J. Math. Soc. Japan
Vol. 39, No. 1, 1987

Contact structures on twistor spaces

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(Received July 19, 1985)

Introduction.

A quaternionic Kähler manifold (M, g, H) is a Riemannian manifold (M, g) together with a coefficient bundle H of quaternions. The twistor space Z of (M, g, H), which is a complex manifold fibring over M, has a natural complex contact structure γ and a natural Einstein pseudo-Kähler metric $\bar{g}$, provided that (M, g) has non-zero scalar curvature (Salamon [8]). In this note we shall study the automorphism groups of these structures.

Let $\mathrm{Aut}(M, g, H)$ and $\mathrm{Aut}(Z, \gamma, \bar{g})$ denote the group of automorphisms of (M, g, H) and the one of isometric contact automorphisms of $(Z, \gamma, \bar{g})$ respectively. Then each element in $\mathrm{Aut}(M, g, H)$ can be lifted to an element in $\mathrm{Aut}(Z, \gamma, \bar{g})$ in a natural way. We show first that *the lifting homomorphism* $\mathrm{Aut}(M, g, H) \rightarrow \mathrm{Aut}(Z, \gamma, \bar{g})$ *is an isomorphism* (Theorem 3.1).

Let $\mathfrak{a}(Z, \gamma)$ and $\mathfrak{a}(Z, \gamma, \bar{g})$ be the Lie algebra of infinitesimal contact automorphisms of (Z, γ) and the one of infinitesimal isometric contact automorphisms of $(Z, \gamma, \bar{g})$ respectively. We prove next that $\mathfrak{a}(Z, \gamma)$ *is the complexification of* $\mathfrak{a}(Z, \gamma, \bar{g})$ (Corollary 2 to Theorem 3.2). This may be viewed as an analogue to the theorem of Matsushima to the effect that the Lie algebra of holomorphic vector fields on a compact Einstein Kähler manifold is the complexification of the Lie algebra of Killing vector fields.

Lastly we study a certain uniqueness of quaternionic Kähler structures. We prove: *Suppose that a compact complex contact manifold M admits a Kähler metric and has the vanishing first integral homology. Then a complex contact structure on M is unique up to automorphisms of M* (Theorem 1.7). Making use of this and previous results we show the following uniqueness: *Let compact quaternionic Kähler manifolds (M, g, H) and (M', g', H') with positive scalar curvatures have the same twistor space Z. Suppose that Z is a kählerian C-space of Boothby type* (see §1 for the definition). *Then (M, g, H) and (M', g', H') are equivalent to each other* (Theorem 4.2). Note that all the known examples of twistor spaces of compact quaternionic Kähler manifolds with positive scalar curvature are kählerian C-spaces of Boothby type.

The first named author would like to express his hearty thanks to Professors

H. Ozeki, T. Mabuchi and Y. Sakane for their constant encouragement and instruction.

§ 1. Complex contact structures.

In this section we recall the basic results on complex contact structures and prove a uniqueness theorem for complex contact structures on certain kählerian manifolds.

Let M be a (connected) complex manifold of odd dimension $2n+1$. By a *system of contact forms* on M we mean an open cover $\{U_i\}$ of M together with a system of holomorphic 1-forms $\{\gamma_i\}$ such that

(1.1) each γ_i is defined on U_i and $\gamma_i \wedge (d\gamma_i)^n \neq 0$ everywhere on U_i; and

(1.2) we have $\gamma_i = f_{ij}\gamma_j$ on $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.

Two systems of contact forms $\{U_i, \gamma_i\}$ and $\{V_\lambda, \delta_\lambda\}$ are said to define the same *complex contact structure* if $\gamma_i = h_{i\lambda}\delta_\lambda$ on $U_i \cap V_\lambda$, where $h_{i\lambda}$ is a holomorphic function on $U_i \cap V_\lambda$. A pair (M, γ) of a complex manifold M and a complex contact structure γ on M is called a *complex contact manifold.* Given a complex contact structure γ on M defined by contact forms $\{U_i, \gamma_i\}$, we define a holomorphic subbundle E_γ of the (holomorphic) tangent bundle TM by

$$(E_\gamma)_x = \{X \in T_x M \,;\, \gamma_i(X) = 0\} \qquad \text{if} \quad x \in U_i .$$

Let $L_\gamma = TM/E_\gamma$ be the quotient line bundle and $\varpi_\gamma : TM \to L_\gamma$ the natural projection, so that we have an exact sequence

$$0 \longrightarrow E_\gamma \longrightarrow TM \xrightarrow{\varpi_\gamma} L_\gamma \longrightarrow 0 \tag{1.3}$$

of holomorphic vector bundles over M.

Let (M, γ) and (M', γ') be complex contact manifolds. By a *contact isomorphism* $\phi : (M, \gamma) \to (M', \gamma')$ we mean a holomorphic diffeomorphism $\phi : M \to M'$ such that the differential $\phi_* : TM \to TM'$ induces an isomorphism $E_\gamma \to E_{\gamma'}$. For a complex contact manifold (M, γ) a contact isomorphism of (M, γ) onto itself is called a *contact automorphism.* Denote by $\mathrm{Aut}(M, \gamma)$ the group of all contact automorphisms of (M, γ). It is a complex Lie group if M is compact (Boothby [**4**]). A holomorphic vector field on M is called an *infinitesimal contact automorphism* of (M, γ) if it generates a local flow of (local) contact automorphisms of (M, γ). Let $\mathfrak{a}(M, \gamma)$ denote the complex Lie algebra of all infinitesimal contact automorphisms of (M, γ). If M is compact, $\mathfrak{a}(M, \gamma)$ is identified with $\mathrm{Lie}\,\mathrm{Aut}(M, \gamma)$, the Lie algebra of $\mathrm{Aut}(M, \gamma)$.

In what follows we recall some basic results on complex contact structures (cf. Kobayashi [6], [7], Boothby [4]). Let (M, γ) be a complex contact manifold of dimension $2n+1$ with γ defined by contact forms $\{U_i, \gamma_i\}$.

THEOREM 1.1. *The system* $\{U_i, d\gamma_i\}$ *induces a non-degenerate alternating holomorphic pairing:*

$$E_\gamma \times E_\gamma \longrightarrow L_\gamma .$$

THEOREM 1.2. *The canonical line bundle* $K_M = \wedge^{2n+1} T^*M$ *of* M *is holomorphically isomorphic to* $L_\gamma^{-(n+1)}$. *In particular, first Chern classes* c_1 *satisfy*

$$c_1(M) = (n+1)c_1(L_\gamma) .$$

Let $\hat{\pi}: P_\gamma \to M$ be the holomorphic C^*-bundle associated to L_γ, i.e., the bundle of frames of L_γ, and denote by R_a the right action of $a \in C^*$ on P_γ. We define a holomorphic 1-form θ_γ on P_γ by

$$\theta_\gamma(X) = u^{-1} \varpi_\gamma(\hat{\pi}_* X) \qquad \text{for} \quad X \in T_u P_\gamma , \ u \in P_\gamma ,$$

and call it the *canonical* 1-*form* on P_γ. We put $\Theta_\gamma = d\theta_\gamma$.

THEOREM 1.3. *We write* P, θ *for* P_γ, θ_γ. *We have then*

(1.4) θ *is semi-basic, i.e.,* θ *is annihilated by the contraction of any vertical vector of* P;

(1.5) $R_a^* \theta = a^{-1}\theta$ *for each* $a \in C^*$; *and*

(1.6) $\Theta = d\theta$ *is a symplectic* 2-*form on* P.

Now we fix a complex manifold M of odd dimension and denote by $\mathcal{C}(M)$ the set of all complex contact structures on M. Note that the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of M acts on $\mathcal{C}(M)$ in a natural way. Next let us consider a pair (P, θ) of holomorphic C^*-bundle P over M and a holomorphic 1-form θ on P satisfying (1.4), (1.5) and (1.6). Two such pairs (P, θ) and (P', θ') are said to be *equivalent* if there exists a holomorphic C^*-bundle isomorphism $\Phi: P \to P'$ inducing the identity on M such that $\Phi^*\theta' = \theta$. The set of all equivalence classes of such pairs is denoted by $\mathcal{P}(M)$.

THEOREM 1.4. *The correspondence* $\gamma \mapsto (P_\gamma, \theta_\gamma)$ *induces a bijection* $\mathcal{C}(M) \to \mathcal{P}(M)$.

Let (M, γ) be a complex contact manifold. For each $\phi \in \mathrm{Aut}(M, \gamma)$ there exists uniquely a holomorphic bundle automorphism $\tilde{\phi}_*$ of L_γ such that $\varpi_\gamma \circ \phi_* = \tilde{\phi}_* \circ \varpi_\gamma$ on TM. We define a map $\hat{\phi}: P_\gamma \to P_\gamma$ by

$$\hat{\phi}(u) = \tilde{\phi}_* \circ u \qquad \text{for} \quad u \in P_\gamma ,$$

which is a holomorphic C^*-bundle automorphism of P_γ such that $\hat{\pi}\circ\hat{\phi}=\phi\circ\hat{\pi}$. This is called the *prolongation* of ϕ. We denote by $\mathrm{Aut}(P_\gamma, \theta_\gamma)$ the group of all holomorphic C^*-bundle automorphisms Φ of P_γ (not necessarily inducing the identity on M) with $\Phi^*\theta_\gamma=\theta_\gamma$. Moreover let $\mathfrak{a}(P_\gamma, \theta_\gamma)$ be the complex Lie algebra of all holomorphic vector fields X on P_γ such that the local flow generated by X is contained (locally) in $\mathrm{Aut}(P_\gamma, \theta_\gamma)$, which is the same as that $R_{a*}X=X$ for each $a\in C^*$ and $\mathcal{L}_X\theta_\gamma=0$, where $\mathcal{L}$ denotes the Lie derivation. If M is compact, $\mathrm{Aut}(P_\gamma, \theta_\gamma)$ is a complex Lie group with $\mathrm{Lie\,Aut}(P_\gamma, \theta_\gamma)=\mathfrak{a}(P_\gamma, \theta_\gamma)$. For each $\phi\in\mathrm{Aut}(M, \gamma)$ the prolongation $\hat{\phi}$ belongs to $\mathrm{Aut}(P_\gamma, \theta_\gamma)$, and hence we have a homomorphism $\mathrm{Aut}(M, \gamma)\hat{\to}\mathrm{Aut}(P_\gamma, \theta_\gamma)$, which is called the *prolongation*. This gives rise to a homomorphism $\mathfrak{a}(M, \gamma)\hat{\to}\mathfrak{a}(P_\gamma, \theta_\gamma)$, which is called the *infinitesimal prolongation.*

THEOREM 1.5. *The prolongation* $\mathrm{Aut}(M, \gamma)\hat{\to}\mathrm{Aut}(P_\gamma, \theta_\gamma)$ *is an isomorphism, which is a complex Lie isomorphism if M is compact. Therefore the infinitesimal prolongation* $\mathfrak{a}(M, \gamma)\hat{\to}\mathfrak{a}(P_\gamma, \theta_\gamma)$ *is also an isomorphism.*

THEOREM 1.6. *Let $\Gamma(L_\gamma)$ be the space of all holomorphic sections of L_γ, and define a linear map $\varpi_\gamma: \mathfrak{a}(M, \gamma)\to\Gamma(L_\gamma)$ by*

$$\varpi_\gamma(X)(x)=\varpi_\gamma(X_x) \qquad \textit{for} \quad X\in\mathfrak{a}(M, \gamma),\ x\in M.$$

Then ϖ_γ is an isomorphism.

COROLLARY. *Let $F(P_\gamma)$ be the space of all holomorphic functions σ on P_γ such that*

$$\sigma(u\cdot a)=a^{-1}\sigma(u) \qquad \textit{for each} \quad u\in P_\gamma,\ a\in C^*.$$

For each $\sigma\in F(P_\gamma)$ a holomorphic vector field X on P_γ is uniquely determined by

$$\iota(X)\Theta_\gamma+d\sigma=0,$$

where $\iota(X)$ denotes the contraction by X. Then the correspondence $\sigma\mapsto X$ gives a linear isomorphism $F(P_\gamma)\to\mathfrak{a}(P_\gamma, \theta_\gamma)$.

PROOF. For each $s\in\Gamma(L_\gamma)$ the holomorphic function σ on P_γ defined by $\sigma(u)=u^{-1}s(\hat{\pi}(u))$ is a function in $F(P_\gamma)$, and the correspondence $s\mapsto\sigma$ gives a linear isomorphism $\Gamma(L_\gamma)\to F(P_\gamma)$. Therefore the corollary follows from Theorems 1.5, 1.6 and the familiar identity $\iota(X)\circ d+d\circ\iota(X)=\mathcal{L}_X$. q. e. d.

A complex contact manifold (M, γ) is said to be *homogeneous* if $\mathrm{Aut}(M, \gamma)$ acts transitively on M. A complex manifold is said to be *kählerian* if it admits a Kähler metric.

EXAMPLE 1.1 (Boothby [4], [5]). Let $\mathfrak{g}$ be a complex simple Lie algebra.

Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and identify the root system of $\mathfrak{g}$ with a subset of the real part $\mathfrak{h}_{\boldsymbol{R}}$ of $\mathfrak{h}$ by means of the Killing form $(\ ,\ )$ of $\mathfrak{g}$. Let $\alpha_0 \in \mathfrak{h}_{\boldsymbol{R}}$ be the highest root with respect to a linear order on $\mathfrak{h}_{\boldsymbol{R}}$ and put $H_0 = (2/(\alpha_0, \alpha_0))\alpha_0$. Denoting by $\mathfrak{g}_\lambda$ the λ-eigenspace of $\mathrm{ad}(H_0)$ in $\mathfrak{g}$, we define a subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ by

$$\mathfrak{u} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 .$$

Let G be the connected complex Lie group with the trivial center such that $\mathrm{Lie}\, G = \mathfrak{g}$, and U the normalizer of $\mathfrak{u}$ in G. Then the quotient complex manifold

$$M = G/U$$

is compact, simply connected and kählerian. It is called a *kählerian C-space of Boothby type.* It is shown that M has always odd dimension and that $\mathfrak{g}_2 \neq \{0\}$. Choose $E \in \mathfrak{g}_2$ with $E \neq 0$ and define a linear form θ^* on $\mathfrak{g}$ by

$$\theta^*(X) = (E, X) \qquad \text{for} \quad X \in \mathfrak{g} .$$

Put

$$U_0 = \{a \in G \ ;\ \mathrm{Ad}(a)E = E\}.$$

Then U_0 is a normal subgroup of U and the quotient group U/U_0 is identified with $\boldsymbol{C}^*$. Thus the quotient complex manifold

$$P = G/U_0$$

has a natural structure of a holomorphic $\boldsymbol{C}^*$-bundle over M. And there exists a holomorphic 1-form θ on P whose pull back to G is equal to θ^*, θ^* being regarded as a left G-invariant 1-form on G. Furthermore it is verified that θ satisfies (1.4), (1.5) and (1.6). Thus by Theorem 1.4 θ determines a complex contact structure γ on M. Our correspondence

$$\mathfrak{g} \longmapsto (M, \gamma)$$

induces a bijection from the set of all isomorphism classes of complex simple Lie algebras onto the set of all contact isomorphism classes of compact simply connected homogeneous complex contact manifolds. Actually we have $\mathfrak{g} = \mathrm{Lie}\,\mathrm{Aut}(M, \gamma)$ in the above construction.

A complex contact structure on a kählerian C-space of Boothby type is essentially unique, because we have the following theorem.

THEOREM 1.7. *Let (M, γ_0) be a compact complex contact manifold. Suppose that M is kählerian and $H_1(M, \boldsymbol{Z}) = \{0\}$. Then* $\mathrm{Aut}(M)$ *acts transitively on $\boldsymbol{C}(M)$.*

PROOF. Let $h^{p,q} = \dim H^q(M, \Omega^p)$ and $b_r =$ the r-th Betti number of M. Since M is compact kählerian, we have

$$b_r = \sum_{p+q=r} h^{p,q}, \qquad h^{p,q} = h^{q,p}.$$

In particular, by $H_1(M, \boldsymbol{Z})=\{0\}$ we get $b_1=0$ and hence

$$h^{0,1} = 0. \tag{1.7}$$

We put

$$L = L_{\gamma_0}, \qquad P = P_{\gamma_0}.$$

Take an arbitrary $\gamma \in \mathcal{C}(M)$. By Theorem 1.2 we have $(n+1)c_1(L_\gamma)=c_1(M)$, where $\dim M=2n+1$. Therefore we have

$$(n+1)c_1(L_\gamma) = (n+1)c_1(L). \tag{1.8}$$

On the other hand, the exact sequence

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow \boldsymbol{C} \longrightarrow \boldsymbol{C}^* \longrightarrow 0$$

of abelian groups yields the cohomology exact sequence

$$H^1(M, \mathcal{O}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \boldsymbol{Z}),$$

where $\dim H^1(M, \mathcal{O})=h^{0,1}=0$ by (1.7). Thus we get the exact sequence

$$0 \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \boldsymbol{Z}). \tag{1.9}$$

Moreover the assumption $H_1(M, \boldsymbol{Z})=\{0\}$ together with the universal coefficient theorem implies that $H^2(M, \boldsymbol{Z})$ has no torsion. Hence by (1.8) and (1.9) we have

$$L_\gamma \cong L \qquad \text{for each} \quad \gamma \in \mathcal{C}(M). \tag{1.10}$$

Next let $\hat{\mathcal{U}}$ be the set of all holomorphic 1-forms θ on P satisfying (1.4), (1.5) and (1.6). Then $\hat{\mathcal{U}}$ is invariant under the multiplication by $\boldsymbol{C}^*$, and the quotient

$$\mathcal{U}(M) = \hat{\mathcal{U}}/\boldsymbol{C}^*$$

is identified with $\mathcal{C}(M)$ by Theorem 1.4, (1.10) and the compactness of M. We identify the space of all holomorphic 1-forms θ on P satisfying (1.4) and (1.5) with the space $\Gamma(M, T^*M \otimes L)$ of all holomorphic sections of $T^*M \otimes L$, and thus $\hat{\mathcal{U}} \subset \Gamma(M, T^*M \otimes L)$. In the same way we identify $\Gamma(M, \wedge^{2n+1} T^*M \otimes L^{n+1})$ with the space of all semi-basic holomorphic $(2n+1)$-forms α on P such that $R_a^* \alpha = a^{-(n+1)}\alpha$ for each $a \in \boldsymbol{C}^*$. We fix an isomorphism κ from the anticanonical line bundle K_M^* onto L^{n+1} (cf. Theorem 1.2). We have then

$$\Gamma(M, \wedge^{2n+1}T^*M\otimes L^{n+1}) = \Gamma(M, K_M\otimes L^{n+1})$$
$$\xrightarrow[\kappa^*]{\cong} \Gamma(M, K_M\otimes K_M^*) = \Gamma(M, 1) = \boldsymbol{C}.$$

Here κ^* is given as follows. Fix a point $x_0\in M$ and $v_0\in(K_M^*)_{x_0}$ with $v_0\neq 0$. Let $\xi_0\in(K_M)_{x_0}$ satisfy $\langle v_0, \xi_0\rangle=1$, where $\langle\ ,\ \rangle$ is the pairing between K_M^* and K_M. Then for $\alpha\in\Gamma(M, K_M\otimes L^{n+1})$, regarded as a homomorphism $\alpha: K_M^*\to L^{n+1}$, we have

$$\kappa^*(\alpha) = \langle\kappa^{-1}(\alpha(v_0)), \xi_0\rangle.$$

Now we define a map $F: \Gamma(M, T^*M\otimes L)\to\boldsymbol{C}$ by

$$F(\theta) = \theta\wedge(d\theta)^n \in \Gamma(M, \wedge^{2n+1}T^*M\otimes L^{n+1}) \overset{\kappa^*}{=} \boldsymbol{C},$$

explicitly, by

$$F(\theta) = \langle\kappa^{-1}[(\theta\wedge(d\theta)^n)(v_0)], \xi_0\rangle \qquad \text{for} \quad \theta\in\Gamma(M, T^*M\otimes L).$$

It is a homogeneous holomorphic function on $\Gamma(M, T^*M\otimes L)$ of degree $n+1$ such that $F(\theta)\neq 0$ if and only if $d\theta$ is a symplectic 2-form on P. Therefore $F\neq 0$ (since $F(\theta_{\gamma_0})\neq 0$) and $\hat{\mathcal{U}}$ is given by

$$\hat{\mathcal{U}} = \{\theta\in\Gamma(M, T^*M\otimes L)\ ;\ F(\theta)\neq 0\}.$$

Hence $\hat{\mathcal{U}}$ is a $\boldsymbol{C}^*$-invariant open connected subset of $\Gamma(M, T^*M\otimes L)$. It follows that $\mathcal{U}(M)$ is identified with an open connected subset of the projective space $P(\Gamma(M, T^*M\otimes L))$ associated to $\Gamma(M, T^*M\otimes L)$. Let $\mathrm{Aut}(L)$ denote the complex Lie group of all holomorphic bundle automorphisms of L (not necessarily inducing the identity on M). Then we have an exact sequence

$$1 \longrightarrow \boldsymbol{C}^* \longrightarrow \mathrm{Aut}(L) \xrightarrow{\ \rho\ } \mathrm{Aut}(M) \longrightarrow 1$$

of complex Lie groups, where ρ is the natural homomorphism. Here the surjectivity of ρ follows from (1.10). For each $g\in\mathrm{Aut}(M)$ we choose an element $s_g\in\mathrm{Aut}(L)$ with $\rho(s_g)=g$. Then the tensor product of the natural action of g on T^*M with s_g induces a linear action of g on $\Gamma(M, T^*M\otimes L)$, and hence it induces a projective action of g on $P(\Gamma(M, T^*M\otimes L))$, which leaves $\mathcal{U}(M)$ invariant. Under our identification of $\mathcal{U}(M)$ with $\mathcal{C}(M)$, the action of g on $\mathcal{U}(M)$ defined in the above way corresponds to the natural action of g on $\mathcal{C}(M)$. Note that our action of $\mathrm{Aut}(M)$ on $\mathcal{U}(M)$ is holomorphic, since the bundle $\rho: \mathrm{Aut}(L)\to\mathrm{Aut}(M)$ has a local holomorphic section $g\mapsto s_g$ around each point of $\mathrm{Aut}(M)$. Therefore it suffices to show the transitivity of $\mathrm{Aut}(M)$ on $\mathcal{U}(M)$.

Let $\gamma\in\mathcal{C}(M)$ be arbitrary. Dualizing the exact sequence (1.3) and tensoring L_γ, we get an exact sequence

$$0 \longrightarrow 1 \longrightarrow T^*M \otimes L_\gamma \longrightarrow E_\gamma^* \otimes L_\gamma \longrightarrow 0.$$

In the associated cohomology exact sequence

$$0 \longrightarrow \Gamma(M, 1) \longrightarrow \Gamma(M, T^*M \otimes L_\gamma) \longrightarrow \Gamma(M, E_\gamma^* \otimes L_\gamma) \longrightarrow H^1(M, \mathcal{O}),$$

we have $\Gamma(M, 1) = \boldsymbol{C}$ and $H^1(M, \mathcal{O}) = \{0\}$ by (1.7). Therefore

$$\dim \Gamma(M, T^*M \otimes L_\gamma) = \dim \Gamma(M, E_\gamma^* \otimes L_\gamma) + 1,$$

and hence $\dim \mathcal{U}(M) = \dim \Gamma(M, E_\gamma^* \otimes L_\gamma)$. But, by tensoring L_γ^{-1} to the pairing $E_\gamma \times E_\gamma \to L_\gamma$ in Theorem 1.1, we get a non-degenerate holomorphic pairing $E_\gamma \otimes (E_\gamma \otimes L_\gamma^{-1}) \to 1$. Thus $E_\gamma \cong (E_\gamma \otimes L_\gamma^{-1})^* = E_\gamma^* \otimes L_\gamma$. Therefore we have

$$\dim \mathcal{U}(M) = \dim \Gamma(M, E_\gamma). \tag{1.11}$$

On the other hand, in the cohomology exact sequence

$$0 \longrightarrow \Gamma(M, E_\gamma) \longrightarrow \Gamma(M, TM) \xrightarrow{\varpi_\gamma} \Gamma(M, L_\gamma)$$

associated to (1.3), we have $\Gamma(M, TM) \cong \operatorname{Lie} \operatorname{Aut}(M)$, and by Theorem 1.6 $\Gamma(M, L_\gamma) \cong \operatorname{Lie} \operatorname{Aut}(M, \gamma)$ and ϖ_γ is surjective. Hence, together with (1.11) we get

$$\dim \mathcal{U}(M) = \dim \operatorname{Aut}(M) - \dim \operatorname{Aut}(M, \gamma) \qquad \text{for each} \quad \gamma \in \mathcal{C}(M).$$

It follows that $\operatorname{Aut}(M)$ acts on $\mathcal{U}(M)$ locally transitively at each point of $\mathcal{U}(M)$. Since $\mathcal{U}(M)$ is connected, we obtain the transitivity of $\operatorname{Aut}(M)$ on $\mathcal{U}(M)$.

q. e. d.

REMARK. As is seen from the proof, the assumptions in the theorem may be replaced by "$H_1(M, \boldsymbol{Z})$ has no torsion and $H^1(M, \mathcal{O}) = \{0\}$".

§ 2. Quaternionic Kähler manifolds.

In this section we give the definition of quaternionic Kähler manifolds and recall some basic results on them.

We denote by $\boldsymbol{H}$ the algebra of real quaternions and by $\mathcal{I}m\boldsymbol{H} \subset \boldsymbol{H}$ the subspace of pure imaginary quaternions. Let (M, g) be a Riemannian manifold of dimension $4n$. A subalgebra bundle H of the bundle $\operatorname{End}(TM)$ of endomorphisms of the tangent bundle TM is called a *quaternionic Kähler structure* on (M, g) if

(2.1) for each point $x \in M$ there is an open set U of M with $x \in U$ such that the restriction $H|U$ to U is isomorphic to the product bundle $U \times \boldsymbol{H}$ as algebra bundles;

(2.2) denoting by $\mathcal{I}m H$ the subbundle of H which corresponds to $U\times\mathcal{I}m\boldsymbol{H}$ under isomorphisms in (2.1), we have

$$g(zX, Y)+g(X, zY)=0 \qquad \text{for} \quad X, Y\in T_xM,\ z\in(\mathcal{I}mH)_x\,;$$

and

(2.3) H is a parallel subbundle of $\mathrm{End}(TM)$ with respect to the connection induced by the Riemannian connection ∇ of (M, g).

Given such a structure H, the set $P(H)$ of all orthonormal frames of (M, g) of the form

$$\{e_1, Ie_1, Je_1, Ke_1, \cdots, e_n, Ie_n, Je_n, Ke_n\}\,,$$

where $e_i\in T_xM$ and $I, J, K\in(\mathcal{I}mH)_x$ with $I^2=J^2=-1$, $IJ=-JI=K$, is a subbundle of the orthonormal frame bundle $O(M)$ with the structure group $Sp(n)Sp(1)\subset O(4n)$. Here $Sp(n)Sp(1)$ is defined as follows. We identify $\boldsymbol{H}^n$ with $\boldsymbol{R}^{4n}$ and denote by $\rho(Sp(1))$ the subgroup of $O(4n)$ of right multiplications by $Sp(1)=\{h\in\boldsymbol{H}; |h|=1\}$. Let $Sp(n)\subset O(4n)$ be the centralizer of $\rho(Sp(1))$ in $O(4n)$ and define $Sp(n)Sp(1)$ to be the product $Sp(n)\rho(Sp(1))$ in $O(4n)$. Actually $Sp(n)Sp(1)\subset SO(4n)$ and we have an exact sequence

$$1\longrightarrow \boldsymbol{Z}_2\longrightarrow Sp(n)\times Sp(1)\longrightarrow Sp(n)Sp(1)\longrightarrow 1\,.$$

By (2.3) the Riemannian connection ∇ on $O(M)$ reduces to the connection ∇ on $P(H)$, and M has a natural orientation determined by the reduction $P(H)\subset O(M)$.

The triple (M, g, H), where H is a quaternionic Kähler structure on (M, g), is called a *quaternionic Kähler manifold* in case $n\geqq 2$, and in case $n=1$ provided that (M, g) is Einstein and anti-selfdual in the sense of Atiyah-Hitchin-Singer [**3**] with respect to the natural orientation. It is known (Alekseevskii [**1**]) that for a quaternionic Kähler manifold (M, g, H), (M, g) is Einstein also in the case $n\geqq 2$ and it is irreducible in the case where the (constant) scalar curvature $t\neq 0$. Let (M, g, H) and (M', g', H') be quaternionic Kähler manifolds. By an *isomorphism* (resp. *equivalence*) $\phi:(M, g, H)\to(M', g', H')$ we mean an isometry (resp. homothety) $\phi:(M, g)\to(M', g')$ such that $\phi_*H\phi_*^{-1}=H'$. An *automorphism* of (M, g, H), $\mathrm{Aut}(M, g, H)$ and $\mathfrak{a}(M, g, H)$ are defined in the analogous way to contact structures. The group $\mathrm{Aut}(M, g, H)$ is a Lie group, since it is a closed subgroup of the group $K(M, g)$ of all isometries of (M, g). If g is complete, we have $\mathrm{Lie\,Aut}(M, g, H)=\mathfrak{a}(M, g, H)$. In the following in this section, (M, g, H) will denote a quaternionic Kähler manifold of dimension $4n$.

We recall first another description of quaternionic Kähler structures by Salamon [**8**]. If a complex $Sp(n)\times Sp(1)$-module V is given, we get a vector bundle over M associated to $P(H)$ with the induced connection, which will be denoted by the corresponding boldface $\boldsymbol{V}$. In general $\boldsymbol{V}$ is locally defined, but it is globally defined if, for example, the action of $Sp(n)\times Sp(1)$ factors through $Sp(n)Sp(1)$. If V has an $Sp(n)\times Sp(1)$-invariant structure, it carries over to the fibres of $\boldsymbol{V}$. For example, if V has an $Sp(n)\times Sp(1)$-invariant real structure $v\mapsto\tilde{v}$, $\boldsymbol{V}$ has the induced real structure $v\mapsto\tilde{v}$. In this case the set of fixed points of $v\mapsto\tilde{v}$ in $\boldsymbol{V}$ is a real vector bundle over M, which will be denoted by $\boldsymbol{V}_{\boldsymbol{R}}$.

Let E be the standard complex $Sp(n)$-module of dimension $2n$. It has an $Sp(n)$-invariant antilinear map $v\mapsto\tilde{v}$ with $\tilde{\tilde{v}}=-v$ and an $Sp(n)$-invariant non-degenerate alternating bilinear form $\omega_E\in\wedge^2E^*$ such that $\omega_E(\tilde{u}, \tilde{v})=\overline{\omega_E(u, v)}$ and that $\langle\langle u, v\rangle\rangle=\omega_E(u, \tilde{v})$ is a hermitian inner product on E. By a *standard basis* of E we mean a unitary basis $\{e_1, \cdots, e_{2n}\}$ with respect to $\langle\langle\, ,\rangle\rangle$ such that $\tilde{e}_i=e_{n+i}$ $(1\leqq i\leqq n)$. We shall often identify E with E^* by the map

$$v\longmapsto\omega_E(\cdot, v)\,. \tag{2.4}$$

Under this identification we have

$$\omega_E=\sum_{i=1}^{n}e_i\wedge\tilde{e}_i \qquad \text{for a standard basis } \{e_i\}. \tag{2.5}$$

In case $n=1$ we write F for E. We regard E and F as $Sp(n)\times Sp(1)$-modules. Then $E\otimes F$ has an $Sp(n)\times Sp(1)$-invariant real structure defined by $\widetilde{e\otimes h}=\tilde{e}\otimes\tilde{h}$. We put

$$T^*=\{v\in E\otimes F\,;\ \tilde{v}=v\}\,,$$
$$g=\omega_E\otimes\omega_F\in S^2(E\otimes F)=S^2((T^*)^C)\,,$$

where S^2V denotes the symmetric product $V\vee V$. Then g defines an $Sp(n)\times Sp(1)$-invariant inner product on T^*, and we have a global identification

$$T^*M=(\boldsymbol{E}\otimes\boldsymbol{F})_{\boldsymbol{R}}$$

including the metrics. Next we put $T^C=E^*\otimes F^*$ and define an action of S^2F on T^C by the composition

$$\begin{aligned} S^2F\otimes T^C \hookrightarrow (F\otimes F)\otimes(E^*\otimes F^*) &\longrightarrow (\underbrace{F\otimes F^*)\otimes(E^*\otimes F^*}) \\ &\longrightarrow E^*\otimes F^*=T^C \end{aligned} \tag{2.6}$$

of the identification (2.4) and the indicated contraction. We denote by $(S^2F)_{\boldsymbol{R}}$ and T the real forms of S^2F and T^C with respect to the real structures defined by $v\mapsto\tilde{v}$. Then $(S^2F)_{\boldsymbol{R}}$ leaves T invariant and we have a global identification

$$\mathcal{I}mH=(S^2\boldsymbol{F})_{\boldsymbol{R}} \tag{2.7}$$

as subbundles of $\mathrm{End}(TM)$.

We define the *twistor space* Z of (M, g, H) by

$$Z = \{z \in H\ ;\ z^2 = -1\} = \{z \in \mathcal{I}mH\ ;\ |z|^2 = 4n\}\,,$$

where $|\cdot|$ means the fibre norm of End(TM) defined by g. Denote by $q: Z \to M$ the projection induced by the one of End(TM). The twistor space Z is an S^2-bundle over M with the induced connection. Also it is described as follows. Let

$$\boldsymbol{F}_0 = \boldsymbol{F} - \{\text{zero section}\},$$

and denote by $p: \boldsymbol{F}_0 \to M$ the projection. We may define a (locally defined) map $\pi: \boldsymbol{F}_0 \to Z$ with $q \circ \pi = p$ by

$$\pi(h) = \frac{1}{\omega_F(h, \tilde{h})} \sqrt{-1}\, h \vee \tilde{h} \qquad \text{for} \quad h \in \boldsymbol{F}_0,$$

under the identification (2.7). We have $\pi(h) = \pi(h')$ if and only if there exists $a \in \boldsymbol{C}^*$ with $h' = h \cdot a$, and therefore Z is identified with the projective bundle $P(\boldsymbol{F}) = \boldsymbol{F}_0/\boldsymbol{C}^*$ associated to $\boldsymbol{F}$, which is a globally defined manifold. Under this identification, $Z_x = (\boldsymbol{F}_0)_x/\boldsymbol{C}^* = P(\boldsymbol{F}_x) \cong P_1(\boldsymbol{C})$. Note that $\pi: \boldsymbol{F}_0 \to Z$ is a (locally defined) smooth $\boldsymbol{C}^*$-bundle. We define a complex structure on Z in the following way. Decompose $T\boldsymbol{F}_0$ as the sum

$$T\boldsymbol{F}_0 = \hat{\mathcal{H}} \oplus \hat{\mathcal{V}}$$

of the horizontal distribution $\hat{\mathcal{H}}$ and the vertical distribution $\hat{\mathcal{V}}$ for the induced connection. For $h \in \boldsymbol{F}_0$ with $x = p(h)$, $\hat{\mathcal{H}}_h$ is isomorphic to T_xM through p_* and $\hat{\mathcal{V}}_h = T_h(\boldsymbol{F}_0)_x \cong \boldsymbol{F}_x$. Getting together the complex structure on $\hat{\mathcal{H}}_h$ corresponding to $\pi(h)$ through p_* and the natural complex structure on $\hat{\mathcal{V}}_h$, we obtain a complex structure $\hat{J}_h$ on $T_h\boldsymbol{F}_0$. It is known that the almost complex structure $\hat{J}$ on $\boldsymbol{F}_0$ thus obtained is integrable (Salamon [8]). Since $\hat{J}$ is invariant under the action of $\boldsymbol{C}^*$, it can be pushed down to a globally defined integrable almost complex structure $\check{J}$ on $Z = P(\boldsymbol{F})$, in such a way that $\pi: \boldsymbol{F}_0 \to Z$ becomes a (locally defined) holomorphic $\boldsymbol{C}^*$-bundle and that each Z_x is a complex submanifold biholomorphic to $P_1(C)$. Note that the space $\wedge^{1,0}(\hat{\mathcal{H}}_h)$ of type (1, 0) forms on $\hat{\mathcal{H}}_h$ is given by

$$\wedge^{1,0}(\hat{\mathcal{H}}_h) = p^*(\boldsymbol{E}_x \otimes \boldsymbol{C}h)\,. \tag{2.8}$$

The antilinear map $h \mapsto \tilde{h}$ of $\boldsymbol{F}_0$ satisfies $\widetilde{h \cdot a} = \tilde{h} \cdot \bar{a}$ for each $a \in \boldsymbol{C}^*$. Therefore it induces an antiholomorphic involution τ of Z with $q \circ \tau = q$. If Z is regarded as $Z \subset \mathcal{I}mH$, τ is nothing but the antipodal map on each fibre $Z_x \cong S^2$. It is called the *canonical involution* of Z. Let Aut(Z) denote the group of all holomorphic automorphisms of Z, and put

$$\mathrm{Aut}(Z, \tau) = \{\phi \in \mathrm{Aut}(Z)\ ;\ \phi\tau = \tau\phi\}\,.$$

For $\phi\in\mathrm{Aut}(M, g, H)$ we define the *lift* $\bar{\phi}$ of ϕ by

$$\bar{\phi}(z)=\phi_{*}z\phi_{*}^{-1}\qquad\text{for } z\in Z\subset\mathcal{I}_m H\subset\mathrm{End}(TM).$$

Then $\bar{\phi}$ belongs to $\mathrm{Aut}(Z, \tau)$ and satisfies $q\circ\bar{\phi}=\bar{\phi}\circ q$, and moreover $\bar{\phi}_{*}$ leaves invariant the horizontal distribution $\mathcal{H}$ in TZ. The homomorphism $\mathrm{Aut}(M, g, H)\xrightarrow{-}\mathrm{Aut}(Z, \tau)$ is injective and called the *lifting*.

Let L^{-1} denote the (locally defined) tautological holomorphic line bundle over $Z=P(\boldsymbol{F})$. Then L^{2} is a globally defined holomorphic line bundle over Z. Let $\boldsymbol{Z}_{2}=\{\pm 1\}\subset\boldsymbol{C}^{*}$ and define

$$P=\boldsymbol{F}_{0}/\boldsymbol{Z}_{2},$$

which is a globally defined complex manifold. Let $\hat{\pi}: P\rightarrow Z$ denote the projection induced by π. Denoting by $\{h\}\in P$ the class of $h\in\boldsymbol{F}_{0}$, we define a free holomorphic action of $\boldsymbol{C}^{*}$ on P by

$$\{h\}\cdot a=\{h\cdot a^{-1/2}\}\qquad\text{for}\quad h\in\boldsymbol{F}_{0},\ a\in\boldsymbol{C}^{*}.$$

Then $\hat{\pi}: P\rightarrow Z$ is identified with the $\boldsymbol{C}^{*}$-bundle $P(L^{2})$ associated to L^{2}. In fact, for $h\in\boldsymbol{F}_{0}$ with $z=\pi(h)$, let $\xi\in(\boldsymbol{C}h)^{*}=L_{z}$ satisfy $\xi(h)=1$. Then the correspondence $\{h\}\mapsto\xi\otimes\xi$ gives the required holomorphic $\boldsymbol{C}^{*}$-bundle isomorphism $P\rightarrow P(L^{2})$. Next we define a (locally defined) smooth $\boldsymbol{C}$-valued 1-form θ on $\boldsymbol{F}_{0}$ by

$$\theta(X)=\omega_{F}(c_{\nabla}(X), h)\qquad\text{for}\quad X\in T_{h}\boldsymbol{F}_{0},\ h\in\boldsymbol{F}_{0},$$

where $c_{\nabla}: T_{h}\boldsymbol{F}\rightarrow\boldsymbol{F}_{x}$, $x=p(h)$, is the connection map for the induced connection ∇ on $\boldsymbol{F}$. It is invariant under the action of $\boldsymbol{Z}_{2}$ on $\boldsymbol{F}_{0}$, and hence it is pushed down to a globally defined 1-form θ on P. Then θ is holomorphic on P, i.e., it is a type (1, 0) form with $d''\theta=0$ (Salamon [8], p. 154) and satisfies (1.4), (1.5). Furthermore, if the scalar curvature t of (M, g) is not zero, θ satisfies also (1.6). This follows from the formula (cf. Salamon [8], p. 155)

$$\Theta=d\theta=-2\nu p^{*}(\omega_{E}\otimes h\otimes h)-2dz^{1}\wedge dz^{2}\qquad\text{at}\quad h\in\boldsymbol{F}_{0},\tag{2.9}$$

where $\nu=t/8n(n+2)$ and (z^{1}, z^{2}) is the fibre coordinate of $\boldsymbol{F}$ with respect to a local standard basis $\{h_{1}, h_{2}\}$ of $\boldsymbol{F}$ around $x=p(h)$ with $(\nabla h_{i})_{x}=0$. Therefore, if $t\neq 0$, by Theorem 1.4 θ defines a complex contact structure γ on Z with $L_{\gamma}\cong L^{2}$. Note that we have $E_{\gamma}=\mathcal{H}$ by definition of θ. It is called the *canonical complex contact structure* on Z. We put

$$\mathrm{Aut}(Z, \gamma, \tau)=\mathrm{Aut}(Z, \gamma)\cap\mathrm{Aut}(Z, \tau).$$

Then the lifting is an injective homomorphism $\mathrm{Aut}(M, g, H)\rightarrow\mathrm{Aut}(Z, \gamma, \tau)$, since the lift of $\phi\in\mathrm{Aut}(M, g, H)$ leaves $\mathcal{H}=E_{\gamma}$ invariant.

Suppose again that (M, g) has non-zero scalar curvature t. The hermitian fibre metric on L^{-1} induced by the metric $\omega_F(h, \tilde{h})$ on $\boldsymbol{F}$ determines a globally defined hermitian fibre metric α on L^2. We put

$$\bar{\omega}_0 = -\frac{\sqrt{-1}}{2}(\text{curvature form of } \alpha),$$

which is a real closed 2-form on Z of type (1, 1), and then define a symmetric 2-tensor $\bar{g}_0$ on Z by

$$\bar{g}_0(X, Y) = \bar{\omega}_0(\bar{J}X, Y) \qquad \text{for} \quad X, Y \in T_z Z.$$

Finally we normalize it as

$$\bar{g} = \frac{1}{\nu}\bar{g}_0.$$

We have then (Salamon [8])

(2.10) $\bar{g}$ is a pseudo-Kähler metric on Z of signature $(2n+1, 0)$ (resp. $(2n, 1)$) if $t>0$ (resp. $t<0$);

(2.11) $\bar{g}$ is invariant under the lifting of $\mathrm{Aut}(M, g, H)$;

(2.12) the horizontal distribution $\mathcal{H}$ and the vertical distribution $\mathcal{V}$ in TZ are orthogonal to each other with respect to $\bar{g}$;

(2.13) $\bar{g}|Z_x$ is the multiple by a constant c_x of the Fubini-Study metric of $Z_x = P_1(\boldsymbol{C})$, where c_x is positive (resp. negative) if $t>0$ (resp. $t<0$) and independent of $x \in M$; and

(2.14) $q:(Z, \bar{g}) \to (M, g)$ is a pseudo-Riemannian submersion.

The pseudo-Kähler metric $\bar{g}$ is uniquely determined by properties (2.12), (2.13) and (2.14). We call $\bar{g}$ the *canonical pseudo-Kähler metric* on Z. Actually $\bar{g}$ is an Einstein pseudo-Kähler metric. In fact, as in the proof of Theorem 1.7 we regard $\kappa = \theta \wedge (d\theta)^n$ as an isomorphism $K_Z^* \to L^{2(n+1)}$. Let $|\cdot|_k^2$ be the hermitian fibre metric on K_Z^* corresponding to the pseudo-Kähler volume element of $(Z, \bar{g}_0)$ and $|\cdot|_l^2$ the fibre metric on $L^{2(n+1)}$ determined by α. We have then

$$|\kappa(v)|_l = 2^n n! |v|_k \qquad \text{for each} \quad v \in K_Z^*.$$

Recalling that the Ricci form $\bar{\rho}_0$ of $\bar{g}_0$ is given by $\sqrt{-1}\,\bar{\rho}_0 =$curvature form of $|\cdot|_k^2$, we get $\bar{\rho}_0 = 2(n+1)\bar{\omega}_0$. Therefore $\bar{g}_0$ is Einstein, and so $\bar{g}$ is also Einstein.

Finally we recall a linear map $\lambda : \Gamma(L^2)_{\boldsymbol{R}} \to \mathfrak{k}(M, g)$ defined by Salamon, where $\Gamma(L^2)_{\boldsymbol{R}}$ is a real form of $\Gamma(L^2)$ and $\mathfrak{k}(M, g)$ is the Lie algebra of all Killing vector fields on (M, g). Recall that $Z_x = P(\boldsymbol{F}_x)$ and $\Gamma(Z_x, L^2|Z_x) \cong S^2\boldsymbol{F}_x^* \cong S^2\boldsymbol{F}_x$. So the restriction to fibres gives an injective linear map $\varphi : \Gamma(L^2) \to C^\infty(S^2\boldsymbol{F})$, where $C^\infty(\cdot)$ designates the space of smooth sections. We define

$$\Gamma(L^2)_R = \varphi^{-1}[\varphi(\Gamma(L^2))\cap C^\infty((S^2\boldsymbol{F})_R)]\,.$$

Next we define a differential operator $\delta: C^\infty(S^2\boldsymbol{F})\to C^\infty((T^*M)^C)$ to be the covariant derivation $\nabla: C^\infty(S^2\boldsymbol{F})\to C^\infty(S^2\boldsymbol{F}\otimes(T^*M)^C)$ followed by the contraction

$$S^2\boldsymbol{F}\otimes(T^*M)^C \hookrightarrow (\boldsymbol{F}\otimes\boldsymbol{F}^*)\otimes(\boldsymbol{E}\otimes\boldsymbol{F}) \longrightarrow \boldsymbol{E}\otimes\boldsymbol{F} = (T^*M)^C\,.$$

It is proved (Salamon [8]) that then the composition

$$\lambda : \Gamma(L^2)_R \xrightarrow{\varphi} C^\infty((S^2\boldsymbol{F})_R) \xrightarrow{\delta} C^\infty(T^*M) \longrightarrow C^\infty(TM)\,,$$

where the last map is the duality by means of g, sends $\Gamma(L^2)_R$ into $\mathfrak{k}(M, g)$.

§3. Contact automorphisms of twistor spaces.

In this section (M, g, H) will be always a quaternionic Kähler manifold of dimension $4n$ with scalar curvature $t\neq 0$, and τ, γ, $\bar{g}$ be the canonical involution, the canonical complex contact structure, the canonical pseudo-Kähler metric on the twistor space Z respectively. We put

$$\mathrm{Aut}(Z, \gamma, \bar{g}) = \mathrm{Aut}(Z, \gamma)\cap K(Z, \bar{g})\,,$$

where $K(Z, \bar{g})$ denotes the group of all isometries of $(Z, \bar{g})$. This is a Lie group, since it is a closed subgroup of $K(Z, \bar{g})$. In this section we shall study the relationship between the groups $\mathrm{Aut}(M, g, H)$, $\mathrm{Aut}(Z, \gamma, \bar{g})$ and $\mathrm{Aut}(Z, \gamma, \tau)$.

Recall the $\mathfrak{a}(Z, \gamma)$ is the complex Lie algebra of all holomorphic vector fields X on Z such that the local flow generated by X is contained in $\mathrm{Aut}(Z, \gamma)$. Here by a holomorphic vector field X on Z we mean a smooth vector field X on Z such that $\mathcal{L}_X\bar{J}=0$, and the complex structure of $\mathfrak{a}(Z, \gamma)$ is given by $X\mapsto \bar{J}X$. Let $\mathfrak{a}(Z, \gamma, \bar{g})$ (resp. $\mathfrak{a}(Z, \gamma, \tau)$) be the Lie algebra of all smooth vector fields X on Z such that the local flow generated by X is contained in $\mathrm{Aut}(Z, \gamma, \bar{g})$ (resp. in $\mathrm{Aut}(Z, \gamma, \tau)$). They are real subalgebras of $\mathfrak{a}(Z, \gamma)$. If $\bar{g}$ is complete, we have $\mathrm{Lie\,Aut}(Z, \gamma, \bar{g})=\mathfrak{a}(Z, \gamma, \bar{g})$. The lifting is an injective homomorphism $\mathrm{Aut}(M, g, H)\to\mathrm{Aut}(Z, \gamma, \bar{g})$ in virtue of (2.11).

THEOREM 3.1. *The lifting* $\mathrm{Aut}(M, g, H)\xrightarrow{\sim}\mathrm{Aut}(Z, \gamma, \bar{g})$ *is an isomorphism.*

PROOF. It suffices to show the surjectivity. Take an arbitrary $\phi\in\mathrm{Aut}(Z, \gamma, \bar{g})$. Since $\phi\in\mathrm{Aut}(Z, \gamma)$, ϕ_* leaves invariant the horizontal distribution $\mathcal{H}=E_\gamma$. Therefore, by (2.12) together with that $\phi\in K(Z, \bar{g})$, ϕ_* leaves invariant also the vertical distribution $\mathcal{V}$. Hence ϕ sends each fibre of q into a fibre of q. Thus there exists a diffeomorphism ψ of M with $q\circ\phi=\psi\circ q$. By (2.14) ψ is an isometry of (M, g). We shall show that $\psi\in\mathrm{Aut}(M, g, H)$ and $\bar{\psi}=\phi$.

Let $\hat{p}=q\circ\hat{\pi}: P\to M$ and $\hat{\phi}: P\to P$ the prolongation of ϕ so that $\hat{p}\circ\hat{\phi}=\psi\circ\hat{p}$.

Take an arbitrary point of M and choose a sufficiently small neighbourhood U of this point. Then there exists a holomorphic map $\Phi: p^{-1}(U)\to p^{-1}(\phi(U))$ with $p\circ\Phi=\phi\circ p$ such that Φ induces $\hat{\phi}: \hat{p}^{-1}(U)\to\hat{p}^{-1}(\phi(U))$. For $x\in U$ we have $\hat{\phi}(\{h\}\cdot a^{-2})=\hat{\phi}(\{h\})\cdot a^{-2}$ for each $h\in(\boldsymbol{F}_0)_x$ and $a\in\boldsymbol{C}^*$. This means $\{\Phi(h\cdot a)\}=\{\Phi(h)\cdot a\}$, i.e., $\Phi(h\cdot a)=\pm\Phi(h)\cdot a$. Since $\boldsymbol{C}^*$ is connected, we have

$$\Phi(h\cdot a)=\Phi(h)\cdot a \qquad \text{for} \quad h\in(\boldsymbol{F}_0)_x,\ a\in\boldsymbol{C}^*.$$

Noting that $\Phi: p^{-1}(x)=(\boldsymbol{F}_0)_x\to p^{-1}(\phi(x))=(\boldsymbol{F}_0)_{\phi(x)}$ is holomorphic, we conclude that $\Phi:(\boldsymbol{F}_0)_x\to(\boldsymbol{F}_0)_{\phi(x)}$ is $\boldsymbol{C}$-linear at each $x\in U$. That is, $\Phi: p^{-1}(U)\to p^{-1}(\phi(U))$ is obtained as the restriction to $p^{-1}(U)$ of a smooth complex vector bundle homomorphism $\boldsymbol{F}|U\to\boldsymbol{F}|\phi(U)$, which will be denoted again by Φ. We prove

$$\omega_F(\Phi(h),\Phi(k))=\omega_F(h,k) \qquad \text{for} \quad h,k\in\boldsymbol{F}_x,\ x\in U. \tag{3.1}$$

Since $\hat{\phi}\in\mathrm{Aut}(P,\theta)$, $\hat{\phi}$ leaves invariant $d\theta$ on P. Therefore Φ leaves invariant $d\theta$ on $\boldsymbol{F}_0$. We fix $l\in(\boldsymbol{F}_0)_x$. By (2.9) we have $(d\theta)_l(h,k)=-2\omega_F(h,k)$ for each $h,k\in T_l(\boldsymbol{F}_0)_x\cong\boldsymbol{F}_x$. But, since Φ is linear on $\boldsymbol{F}_x$ we have the commutative diagram

$$\begin{array}{ccc} T_l(\boldsymbol{F}_0)_x & \xrightarrow{\Phi_*} & T_{\Phi(l)}(\boldsymbol{F}_0)_{\phi(x)} \\ \wr\Vert & & \wr\Vert \\ \boldsymbol{F}_x & \xrightarrow[\Phi]{} & \boldsymbol{F}_{\phi(x)}\,. \end{array}$$

Thus the invariance of $d\theta$ reads $-2\omega_F(\Phi(h),\Phi(k))=-2\omega_F(h,k)$, which completes the proof. The linear map $\Phi:(\boldsymbol{F}_0)_x\to(\boldsymbol{F}_0)_{\phi(x)}$ with (3.1) induces $\phi: Z_x=P(\boldsymbol{F}_x)\to Z_{\phi(x)}=P(\boldsymbol{F}_{\phi(x)})$, which is a holomorphic isometry of the Fubini-Study metric by (2.13). Therefore we have

$$\Phi(\tilde{h})=\widetilde{\Phi(h)} \qquad \text{for} \quad h\in(\boldsymbol{F}_0)_x,\ x\in U. \tag{3.2}$$

Now seeing (2.9) we know that the horizontal distribution $\hat{\mathcal{H}}$ and the vertical distribution $\hat{\mathcal{V}}$ for $p:\boldsymbol{F}_0\to M$ are orthogonal to each other with respect to $d\theta$ and that $d\theta|\hat{\mathcal{V}}\times\hat{\mathcal{V}}$ is non-degenerate. Hence the invariance of $\hat{\mathcal{V}}$ under Φ_* implies that of $\hat{\mathcal{H}}$ under Φ_*. Therefore, for a fixed $h\in(\boldsymbol{F}_0)_x$ we have the commutative diagram

$$\begin{array}{ccc} (\hat{\mathcal{H}}^*_h)^C & \xrightarrow{(\Phi^*)^{-1}} & (\hat{\mathcal{H}}^*_{\Phi(h)})^C \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ (T^*_xM)^C & \xrightarrow[(\phi^*)^{-1}]{} & (T^*_{\phi(x)}M)^C\,. \end{array}$$

Since Φ is holomorphic, by (2.8) we obtain the commutative diagram

$$\tag{3.3}\begin{tikzcd}\wedge^{1,0}(\hat{\mathcal{H}}_h) \arrow[r, "(\Phi^*)^{-1}"] & \wedge^{1,0}(\hat{\mathcal{H}}_{\Phi(h)}) \\ E_x\otimes Ch \arrow[u, "p^*"] \arrow[r, "(\phi^*)^{-1}"'] & E_{\phi(x)}\otimes C\Phi(h)\,. \arrow[u, "p^*"']\end{tikzcd}$$

Therefore there is a unique linear isomorphism $\Psi: E_x \to E_{\phi(x)}$ such that

$$\tag{3.4}(\phi^*)^{-1}(e\otimes h)=\Psi(e)\otimes\Phi(h)\qquad\text{for each } e\in E_x\,.$$

We prove

$$\tag{3.5}\omega_E(\Psi(e),\Psi(f))=\omega_E(e,f)\qquad\text{for }\ e,f\in E_x\,.$$

By the invariance of $d\theta$ under Φ and (2.9), we get $(\omega_E)_{\phi(x)}\otimes\Phi(h)\otimes\Phi(h)=(\phi^*)^{-1}((\omega_E)_x\otimes h\otimes h)=\Psi((\omega_E)_x)\otimes\Phi(h)\otimes\Phi(h)$. Thus we have $\Psi((\omega_E)_x)=(\omega_E)_{\phi(x)}$, which means (3.5). Next we show

$$\tag{3.6}\omega_E(\Psi(e),\widetilde{\Psi(f)})=\omega_E(e,\tilde{f})\qquad\text{for }\ e,f\in E_x\,.$$

In the same way as (3.3) we have the commutative diagram

$$\begin{tikzcd}\wedge^{1,0}(\mathcal{H}_{\pi(h)}) \arrow[r, "(\phi^*)^{-1}"] & \wedge^{1,0}(\mathcal{H}_{\phi(\pi(h))}) \\ E_x\otimes Ch \arrow[u, "q^*"] \arrow[r, "(\phi^*)^{-1}"'] & E_{\phi(x)}\otimes C\Phi(h)\,. \arrow[u, "q^*"']\end{tikzcd}$$

Since ϕ is a holomorphic isometry of $(Z,\bar{g})$ and q is a pseudo-Riemannian submersion (2.14), for each $\xi,\eta\in E_x\otimes Ch$ we have

$$\begin{aligned}g((\phi^*)^{-1}(\xi),\overline{(\phi^*)^{-1}(\eta)})&=\bar{g}(q^*(\phi^*)^{-1}(\xi),\overline{q^*(\phi^*)^{-1}(\eta)})\\&=\bar{g}((\phi^*)^{-1}q^*(\xi),\overline{(\phi^*)^{-1}q^*(\eta)})=\bar{g}(q^*(\xi),q^*(\bar{\eta}))=g(\xi,\bar{\eta})\,.\end{aligned}$$

Therefore we have

$$g(\Psi(e)\otimes\Phi(h),\overline{\Psi(f)\otimes\Phi(h)})=g(e\otimes h,\overline{f\otimes h})\,.$$

But, the left hand side equals

$$\begin{aligned}g(\Psi(e)\otimes\Phi(h),\widetilde{\Psi(f)}\otimes\widetilde{\Phi(h)})&=\omega_E(\Psi(e),\widetilde{\Psi(f)})\omega_F(\Phi(h),\widetilde{\Phi(h)})\\&=\omega_E(\Psi(e),\widetilde{\Psi(f)})\omega_F(h,\tilde{h})\end{aligned}$$

by (3.1), (3.2), and the right hand side equals $\omega_E(e,\tilde{f})\omega_F(h,\tilde{h})$, which implies (3.6).

Now we have also a linear isomorphism $\Psi_1: E_x\to E_{\phi(x)}$ such that $(\phi^*)^{-1}(e\otimes\tilde{h})=\Psi_1(e)\otimes\Phi(\tilde{h})$ for each $e\in E_x$. Since $(\phi^*)^{-1}$ is R-linear, we have $(\phi^*)^{-1}\overline{(e\otimes h)}$

$=\overline{(\phi^*)^{-1}(e\otimes h)}$, and hence $\Psi_1(\tilde{e})\otimes\Phi(\tilde{h})=\widetilde{\Psi(e)}\otimes\widetilde{\Phi(h)}=\widetilde{\Psi(e)}\otimes\Phi(\tilde{h})$ by (3.2). On the other hand, (3.5) and (3.6) imply $\Psi(\tilde{e})=\widetilde{\Psi(e)}$. Thus we get $\Psi_1=\Psi$. Therefore we have

$$(\phi^*)^{-1}(e\otimes\tilde{h})=\Psi(e)\otimes\Phi(\tilde{h}) \qquad \text{for each } e\in \boldsymbol{E}_x. \tag{3.7}$$

From (3.4) and (3.7) we conclude that

$$(\phi^*)^{-1}=\Psi\otimes\Phi \qquad \text{on} \quad (T_x^*M)^C=\boldsymbol{E}_x\otimes\boldsymbol{F}_x.$$

This means that if we identify $(TM)^C$ with $\boldsymbol{E}\otimes\boldsymbol{F}$ by the identifications (2.4) for E and F, we have

$$\phi_*=\Psi\otimes\Phi \qquad \text{on} \quad (T_xM)^C=\boldsymbol{E}_x\otimes\boldsymbol{F}_x.$$

Moreover the action $S^2F\otimes T^C\to T^C$ in (2.6) is given explicitly by

$$(h\vee k)(e\otimes l)=\omega_F(h,\, l)e\otimes k+\omega_F(k,\, l)e\otimes h$$

for $h,\, k,\, l\in F$ and $e\in E$. These yield

$$\phi_*(h\vee k)\phi_*^{-1}=\Phi(h)\vee\Phi(k) \qquad \text{on } (T_xM)^C \quad \text{for } h,\, k\in\boldsymbol{F}_x. \tag{3.8}$$

In fact, for each $e\in\boldsymbol{E}_x$, $l\in\boldsymbol{F}_x$ we have

$$\phi_*((h\vee k)(e\otimes l))=\omega_F(h,\, l)\Psi(e)\otimes\Phi(k)+\omega_F(k,\, l)\Psi(e)\otimes\Phi(h)\,,$$

$$\begin{aligned}(\Phi(h)\vee\Phi(k))(\phi_*(e\otimes l))&=\omega_F(\Phi(h),\, \Phi(l))\Psi(e)\otimes\Phi(k)\\&\quad+\omega_F(\Phi(k),\, \Phi(l))\Psi(e)\otimes\Phi(h)\,,\end{aligned}$$

which are the same by (3.1). Now it follows from (3.8) that $\phi_*(S^2\boldsymbol{F})_{\boldsymbol{R}}\phi_*^{-1}=(S^2\boldsymbol{F})_{\boldsymbol{R}}$, which means $\phi_*(\mathcal{J}_mH)\phi_*^{-1}=\mathcal{J}_mH$ by (2.7). Therefore we have $\phi_*H\phi_*^{-1}=H$, and hence $\phi\in\mathrm{Aut}(M,\, g,\, H)$. That $\bar{\phi}=\phi$ follows also by (3.8). q. e. d.

Recalling that the lifting sends $\mathrm{Aut}(M,\, g,\, H)$ into $\mathrm{Aut}(Z,\, \gamma,\, \tau)$, we obtain the following corollary.

COROLLARY. $\mathrm{Aut}(Z,\, \gamma,\, \bar{g})\subset\mathrm{Aut}(Z,\, \gamma,\, \tau)$.

REMARK. In the same way we can prove the following: Let $(M_1,\, g_1,\, H_1)$ and $(M_2,\, g_2,\, H_2)$ be quaternionic Kähler manifolds with non-zero scalar curvature, and $(Z_1,\, \gamma_1,\, \bar{g}_1)$ and $(Z_2,\, \gamma_2,\, \bar{g}_2)$ their twistor spaces together with canonical complex contact structure and canonical pseudo-Kähler metric. Then for any isometric contact isomorphism $\phi:(Z_1,\, \gamma_1,\, \bar{g}_1)\to(Z_2,\, \gamma_2,\, \bar{g}_2)$ there exists a unique isomorphism $\psi:(M_1,\, g_1,\, H_1)\to(M_2,\, g_2,\, H_2)$ such that $\phi(z)=\psi_*z\psi_*^{-1}$ for each $z\in Z_1\subset\mathcal{J}_mH_1$.

THEOREM 3.2. $\mathfrak{a}(Z,\, \gamma,\, \tau)=\mathfrak{a}(Z,\, \gamma,\, \bar{g})$.

PROOF. Since by the above Corollary we have $\mathfrak{a}(Z, \gamma, \bar{g})\subset\mathfrak{a}(Z, \gamma, \tau)$, it suffices to show $\mathfrak{a}(Z, \gamma, \tau)\subset\mathfrak{a}(Z, \gamma, \bar{g})$. Note first that $\tau\phi\tau^{-1}\in\mathrm{Aut}(Z, \gamma)$ for each $\phi\in\mathrm{Aut}(Z, \gamma)$. It follows that $X\mapsto\tau_*X$ is a real structure of $\mathfrak{a}(Z, \gamma)$, and $\mathfrak{a}(Z, \gamma, \tau)$ is a real form of $\mathfrak{a}(Z, \gamma)$ given by

$$\mathfrak{a}(Z, \gamma, \tau) = \{X\in\mathfrak{a}(Z, \gamma) \; ; \; \tau_*X=X\}.$$

Let $\hat{\tau}$ denote the antiholomorphic involution of P induced by the antilinear map $h\mapsto\tilde{h}$ of $\boldsymbol{F}_0$. We have then

$$\hat{\tau}(u\cdot a) = \hat{\tau}(u)\cdot\bar{a} \quad \text{for} \quad u\in P,\ a\in\boldsymbol{C}^*, \tag{3.9}$$

$$\theta(\hat{\tau}_*X) = \overline{\theta(X)} \quad \text{for} \quad X\in\boldsymbol{C}^\infty(TP). \tag{3.10}$$

For a holomorphic function σ on P, i. e., a smooth $\boldsymbol{C}$-valued function σ on P with $d''\sigma=0$, we define a holomorphic function $\hat{\tau}(\sigma)$ on P by

$$\hat{\tau}(\sigma)(u) = \overline{\sigma(\hat{\tau}(u))} \quad \text{for} \quad u\in P.$$

Then we have

$$(\hat{\tau}_*X)\cdot\hat{\tau}(\sigma) = \overline{X\cdot\sigma} \quad \text{for} \quad X\in C^\infty(TP). \tag{3.11}$$

Now let $X\in\mathfrak{a}(Z, \gamma)$ and ϕ_t the local flow generated by X. Then the prolongations $\hat{\phi}_t$ give the local flow generated by the infinitesimal prolongation $\hat{X}\in\mathfrak{a}(P, \theta)$. From (3.9) and (3.10) it follows that $\hat{\tau}\hat{\phi}_t\hat{\tau}^{-1}$ belong to $\mathrm{Aut}(P, \theta)$ and $\hat{\tau}\hat{\phi}_t\hat{\tau}^{-1}$ $=\widehat{\tau\phi_t\tau^{-1}}$. This implies that $\hat{\tau}_*\hat{X}\in\mathfrak{a}(P, \theta)$ and $\hat{\tau}_*\hat{X}=\widehat{\tau_*X}$. Under the notation in Corollary to Theorem 1.6, suppose that $\sigma\in F(P)$ corresponds to $\hat{X}$, i. e.,

$$\iota(\hat{X})\Theta+d\sigma = 0. \tag{3.12}$$

Then by (3.9) $\hat{\tau}(\sigma)$ also belongs to $F(P)$ and satisfies $\iota(\hat{\tau}_*\hat{X})\Theta+d\hat{\tau}(\sigma)=0$ by (3.10), (3.11). Thus $\hat{\tau}(\sigma)$ corresponds to $\hat{\tau}_*\hat{X}$.

Now suppose that $X\in\mathfrak{a}(Z, \gamma, \tau)$. By the above arguments, this is equivalent to

$$\hat{\tau}(\sigma) = \sigma. \tag{3.13}$$

Take an arbitrary point of M and choose a neighbourhood U of this point so small that there exists a local standard basis $\{h_1, h_2\}$ of $\boldsymbol{F}$ over U. Let (z^1, z^2) be the fibre coordinate of $\boldsymbol{F}$ with respect to this basis $\{h_1, h_2\}$. For a fixed $x\in U$, $\hat{p}^{-1}(x)=(\boldsymbol{F}_0)_x/\boldsymbol{Z}_2\cong(\boldsymbol{C}^2-\{0\})/\boldsymbol{Z}_2$, and so we may regard $\sigma|\hat{p}^{-1}(x)$ as a holomorphic function $\sigma(z^1, z^2)$ on $\boldsymbol{C}^2$ with $\sigma(-z^1, -z^2)=\sigma(z^1, z^2)$. From the equality $\sigma(u\cdot a)=a^{-1}\sigma(u)$ $(u\in P,\ a\in\boldsymbol{C}^*)$ we obtain $\sigma(a^{-1/2}z^1, a^{-1/2}z^2)=a^{-1}\sigma(z^1, z^2)$ for each $a\in\boldsymbol{C}^*$. It follows that $\sigma(z^1, z^2)$ is of the form $\sigma(z^1, z^2)=a(z^1)^2+2bz^1z^2$ $+c(z^2)^2$. Therefore σ is written as

$$\sigma = a(z^1)^2 + 2bz^1z^2 + c(z^2)^2 \qquad \text{on} \quad \hat{p}^{-1}(U),$$

by $\boldsymbol{C}$-valued smooth functions a, b, c on U. Since $\hat{\tau}(\sigma) = \bar{a}(z^2)^2 - 2\bar{b}z^1z^2 + \bar{c}(z^1)^2$, the condition (3.13) is equivalent to

$$\bar{a} = c, \qquad b + \bar{b} = 0. \tag{3.14}$$

Now we fix $x \in U$ and choose $\{h_1, h_2\}$ in such a way that $(\nabla h_i)_x = 0$. Then the holomorphy of σ implies (Salamon [8]) that there exist e, $e' \in \boldsymbol{E}_x$ such that

$$da = e \otimes h_1, \qquad 2(db) = e' \otimes h_1 + e \otimes \tilde{h}_1, \qquad dc = e' \otimes \tilde{h}_1 \tag{3.15}$$

at x. Furthermore by (3.14) we have

$$e' = \tilde{e}. \tag{3.16}$$

We want to write down the formula (3.12) explicitly. Regard $\hat{X}$ as a $\boldsymbol{Z}_2$-invariant vector field on $\boldsymbol{F}_0$ and decompose it as

$$\hat{X}_h = \hat{H}_h + \hat{V}_h, \qquad \hat{H}_h \in \hat{\mathcal{H}}_h, \quad \hat{V}_h \in \hat{\mathcal{V}}_h, \quad h \in (\boldsymbol{F}_0)_x.$$

Let $W_h = p_* \hat{H}_h \in T_x M$, $\hat{V}_h = v^1 h_1 + v^2 h_2 \in \hat{\mathcal{V}}_h \cong \boldsymbol{F}_x$ and choose a standard basis $\{e_i\}$ of $\boldsymbol{E}_x$. If $h = z^1 h_1 + z^2 h_2$, by (2.5) we have

$$\begin{aligned}
\omega_E \otimes h \otimes h = {} & (z^1)^2 \sum_{i=1}^{n} (e_i \otimes h_1) \wedge (\tilde{e}_i \otimes h_1) \\
& + z^1 z^2 \sum_{i=1}^{n} \{(e_i \otimes h_1) \wedge (\tilde{e}_i \otimes h_1) + (e_i \otimes \tilde{h}_1) \wedge (\tilde{e}_i \otimes h_1)\} \\
& + (z^2)^2 \sum_{i=1}^{n} (e_i \otimes \tilde{h}_1) \wedge (\tilde{e}_i \otimes \tilde{h}_1).
\end{aligned}$$

Therefore, by (2.9) the formula (3.12) is written as

$$\begin{aligned}
& -2\nu (z^1)^2 \sum (\langle e_i \otimes h_1, W_h \rangle \tilde{e}_i \otimes h_1 - \langle \tilde{e}_i \otimes h_1, W_h \rangle e_i \otimes h_1) \\
& -2\nu z^1 z^2 \sum (\langle e_i \otimes h_1, W_h \rangle \tilde{e}_i \otimes \tilde{h}_1 - \langle e_i \otimes \tilde{h}_1, W_h \rangle e_i \otimes h_1 \\
& \qquad\qquad + \langle e_i \otimes \tilde{h}_1, W_h \rangle \tilde{e}_i \otimes h_1 - \langle \tilde{e}_i \otimes h_1, W_h \rangle e_i \otimes \tilde{h}_1) \\
& -2\nu (z^2)^2 \sum (\langle e_i \otimes \tilde{h}_1, W_h \rangle \tilde{e}_i \otimes \tilde{h}_1 - \langle \tilde{e}_i \otimes \tilde{h}_1, W_h \rangle e_i \otimes \tilde{h}_1) \\
& -2(v^1 dz^2 - v^2 dz^1) \\
& + (z^1)^2 da + 2z^1 z^2 db + (z^2)^2 dc + 2az^1 dz^2 + 2b(z^2 dz^1 + z^1 dz^2) + 2cz^2 dz^2 \\
= {} & 0,
\end{aligned}$$

where p^* is omitted and $\langle \, , \rangle$ denotes the pairing of $(T_x^* M)^C$ and $(T_x M)^C$. By (3.15) the last line becomes

$$((z^1)^2 e + z^1 z^2 e') \otimes h_1 + (z^1 z^2 e + (z^2)^2 e') \otimes \tilde{h}_1 + 2(az^1 + bz^2) dz^1 + 2(bz^1 + cz^2) dz^2.$$

Therefore (3.12) is equivalent to the following equations:

$$(3.17)\qquad \begin{cases} 2\nu((z^1)^2A+z^1z^2\tilde{A}) = (z^1)^2e+z^1z^2e', \\ 2\nu(z^1z^2A+(z^2)^2\tilde{A}) = z^1z^2e+(z^2)^2e', \end{cases}$$

and

$$(3.18)\qquad v^1 = bz^1+cz^2\,, \qquad v^2 = -az^1-bz^2\,,$$

where

$$A = \sum (\langle e_i\otimes h_1,\, W_h\rangle \tilde{e}_i - \langle \tilde{e}_i\otimes h_1,\, W_h\rangle e_i)\,.$$

In particular, (3.17) characterizes the horizontal component $\hat{H}_h$. Now, from the non-degeneracy of ω_E on E_x and $\nu\neq 0$, there exists uniquely $Y\in T_xM$ such that

$$(3.19)\qquad 2\nu\sum (\langle e_i\otimes h_1,\, Y\rangle \tilde{e}_i - \langle e_i\otimes h_1,\, Y\rangle e_i) = e\,.$$

We put

$$B = \sum (\langle e_i\otimes h_1,\, Y\rangle \tilde{e}_i - \langle \tilde{e}_i\otimes h_1,\, Y\rangle e_i)\,.$$

Then, by (3.16) B satisfies (3.17) instead of A. This means that the horizontal lift of Y along $p^{-1}(x)$ coincides with the horizontal component $\hat{H}$ of $\hat{X}$ along $p^{-1}(x)$. Since $x\in U$ is arbitrary, it follows that there exists a unique smooth vector field Y on M such that the horizontal lift of Y coincides with $\hat{H}$. Then we have

$$\hat{p}_*\hat{X}_u = Y_{\hat{p}(u)} \qquad \text{for each } u\in P.$$

Therefore the local flow ψ_t on M generated by Y satisfies $\hat{p}\circ\hat{\phi}_t=\psi_t\circ\hat{p}$. Hence $\hat{\phi}_t$ sends each fibre of $\hat{p}$ into a fibre of $\hat{p}$. So ϕ_t sends each fibre of q into a fibre of q. Since $\phi_t\tau=\tau\phi_t$, $\phi_t: q^{-1}(x)=P(\boldsymbol{F}_x)\rightarrow q^{-1}(\psi_t(x))=P(\boldsymbol{F}_{\psi_t(x)})$ is an isometry of the Fubini-Study metric, and hence it is an isometry of $\bar{g}$ in virtue of (2.13). We shall show that ψ_t is an isometry of (M, g). Then, since $\phi_t\in \mathrm{Aut}(\boldsymbol{Z}, \gamma)$ leaves $\mathcal{H}$ invariant and q is a pseudo-Riemannian submersion (2.14), ϕ_t induces an isometry of $\mathcal{H}$. Thus by (2.12) ϕ_t is an isometry of $(Z, \bar{g})$, which means $X\in\mathfrak{a}(Z, \gamma, \bar{g})$. This completes the proof of $\mathfrak{a}(Z, \gamma, \tau)\subset\mathfrak{a}(Z, \gamma, \bar{g})$.

For this purpose let us consider the isomorphism $\varpi_\gamma: \mathfrak{a}(Z, \gamma)\rightarrow\Gamma(L^2)$ in Theorem 1.6 and let $s\in\Gamma(L^2)$ corresponds to $\sigma\in F(P)$. Then $\varpi_\gamma(X)=s$, and under the notation in the last part of § 2 we have

$$\varphi(s) = a\tilde{h}_1\otimes\tilde{h}_1 - bh_1\vee\tilde{h}_1 + ch_1\otimes h_1\,,$$
$$\overline{\varphi(s)} = \bar{a}h_1\otimes h_1 + \bar{b}h_1\vee\tilde{h}_1 + \bar{c}\tilde{h}_1\otimes\tilde{h}_1\,.$$

Thus by (3.14) we know that ϖ_γ induces an isomorphism $\mathfrak{a}(Z, \gamma, \tau)\rightarrow\Gamma(L^2)_{\boldsymbol{R}}$. We define a linear map $q_\#: \mathfrak{a}(Z, \gamma, \tau)\rightarrow\mathfrak{k}(M, g)$ by $q_\#=\lambda\circ\varpi_\gamma$. We shall prove

$$(3.20)\qquad q_\#(X) = 3\nu Y \qquad \text{for our } X \text{ and } Y.$$

This will imply $Y\in\mathfrak{k}(M, g)$ since $\nu\neq 0$, and hence ψ_t is an isometry. Under the previous situation, at $x\in U$ we have

$$\begin{aligned}\nabla\varphi(s) &= \tilde{h}_1\otimes\tilde{h}_1\otimes da-h_1\vee\tilde{h}_1\otimes db+h_1\otimes h_1\otimes dc\\ &= \tilde{h}_1\otimes\tilde{h}_1\otimes e\otimes h_1-\frac{1}{2}(h_1\otimes\tilde{h}_1+\tilde{h}_1\otimes h_1)\otimes(\tilde{e}\otimes h_1+e\otimes\tilde{h}_1)\\ &\quad +h_1\otimes h_1\otimes\tilde{e}\otimes\tilde{h}_1,\end{aligned}$$

by (3.15), (3.16). Therefore we get

$$\begin{aligned}\delta\varphi(s) &= e\otimes\tilde{h}_1-\frac{1}{2}\tilde{e}\otimes h_1+\frac{1}{2}e\otimes\tilde{h}_1-\tilde{e}\otimes h_1\\ &= \frac{3}{2}(-\tilde{e}\otimes h_1+e\otimes\tilde{h}_1)\\ &= 3\nu\sum_{i=1}^{n}(-\langle e_i\otimes\tilde{h}_1, Y\rangle\tilde{e}_i\otimes h_1+\langle\tilde{e}_i\otimes\tilde{h}_1, Y\rangle e_i\otimes h_1\\ &\qquad +\langle e_i\otimes h_1, Y\rangle\tilde{e}_i\otimes\tilde{h}_1-\langle\tilde{e}_i\otimes h_1, Y\rangle e_i\otimes\tilde{h}_1),\end{aligned}$$

by substituting (3.19). If we put $\xi^i=e_i\otimes h_1$, $\xi^{n+i}=\tilde{e}_i\otimes h_1$ $(1\leqq i\leqq n)$, then $g(\xi^i, \xi^j)=g(\bar{\xi}^i, \bar{\xi}^j)=0$, $g(\xi^i, \bar{\xi}^j)=\delta_{ij}$ $(1\leqq i, j\leqq 2n)$ and

$$\delta\varphi(s) = 3\nu\sum_{i=1}^{2n}(\langle\xi^i, Y\rangle\bar{\xi}^i+\langle\bar{\xi}^i, Y\rangle\xi^i).$$

Therefore under the duality $\delta\varphi(s)$ corresponds to $3\nu Y$, which completes the proof of (3.20). q. e. d.

COROLLARY 1. *If $\bar{g}$ is complete,* $\mathrm{Aut}(Z, \gamma, \tau)$ *has a unique Lie group structure such that* $\mathrm{Aut}^0(Z, \gamma, \tau)=\mathrm{Aut}^0(Z, \gamma, \bar{g})$, *where* $\mathrm{Aut}^0(\cdot)$ *designates the identity component of* $\mathrm{Aut}(\cdot)$.

PROOF. Since $\bar{g}$ is complete, we have $\mathrm{Lie\,Aut}(Z, \gamma, \bar{g})=\mathfrak{a}(Z, \gamma, \bar{g})$. Thus the corollary follows from Theorem 3.2. q. e. d.

COROLLARY 2. $\mathfrak{a}(Z, \gamma)=\mathfrak{a}(Z, \gamma, \bar{g})^C\cong\mathfrak{a}(M, g, H)^C$.

PROOF. Since $\mathfrak{a}(Z, \gamma, \tau)$ is a real form of $\mathfrak{a}(Z, \gamma)$, by Theorem 3.2 $\mathfrak{a}(Z, \gamma)$ is the complexification of $\mathfrak{a}(Z, \gamma, \bar{g})$. The second isomorphism follows from Theorem 3.1. q. e. d.

COROLLARY 3. *If g is complete, we have*

$$\dim_R\mathrm{Aut}(M, g, H) = \dim_C\Gamma(L^2).$$

PROOF. Since g is complete, we have $\mathrm{Lie\,Aut}(M, g, H)=\mathfrak{a}(M, g, H)$. Thus by Corollary 2 $\dim_R\mathrm{Aut}(M, g, H)$ is equal to $\dim_C\mathfrak{a}(Z, \gamma)$, which is the same as $\dim_C\Gamma(L^2)$ by Theorem 1.6. q. e. d.

COROLLARY 4. *If* (M, g, H) *is a compact quaternionic Kähler manifold with positive scalar curvature, then* $\mathrm{Aut}^0(Z, \gamma, \bar{g})$ *and* $\mathrm{Aut}^0(M, g, H)$ *are compact, and*

$$\mathrm{Aut}^0(Z, \gamma) = (\mathrm{Aut}^0(Z, \gamma, \bar{g}))^C \cong (\mathrm{Aut}^0(M, g, H))^C . \tag{3.21}$$

In particular, $\mathrm{Aut}^0(Z, \gamma)$ *is a reductive complex Lie group.*

PROOF. In this case Z is compact and $\bar{g}$ is a Kähler metric. Thus both g and $\bar{g}$ are complete, and hence we have $\mathrm{Lie}\,\mathrm{Aut}^0(Z, \gamma, \bar{g})=\mathfrak{a}(Z, \gamma, \bar{g})$ and $\mathrm{Lie}\,\mathrm{Aut}^0(M, g, H)=\mathfrak{a}(M, g, H)$. Hence (3.21) follows from Corollary 2. The compactness of $\mathrm{Aut}^0(Z, \gamma, \bar{g})$ and $\mathrm{Aut}^0(M, g, H)$ follows from that of $K(Z, \bar{g})$ and $K(M, g)$. q. e. d.

§ 4. Uniqueness of quaternionic Kähler manifolds with certain twistor spaces.

In this section we prove the uniqueness of a quaternionic Kähler manifold whose twistor space is a kählerian C-space of Boothby type.

We recall first the following result.

THEOREM 4.1 (Wolf [10]). *The set of all equivalence classes of quaternionic Kähler manifolds* (M, g, H) *such that* (M, g) *is a compact symmetric space with positive scalar curvature is in a bijective correspondence with the set of all contact isomorphism classes of compact simply connected homogeneous complex contact manifolds, by the assignment for* (M, g, H) *of its twistor space* (Z, γ) *with the canonical complex contact structure* γ.

REMARK. Actually Wolf [10] dealt with compact symmetric quaternionic Kähler manifolds, whose " holonomy has quaternion scalar part ". But this is equivalent to " with positive scalar curvature " in virtue of Alekseevskii [2].

THEOREM 4.2. *Let* (M_1, g_1, H_1) *and* (M_2, g_2, H_2) *be compact quaternionic Kähler manifolds with positive scalar curvature, and* Z_1 *and* Z_2 *their twistor spaces. Suppose that* Z_1 *is a kählerian C-space of Boothby type. Then, if* Z_1 *and* Z_2 *are biholomorphic,* (M_1, g_1, H_1) *and* (M_2, g_2, H_2) *are equivalent.*

PROOF. Let γ_i be the canonical complex contact structure on Z_i, $\bar{g}_i$ the canonical Einstein Kähler metric on Z_i and $G_i=\mathrm{Aut}^0(Z_i, \gamma_i, \bar{g}_i)$ for $i=1, 2$. Since Z_i is kählerian and $H_1(Z_i, \boldsymbol{Z})=\{0\}$ in virtue of the simply connectedness, by Theorem 1.7 there exists a contact isomorphism $\phi_1 : (Z_1, \gamma_1)\rightarrow(Z_2, \gamma_2)$. The same theorem and Example 1.1 also imply the transitivity of $\mathrm{Aut}^0(Z_i, \gamma_i)$ on Z_i. Put $\bar{g}'_1=\phi_1^*\bar{g}_2$ and $G'_1=\mathrm{Aut}^0(Z_1, \gamma_1, \bar{g}'_1)$. We have then

$$K(Z_1, \bar{g}'_1) = \phi_1^{-1}K(Z_2, \bar{g}_2)\phi_1 , \qquad \mathrm{Aut}(Z_1, \gamma_1) = \phi_1^{-1}\mathrm{Aut}(Z_2, \gamma_2)\phi_1 .$$

It follows that $G_1'=\phi_1^{-1}G_2\phi_1$. Since G_2 is a maximal compact subgroup of $\mathrm{Aut}^0(Z_2, \gamma_2)$ by Corollary 4 to Theorem 3.2, G_1' is a maximal compact subgroup of $\mathrm{Aut}^0(Z_1, \gamma_1)$. By the same reasoning G_1 is also a maximal compact subgroup of $\mathrm{Aut}^0(Z_1, \gamma_1)$. Therefore there exists $\phi_2\in\mathrm{Aut}^0(Z_1, \gamma_1)$ such that $\phi_2G_1\phi_2^{-1}=G_1'$. We put $\phi=\phi_1\circ\phi_2$. This is a contact isomorphism $\phi:(Z_1, \gamma_1)\to(Z_2, \gamma_2)$ with $\phi G_1\phi^{-1}=G_2$. Hence both $\bar{g}_1$ and $\phi^*\bar{g}_2$ are G_1-invariant Einstein Kähler metrics on Z_1. Moreover $G_1^C=\mathrm{Aut}^0(Z_1, \gamma_1)$ is semi-simple by Example 1.1. It follows from Takeuchi [9] that there exists $c>0$ such that $c\phi^*\bar{g}_2=\bar{g}_1$. Therefore $\phi:(Z_1, \gamma_1, \bar{g}_1)\to(Z_2, \gamma_2, c\bar{g}_2)$ is an isometric contact isomorphism. Thus by Remark in §3 there exists an isomorphism $\psi:(M_1, g_1, H_1)\to(M_2, cg_2, H_2)$. Hence (M_1, g_1, H_1) and (M_2, g_2, H_2) are equivalent. q. e. d.

COROLLARY. *Let Z be a kählerian C-space of Boothby type. Then there exists a compact quaternionic Kähler manifold (M, g, H) with positive scalar curvature such that its twistor space is biholomorphic to Z, which is unique up to equivalence. The underlying Riemannian manifold (M, g) of (M, g, H) is always a symmetric space.*

PROOF. By Example 1.1 Z has a homogeneous complex contact structure. Thus Theorem 4.1 implies the existence of (M, g, H) as above. The uniqueness follows from Theorem 4.2. q. e. d.

EXAMPLE 4.1. The complex projective $(2n+1)$-space $P_{2n+1}(C)$ is a kählerian C-space of Boothby type. The corresponding quaternionic Kähler manifold is the quaternionic projective n-space $P_n(\boldsymbol{H})$ with the canonical quaternionic Kähler structure.

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