J. Math. Soc. Japan
Vol. 39, No. 1, 1987

Stability of foliations of 3-manifolds by circles

Dedicated to Professor Itiro Tamura on his 60th birthday

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(Received March 29, 1985)
(Revised July 5, 1985)

Introduction.

Let $\mathrm{Fol}_q(M)$ denote the set of codimension q C^∞-foliations of a closed m-manifold M. $\mathrm{Fol}_q(M)$ carries a natural weak C^r-topology $(0 \leqq r \leqq \infty)$, which is described in [**5**]. We denote this space by $\mathrm{Fol}_q^r(M)$. We say a foliation F is C^r-stable if there exists a neighborhood V of F in $\mathrm{Fol}_q^r(M)$ such that every foliation in V has a compact leaf. We say F is C^r-unstable if not. We simply say F is (un-)stable if F is C^1-(un-)stable. It seems to be of interest to determine if F is C^r-stable or not.

Let L be a compact leaf of F. Thurston [**13**] and Langevin-Rosenberg [**6**] showed, generalizing the Reeb stability theorem [**9**] that if $H^1(L;\boldsymbol{R})=0$, then F is stable. Let $\pi_1(L) \rightarrow GL(q, \boldsymbol{R})$ be the action determined by the linear holonomy of L, where q is the codimension of F. Then generalizing the results of Hirsch [**5**] and Thurston [**13**], Stowe [**12**] showed that if the cohomology group $H^1(\pi_1(L);\boldsymbol{R}^q)$ is trivial, then F is stable.

Let F be a foliation of an orientable S^1-bundle over a closed surface B by fibres. Seifert [**11**] showed that F is C^0-stable if $\chi(B) \neq 0$, where $\chi(B)$ is the euler characteristic of B. The result was generalized by Fuller [**4**] to orientable circle bundles over arbitrary closed manifolds B with $\chi(B) \neq 0$. Let $\pi: M \rightarrow B$ be a fibration with fibre L. Langevin-Rosenberg [**7**] showed that the foliation of M by fibres is C^0-stable provided that 1) $\pi_1(L) \cong \boldsymbol{Z}$, 2) B is a closed surface with $\chi(B) \neq 0$ and 3) $\pi_1(B)$ acts trivially on $\pi_1(L)$. The author [**3**] generalized the above result to compact codimension two foliations. Furthermore Plante [**8**] gave a necessary and sufficient condition for a transversely orientable foliation of a closed 3-manifold by closed orientable surfaces to be C^0-stable.

We study here the stability of all foliations of closed 3-manifolds by circles and give a necessary and sufficient condition for such a foliation to be stable. Indeed, we have the following theorem.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 60540071), Ministry of Education, Science and Culture.

THEOREM. *Let F be a foliation of a closed 3-manifold M by circles. Then F is stable if and only if one of the followings holds;*

(i) $\chi(M/F)^2+\chi_V(M/F)^2\neq 0$,

(ii) *the union of all reflection leaves of F contains a subset homeomorphic to a Klein bottle,*

where $\chi_V(M/F)$ is the V-euler characteristic of the leaf space M/F (for definitions, see § 1).

We prove the sufficient part (Theorems 2 and 4) of Theorem in § 1, and the necessary part (Theorems 9 and 15) of Theorem in §§ 2 and 3. All foliations we consider here are smooth of class C^∞ and of codimension two.

1. Sufficient condition for F to be stable.

Let D^2 be the unit disk and G a finite cyclic subgroup of $O(2)$. We foliate $S^1\times D^2$ with leaves of the form $S^1\times\{\mathrm{pt}\}$. This foliation is preserved by the diagonal action of G, defined by $g(x, y)=(x\cdot g^{-1}, g\cdot y)$ for $g\in G$, $x\in S^1$ and $y\in D^2$, where G acts linearly on D^2 and freely on S^1 on the right. So we have a foliation induced on $S^1\times_G D^2$. Note that the central leaf in this foliation corresponding to $y=0$ has the holonomy group G.

PROPOSITION 1 (Epstein [**1**]). *Let F be a foliation of a closed 3-manifold M by circles. Then each leaf in F has a neighborhood diffeomorphic to such a foliation of $S^1\times_G D^2$.*

Proposition 1 implies that all leaves in F have the holonomy groups isomorphic to finite cyclic groups. Hence, G is either a subgroup of $SO(2)$ which consists of k rotations and is denoted by $\boldsymbol{Z}_k$ or a subgroup of $O(2)$ which consists of a reflection and the identity and is denoted by $\boldsymbol{D}$. We say a leaf with non-trivial holonomy is a rotation leaf or a reflection leaf if the holonomy group is $\boldsymbol{Z}_k$ $(k>1)$ or $\boldsymbol{D}$. Notice that F has only a finite number of rotation leaves because of the compactness of M. Furthermore the leaf space M/F is homeomorphic to a compact 2-dimensional V-manifold (which is equivalently called an orbifold) and we can define the V-euler characteristic of M/F, $\chi_V(M/F)\in\boldsymbol{Q}$ (see Satake [**10**] for definitions). In this case, M/F is also a topological manifold and the union $R(F)$ of all reflection leaves corresponds to the boundary of M/F. Let $L_1, \cdots, L_n$ be all rotation leaves in F, whose holonomy groups are $\boldsymbol{Z}_{k_1}, \cdots, \boldsymbol{Z}_{k_n}$ respectively. Then the V-euler characteristic of M/F is given by

$$\chi_V(M/F)=\chi(M/F)+\sum_{i=1}^{n}(1/k_i-1),$$

where $\chi(M/F)$ is the euler characteristic of M/F.

Now we give a sufficient condition for F to be stable in the following which is a corollary to the results of Hirsch [**5**], Seifert [**11**] and Fukui [**3**].

THEOREM 2. *Let F be a foliation of a closed 3-manifold M by circles. If $\chi(M/F)^2+\chi_V(M/F)^2\neq 0$, then F is stable.*

PROOF. Let τM be the tangent bundle of M and τF the subbundle of τM which consists of the vectors tangent to the leaves of F. Suppose that τF is not orientable. Take the unit vector subbundle $\tilde{M}$ of τF. Then $\tilde{M}$ is a double cover of M and for the foliation $\tilde{F}$ induced on $\tilde{M}$, $\tau\tilde{F}$ is orientable. Moreover we see that $\chi(\tilde{M}/\tilde{F})=2\chi(M/F)$ and $\chi_V(\tilde{M}/\tilde{F})=2\chi_V(M/F)$. Let F' be a foliation of M which is C^1-close to F. The foliation $\tilde{F}'$ on $\tilde{M}$ is also C^1-close to $\tilde{F}$. Then we can see that if $\tilde{F}'$ has a compact leaf, then F' also has a compact leaf. Hence it is sufficient to prove Theorem 2 when τF is orientable. First we prove the case $n\neq 0$. Then F has a rotation leaf. Since 1 is not an eigenvalue of the linear holonomy of every rotation leaf, the proof follows from the result of Hirsch ([**5**], Theorem 1.1). Next we prove the case $n=0$. Then F satisfies the conditions of Corollary 5 of [**3**], hence F is stable. This completes the proof.

REMARK 3. We can define another topology on $\mathrm{Fol}_q(M)$ as follows. Given a foliation F, we associate to each point x of M the plane tangent to F at x. This gives a section of the bundle over M whose fibre over x is the Grassmannian of all $(m-q)$-dimensional planes of the tangent space of M at x, where m is the dimension of M. The C^r-topology $(0\leqq r\leqq\infty)$ on the space of all sections of this bundle topologizes $\mathrm{Fol}_q(M)$. We denote this space by $\overline{\mathrm{Fol}}{}_q^r(M)$. Note that the identity: $\mathrm{Fol}_q^r(M)\to\overline{\mathrm{Fol}}{}_q^{r-1}(M)$ is continuous but in general not homeomorphic (see Epstein [**2**]). Under this topology Theorem 1.1 of [**5**] is still true except for uniqueness in our case. For, let $h(F):(\boldsymbol{R}^2, 0)\to(\boldsymbol{R}^2, 0)$ be a representation of the holonomy of a rotation leaf of F. Then we can take a neighborhood $\bar{V}$ of F in $\overline{\mathrm{Fol}}{}_2^0(M)$ such that for any $F'\in\bar{V}$, $h(F'):\boldsymbol{R}^2\to\boldsymbol{R}^2$ has a (not necessarily unique) fixed point since $h(F')-\mathrm{id}_{\boldsymbol{R}^2}:(\boldsymbol{R}^2, 0)\to(\boldsymbol{R}^2, 0)$ is diffeomorphic, where $h(F')$ is the perturbed holonomy map associated to F' (see [**5**]). Hence Theorem 2 is also true under this topology.

THEOREM 4. *Let F be a foliation of a closed 3-manifold M by circles. If $R(F)$ contains a subset homeomorphic to a Klein bottle, then F is stable.*

PROOF. We may assume that $R(F)$ is connected because the argument is local. The bundle $q:R(F)\to R(F)/F$ is equivalent to the non-trivial S^1-bundle $p:K^2\to S^1$, where K^2 is the Klein bottle. Let N be a saturated tubular neighborhood of $R(F)$ in M. We consider the quotient space N_0 obtained in the product $[0, 1]\times[0, 1]\times(-1, 1)$ with coordinate (s, t, u), $s, t\in[0, 1]$ and $u\in$

$(-1, 1)$ by making the following identifications; $(0, t, u)\sim(1, t, -u)$ and $(s, 0, u)\sim(1-s, 1, u)$ for all s, t and u. We foliate $[0, 1]\times[0, 1]\times(-1, 1)$ with leaves of form $[0, 1]\times\{t\}\times\{u\}$ to obtain a foliation F_0 induced on N_0. Then we may assume, taking a double cover of N if necessary, that (N, F) is diffeomorphic to (N_0, F_0). So they are identified. Note that $R(F)$ corresponds to the subset $\{(s, t, 0); 0\leqq s, t\leqq 1\}/\sim$ which is homeomorphic to K^2. We define a section c of the bundle $p: K^2\to S^1$ by $c(t)=(1/2, t, 0)$. Let $\alpha(0)$ be a closed curve on $L_{c(0)}$ with base point $c(0)$ and orientation $\partial/\partial s$ and $\alpha(t)$ $(0\leqq t\leqq 1)$ translations of $\alpha(0)$ along $c(t)$, where $L_{c(0)}$ is a leaf of F through $c(0)$. Note that $\alpha(1)=-\alpha(0)$. Let F' be a foliation which is sufficiently C^1-close to F. We can construct the perturbed holonomy map $H(F', \alpha(t))$ of $\{(1/2, t, u) ; -\delta<u<\delta\}$ into an annulus $A=\{(1/2, t, u) ; 0\leqq t\leqq 1, -1<u<1\}/\sim$ for each t, where δ is a small number (see Hirsch [5] and Langevin-Rosenberg [7]). We take the product $[0, 2]\times(-k, k)$ with coordinate (t', u), $t'\in[0, 2]$, $u\in(-k, k)$ and identify $(0, u)$ and $(2, u)$ to obtain an annulus A_k. The map $\pi: A_1\to A$ defined by $\pi(t', u)=(1/2, t' \bmod 1, u)$ is a double covering. Then we define a map H: $\{(t', u); -\delta<u<\delta\}\to A_1$ by

$$H(t', u)=\begin{cases} H(F', \alpha(t'))(t', u) & (0\leqq t'\leqq 1) \\ H(F', -\alpha(t'-1))(t'-1, u)+(1, 0) & (1\leqq t'\leqq 2). \end{cases}$$

Note that $H(0, u)=H(2, u)$ for each u and H is a diffeomorphism of A_δ into A_1. We put $H(F', \alpha(t))(t, u)=(f_1(t, u), f_2(t, u))$ and $H(t', u)=(\bar{f}_1(t', u), \bar{f}_2(t', u))$. There exists a unique $u(t')$ with $u(t')=\bar{f}_2(t', u(t'))$ near $u=0$ for each t' since the holonomy group of each $L_{c(t)}$ is isomorphic to $\boldsymbol{D}$. Then $l: [0, 2]\to A_1$ defined by $l(t')=(t', u(t'))$ is a loop. Therefore we may assume, changing the coordinate t if necessary, that $\pi\circ l\circ i: [0, 1]\to A$ is a loop, where $i: [0, 1]\to[0, 2]$ is the inclusion. We define $v(l(t'))$ to be the vector tangent to A_1 joining $l(t')$ and $H(l(t'))$. π_* projects $v(l\circ i(t))$ to a vector $v(\pi\circ l\circ i(t))$ on $\pi\circ l\circ i(t)$, whose $\partial/\partial t$-component is $f_1(t, u(t))-t$, where $\pi\circ l\circ i(t)=(t, u(t))$. If $f_1(0, u(0))>0$, then $f_1(1, u(1))-1<0$ since $H(F', \alpha(1))=H(F', -\alpha(0))$. Hence by the mean value theorem, there exists t_0 $(0<t_0<1)$ such that $v(p(t_0))=0$. This means that $H(F', \alpha(t_0))(t_0, u(t_0))=(t_0, u(t_0))$, hence $L'_{p(t_0)}$ is compact, where $L'_{p(t_0)}$ is a leaf of F' through $p(t_0)$.

Remark 5. Let F be a foliation of a closed 3-manifold M by circles such that M/F is homeomorphic to a cylinder. Then τF is orientable if and only if $q: R(F)\to R(F)/F$ is an orientable S^1-bundle.

2. Necessary condition for F to be stable, orientable case.

The equation $\chi(M/F)^2+\chi_V(M/F)^2=0$ implies the equations $\chi(M/F)=0$ and $n=0$. Therefore F has not any rotation leaves and M/F is homeomorphic to one of the following four manifolds: (1) torus T^2, (2) Klein bottle K^2, (3) cylinder $S^1\times[0, 1]$ and (4) Möbius band B. We assume that τF is orientable in this section.

First we consider the case that M/F is homeomorphic to T^2 or K^2. We regard T^2 (resp. K^2) to be the quotient space obtained in the product $[0,1]\times[0,1]$ with coordinate (s, t), $s, t\in[0, 1]$, by the following identifications; $(0, t)\sim(1, t)$, $(s, 0)\sim(s, 1)$ (resp. $(0, t)\sim(1, 1-t)$, $(s, 0)\sim(s, 1)$). We denote by $\{s\}\times S_t^1$ the quotient subspace obtained in $\{s\}\times[0, 1]$ by identifying $(s, 0)$ and $(s, 1)$. Since the quotient map $q: M\to M/F$ is an orientable S^1-bundle, $q^{-1}((1/2-3\varepsilon, 1/2+3\varepsilon)\times S_t^1)$ is diffeomorphic to $S^1\times(1/2-3\varepsilon, 1/2+3\varepsilon)\times S_t^1$ with coordinate (θ, s, t), where $0<\varepsilon<1/6$. Let $\varphi, \phi: [0, 1]\to \boldsymbol{R}$ be C^∞-functions such that 1) $0\leqq\varphi(s)\leqq1$, $0\leqq\phi(s)\leqq1$, 2) $\varphi(1/2)=0$ and $\varphi(s)\neq0$ for $s\neq1/2$ and 3) $\phi(s)=0$ for $s\in[0, 1/2-2\varepsilon)\cup(1/2+2\varepsilon, 1]$ and $\phi(s)=1$ for $s\in(1/2-\varepsilon, 1/2+\varepsilon)$. Then we can define a vector field $\bar{Y}$ on T^2 (resp. K^2) by $\bar{Y}=\varphi(s)\partial/\partial s+\phi(s)\partial/\partial t$. We can lift $\bar{Y}$ to a vector field Y on M such that Y on $q^{-1}((1/2-3\varepsilon, 1/2+3\varepsilon)\times S_t^1)$ is given by $\varphi(s)\partial/\partial s+\phi(s)\partial/\partial t$ using the coordinate (θ, s, t). From the assumption, F gives rise to a non-singular vector field X of M. We may assume that X on $q^{-1}((1/2-3\varepsilon, 1/2+3\varepsilon)\times S_t^1)$ is given by $X=\partial/\partial\theta$ using the same coordinate. Then a vector field $X+\lambda Y$ is non-singular for a small number λ. We define a foliation $F_\lambda(Y)$ to be the set of the integral curves of $X+\lambda Y$. It is easy to see that F and $F_\lambda(Y)$ are sufficiently C^r-close if λ is sufficiently small. Furthermore $F_\lambda(Y)$ has not any compact leaves if λ is irrational. Indeed, the $\partial/\partial s$-component of $X+\lambda Y$ is not zero on $M-q^{-1}(\{1/2\}\times S_t^1)$, hence all leaves of $F_\lambda(Y)$ in $M-q^{-1}(\{1/2\}\times S_t^1)$ are not closed. On the other hand, $X+\lambda Y$ is equal to $\partial/\partial\theta+\lambda\partial/\partial t$ on $q^{-1}(\{1/2\}\times S_t^1)$. Hence the leaves of $F_\lambda(Y)$ in $q^{-1}(\{1/2\}\times S_t^1)=S^1\times\{1/2\}\times S_t^1$ consist of the linear curves of slope λ. Since λ is irrational, these leaves are not compact. So we have the following proposition because we can take a sequence of irrational numbers $\{\lambda_i\}_{i\in N}$ with $\lambda_i\to0$ $(i\to\infty)$.

PROPOSITION 6. *Let F be a foliation of a closed 3-manifold M by circles such that M/F is homeomorphic to T^2 or K^2. Suppose that τF is orientable. Then F is C^r-unstable $(r\geqq0)$.*

Next we consider the case that M/F is homeomorphic to a cylinder. Let (s, t) be a coordinate of the cylinder $S_s^1\times[0, 1]$, $s\in S_s^1$, $t\in[0, 1]$. The union $R(F)=q^{-1}(S_s^1\times\{0, 1\})$ of reflection leaves of F is the total space of an orientable S^1-bundle $q: R(F)\to S_s^1\times\{0, 1\}$ by Remark 5, hence it is diffeomorphic to $S^1\times S_s^1\times\{0, 1\}$ with coordinate (θ, s), so we identify $R(F)$ with $S^1\times S_s^1\times\{0, 1\}$. Let N be a tubular neighborhood of $R(F)$ in M which is diffeomorphic to $q^{-1}(S_s^1\times\{[0, \varepsilon)\cup(1-\varepsilon, 1]\})$ and $\pi: N\to S^1\times S_s^1\times\{0, 1\}$ its projection. Let $\varphi: [0,1]\to\boldsymbol{R}$ be a C^∞-function such that $0\leqq\varphi(t)\leqq1$ for $t\in[0, 1]$, $\varphi(t)=1$ for $t\in(\varepsilon, 1-\varepsilon)$,

$\varphi(t)\neq 0$ for $t\neq 0, 1$ and φ is infinitely tangent to the zero map at $t=0, 1$. Let $\varphi(t)\partial/\partial t$ and $\psi(t)\partial/\partial s$ be vector fields on $S_s^1\times[0, 1]$, where $\psi(t)=1-\varphi(t)$. From the assumption, F gives rise to a non-singular vector field X. We may assume that X on $q^{-1}(S_s^1\times\{0, 1\})$ is given by $\partial/\partial\theta$ using the coordinate (θ, s). Since $q: M-R(F)\rightarrow S_s^1\times(0, 1)$ is an orientable S^1-bundle, we can lift $\varphi(t)\partial/\partial t$ to a vector field Y on M. Moreover we can lift $\psi(t)\partial/\partial s$ to a vector field Z on M. A vector field $X+\lambda(Y+Z)$ is non-singular for a small number λ. So we define a foliation $F_\lambda(Y, Z)$ to be the set of the integral curves of $X+\lambda(Y+Z)$. We see that F and $F_\lambda(Y, Z)$ are sufficiently C^r-close if λ is sufficiently small. Furthermore we can see that $F_\lambda(Y, Z)$ has not any compact leaves if λ is irrational. Hence we have the following.

PROPOSITION 7. *Let F be a foliation of M by circles such that M/F is homeomorphic to a cylinder. Suppose that τF is orientable. Then F is C^r-unstable* $(r\geqq 0)$.

Finally we consider the case that M/F is homeomorphic to a Möbius band. We regard the Möbius band B to be the quotient space obtained in the product $[0, 1]\times[-1, 1]$ with coordinate (s, t), $s\in[0, 1]$, $t\in[-1, 1]$ by identifying the segments $\{0\}\times[-1, 1]$ and $\{1\}\times[-1, 1]$ by the involution $\tau(t)=-t$. We denote by $S_s^1\times\{t\}$ the quotient subspace obtained in $[0, 1]\times\{t, -t\}$ by the above identification. Let $\varphi, \psi: [-1, 1]\rightarrow\boldsymbol{R}$ be C^∞-functions such that 1) $\varphi(-t)=-\varphi(t)$, $\psi(-t)=\psi(t)$, 2) φ is infinitely tangent to the zero map at $t=-1, 0, 1$ and is never zero for $t\neq -1, 0, 1$ and 3) $\psi(t)=0$ for $|t|\in[2\varepsilon, 1-2\varepsilon]$, $\psi(t)=1$ for $|t|\in[0, \varepsilon]\cup[1-\varepsilon, 1]$, where $0<\varepsilon<1/6$. Since $\varphi(t)\partial/\partial t$ is a τ-invariant vector field on $[0, 1]\times[-1, 1]$, we can define a vector field $\varphi(t)\partial/\partial t$ on B. We assume that the union $R(F)=q^{-1}(\partial B)$ of the reflection leaves of F is the total space of an orientable S^1-bundle $q: R(F)\rightarrow\partial B\cong S_s^1\times\{1\}$, hence it is diffeomorphic to $S^1\times S_s^1\times\{1\}$ with coordinate (θ, s). So we identify $R(F)$ with $S^1\times S_s^1\times\{1\}$. Let N be a tubular neighborhood of $R(F)$ in M which is diffeomorphic to $q^{-1}(\bigcup_{1-2\varepsilon<t\leq 1}S_s^1\times\{t\})$ and $\pi: N\rightarrow R(F)$ its projection. From the assumption, F gives rise to a non-singular vector field X. We may assume that X on $R(F)$ is given by $\partial/\partial\theta$ using the coordinate (θ, s). Since $q: M-R(F)\rightarrow B-\partial B$ is an orientable S^1-bundle, we can lift $\varphi(t)\partial/\partial t$ to a vector field Y on M. Moreover we can lift $\psi(t)\partial/\partial s$ to a vector field Z on M. A vector field $X+\lambda(Y+Z)$ is non-singular for a small number λ. So we define a foliation $F_\lambda(Y, Z)$ to be the set of the integral curves of $X+\lambda(Y+Z)$. We see that F and $F_\lambda(Y, Z)$ are sufficiently C^r-close if λ is sufficiently small. Furthermore we can see that $F_\lambda(Y, Z)$ has not any compact leaves if λ is irrational. Hence we have the following.

PROPOSITION 8. *Let F be a foliation of M by circles such that M/F is homeomorphic to a Möbius band. Suppose that τF is orientable. Then F is C^r-unstable* $(r\geqq 0)$.

Combining Propositions 6, 7 and 8, we have the following theorem which gives a necessary condition for F to be stable in case that τF is orientable.

THEOREM 9. *Let F be a foliation of a closed 3-manifold M by circles. Suppose that τF is orientable. If $\chi(M/F)^2+\chi_V(M/F)^2=0$, then F is C^r-unstable ($r\geqq 0$).*

3. Necessary condition for F to be stable, non-orientable case.

We assume that τF is non-orientable in this section. First we consider the case that M/F is homeomorphic to a torus. We can regard M to be the quotient space obtained in the product $S^1\times[0, 1]\times[0, 1]$ with coordinate (θ, s, t), $\theta\in S^1$, $s, t\in[0, 1]$ by making the following identifications; $(\theta, 0, t)\sim(f(t)\cdot\theta, 1, t)$, and $(\theta, s, 0)\sim(g(s)\cdot\theta, s, 1)$ for all θ, s and t, where $f, g:[0, 1]\to O(2)$ are C^∞-maps with $f(0)=f(1)$, $g(0)=g(1)$ and $g(1)f(0)=f(1)g(0)$. Moreover we may assume that $f(t)\notin SO(2)$ and $g(s)\in SO(2)$ for all s, t. Let $\varphi:[0, 1]\to\boldsymbol{R}$ be a C^∞-function such that $\varphi(0)=\varphi(1)=0$ and $\varphi(s)>0$ for $s\in(0, 1)$. Let $q_t:[0, 1]\times[0, 1]\to[0, 1]\times S^1$ and $q_s:[0, 1]\times S^1\to T^2$ be the quotient maps induced from the above identifications. Then the pullback $q:q_s^*(M)\to[0, 1]\times S^1$ is an orientable S^1-bundle. Hence the foliation q^*F induced on $q_s^*(M)$ gives rise to a non-singular vector field X. We can lift a vector field $\varphi(s)\partial/\partial s$ to a vector field on $q_s^*(M)$ which is denoted by the same letter. A vector field $X+\lambda\varphi(s)\partial/\partial s$ is non-singular and it defines a foliation F_1 on $q_s^*(M)$ to be the set of the integral curves of $X+\lambda\varphi(s)\partial/\partial s$. Since F_1 on $q^{-1}(\{0, 1\}\times S^1)$ has the same compact leaves as F, we can define a foliation of M which is denoted by the same letter. Let N (resp. N') be an ε (resp. $\varepsilon/2$)-neighborhood of $\{0\}\times S^1$ in T^2. Since $q:q^{-1}(N)\to N$ is orientable, F_1 on $q^{-1}(N)$ gives rise to a non-singular vector field X_1. Let $\psi:(-\varepsilon, \varepsilon)\to\boldsymbol{R}$ be a C^∞-function such that $\psi(0)=1$ and the support of ψ is contained in $(-\varepsilon/2, \varepsilon/2)$. Then we can lift a vector field $\psi(s)\partial/\partial t$ to a vector field on $q^{-1}(N)$ which is denoted by the same letter. A vector field $X_1+\lambda\psi(s)\partial/\partial t$ on $q^{-1}(N)$ is non-singular. We can define a foliation F_2 on $q^{-1}(N)$ to be the set of the integral curves of $X_1+\lambda\psi(s)\partial/\partial t$. Since F_1 is equal to F_2 on $q^{-1}(N-N')$, we can define a foliation F' by $F'=F_1$ outside of $q^{-1}(N')$ and $F'=F_2$ on $q^{-1}(N)$. It is easy to see that F and F' are C^r-close if λ is small. We may assume that X_1 on $q^{-1}(\{0\}\times S^1)$ is given by $\partial/\partial\theta$. So we can see that F' has not any compact leaves if λ is irrational. Hence we have the following.

PROPOSITION 10. *Let F be a foliation of a closed 3-manifold M by circles such that τF is non-orientable and M/F is homeomorphic to a torus. Then F is C^r-unstable ($r\geqq 0$).*

Next we consider the case M/F is homeomorphic to K^2. Then we can regard M to be the quotient space obtained in the product $S^1\times[0, 1]\times[0, 1]$

with coordinate (θ, s, t), $\theta \in S^1$, $s, t \in [0, 1]$ by making the following identifications; $(\theta, 0, t) \sim (g(t)\cdot\theta, 1, 1-t)$ and $(\theta, s, 0) \sim (f(s)\cdot\theta, s, 1)$ for all θ, s and t, where $f, g : [0, 1] \to O(2)$ be C^∞-maps with $f(0)=f(1)$, $g(0)=g(1)$ and $f(1)g(1)f(0)=g(0)$. Then we have the following three cases. (1) $f(s) \in SO(2)$, $g(t) \notin SO(2)$, (2) $f(s) \notin SO(2)$, $g(t) \in SO(2)$ and (3) $f(s) \notin SO(2)$, $g(t) \notin SO(2)$.

LEMMA 11. *Let F be a foliation of a closed 3-manifold M by circles such that M/F is homeomorphic to K^2 and F is in* (1). *Then F is C^r-unstable* ($r \geqq 0$).

PROOF. The proof is similar as in the proof of Proposition 9 since the pullback $q : q_s^*(M) \to [0, 1] \times S^1$ is an orientable S^1-bundle, where $q_s : [0, 1] \times S^1 \to K^2$ is the quotient map induced from the above identifications.

LEMMA 12. *Let F be a foliation of a closed 3-manifold M by circles such that M/F is homeomorphic to K^2 and F is in* (2). *Then F is C^r-unstable* ($r \geqq 0$).

PROOF. We regard K^2 to be the quotient space obtained in the product $[0, 1] \times [0, 1]$ with coordinate (s, t), $s, t \in [0, 1]$ by making the following identifications; $(0, t) \sim (1, 1-t)$, $(s, 0) \sim (s, 1)$ for all s, t. The circles $\{(s, 0) ; 0 \leqq s \leqq 1\}/\sim$ and $\{(s, 1/2) ; 0 \leqq s \leqq 1\}/\sim$ are denoted by S_0, S_1 respectively. Note that $K^2 - S_0$ is an open Möbius band and $q : q^{-1}(K^2 - S_0) \to K^2 - S_0$ is an orientable S^1-bundle. Let $\varphi : [0, 1] \to \boldsymbol{R}$ be a C^∞-function such that 1) $\varphi(t) = -\varphi(1-t)$, 2) $\varphi(0) = \varphi(1/2) = \varphi(1) = 0$, 3) $\varphi(t) > 0$ for $t \in (0, 1/2)$ and 4) φ is infinitely tangent to the zero map at $t = 0, 1/2, 1$. We can lift a vector field $\varphi(t)\partial/\partial t$ on K^2 to a vector field on M which is denoted by the same letter. F on $q^{-1}(K^2 - S_0)$ gives rise to a non-singular vector field X since $q : q^{-1}(K^2 - S_0) \to K^2 - S_0$ is orientable. A vector field $X + \lambda\varphi(t)\partial/\partial t$ is non-singular. We can define a foliation F_1 of M to be the set of the integral curves of $X + \lambda\varphi(t)\partial/\partial t$ on $q^{-1}(K^2 - S_0)$ and the set of $\{q^{-1}(x) ; x \in S_0\}$ on $q^{-1}(S_0)$. Then F_1 has not any compact leaves on $q^{-1}(K^2 - S_0 \cup S_1)$. Let N_0, N_1 be disjoint tubular neighborhoods of S_0, S_1 in K^2 which are induced from $\{(s, t) ; 0 \leqq t < \varepsilon \text{ or } 1-\varepsilon < t \leqq 1\}$, $\{(s, t) ; 1/2 - \varepsilon < t < 1/2 + \varepsilon\}$ $(0 < \varepsilon < 1/4)$ respectively. Let $\psi : [0, 1] \to \boldsymbol{R}$ be a C^∞-function such that 1) $\psi(t) = \psi(1-t)$, $\psi(0) = \psi(1/2) = 1$, 2) the support of ψ is contained in $[0, \varepsilon/2] \cup [1/2 - \varepsilon/2, 1/2 + \varepsilon/2] \cup [1 - \varepsilon/2, 1]$ and 3) ψ is infinitely tangent to the constant 1 map at $t = 0$. Then we can lift a vector field $\psi(t)\partial/\partial s$ on K^2 to a vector field on M which is denoted by the same letter. Since $q : q^{-1}(N_0 \cup N_1) \to N_0 \cup N_1$ is orientable, $q^{-1}(N_0 \cup N_1)$ is diffeomorphic to $S^1 \times (N_0 \cup N_1)$ with coordinate (θ, s, t). F_1 on $q^{-1}(N_0 \cup N_1)$ gives rise to a non-singular vector field X_1. Then we may assume that X_1 on $q^{-1}(S_0 \cup S_1)$ is given by $\partial/\partial\theta$ using the coordinate (θ, s, t). A vector field $X_1 + \lambda\psi(t)\partial/\partial s$ is non-singular, hence it defines a foliation F' of M by its integral curves. We can see that F and F' are C^r-close if λ is small and F' has not any compact leaves if λ is irrational. This completes the proof.

Since each foliation in (3) reduces to a foliation in (2), we have the following proposition from Lemmas 11 and 12.

PROPOSITION 13. *Let F be a foliation of a closed 3-manifold M by circles such that τF is non-orientable and M/F is homeomorphic to a Klein bottle. Then F is C^r-unstable* ($r \geqq 0$).

Finally we consider the case that M/F is homeomorphic to a Möbius band. Let B be the Möbius band obtained in the product $[0, 1]\times[-1, 1]$ with coordinate (s, t), $s\in[0, 1]$, $t\in[-1, 1]$ by making the following identifications; $(0, t)\sim(1, -t)$. We identify M/F with B. So $q: M\to B$ is the quotient map and $q^{-1}(\partial B)$ is the union of reflection leaves of F. Let $\varphi:[-1, 1]\to\boldsymbol{R}$ and $\psi:[0, 1]\to\boldsymbol{R}$ be C^∞-functions such that 1) $\varphi(t)>0$ for $t\in(-1, 1)$, $\varphi(-t)=\varphi(t)$, $\varphi(t)=1$ for $t\in[-1+\varepsilon, 1-\varepsilon]$ and φ is infinitely tangent to the zero map at $t=-1, 1$ and 2) $\psi(s)=1$ for $s\in(1/2-\varepsilon, 1/2+\varepsilon)$ and $\psi(s)=0$ for $s\notin(1/2-2\varepsilon, 1/2+2\varepsilon)$ $(0<\varepsilon<1/4)$. Then we can lift vector fields $\varphi(t)\psi(s)\partial/\partial t$ and $\varphi(t)(1-\psi(s))\partial/\partial s$ on $[0, 1]\times[-1, 1]$ to vector fields on M which are denoted by the same letters.

$$q : q^{-1}([1/2-2\varepsilon, 1/2+2\varepsilon]\times(-1, 1)) \longrightarrow [1/2-2\varepsilon, 1/2+2\varepsilon]\times(-1, 1)$$

and

$$q : q^{-1}(\{[0, 1/2-\varepsilon]\cup[1/2+\varepsilon, 1]\}\times(-1, 1)/\sim)$$
$$\longrightarrow \{[0, 1/2-\varepsilon]\cup[1/2+\varepsilon, 1]\}\times(-1, 1)/\sim$$

are trivial S^1-bundles. Hence F on $q^{-1}([1/2-2\varepsilon, 1/2+2\varepsilon]\times(-1, 1))$ gives rise to a non-singular vector field X. A vector field $X+\lambda\varphi(t)\psi(s)\partial/\partial t$ is non-singular on $q^{-1}([1/2-2\varepsilon, 1/2+2\varepsilon]\times(-1, 1))$. We can define a foliation F_1 of M to be the set of the integral curves of $X+\lambda\varphi(t)\psi(s)\partial/\partial t$ on $q^{-1}([1/2-2\varepsilon, 1/2+2\varepsilon]\times(-1, 1))$ and the set of the leaves of F otherwise. F_1 on $q^{-1}(\{[0,1/2-\varepsilon]\cup[1/2+\varepsilon,1]\}\times(-1,1)/\sim)$ also gives rise to a non-singular vector field X_1. A vector field $X_1+\lambda\varphi(t)(1-\psi(s))\partial/\partial s$ is non-singular. So we can define a foliation F_2 of M to be the set of the integral curves of $X_1+\lambda\varphi(t)(1-\psi(s))\partial/\partial s$ on $q^{-1}(\{[0,1/2-\varepsilon]\cup[1/2+\varepsilon,1]\}\times(-1,1)/\sim)$ and the set of the leaves of F_1 otherwise. It is easy to see that F_2 has not any compact leaves on $q^{-1}(B-\partial B)$ and $F_2=F$ on $q^{-1}(\partial B)$. We assume that $q: R(F)=q^{-1}(\partial B)\to\partial B$ is an orientable S^1-bundle. So $q^{-1}(\partial B)$ is diffeomorphic to $S^1\times\partial B$ with coordinate (θ, s), $\theta\in S^1$, $s\in\partial B$. Let N be an ε-tubular neighborhood of $q^{-1}(\partial B)$ in M. F_2 on N gives rise to a non-singular vector field X_2 which can be given by $\partial/\partial\theta$ on $q^{-1}(\partial B)$. A vector field $X_2+\lambda(1-\varphi(t))\partial/\partial s$ on N is non-singular and can be extended to a vector field on M, which is denoted by the same letter. We define a foliation F' of M to be the set of the integral curves of $X_2+\lambda(1-\varphi(t))\partial/\partial s$ on N and the set of the leaves of F_2 outside of N. We can easily see that F and F' are C^r-close if λ is small and F' has not any compact leaves if λ is irrational. Thus we have the following proposition.

PROPOSITION 14. *Let F be a foliation of a closed 3-manifold M by circles such that M/F is homeomorphic to a Möbius band. Suppose that τF is non-orientable and $q: R(F)\rightarrow R(F)/F$ is an orientable S^1-bundle. Then F is C^r-unstable* ($r\geqq 0$).

Combining Propositions 10, 13, 14 and Remark 5, we have the following theorem which gives a necessary condition for F to be stable in case that τF is non-orientable.

THEOREM 15. *Let F be a foliation of a closed 3-manifold M by circles such that τF is non-orientable. Suppose that M/F is not homeomorphic to a cylinder and $q: R(F)\rightarrow R(F)/F$ is an orientable S^1-bundle if M/F is homeomorphic to a Möbius band. If $\chi(M/F)^2+\chi_V(M/F)^2=0$, then F is C^r-unstable* ($r\geqq 0$).

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