^n),\, L_{-\alpha}^2(\boldsymbol{R}^n))$-valued continuous function.

LEMMA 2.4. *K_0 can be represented as*

$$(2.20)\qquad (\mathcal{F}_{t\to\mu}K_0\Psi)_\mu=\Big(H_0+\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Psi)_\mu,\qquad \Psi\in D(K_0).$$

Further, if we define

$$F(\lambda)\ :\ L^2(\boldsymbol{T},\, L_\alpha^2)\longrightarrow \bigoplus_{\mu<(\omega/2\pi)\lambda}\mathcal{X}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)\equiv\mathcal{Y}(\lambda)$$

by

$$(F(\lambda)\Psi)_\mu=\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Psi)_\mu,$$

then $(F(\lambda),\, \mathcal{Y}(\lambda),\, d\lambda)$ provides a spectral representation of K_0 i.e.

$$(2.21)\qquad P_\Omega(K_0)=\int_{\lambda\in\Omega}F(\lambda)^*F(\lambda)d\lambda\qquad (\Omega:\text{ a Borel set of } \boldsymbol{R})$$

where the integral is a Riemann integral of $\mathrm{B}(L^2(\boldsymbol{T},\, L_\alpha^2),\, L^2(\boldsymbol{T},\, L_{-\alpha}^2))$*-valued continuous function.*

PROOF. (2.20) follows easily from the definition (2.9). From (2.20) we have

$$(2.22)\qquad \mathcal{F}_{t\to\mu}f(K_0)=f\Big(H_0+\frac{2\pi}{\omega}\mu\Big)\mathcal{F}_{t\to\mu},$$

for any bounded Borel function f on $\boldsymbol{R}$. So we see

$$(2.23)\quad (\Phi, P_\Omega(K_0)\Psi)=\sum_\mu\Big((\mathcal{F}_{t\to\mu}\Phi)_\mu,\ P_\Omega\Big(H_0+\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Psi)_\mu\Big)$$
$$=\sum_\mu\int_{\lambda+(2\pi/\omega)\mu\in\Omega}((\mathcal{F}_{t\to\mu}\Phi)_\mu,\ \tilde{F}(\lambda)^*\tilde{F}(\lambda)(\mathcal{F}_{t\to\mu}\Psi)_\mu)d\lambda$$
$$=\int_{\lambda\in\Omega}\sum_\mu\Big(\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Phi)_\mu,\ \tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Psi)_\mu\Big)d\lambda$$
$$=\int_{\lambda\in\Omega}(F(\lambda)\Phi,\ F(\lambda)\Psi)d\lambda,$$

for $\Phi, \Psi\in L^2(\boldsymbol{T}, L^2_\alpha)$. This proves (2.21). □

Next we consider the potential function $V(t, x)$. We define an operator V on $\mathcal{K}$ by $(Vf)(t, x)=V(t, x)f(t, x)$, and set

$$(2.24)\qquad V_\mu(x)=\omega^{1/2}(\mathcal{F}_{t\to\mu}V)_\mu=\int_0^\omega e^{-i2\pi\mu t/\omega}V(t, x)dt.$$

LEMMA 2.5. *For γ such that $n/p<\gamma<1$, and for $\varepsilon<\beta$,*

$$(2.25)\qquad \sum_{\mu\in\boldsymbol{Z}}\langle\mu\rangle^\varepsilon\|\langle x\rangle^{2\alpha}V_\mu\|_{B(H^\gamma, L^2)}<\infty,$$

and for $\Psi\in L^2(\boldsymbol{T}, H^\gamma)$,

$$(2.26)\qquad (\mathcal{F}_{t\to\mu}V\Psi)_\mu=\sum_{\nu\in\boldsymbol{Z}}V_{\mu-\nu}(\mathcal{F}_{t\to\mu}\Psi)_\nu.$$

PROOF. By Assumption $(A)_\beta$, we have

$$(2.27)\qquad \sup_{\mu\in\boldsymbol{Z}}\langle\mu\rangle^{1+\beta}\|\langle x\rangle^{2\alpha}V_\mu(x)\|_{L^p+L^\infty}<\infty.$$

Thus,

$$(2.28)\qquad \sum_{\mu\in\boldsymbol{Z}}\langle\mu\rangle^\varepsilon\|\langle x\rangle^{2\alpha}V_\mu(x)\|_{L^p+L^\infty}<\infty.$$

Since for any multiplication operator $F=F(x)$ and for $n/p<\gamma<1$, $\|F\|_{B(H^\gamma, L^2)}\leqq C\|F\|_{L^p+L^\infty}$ by Sobolev embedding theorem, we obtain (2.25). (2.26) is obvious. □

We fix γ in the above lemma.

LEMMA 2.6 (Yajima-Kitada [16]). *V is K_0-bounded with zero K_0-bound and $K=K_0+V$.*

PROOF. The lemma follows from (2.26) and the fact that $\langle x\rangle^{-\alpha}(H_0+1)^{(1/2)\gamma}$ is K_0-bounded with zero K_0-bound (see Lemma 4.2 of Yajima-Kitada [16], see also Proposition 2.4 of Nakamura [11]). □

LEMMA 2.7. *$G_-(\lambda)=\lim_{\varepsilon\downarrow 0}(1+(K_0-(\lambda+i\varepsilon))^{-1}V)^{-1}$ exists for $\lambda\notin\mathcal{E}$ in $B(L^2(\boldsymbol{T},H^\gamma_{-\alpha}(\boldsymbol{R}^n)))$, and $G_-(\lambda)$ is uniformly bounded in $\lambda\in J$ if J satisfies the as-*

sumptions of Theorem 1.

For the proof, see Nakamura [**11**], or Appendix of this paper.

We define $T(\lambda)=VG_-(\lambda)$, then $T(\lambda)$ is a bounded operator from $L^2(\boldsymbol{T}, H^{\gamma}_{-\alpha})$ to $L^2(\boldsymbol{T}, L^2_{\alpha})$ if $\lambda\notin\mathcal{E}$, by Lemma 2.5 and Lemma 2.7. We denote by $T(\lambda)_{\mu\nu}$ the (μ, ν)-matrix element of $\mathcal{F}_{t\to\mu}T(\lambda)\mathcal{F}^{-1}_{t\to\mu}$. Now we can state our main result of this section.

THEOREM 2. *Suppose* ϕ, $\psi\in C_0^\infty(\boldsymbol{R}^n)$ *and* E-supp$\psi\cap\mathcal{E}=\emptyset$, *then*

$$(2.29)\quad (\phi, S(s)\psi)=(\phi, \psi) -2\pi i\sum_{\mu\in\boldsymbol{Z}}e^{is(2\pi/\omega)\mu}\int_0^\infty d\lambda\Big(\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^*\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)\phi,\ T(\lambda)_{\mu o}\tilde{F}(\lambda)^*\tilde{F}(\lambda)\psi\Big)$$

or, in the form of S-matrices defined by (1.6),

$$(2.30)\quad \tilde{S}_{\mu}(\lambda)=\delta_{\mu o}-2\pi i\, e^{is(2\pi/\omega)\mu}\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)T(\lambda)_{\mu o}\tilde{F}(\lambda)^*$$
$$=\delta_{\mu o}-\pi i\, e^{is(2\pi/\omega)\mu}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^{-1/4}\gamma\Big(\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^{1/2}\Big)\mathcal{F}_{x\to\xi}T(\lambda)_{\mu o}\mathcal{F}^{-1}_{x\to\xi}\gamma(\lambda^{1/2})^*\lambda^{-1/4}$$

where $\delta_{\mu o}$ *is the Kronecker's delta symbol.*

REMARK. The integrant in the right hand side of (2.29) is continuous in λ because $\tilde{F}(\lambda)^*\tilde{F}(\lambda)$ is a continuous $B(L^2_{\alpha}, H^{\gamma}_{-\alpha})$-valued function as is easily seen from the definition of $\tilde{F}(\lambda)$, and $T(\lambda)$ is continuous in $\lambda\notin\mathcal{E}$.

PROOF. If we set

$$(2.31)\qquad \tilde{\mathcal{S}}(\lambda)=1-2\pi i\, F(\lambda)T(\lambda)F(\lambda)^*,$$

Theorem 6.3 of Kuroda [**9**] yields

$$(2.32)\qquad F(\lambda)\mathcal{S}\Phi=\tilde{\mathcal{S}}(\lambda)F(\lambda)\Phi \qquad (\text{a.e. } \lambda)$$

for $\Phi\in L^2(\boldsymbol{T}, L^2_{\alpha})$. Hence if Φ and $\Psi\in L^2(\boldsymbol{T}, H^{\gamma}_{\alpha})$ satisfy dist$(\bigcup_\mu \text{E-supp}(\mathcal{F}_{t\to\mu}\Psi)_\mu, \mathcal{E})>0$, we see

$$(2.33)\quad (\Phi, (\mathcal{S}-1)\Psi)=-2\pi i\int d\lambda(F(\lambda)\Phi, (\tilde{\mathcal{S}}(\lambda)-1)F(\lambda)\Psi)$$
$$=-2\pi i\sum_{\mu,\nu}\int d\lambda\Big(\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^*\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)(\mathcal{F}_{t\to\mu}\Phi)_\mu,$$
$$T(\lambda)_{\mu\nu}\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\nu\Big)^*\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\nu\Big)(\mathcal{F}_{t\to\mu}\Psi)_\nu\Big).$$

Note that $\mathcal{F}_{t\to\mu}\mathcal{U}_{0s}\phi$ is given by

$$(2.34)\qquad (\mathcal{F}_{t\to\mu}\mathcal{U}_{0s}\phi)_\mu=i\omega^{-1/2}\Big(H_0+\frac{2\pi}{\omega}\mu\Big)^{-1}(e^{-i\omega H_0}-1)e^{isH_0}\phi.$$

Combining (2.15), (2.33) and (2.34), we have

$$(2.35)\qquad (\phi, (S(s)-1)\psi)=\omega^{-1}(\mathcal{U}_{0s}\phi, (\mathcal{S}-1)\mathcal{U}_{0s}\psi)$$
$$=-\frac{2\pi i}{\omega^2}\sum_{\mu,\nu}\int d\lambda\frac{|e^{i\omega\lambda}-1|^2}{\lambda^2}e^{is(2\pi/\omega)(\mu-\nu)}$$
$$\times\left(\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\mu\right)^*\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\mu\right)\phi, T(\lambda)_{\mu\nu}\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\nu\right)^*\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\nu\right)\psi\right).$$

Then observing the property: $T(E)_{\mu\nu}=T(E-(2\pi/\omega)m)_{\mu-m,\nu-m}$, we can write the right hand side of (2.35) as

$$(2.36)\qquad -2\pi i\sum_{\mu}\int d\lambda\left\{\frac{1}{\omega^2}\sum_{\nu}\frac{|e^{i\omega\lambda}-1|^2}{(\lambda+(2\pi/\omega)\nu)^2}\right\}e^{is(2\pi/\omega)\mu}$$
$$\times\left(\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\mu\right)^*\tilde{F}\left(\lambda-\frac{2\pi}{\omega}\mu\right)\phi, T(\lambda)_{\mu 0}\tilde{F}(\lambda)^*\tilde{F}(\lambda)\psi\right).$$

Note $\frac{1}{\omega^2}\sum_{\nu}\frac{|e^{i\omega\lambda}-1|^2}{(\lambda+(2\pi/\omega)\nu)^2}\equiv 1$ by Plancherel's theorem since $-i\omega^{-1/2}\frac{e^{i\omega\lambda}-1}{\lambda-(2\pi/\omega)\nu}$ are Fourier coefficients of $e^{it\lambda}$. Thus the theorem is proved. □

REMARK. Theorem 2 holds under a slightly weaker assumption:

ASSUMPTION (A)′. *For some* $p>n$ *and* $\alpha>1/2$, $t\to\langle x\rangle^{2\alpha}V(t, x)$ *is an* $(L^p(\boldsymbol{R}^n)+L^\infty(\boldsymbol{R}^n))$*-valued absolutely continuous function.*

In fact, Lemma 2.6 and Lemma 2.7 still remains valid under (A)′. Moreover if we employ a different formulation of $T(\lambda)$, we can obtain an analogous representation formula of $S(s)$ under

ASSUMPTION (A)″. *For some* $p>n/2$ *and* $\alpha>1/2$, $t\to\langle x\rangle^{2\alpha}V(t, x)$ *is an* $(L^p(\boldsymbol{R}^n)+L^\infty(\boldsymbol{R}^n))$*-valued absolutely continuous function.* □

§ 3. Proof of Theorem 1.

In this section, we assume $(A)_\beta$ and suppose that J and ε satisfy the assumption of the theorem. We begin with

LEMMA 3.1. *Let* W *be a multiplication operator by* $W(x)$. *Suppose* $\langle x\rangle^{2\alpha}W(x)\in L^p(\boldsymbol{R}^n)+L^\infty(\boldsymbol{R}^n)$ *for* $\alpha>1/2$ *and* $p>n$. *Then for* $n/p<\gamma<1$ *and* $\delta>0$, *there exists* $\varepsilon_1>0$ *such that*

$$(3.1)\qquad \|(H_0-\zeta)^{-1}W\|_{B(H^\gamma_{-\alpha}(R^n))}\leqq C|\zeta|^{-\varepsilon_1}\|\langle x\rangle^{2\alpha}W\|_{L^p+L^\infty}\qquad (|\zeta|>\delta).$$

PROOF. It is sufficient to prove

$$(3.2)\qquad \|(H_0+1)^{\gamma/2}\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}W\langle x\rangle^{\alpha}(H_0+1)^{-\gamma/2}\|_{B(L^2)}$$
$$\leqq C|\zeta|^{-\varepsilon_1}\|\langle x\rangle^{2\alpha}W\|_{L^p+L^\infty}\qquad (|\zeta|>\delta).$$

Now we have

$$(3.3)\qquad (H_0+1)^{\gamma/2}\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}W\langle x\rangle^{\alpha}(H_0+1)^{-\gamma/2}$$
$$=\{(H_0+1)^{\gamma/2}\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\}\{\langle x\rangle^{2\alpha}W(H_0+1)^{-\gamma/2}\}.$$

By Sobolev embedding theorem, we see

$$(3.4)\qquad \|(\langle x\rangle^{2\alpha}W)(H_0+1)^{-\gamma/2}\|\leqq C\|\langle x\rangle^{2\alpha}W\|_{L^p+L^\infty}.$$

Hence it remains only to prove

$$(3.5)\qquad \|(H_0+1)^{\gamma/2}\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|\leqq C|\zeta|^{-\varepsilon_1}\qquad (|\zeta|>\delta).$$

If $\gamma=0$, by Proposition 2.3.2 of Ginibre-Moulin [**4**],

$$(3.6)\qquad \|\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|\leqq C|\zeta|^{-1/2}.$$

On the other hand, if $\gamma=2$,

$$(3.7)\qquad \|(H_0+1)\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|$$
$$\leqq\|[H_0,\langle x\rangle^{-\alpha}](H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|+\|\langle x\rangle^{-\alpha}\{1+(\zeta+1)(H_0-\zeta)^{-1}\}\langle x\rangle^{-\alpha}\|$$
$$\leqq C\|\langle x\rangle^{-\alpha-1}\nabla_x(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|+C(|\zeta|+1)\|\langle x\rangle^{-\alpha}(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|+C$$
$$\leqq C|\zeta|^{1/2}\qquad (|\zeta|>\delta),$$

where we have used the relation $\|\langle x\rangle^{-\alpha}\nabla_x(H_0-\zeta)^{-1}\langle x\rangle^{-\alpha}\|\leqq C$ for any ζ. Using interpolation theorem of Calderón-Lions (Reed-Simon [**12**], Theorem IX-20) between (3.6) and (3.7), we obtain (3.5). □

We denote by $G_-(\lambda)_{\mu\nu}$ the (μ,ν)-matrix component of $\mathcal{F}_{t\to\mu}G_-(\lambda)\mathcal{F}^{-1}_{t\to\mu}$. Then $\{G_-(\lambda)_{\mu0}\}$ are estimated as follows:

PROPOSITION 3.2. *For $\phi\in H^\gamma_{-\alpha}(\boldsymbol{R}^n)$ and $\lambda\in J$,*

$$(3.8)\qquad \Big(\sum_{\mu\in\boldsymbol{Z}}\langle\mu\rangle^{2\varepsilon}\|G_-(\lambda)_{\mu0}\phi\|^2_{H^\gamma_{-\alpha}}\Big)^{1/2}\leqq C\|\phi\|_{H^\gamma_{-\alpha}}.$$

PROOF. Let $\Phi\in L^2(\boldsymbol{T},H^\gamma_{-\alpha})$ be $\Phi(t)\equiv\phi\in H^\gamma_{-\alpha}$, and let $\Psi=G_-(\lambda)\Phi$. We denote $(\mathcal{F}_{t\to\mu}\Psi)_\mu=G_-(\lambda)_{\mu0}\phi$ by $\psi_\mu\in H^\gamma_{-\alpha}$. Then obviously

$$(3.9)\qquad \psi_\mu=\delta_{\mu0}\phi-\sum_{\nu\in\boldsymbol{Z}}\Big(H_0-(\lambda+i0)+\frac{2\pi}{\omega}\mu\Big)^{-1}V_{\mu-\nu}\psi_\nu.$$

On the other hand Lemma 2.7 implies

$$(3.10)\qquad \sum_{\mu\in\boldsymbol{Z}}\|\psi_\mu\|^2_{H^\gamma_{-\alpha}}\leqq C\|\phi\|^2_{H^\gamma_{-\alpha}}.$$

By Lemma 3.1 and (3.9), we see

$$(3.11)\qquad \|\psi_\mu\|_{H^\gamma_{-\alpha}}\leqq\delta_{\mu0}\|\phi\|_{H^\gamma_{-\alpha}}+C\langle\mu\rangle^{-\varepsilon_1}\sum_{\nu\in\boldsymbol{Z}}\|\langle x\rangle^{2\alpha}V_{\mu-\nu}\|_{L^p+L^\infty}\|\psi_\nu\|_{H^\gamma_{-\alpha}}.$$

Using Young's inequality, (3.10) and (3.11) we obtain

$$(3.12)\qquad (\sum_{\mu\in Z}\langle\mu\rangle^{2\varepsilon_1}\|\psi_\mu\|^2_{H^\gamma_{-\alpha}})^{1/2}\leqq C(1+\sum_{\nu\in Z}\|\langle x\rangle^{2\alpha}V_\nu\|_{L^p+L^\infty})\|\phi\|_{H^\gamma_{-\alpha}} \leqq C\|\phi\|_{H^\gamma_{-\alpha}}.$$

Similarly, for $\varepsilon_1<\varepsilon$ and $2\varepsilon_1<\beta$, using Young's inequality, (3.11) and (3.12), we obtain

$$(3.13)\qquad (\sum_{\mu\in Z}\langle\mu\rangle^{4\varepsilon_1}\|\psi_\mu\|^2_{H^\gamma_{-\alpha}})^{1/2} \leqq C(1+\sum_{\nu\in Z}\|\langle\nu\rangle^{\varepsilon_1}\langle x\rangle^{2\alpha}V_\nu\|_{L^p+L^\infty})\|\phi\|_{H^\gamma_{-\alpha}} \leqq C\|\phi\|_{H^\gamma_{-\alpha}}.$$

Repeating this procedure, we have

$$(3.14)\qquad (\sum_{\mu\in Z}\langle\mu\rangle^{2m\varepsilon_1}\|\psi_\mu\|^2_{H^\gamma_{-\alpha}})^{1/2}\leqq C\|\phi\|_{H^\gamma_{-\alpha}}$$

provided $m\varepsilon_1<\beta$. Replacing ε_1 by smaller one if necessary, we can find $m\in N$ such that $\varepsilon<m\varepsilon_1<\beta$. This completes the proof. □

PROPOSITION 3.3. *For $\phi\in H^\gamma_{-\alpha}(R^n)$ and $\lambda\in J$,*

$$(3.15)\qquad (\sum_{\mu\in Z}\langle\mu\rangle^{2\varepsilon}\|T(\lambda)_{\mu 0}\phi\|^2_{L^2_\alpha})^{1/2}\leqq C\|\phi\|_{H^\gamma_{-\alpha}}.$$

PROOF. If we observe that

$$(3.16)\qquad \|T(\lambda)_{\mu 0}\phi\|_{L^2_\alpha}\leqq C\langle\mu\rangle^{-\varepsilon}\sum_{\nu\in Z}(\langle\mu-\nu\rangle^{\varepsilon}\|V_{\mu-\nu}\|_{L^p+L^\infty})\langle\nu\rangle^{\varepsilon}\|G_-(\lambda)_{\nu 0}\phi\|_{H^\gamma_{-\alpha}},$$

the proposition follows from Young's inequality and Proposition 3.2. □

PROOF OF THEOREM 1. We may assume without loss of generality that $J\subset((2\pi/\omega)k, (2\pi/\omega)(k+1))$ for some $k\in N$, then the ranges of $\{S_\mu P_J(H_0)\}$ are mutually orthogonal where S_μ is defined by (1.6). By Theorem 2, $S_\mu P_J(H_0)$ has a representation:

$$(3.17)\qquad (\phi, S_\mu P_J(H_0)\psi)=\delta_{\mu 0}(\phi, P_J(H_0)\psi)-\pi i\, e^{is(2\pi/\omega)\mu}\int_{\lambda\in J}d\lambda\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^{-1/4}\lambda^{-1/4} \times\Big(\mathcal{F}^{-1}_{x\to\xi}\gamma\Big(\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)^{1/2}\Big)^*\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)\phi,\ T(\lambda)_{\mu 0}\mathcal{F}^{-1}_{x\to\xi}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\Big)$$

for $\phi, \psi\in L^2_\alpha(R^n)$. Because $\gamma(\rho)$ is uniformly bounded in ρ as operators from $H^\alpha(R^n)$ to $L^2(\rho S^{n-1})$, we see

$$(3.18)\qquad |(\phi, S_\mu P_J(H_0)\psi)| \leqq\delta_{\mu 0}\|\phi\|\|\psi\| +C\langle\mu\rangle^{-1/4}\int_{\lambda\in J}d\lambda\Big\|\tilde{F}\Big(\lambda-\frac{2\pi}{\omega}\mu\Big)\phi\Big\|_{\mathcal{X}(\lambda-(2\pi/\omega)\mu)}\|T(\lambda)_{\mu 0}\mathcal{F}^{-1}_{x\to\xi}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\|_{L^2_\alpha}$$

$$\leqq \delta_{\mu o}\|\phi\|\|\psi\| + C\langle\mu\rangle^{-(1/4+\varepsilon)}\|\phi\|\left\{\int_{\lambda\in J} d\lambda\langle\mu\rangle^{2\varepsilon}\|T(\lambda)_{\mu o}\mathcal{F}_{x\to\xi}^{-1}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\|_{L_\alpha^2}^2\right\}^{1/2}$$

by Schwartz inequality.

Hence we obtain for $m>k$ and $\psi\in L_\alpha^2(\boldsymbol{R}^n)$,

$$\begin{aligned}(3.19)\quad &\|P_{\{\lambda:\lambda>(2\pi/\omega)m\}}(H_0)S_\mu P_J(H_0)\psi\|^2\\ &=\sum_{-\mu\geqq m-k}\|S_\mu P_J(H_0)\psi\|^2\\ &\leqq C\sum_{-\mu\geqq m-k}\langle\mu\rangle^{-2(1/4+\varepsilon)}\int_{\lambda\in J} d\lambda\langle\mu\rangle^{2\varepsilon}\|T(\lambda)_{\mu o}\mathcal{F}_{x\to\xi}^{-1}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\|_{L_\alpha^2}^2 \quad \text{(by (3.18))}\\ &\leqq C\langle m\rangle^{-2(1/4+\varepsilon)}\int_{\lambda\in J} d\lambda\{\sum_\mu\langle\mu\rangle^{2\varepsilon}\|T(\lambda)_{\mu o}\mathcal{F}_{x\to\xi}^{-1}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\|_{L_\alpha^2}^2\}\\ &\leqq C\langle m\rangle^{-2(1/4+\varepsilon)}\int_{\lambda\in J} d\lambda\|\mathcal{F}_{x\to\xi}^{-1}\gamma(\lambda^{1/2})^*\tilde{F}(\lambda)\psi\|_{H_{-\alpha}^\gamma}^2 \quad \text{(by Proposition 3.3)}\\ &\leqq C\langle m\rangle^{-2(1/4+\varepsilon)}\|\psi\|_{L^2}^2.\end{aligned}$$

This proves the theorem. □

Appendix.

Here we sketch the proof of Lemma 2.7.

At first, we prove the existence of $G_-(\lambda)$ for almost every λ by Lemma 3.1.

LEMMA A.1. *There exists a closed null set $\mathcal{E}'\subset\boldsymbol{R}$ such that for $\lambda\notin\mathcal{E}'$, $G_-(\lambda)=\lim_{\varepsilon\downarrow 0}(1+(K_0-(\lambda+i\varepsilon))^{-1}V)^{-1}$ exists in* $\mathrm{B}(L^2(\boldsymbol{T}, H_{-\alpha}^\gamma(\boldsymbol{R}^n)))$. *Moreover $G_-(\lambda)$ is locally uniformly bounded in $\boldsymbol{R}\setminus\mathcal{E}'$.*

PROOF. By Lemma 3.1, it is obvious that $(K_0-\zeta)^{-1}V$ is bounded in $L^2(\boldsymbol{T},H_{-\alpha}^\gamma(\boldsymbol{R}^n))$ if $\zeta\in\boldsymbol{C}\setminus\boldsymbol{R}$, and is uniformly bounded if ζ is away from $(2\pi/\omega)\boldsymbol{Z}$. Further, since $(H_0-\zeta)^{-1}V_\mu$ is compact for each μ, Lemma 3.1 implies the compactness of $(K_0-\zeta)^{-1}V$. Then the theorem of F. and M. Riesz can be applied and we obtain the assertion. □

Now it remains only to prove $\mathcal{E}\supset\mathcal{E}'$. To prove that, we follow the trace method due to Agmon (see Reed-Simon [**12**], § XIII-8).

LEMMA A.2. *Let $\delta>1/2$ and suppose that $\psi\in L^2(\boldsymbol{T}, L_\delta^2(\boldsymbol{R}^n))$ satisfies*

$$\text{(A.1)}\qquad \mathrm{Im}(\psi, (K_0-(\lambda+i0))^{-1}\psi)=0$$

for $\lambda\in\boldsymbol{R}\setminus(2\pi/\omega)\boldsymbol{Z}$. Then $(K_0-(\lambda+i0))^{-1}\psi\in L^2(\boldsymbol{T}, H_{\delta-1}^{1/2}(\boldsymbol{R}^n))$.

Lemma A.2 can be proved analogously to Lemma 3.1 using Proposition 2.6.1 of Ginibre-Moulin [**4**] and interpolation.

PROOF OF LEMMA 2.7. If $\lambda\in\mathcal{E}'$ and $\lambda\notin(2\pi/\omega)\boldsymbol{Z}$, the Fredholm theorem implies that there exists $\phi\in L^2(\boldsymbol{T}, H^{\gamma}_{-\alpha}(\boldsymbol{R}^n))$ such that

$$\phi+(K_0-(\lambda+i0))^{-1}V\phi=0. \tag{A.2}$$

If we set $\psi=V\phi\in L^2(\boldsymbol{T}, L^2_{\alpha}(\boldsymbol{R}^n))$, we can see easily that ψ satisfies (A.1). Hence $\phi=-(K_0-(\lambda+i0))^{-1}\psi$ is an element of $L^2(\boldsymbol{T}, H^{1/2}_{\alpha-1}(\boldsymbol{R}^n))=L^2(\boldsymbol{T}, H^{1/2}_{-\alpha+(2\alpha-1)}(\boldsymbol{R}^n))$. Repeating this procedure m-times, we have $\phi\in L^2(\boldsymbol{T}, H^{1/2}_{-\alpha+m(2\alpha-1)}(\boldsymbol{R}^n))$, and $\psi\in L^2(\boldsymbol{T}, H^{1/2}_{\alpha+m(2\alpha-1)}(\boldsymbol{R}^n))$. Thus we obtain $\phi\in\mathcal{K}$ and it is clear from (A.2) that ϕ is a λ-eigenfunction of K. This implies $\lambda\in\mathcal{E}$. □

ACKNOWLEDGEMENTS. The author wishes to express his sincere thanks to Professor Kenji Yajima for his guidance and a lot of advice. In particular, the problem was suggested by him. The author also thanks Professor S. T. Kuroda for his unceasing encouragement.

References

[1] E. B. Davies, Time-dependent scattering theory, Math. Ann., **210** (1974), 149-162.
[2] V. Enss, Asymptotic completeness for quantum mechanical potential scattering, Commun. Math. Phys., **61** (1978), 285-291.
[3] V. Enss and K. Veselić, Bound states and propagating states for time-dependent Hamiltonians, Ann. Inst. H. Poincaré, **39** (1983), 159-191.
[4] J. Ginibre and M. Moulin, Hilbert space approach to the quantum mechanical three-body problem, Ann. Inst. H. Poincaré, **21** (1974), 97-145.
[5] J. S. Howland, Stationary scattering theory for time-dependent Hamiltonian, Math. Ann., **207** (1974), 315-335.
[6] J. S. Howland, Scattering theory for Hamiltonians periodic in time, Indiana Univ. Math. J., **28** (1979), 471-494.
[7] T. Kato, Linear evolution equations of "hyperbolic" type II, J. Math. Soc. Japan, **25** (1973), 648-666.
[8] H. Kitada and K. Yajima, A scattering theory for time-dependent long-range potentials, Duke Math. J., **49** (1983), 341-376.
[9] S. T. Kuroda, Scattering theory for differential operators I, Operator theory, J. Math. Soc. Japan., **25** (1973), 75-104.
[10] S. T. Kuroda, An Introduction to Scattering Theory, Lecture Note Series, No. 51, Aarhas Univ., 1978.
[11] S. Nakamura, On scattering theory for time-periodic quantum mechanical systems, Master thesis, Univ. of Tokyo, 1984 (in Japanese).
[12] M. Reed and B. Simon, Methods of Modern Mathematical Physics I-IV, Academic Press, New York, 1972-1979.
[13] E. J. P. G. Schmidt, On scattering by time-dependent perturbations, Indiana Univ. Math. J., **24** (1975), 925-935.
[14] K. Yajima, Scattering theory for Schrödinger equations with potentials periodic in time, J. Math. Soc. Japan, **29** (1977), 729-743.
[15] K. Yajima, Large time behavior of time-periodic quantum systems, Differential Equations, Proc. Int. Conf. Birmingham, Math. Studies, **92**, North-Holland, 1984.

[16] K. Yajima and H. Kitada, Bound states and scattering states for time periodic Hamiltonians, Ann. Inst. H. Poincaré, **39** (1983), 145-157.

Shu NAKAMURA
Department of Pure and Applied Sciences
University of Tokyo
Komaba, Tokyo 153
Japan