J. Math. Soc. Japan
Vol. 39, No. 1, 1987

On Itô's formula for certain fields of geometric objects

By Hiroshi AKIYAMA

(Received Oct. 20, 1984)
(Revised July 1, 1985)

1. Introduction.

In stochastic differential geometry, Itô's formula has been extended for tensor fields on a C^∞ manifold in connection with stochastic Lie transport ([**6**], [**11**], cf. [2]), stochastic parallel displacement ([8], [**11**]), and stochastic flow of tensors induced by tensor derivations ([**1**]).

The purpose of the present paper is to establish a general stochastic formula for C^∞ (local) cross sections of fiber bundles which implies a further extension of Itô's formula to the case of "geometric object fields". As to bundles of geometric objects, we adopt a constructive approach due to Ferraris, Francaviglia and Reina [**3**], [**4**]; thus, by a geometric object field, we mean a C^∞ (local) cross section of a C^∞ fiber bundle associated with the bundle of r-th order frames (over a C^∞ manifold) for some positive integer r. (For a general theory of geometric objects, we refer to Salvioli [**14**]. See also Nijenhuis [**13**], Yano [**15**].) Then, for example, C^∞ tensor fields, C^∞ pseudo-tensor fields, their jet extensions, and linear connections of a C^∞ manifold are geometric object fields. It should be pointed out that the use of frames of higher order contact enables us to treat some geometric structures (for instance, projective structures [**9**]) as geometric object fields.

Now let $\pi_E: E\to M$ be a C^∞ fiber bundle associated with a principal fiber bundle $P(M, G, \pi_P)$ (cf. [**10**]). Let $\varphi_t(p)$ be the solution of the following stochastic differential equation (in the Stratonovich form) on P;

$$d\varphi_t = \sum_{\alpha=1}^{k} A_\alpha(\varphi_t)\circ dN^\alpha(t), \qquad \varphi_0=p\in P, \tag{3.1}$$

where A_α, ($\alpha=1, \cdots, k$), are right G-invariant C^∞ vector fields on P and $N^\alpha(t)$'s are real valued continuous semi-martingales. Suppose we are given a C^∞ (local) cross section σ of E. In this paper, we establish a formula for the $\pi_E^{-1}(x)$-

This research was partially supported by Grant-in-Aid for Scientific Research (No. 59740103), Ministry of Education, Science and Culture.

valued stochastic process $\eta_t^{-1}(\sigma(\theta_t(x)))$, where $\theta_t(x)=\pi_P(\varphi_t(p))$ and $\eta_t=\varphi_t(p)\circ p^{-1}:\pi_E^{-1}(x)\to\pi_E^{-1}(\theta_t(x))$, $p\in\pi_P^{-1}(x)$, $x\in\mathrm{Dom}(\sigma)$. (For details, see § 3, Theorem 3.1.) It is to be noted that such a situation includes known cases of tensor fields (cf. § 4, Example 1).

Our formula can be applied to the study of the stochastic flow θ_t: Suppose M is endowed with a linear connection [resp. a Riemannian metric] and assume θ_t is a flow of diffeomorphisms of M almost surely. Then it can be shown that θ_t is a stochastic flow of affine [resp. conformal] motions if all of the vector fields that appear in the stochastic differential equation governing θ_t are infinitesimal affine [resp. conformal] motions (Theorem 5.1). Another application will be given to the description of the behavior of a stochastically deformed projective structure in § 5.2.

This paper is organized as follows. In § 2, we prepare some notions concerning bundles of geometric objects following [**3**], [**4**]. In § 3, we describe our setting (with some discussion) and state a main theorem (Theorem 3.1). We prove it in an intrinsic (coordinate free) manner to clarify the structure of our formula, using Kunita's results ([**11**], [**12**]) and lifted stochastic differential equations on some fiber bundles. We also study the case where E is a C^∞ vector bundle (Corollary 3.2). In § 4, we discuss the case of geometric object fields and give examples which illustrate Theorem 3.1 and Corollary 3.2. Finally, we treat stochastic flows of affine motions and conformal motions, and the stochastic deformation of a projective structure in § 5.

The author would like to thank the referee for his valuable comments.

2. Bundles of geometric objects.

In this paper, all *manifolds* are finite dimensional, σ-compact and of class C^∞. As to manifolds and stochastic differential equations, we shall use freely concepts and notations in Kobayashi-Nomizu [**10**], Ikeda-Watanabe [**6**], and Kunita [**12**].

Let M be an n-dimensional manifold. Let $PS(\boldsymbol{R}^n)$ be the pseudogroup of all diffeomorphisms Ψ from an open neighborhood of the origin 0 of $\boldsymbol{R}^n$ onto an open neighborhood of $0\in\boldsymbol{R}^n$ such that $\Psi(0)=0$. For a positive integer r, set $G^r(n, \boldsymbol{R})=\{j^r(\Psi);\Psi\in PS(\boldsymbol{R}^n)\}$, where $j^r(\Psi)$ denotes the r-jet of Ψ at 0. The set $G^r(n, \boldsymbol{R})$ is a Lie group with multiplication defined by the usual composition law of jets. Let $L^r(M)$ be the set of all r-th order frames $j^r(h)$, where h is a C^∞ map from some open neighborhood U of $0\in\boldsymbol{R}^n$ into M such that $h:U\to h(U)$ is a diffeomorphism. Define a map $\pi^r:L^r(M)\to M$ by $\pi^r(j^r(h))=h(0)$. Then $L^r(M)$ becomes a principal $G^r(n, \boldsymbol{R})$-bundle over M. Moreover, $L^1(M)$ is isomorphic to the bundle $L(M)$ of linear frames over M.

DEFINITION 2.1 (Ferraris et al. [**3**], [**4**], Kobayashi [**9**]). The bundle $L^r(M)$ is called the *bundle of r-th order frames over M.*

Let F be a manifold on which a Lie group G acts on the left effectively and differentiably, and let $\rho: G^r(n, \boldsymbol{R}) \to G$ be a homomorphism of Lie groups.

DEFINITION 2.2 (Ferraris et al. [**3**], [**4**]). The fiber bundle $B = L^r(M) \times_\rho F$ of type ρ associated with $L^r(M)$ is called a *bundle of geometric objects of type ρ of finite rank* ($\leqq r$). Each C^∞ (local) cross section of B is called a *geometric object field* or a *field of geometric objects.*

For later use, we prepare the notion of lifting of (local) diffeomorphisms of M following [**3**], [**4**] and [**9**]. Every local diffeomorphism θ of M can be lifted canonically to a local diffeomorphism $L^r(\theta)$ of $L^r(M)$ by defining

$$L^r(\theta)[j^r(h)] = j^r(\theta \circ h).$$

Thus every C^∞ vector field X on M can be lifted canonically to a C^∞ vector field $L^r(X)$ on $L^r(M)$.

DEFINITION 2.3. We call $L^r(\theta)$ [resp. $L^r(X)$] the *natural lift of θ* [resp. the *natural lift of X*] *to $L^r(M)$.*

Since $L^r(\theta)$ commutes with the natural right action of $G^r(n, \boldsymbol{R})$ on $L^r(M)$, the *natural lift of θ* [resp. the *natural lift of X*] *to any bundle of geometric objects* of finite rank is also defined in the obvious way. Then the *Lie derivative of a geometric object field σ with respect to X in the sense of Salvioli* [**14**] can be defined (cf. [**3**], [**4**]), which we denote by $\hat{L}_X \sigma$. The (*usual*) *Lie differentiation* (cf. [**10**]) with respect to X is denoted by $\mathcal{L}_X$ (cf. § 4).

3. A stochastic formula for (local) cross sections of fiber bundles.

In order to extend Itô's formula to the case of geometric object fields, we shall derive a general stochastic formula for (local) cross sections of fiber bundles.

3.1. Setting. Let $\pi_E: E \to M$ be a C^∞ fiber bundle with standard fiber F, associated with a principal fiber bundle $P(M, G, \pi_P)$. As usual, we regard the principal map $\chi: P \times F \to E$ as a multiplication, so that $p\xi = \chi(p, \xi)$, $p \in P$, $\xi \in F$. For each $a \in G$, we denote by R_a the right translation $P \to P$, $p (\in P) \to pa$. Let $A_1, \cdots, A_k$ be C^∞ vector fields on P such that $(R_a)_{*,p}(A_\alpha(p)) = A_\alpha(pa)$ for every $p \in P$, $a \in G$, ($\alpha = 1, \cdots, k$), where $(R_a)_{*,p}: T_pP$ (=the tangent space of P at p) $\to T_{pa}P$ denotes the differential of R_a at p, and let $X_1, \cdots, X_k$ be C^∞ vector fields on M such that

$$(\pi_P)_{*,p}(A_\alpha(p)) = X_\alpha(\pi_P(p)), \qquad (\alpha=1, \cdots, k\,;\ p\in P).$$

Let $N^1(t), \cdots, N^k(t)$, $(t\geqq 0)$, be real valued continuous semi-martingales defined on a filtered probability space $(\Omega, \mathcal{F}, \mu\,; (\mathcal{F}_t)_{t\geqq 0})$ satisfying the usual conditions (cf. Ikeda-Watanabe [**6**, p. 45]); $(\Omega, \mathcal{F}, \mu)$ is a complete probability space and $(\mathcal{F}_t)_{t\geqq 0}$ is a right-continuous increasing family of sub σ-fields of $\mathcal{F}$ such that $\mathcal{F}_t$ contains all μ-null sets for every $t\geqq 0$. *Throughout this paper, indices α and β run from* 1 *to* k. Solutions of stochastic differential equations are considered up to their life times, unless otherwise stated.

Consider a stochastic differential equation (SDE) on P;

$$d\varphi_t = \sum_\alpha A_\alpha(\varphi_t)\circ dN^\alpha(t)\,, \tag{3.1}$$

($\circ dN^\alpha(t)$: Stratonovich's stochastic differential of $N^\alpha(t)$). We denote the solution of (3.1) with initial condition $\varphi_0=p\in P$ by $\varphi_t(p)$ or $\varphi_t(p, \omega)$, $(\omega\in\Omega)$. This φ_t is a map from P into itself for each t and almost all ω. Moreover, φ_t is a local diffeomorphism for any t a.s. (see Kunita [**12**]). Since A_α is invariant by R_a for every $a\in G$, it follows that

$$d(R_a\varphi_t) = \sum_\alpha A_\alpha(R_a\varphi_t)\circ dN^\alpha(t)\,.$$

Therefore we see that, for each $x\in M$, the life time of $\varphi_t(p)$ does not depend on the choice of $p\in\pi_P^{-1}(x)$. We write the life time of $\varphi_t(p)$ by $\tau_\varphi(x)$ (or $\tau_\varphi(x, \omega)$, $(\omega\in\Omega)$) if $\pi_P(p)=x$. As is easily seen, $\theta_t(x) := \pi_P(\varphi_t(p))$, $(x\in M,\ p\in\pi_P^{-1}(x))$, is well-defined and satisfies the following SDE on M:

$$d\theta_t = \sum_\alpha X_\alpha(\theta_t)\circ dN^\alpha(t)\,. \tag{3.2}$$

Let $q\in E$. Take $p_0\in\pi_P^{-1}(\pi_E(q))$ and put $p_t=\varphi_t(p_0)$. It is easy to check that the (stochastic) map

$$p_t\circ p_0^{-1}\ :\ \pi_E^{-1}(\pi_E(q)) \longrightarrow \pi_E^{-1}(\theta_t(\pi_E(q)))$$

does not depend on the choice of $p_0\in\pi_P^{-1}(\pi_E(q))$. Here, as usual, each element p in P is regarded as an admissible map $F\to\pi_E^{-1}(\pi_P(p))\subset E$, $\xi\,(\in F)\to p\xi$. We obtain a (stochastic) map $\eta_t: E\to E$ given by $\eta_t(q)=(p_t\circ p_0^{-1})(q)$ for $q\in E$ and $p_0\in\pi_P^{-1}(\pi_E(q))$. Since there exists $\xi\in F$ such that $q=p_0\xi$, defining $f_\xi: P\to E$ by $f_\xi(p)=p\xi$, $(p\in P)$, we have, for $t<\tau_\varphi(\pi_E(q))$,

$$d\eta_t(q) = d(f_\xi p_t) = \sum_\alpha (f_\xi)_{*,p_t}(A_\alpha(p_t))\circ dN^\alpha(t)\,.$$

On the other hand, it holds that

$$(f_\xi)_{*,p}(A_\alpha(p)) = (f_{\xi'})_{*,pa}(A_\alpha(pa))\,,\quad \xi' = a^{-1}\xi\,, \qquad (p\in P,\ a\in G)\,.$$

Therefore each A_α induces a C^∞ vector field Y_α on E such that

$$Y_\alpha(q)=(f_\xi)_{*,p_0}(A_\alpha(p_0)), \qquad (q=p_0\xi\,;\ p_0\in P,\ \xi\in F).$$

Each Y_α is π_E-related to X_α; that is,

$$(\pi_E)_{*,q}(Y_\alpha(q))=X_\alpha(x), \qquad q\in E,\ \pi_E(q)=x.$$

Thus η_t satisfies the following SDE (3.3) on E:

$$d\eta_t=\sum_\alpha Y_\alpha(\eta_t)\circ dN^\alpha(t). \tag{3.3}$$

Moreover, $\pi_E\circ\eta_t(q)=\theta_t\circ\pi_E(q)$, $q\in E$. Note that η_t^{-1} satisfies (cf. Kunita [**12**, Proposition 5.1])

$$d\eta_t^{-1}=-\sum_\alpha((\eta_t^{-1})_*Y_\alpha)(\eta_t^{-1})\circ dN^\alpha(t),$$

where

$$((\eta_t^{-1})_*Y_\alpha)(q)=(\eta_t^{-1})_{*,\eta_t(q)}(Y_\alpha(\eta_t(q))).$$

Now let σ be a C^∞ (local) cross section of E. [We want to treat $\eta_t^{-1}(\sigma(\theta_t(x)))\in\pi_E^{-1}(x)$, $x\in\mathrm{Dom}(\sigma)$ (=the domain of σ).] Set

$$(D(X_\alpha, Y_\alpha)\sigma)(x)=\sigma_{*,x}(X_\alpha(x))-Y_\alpha(\sigma(x))\ (\in T_{\sigma(x)}E), \quad x\in\mathrm{Dom}(\sigma). \tag{3.4}$$

Clearly $D(X_\alpha, Y_\alpha)\sigma: x\to(D(X_\alpha, Y_\alpha)\sigma)(x)$ defines a C^∞ cross section of the pull-back σ^*TE of the tangent bundle TE over E by σ. Let $\tilde{Y}_\alpha$ be the natural lift of Y_α to TE. For arbitrary C^∞ (local) cross section ζ of σ^*TE, we put

$$(\tilde{D}(X_\alpha, \tilde{Y}_\alpha)\zeta)(x)=\zeta_{*,x}(X_\alpha(x))-\tilde{Y}_\alpha(\zeta(x)), \qquad x\in\mathrm{Dom}(\zeta)\subset\mathrm{Dom}(\sigma). \tag{3.5}$$

Then $\tilde{D}(X_\alpha, \tilde{Y}_\alpha)\zeta: x\to(\tilde{D}(X_\alpha, \tilde{Y}_\alpha)\zeta)(x)$ is a C^∞ cross section of $\zeta^*T(\sigma^*TE)$. Put $\bar{Y}_\alpha=L^1(Y_\alpha)$ (the natural lift of Y_α to the bundle $L(E)$ of linear frames over E). Let ψ_t be the solution of the following SDE on $L(E)$:

$$d\psi_t=\sum_\alpha\bar{Y}_\alpha(\psi_t)\circ dN^\alpha(t). \tag{3.6}$$

Since $\bar{Y}_\alpha$ is invariant by R_b for every $b\in GL(\tilde{m}, \boldsymbol{R})$ with $\tilde{m}=\dim E$, for a fixed $q\in E$, the life time of the solution $\psi_t(z)$ of (3.6) with initial condition $\psi_0(z)=z\in\pi_{LE}^{-1}(q)$ does not depend on the choice of $z\in\pi_{LE}^{-1}(q)$, where $\pi_{LE}: L(E)\to E$ denotes the projection. The life time of $\tilde{p}_t=\psi_t(z)$ is denoted by $\tau_\psi(q)$ (or $\tau_\psi(q, \omega)$, $(\omega\in\Omega)$) if $\pi_{LE}(z)=q$. Define $\tilde{\eta}_t: TE\to TE$ by $\tilde{\eta}_t(X)=(\tilde{p}_t\circ\tilde{p}_0^{-1})(X)$ if $X\in T_qE$ and $\tilde{p}_0\in\pi_{LE}^{-1}(q)$. For $q\in E$, we put $\tau_{(\eta;\tilde{\eta})}(q)=\min[\tau_\psi(q), \tau_\varphi(\pi_E(q))]$. Then we have $\tilde{\eta}_tX=\eta_{t*,q}X$ for $t<\tau_{(\eta;\tilde{\eta})}(q)$, $X\in T_qE$. Thus $\tilde{\eta}_t$ is the natural lift of η_t to TE. It satisfies

$$d\tilde{\eta}_t=\sum_\alpha\tilde{Y}_\alpha(\tilde{\eta}_t)\circ dN^\alpha(t). \tag{3.7}$$

Furthermore its inverse $\tilde{\eta}_t^{-1}$ satisfies

$$d\tilde{\eta}_t^{-1} = -\sum_\alpha ((\tilde{\eta}_t^{-1})_* \tilde{Y}_\alpha)(\tilde{\eta}_t^{-1}) \circ dN^\alpha(t). \tag{3.8}$$

For each C^∞ function f defined on an open set U of E, define a C^∞ function G_f on $\pi_{TE}^{-1}(U)$ by

$$G_f(X) = (\boldsymbol{d}f)(X) = X[f] \tag{3.9}$$

for every $X \in \pi_{TE}^{-1}(U)$, where $\pi_{TE}: TE \to E$ denotes the projection and $\boldsymbol{d}f$ stands for the exterior derivative of f.

REMARK. Let $\{\theta_s^{(\alpha)}\}$ [resp. $\{\rho_s^{(\alpha)}\}$] be the local one-parameter group of local transformations generated by X_α [resp. Y_α]. Then

$$(D(X_\alpha, Y_\alpha)\sigma)(x)[f] = \frac{d}{ds} f((\rho_s^{(\alpha)})^{-1} \circ \sigma \circ \theta_s^{(\alpha)}(x))\Big|_{s=0}$$

and

$$(\pi_E)_{*, \sigma(x)}((D(X_\alpha, Y_\alpha)\sigma)(x)) = 0_x \qquad \text{[zero-vector at } x\text{]}$$

for every $x \in \mathrm{Dom}(\sigma)$ and every C^∞ function f on $\pi_E^{-1}(\mathrm{Dom}(\sigma))$.

3.2. A main theorem. We now give a stochastic formula for C^∞ (local) cross sections of fiber bundles. We shall succeed the notations in § 3.1. In the following, $\cdot dN^\alpha(t)$ stands for Itô's stochastic differential of $N^\alpha(t)$, and $(\eta_t^{-1})^* f = f \circ \eta_t^{-1}$, that is, $(\eta_t^{-1}(\cdot, \omega))^* f = f \circ \eta_t^{-1}(\cdot, \omega)$.

THEOREM 3.1. *Let σ be a C^∞ (local) cross section of E. Set*

$$D_\alpha \sigma = D(X_\alpha, Y_\alpha)\sigma, \qquad D_{\alpha\beta}\sigma = \tilde{D}(X_\alpha, \tilde{Y}_\alpha)(D(X_\beta, Y_\beta)\sigma).$$

Then for every $x \in \mathrm{Dom}(\sigma)$ and every C^∞ function f on E, it holds that

$$\begin{aligned} & df(\eta_t^{-1}(\sigma(\theta_t(x)))) \\ &= \sum_\alpha (\eta_t^{-1})_{*, \sigma(\theta_t(x))}((D_\alpha \sigma)(\theta_t(x)))[f] \circ dN^\alpha(t) && \text{(3.10a)} \\ &= \sum_\alpha (\eta_t^{-1})_{*, \sigma(\theta_t(x))}((D_\alpha \sigma)(\theta_t(x)))[f] \cdot dN^\alpha(t) \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} (\tilde{\eta}_t^{-1})_{*, (D_\beta \sigma)(\theta_t(x))}((D_{\alpha\beta}\sigma)(\theta_t(x)))[G_f] dN^\alpha(t) dN^\beta(t) && \text{(3.10b)} \\ &= \sum_\alpha (\eta_t^{-1})_{*, \sigma(\theta_t(x))}((D_\alpha \sigma)(\theta_t(x)))[f] \cdot dN^\alpha(t) \\ &\quad + \frac{1}{2} \sum_{\alpha, \beta} \{X_\alpha(\theta_t(x))[(D_\beta \sigma)[(\eta_t^{-1})^* f]] \\ &\qquad - (D_\beta \sigma)(\theta_t(x))[Y_\alpha[(\eta_t^{-1})^* f]]\} dN^\alpha(t) dN^\beta(t), && \text{(3.10c)} \end{aligned}$$

$$0 \leqq t < \tau(x) := \min[\tau_\varphi(x), \tau_\psi(\sigma(x)), \inf\{t > 0; \theta_t(x) \notin \mathrm{Dom}(\sigma)\}].$$

PROOF. We first remark that

$$d\sigma(\theta_t(x)) = \sum_\alpha \sigma_{*, \theta_t(x)}(X_\alpha(\theta_t(x))) \circ dN^\alpha(t).$$

Using (3.4) and applying Kunita's results ([11], [12]) to $\eta_t^{-1}\circ(\sigma\circ\theta_t)$, we obtain

$$\begin{aligned} &df(\eta_t^{-1}(\sigma(\theta_t(x)))) \\ &= -\sum_\alpha (\eta_t^{-1})_{*,\sigma(\theta_t(x))}(Y_\alpha(\sigma(\theta_t(x))))[f]\circ dN^\alpha(t) \\ &\quad +\sum_\alpha (\eta_t^{-1})_{*,\sigma(\theta_t(x))}\sigma_{*,\theta_t(x)}(X_\alpha(\theta_t(x)))[f]\circ dN^\alpha(t) \\ &= \sum_\alpha (\eta_t^{-1})_{*,\sigma(\theta_t(x))}((D_\alpha\sigma)(\theta_t(x)))[f]\circ dN^\alpha(t)\,. \end{aligned}$$

This proves (3.10a). To derive (3.10b), rewrite the right-hand side of (3.10a) as

$$\begin{aligned} &\sum_\alpha (\eta_t^{-1})_{*,\sigma(\theta_t(x))}((D_\alpha\sigma)(\theta_t(x)))[f]\cdot dN^\alpha(t) \\ &+\frac{1}{2}\sum_\beta d((\eta_t^{-1})_{*,\sigma(\theta_t(x))}((D_\beta\sigma)(\theta_t(x)))[f])\cdot dN^\beta(t)\,. \end{aligned}$$

Set, for each $\omega\in\Omega$,

$$\mathcal{D}_t(\eta,\tilde{\eta},\omega)=\{q\in E\,;\tau_{(\eta;\tilde{\eta})}(q,\omega)>t\}\,.$$

Let β be fixed. Put $\zeta=D_\beta\sigma$. Recall (3.5), (3.8) and (3.9). Then for $x\in\mathrm{Dom}(\sigma)$ ($=\mathrm{Dom}(\zeta)$) and $t<\tau(x)$ we have $\sigma(\theta_t(x,\omega))\in\eta_t(\mathcal{D}_t(\eta,\tilde{\eta},\omega),\omega)$ and

$$\begin{aligned} &d((\eta_t^{-1})_{*,\sigma(\theta_t(x))}(\zeta(\theta_t(x)))[f]) \\ &= -\sum_\alpha (\tilde{\eta}_t^{-1})_{*,\zeta(\theta_t(x))}(\tilde{Y}_\alpha(\zeta(\theta_t(x))))[G_f]\circ dN^\alpha(t) \\ &\quad +\sum_\alpha (\tilde{\eta}_t^{-1})_{*,\zeta(\theta_t(x))}\zeta_{*,\theta_t(x)}(X_\alpha(\theta_t(x)))[G_f]\circ dN^\alpha(t) \\ &= \sum_\alpha (\tilde{\eta}_t^{-1})_{*,\zeta(\theta_t(x))}((\tilde{D}(X_\alpha,\tilde{Y}_\alpha)\zeta)(\theta_t(x)))[G_f]\circ dN^\alpha(t)\,, \end{aligned}$$

from which (3.10b) follows. To show (3.10c), observe that

$$G_f\circ\tilde{\eta}_t^{-1}=G_{f(t)}\,,\qquad (f(t):=(\eta_t^{-1})^*f=f\circ\eta_t^{-1})\,,$$

in the sense that

$$\begin{aligned} (\boldsymbol{d}f)(\tilde{\eta}_t^{-1}(\cdot,\omega)Z) &= (\boldsymbol{d}f)((\eta_t^{-1}(\cdot,\omega))_{*,\eta_t(q)}Z) \\ &= (\boldsymbol{d}(f\circ\eta_t^{-1}(\cdot,\omega)))(Z) \end{aligned}$$

for every $Z\in T_{\eta_t(q)}E$ with $q\in\mathcal{D}_t(\eta,\tilde{\eta},\omega)$, a. s. Accordingly

$$\begin{aligned} &(\tilde{\eta}_t^{-1})_{*,\zeta(\theta_t(x))}\{\zeta_{*,\theta_t(x)}(X_\alpha(\theta_t(x)))-\tilde{Y}_\alpha(\zeta(\theta_t(x)))\}[G_f] \\ &= X_\alpha(\theta_t(x))[G_{f(t)}\circ\zeta]-\tilde{Y}_\alpha(\zeta(\theta_t(x)))[G_{f(t)}]\,, \\ &\qquad\qquad 0\leqq t<\tau(x)\,,\quad x\in\mathrm{Dom}(\sigma)\,. \end{aligned}$$

On the other hand,

$$(G_{f(t)}\circ\zeta)(x)=\zeta(x)[f(t)]=(\zeta[f(t)])(x)\,,\qquad 0\leqq t<\tau(x)\,,\quad x\in\mathrm{Dom}(\sigma)\,.$$

Let $\{\rho_s^{(\alpha)}\}$ [resp. $\{\varphi_s^{(\alpha)}\}$] be the local one-parameter group of local transformations generated by Y_α [resp. $\tilde{Y}_\alpha$]. Set $\mathcal{D}_t(\tilde{\eta}, \omega)=\{u\in TE\ ;\ \tau_{(\eta;\tilde{\eta})}(\pi_{TE}(u), \omega)>t\}$. For almost all ω, we have, at $u\in\tilde{\eta}_t(\mathcal{D}_t(\tilde{\eta}, \omega), \omega)$,

$$\begin{aligned}\tilde{Y}_\alpha(u)[G_{f(t)}] &= \frac{d}{ds}(G_{f(t)}\circ\varphi_s^{(\alpha)}(u))\Big|_{s=0} \\ &= \frac{d}{ds}\{(\boldsymbol{d}(f(t)))((\rho_s^{(\alpha)})_{*,\pi_{TE}(u)}u)\}\Big|_{s=0} \qquad (\text{by } \varphi_s^{(\alpha)}u=(\rho_s^{(\alpha)})_{*,\pi_{TE}(u)}u) \\ &= \frac{d}{ds}\{(\boldsymbol{d}(f(t)\circ\rho_s^{(\alpha)}))(u)\}\Big|_{s=0} \\ &= \boldsymbol{d}(\mathcal{L}_{Y_\alpha}(f(t)))(u) = u[Y_\alpha[f(t)]]\,.\end{aligned}$$

Hence

$$\begin{aligned}&(\tilde{\eta}_t^{-1})_{*,\zeta(\theta_t(x))}\{\zeta_{*,\theta_t(x)}(X_\alpha(\theta_t(x)))-\tilde{Y}_\alpha(\zeta(\theta_t(x)))\}[G_f] \\ &= X_\alpha(\theta_t(x))[\zeta[f(t)]]-\zeta(\theta_t(x))[Y_\alpha[f(t)]] \qquad (3.11)\end{aligned}$$

for $0\leqq t<\tau(x)$ and $x\in\mathrm{Dom}(\sigma)$. Then (3.10b) and (3.11) imply (3.10c). This completes the proof.

Let us consider the case where E is a C^∞ vector bundle. Then it makes sense to treat $\eta_t^{-1}(\sigma(\theta_t(x)))-\sigma(x)$. Noting that $(D_\alpha\sigma)(x)=(D(X_\alpha, Y_\alpha)\sigma)(x)$ is vertical (cf. Remark in §3.1) and using a canonical map $\iota: (VE)_{\sigma(x)}\to\pi_E^{-1}(x)$ which identifies the vertical tangent space $(VE)_{\sigma(x)}$ at $\sigma(x)$ with $\pi_E^{-1}(x)$ in the obvious way, we define a C^∞ (local) cross section $L(X_\alpha, Y_\alpha)\sigma : x\ (\in\mathrm{Dom}(\sigma))\to(L(X_\alpha, Y_\alpha)\sigma)(x)\in E$ by

$$(L(X_\alpha, Y_\alpha)\sigma)(x)=\iota((D(X_\alpha, Y_\alpha)\sigma)(x))\,.$$

Remark that

$$(L(X_\alpha, Y_\alpha)\sigma)(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{(\rho_\varepsilon^{(\alpha)})^{-1}\sigma(\theta_\varepsilon^{(\alpha)}(x))-\sigma(x)\}$$

and

$$(D(X_\alpha, Y_\alpha)\sigma)(x)[f]=\frac{d}{d\varepsilon}f(\sigma(x)+\varepsilon(L(X_\alpha, Y_\alpha)\sigma)(x))\Big|_{\varepsilon=0}$$

for every C^∞ function f on E. From Theorem 3.1, we obtain easily the following.

COROLLARY 3.2. *If E is a C^∞ vector bundle, then*

$$\begin{aligned}&\eta_t^{-1}(\sigma(\theta_t(x)))-\sigma(x) \\ &= \sum_\alpha\int_0^t\eta_s^{-1}((L(X_\alpha, Y_\alpha)\sigma)(\theta_s(x)))\circ dN^\alpha(s) \\ &= \sum_\alpha\int_0^t\eta_s^{-1}((L(X_\alpha, Y_\alpha)\sigma)(\theta_s(x)))\cdot dN^\alpha(s) \\ &\quad +\frac{1}{2}\sum_{\alpha,\beta}\int_0^t\eta_s^{-1}((L(X_\alpha, Y_\alpha)(L(X_\beta, Y_\beta)\sigma))(\theta_s(x)))dN^\alpha(s)dN^\beta(s)\end{aligned}$$

holds a. s. on the set $\{\omega\in\Omega ; \tau(x, \omega)>t\}$, $x\in\mathrm{Dom}(\sigma)$, $t\geqq 0$. (*If, moreover,* $\mathrm{Dom}(\sigma)=M$ *and* $\tau(x)=\infty$ *for all* x, *then the above formula holds for all* $(t, x)\in[0, \infty)\times M$, *a. s.*)

4. Case of geometric object fields and examples.

First, we take as E a bundle of geometric objects of type ρ of finite rank, associated with $L^r(M)$. Then we obtain a stochastic formula for geometric object fields from Theorem 3.1. If each A_α [resp. Y_α] is the natural lift of X_α to $L^r(M)$ [resp. E], then we have $D(X_\alpha, Y_\alpha)\sigma=\hat{L}_{X_\alpha}\sigma$. (Note that the natural lift of X_α to $L^r(M)$ is invariant by R_a for every $a\in G^r(n, \boldsymbol{R})$.) Moreover, $\eta_t^{-1}(\sigma(\theta_t(\cdot)))$ in Theorem 3.1 reduces to the (*stochastically*) *deformed geometric object field* (of σ). In this case, we write $\eta_t=(\theta_t)_\#$ and set

$$(\theta_t^\#\sigma)(x)=(\theta_t)_\#^{-1}(\sigma(\theta_t(x)))\,, \qquad x\in\mathrm{Dom}(\sigma)\,.$$

If the bundle E is a C^∞ vector bundle, we extend $\mathcal{L}_{X_\alpha}$ by setting $(\mathcal{L}_{X_\alpha}\sigma)(x)=\iota((\hat{L}_{X_\alpha}\sigma)(x))$, $(x\in\mathrm{Dom}(\sigma))$, for every geometric object field σ of E.

Next, we give examples of Theorem 3.1 and Corollary 3.2.

EXAMPLE 1. Consider the case $P=L(M)$ and $E=T_q^p(M)$, the tensor bundle of type (p, q) over M.

(1) If A_α is the natural lift of X_α to $L(M)$, then $L(X_\alpha, Y_\alpha)\sigma=\mathcal{L}_{X_\alpha}\sigma$ holds and $\theta_t^\#\sigma: x\to(\theta_t^\#\sigma)(x)$ becomes the pull-back of σ by θ_t (cf. Bismut [2], Ikeda-Watanabe [**6**], Kunita [**12**]).

(2) If M is endowed with a linear connection and if A_α is the horizontal lift of X_α to $L(M)$, then η_t reduces to Itô's stochastic parallel displacement [8] and $L(X_\alpha, Y_\alpha)\sigma=\nabla_{X_\alpha}\sigma$ (the covariant derivative of σ with respect to X_α). (Cf. Kunita [**12**].)

(3) More generally, let $\delta_1, \cdots, \delta_k$ be derivations ([**10**]) of the tensor algebra of M. Choosing X_α and A_α adequately, we can do as $L(X_\alpha, Y_\alpha)\sigma=\delta_\alpha\sigma$ and we get a formula involving tensor derivations $\delta_1, \cdots, \delta_k$. (See [**1**].)

EXAMPLE 2. If E is a C^∞ vector bundle over M and if a linear connection ∇^E is given in E, then, choosing Y_α to be the horizontal lift of X_α to E, we get a formula involving $\nabla^E_{X_\alpha}\sigma$ (the covariant derivative of a C^∞ (local) cross section σ of E with respect to X_α), since $L(X_\alpha, Y_\alpha)\sigma=\nabla^E_{X_\alpha}\sigma$ holds in this case.

EXAMPLE 3. We now take $L^2(M)$ as P. Following Ferraris et al. [**3**], [**4**], consider the affine representation

$$\rho\ :\ G^2(n, \boldsymbol{R})\longrightarrow \mathrm{Diff}[\boldsymbol{T}_2^1(\boldsymbol{R}^n)]$$

defined by

$$(g^i_j,\ g^i_{jm})(\Gamma^\xi_{\lambda\nu}) = \Big(\sum_{\kappa,\lambda,\nu=1}^{n} g^i_\kappa \Gamma^\kappa_{\lambda\nu}\tilde{g}^\lambda_j \tilde{g}^\nu_m + \sum_{\kappa=1}^{n} g^i_\kappa \tilde{g}^\kappa_{jm}\Big),$$

where (g^i_j, g^i_{jm}) and $\Gamma^\xi_{\lambda\nu}$ are canonical coordinates in $G^2(n, \boldsymbol{R})$ and $\boldsymbol{T}^1_2(\boldsymbol{R}^n) = \boldsymbol{R}^n \otimes (\boldsymbol{R}^n)^* \otimes (\boldsymbol{R}^n)^*$, respectively, and $(\tilde{g}^i_j, \tilde{g}^i_{jm}) = (g^i_j, g^i_{jm})^{-1}$. Here $\mathrm{Diff}[\boldsymbol{T}^1_2(\boldsymbol{R}^n)]$ is the diffeomorphisms of $\boldsymbol{T}^1_2(\boldsymbol{R}^n)$ and $\dim M = n$. Each C^∞ cross section Γ of the fiber bundle $\mathrm{Conn}(M) = L^2(M) \times_\rho \boldsymbol{T}^1_2(\boldsymbol{R}^n)$ defines a linear connection of M. This fiber bundle is called the *bundle of linear connections of* M (cf. [**3**], [**4**]). Let us take $\mathrm{Conn}(M)$ as E. If A_α [resp. Y_α] is the natural lift of X_α to $L^2(M)$ [resp. $\mathrm{Conn}(M)$], then $\eta_t^{-1} \circ \Gamma \circ \theta_t$ reduces to the *(stochastically) deformed linear connection* $\theta_t^\# \Gamma$ of Γ and moreover $D(X_\alpha, Y_\alpha)\Gamma = \hat{L}_{X_\alpha}\Gamma$.

5. Applications.

Let M be a connected n-dimensional manifold. Let θ_t be the solution of the SDE (3.2). In this section, we assume for simplicity that $\theta_t(\cdot, \omega)$ is a flow of diffeomorphisms of M a. s.

5.1. Stochastic flows of affine motions and conformal motions. We consider first the case where M is endowed with a linear connection.

DEFINITION 5.1. Given a linear connection Γ (cf. Example 3) of M, we say that the solution θ_t of (3.2) is a *stochastic flow of affine motions in* (M, Γ) if the (stochastically) deformed linear connection $\theta_t^\# \Gamma$ is equal to the original linear connection Γ (that is, $(\theta_t^\# \Gamma)(x) = \Gamma(x)$, $x \in M$ for any t, a. s.).

Next, consider the case where a Riemannian metric g is given on M. Define a vector bundle $Q(M)$ over M by

$$Q(M) = (T^*M \circledcirc T^*M) \otimes |\wedge|^{-2/n}(M),$$

where $T^*M \circledcirc T^*M$ denotes the symmetric tensor product of the cotangent bundle T^*M (over M) with itself, and $|\wedge|^{-2/n}(M)$ stands for the bundle of densities ([**5**]) of order $-2/n$. Obviously, $Q(M)$ is associated with $L(M)$. Let Φ be the C^∞ cross section of $Q(M)$ given in terms of a local coordinate system (x^i) by

$$\Phi = \sum_{i,j} |\det(g_{\kappa\lambda})|^{-1/n} g_{ij} (\boldsymbol{d}x^i \circledcirc \boldsymbol{d}x^j) \otimes |\boldsymbol{d}x^1 \wedge \cdots \wedge \boldsymbol{d}x^n|^{-2/n},$$

where $g = \sum_{i,j} g_{ij} \boldsymbol{d}x^i \circledcirc \boldsymbol{d}x^j$, locally.

DEFINITION 5.2. We say that the solution θ_t of (3.2) is a *stochastic flow of conformal motions in* (M, g) if $(\theta_t^\# \Phi)(x) = \Phi(x)$, $(x \in M)$, for any t, a. s. (See [**15**, pp. 32-33].)

We can now state a sufficient condition for the solution of the SDE (3.2) to be a stochastic flow of affine motions [resp. conformal motions] as follows.

THEOREM 5.1. *Let θ_t be the solution of* (3.2).

(1) *Suppose M is endowed with a linear connection Γ. If $X_1, \cdots, X_k$ are infinitesimal affine motions in (M, Γ), then θ_t is a stochastic flow of affine motions in (M, Γ).*

(2) *Suppose a Riemannian metric g is given on M. If $X_1, \cdots, X_k$ are infinitesimal conformal motions in (M, g), then θ_t is a stochastic flow of conformal motions in (M, g).*

PROOF. We first remark that X_α is an infinitesimal affine [resp. conformal] motion in (M, Γ) [resp. (M, g)] if and only if $\hat{L}_{X_\alpha}\Gamma=0$ [resp. $\mathcal{L}_{X_\alpha}\Phi=0$] (cf. [**15**]). Then part (1) is an obvious consequence of Example 3. Part (2) follows directly from the discussion in §4 if we take $L(M)$, $Q(M)$ and Φ as P, E and σ, respectively.

REMARK 5.1. Let $P=L(M)$ and let γ be the connection form corresponding to Γ. Take A_α to be the natural lift of X_α to $L(M)$ and consider (3.1). We can also prove (1) of Theorem 5.1 with the use of γ, since the condition $\theta_t^\#\Gamma=\Gamma$ in Definition 5.1 can be replaced by saying that $\varphi_t^*\gamma=\gamma$ (cf. [**9**], [**10**]), where $\varphi_t^*\gamma$ is the pull-back of γ by φ_t.

REMARK 5.2. Take $(\Omega, \mathcal{F}, \mu; (\mathcal{F}_t)_{t\geq 0})$ to be the standard $(k-1)$-dimensional Wiener space in the sense of [7]. Let $w(t)=(w^1(t), \cdots, w^{k-1}(t))$ be the canonical realization of Brownian motion on the probability space. Consider the case $(N^1(t), \cdots, N^k(t))=(w^1(t), \cdots, w^{k-1}(t), t)$.

(a) Assume for simplicity that the manifold M is compact. Theorem 5.1 can also be proved by using the approximation theorem (or the support theorem) for stochastic flows (cf. [**6**, Chapter VI, Theorem 7.3], [**7**]). If each X_α is an infinitesimal affine [resp. conformal] motion, then by a polygonal approximation it is shown that the flow θ_t is the almost sure limit of a sequence of affine [resp. conformal] motions. From this, θ_t becomes an affine [resp. a conformal] motion.

(b) The results in Theorem 5.1 can be strengthened to "if and only if". We shall show the "only if" part for (2) of Theorem 5.1, although the proof is analogous to that of [**2**, p. 160, Théorème 1.2], except that Φ is not a tensor field in the ordinary sense. Take a Riemannian fiber metric Ξ in $Q(M)$. Let $x\in M$ be arbitrarily fixed. By Corollary 3.2, the martingale part $J_t(x)=(J_t(x, w))$ of $(\theta_t^\#\Phi)(x)-\Phi(x)$ is given by

$$J_t(x, w) = \sum_{\nu=1}^{k-1}\int_0^t(\theta_s^\#(\mathcal{L}_{X_\nu}\Phi))(x)\cdot dw^\nu(s)\,.$$

If $(\theta_t^\#\Phi)(x)=\Phi(x)$, then $J_t(x)=0$, and thus for each $\alpha=1, \cdots, k-1$,

$$C_t^{(\alpha)}(x, w) := \Xi\Big(\int_0^t (\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x)\cdot dw^\alpha(s)\,,\ J_t(x, w)\Big) = 0\,.$$

Therefore, the bounded variation part of $C_t^{(\alpha)}(x)=(C_t^{(\alpha)}(x, w))$ is

$$\int_0^t \Xi((\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x),\ (\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x))ds = 0\,.$$

Since $\Xi((\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x),\ (\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x))$ is continuous in s, we can conclude that $(\theta_s^\#(\mathcal{L}_{X_\alpha}\Phi))(x)=0$. Hence $\mathcal{L}_{X_\alpha}\Phi=0$, $(\alpha=1, \cdots, k-1)$. Now Corollary 3.2 yields

$$\int_0^t (\theta_s^\#(\mathcal{L}_{X_k}\Phi))(x)ds = 0\,,$$

from which we obtain $\mathcal{L}_{X_k}\Phi=0$.

As for (1), use the connection form γ (Remark 5.1), with a suitable modification of the above discussion. (The "only if" part also follows from the support theorem.)

5.2. Stochastic deformation of a projective structure. We now apply Theorem 3.1 to the study of the behavior of a stochastically deformed projective structure. Let G_0 be the following closed subgroup of the group $SL(n+1, \boldsymbol{R})$/center;

$$G_0 = \left\{\begin{pmatrix} A & 0 \\ v & c \end{pmatrix} \in SL(n+1, \boldsymbol{R})\right\}\Big/\text{center}\,,$$

where $A\in GL(n, \boldsymbol{R})$ and v is a row n-vector ([**9**, p. 132]). By definition, a projective structure on M is a principal subbundle of $L^2(M)$ with structure group G_0 $(\subset G^2(n, \boldsymbol{R}))$ ([**9**, p. 142]). A basic fact is that the cross sections $M\to L^2(M)/G_0$ are in one-to-one correspondence with the projective structures on M ([**9**, p. 147, Proposition 7.1]). Since the quotient bundle $L^2(M)/G_0$ is the fiber bundle with standard fiber $G^2(n, \boldsymbol{R})/G_0$ associated with $L^2(M)$, each projective structure on M can be regarded as a geometric object field.

Now suppose we are given a projective structure on M and let $\sigma: M\to L^2(M)/G_0$ be the cross section corresponding to the projective structure. Then we obtain $\theta_t^\#\sigma$ (cf. §4), which defines the *stochastic deformation of the projective structure* (*corresponding to* σ) *by* θ_t^{-1}. From Theorem 3.1, the behavior of the deformed projective structure is immediately given by the equation

$$\begin{aligned} df((\theta_t^\#\sigma)(x)) = &\sum_\alpha ((\theta_t)_\#)_{*,\sigma(\theta_t(x))}^{-1}((\hat{L}_{X_\alpha}\sigma)(\theta_t(x)))[f]\cdot dN^\alpha(t) \\ &+\frac{1}{2}\sum_{\alpha,\beta}\{X_\alpha(\theta_t(x))[(\hat{L}_{X_\beta}\sigma)[f\circ(\theta_t)_\#^{-1}]] \\ &\qquad -(\hat{L}_{X_\beta}\sigma)(\theta_t(x))[X_\alpha^N[f\circ(\theta_t)_\#^{-1}]]\}dN^\alpha(t)dN^\beta(t) \end{aligned}$$

for every C^∞ function f on $L^2(M)/G_0$, where X_α^N denotes the natural lift of X_α

to $L^2(M)/G_0$.

REMARK 5.3. For a closed subgroup G of $G^r(n, \boldsymbol{R})$, the *stochastic deformation of a G-structure of degree r* ([**9**, p. 37]) on M by θ_t^{-1} can be discussed in a similar manner.

References

[1] H. Akiyama, Differential geometric approach to stochastic differential equations on vector bundles, I, Math. Japon., **26** (1981), 421-432.
[2] J.-M. Bismut, Mécanique Aléatoire, Lecture Notes in Math., **866**, Springer, 1981.
[3] M. Ferraris, M. Francaviglia and C. Reina, A constructive approach to bundles of geometric objects on a differentiable manifold, J. Math. Phys., **24** (1983), 120-124.
[4] M. Ferraris, M. Francaviglia and C. Reina, Sur les fibrés d'objets géométriques et leurs applications physiques, Ann. Inst. Henri Poincaré Sect. A Phys. Théor., **38** (1983), 371-383.
[5] V. Guillemin and S. Sternberg, Geometric Asymptotics, Math. Surveys No. 14, Amer. Math. Soc., Providence, R. I., 1977.
[6] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, Kodansha/North-Holland, Tokyo/Amsterdam, 1981.
[7] N. Ikeda and S. Watanabe, Stochastic flows of diffeomorphisms, Stochastic Analysis and Applications (Ed. by M. A. Pinsky), Adv. Probab. Related Topics, **7**, Marcel Dekker, New York, 1984, pp. 179-198.
[8] K. Itô, Stochastic parallel displacement, Probabilistic Methods in Differential Equations, Lecture Notes in Math., **451**, Springer, 1975, pp. 1-7.
[9] S. Kobayashi, Transformation Groups in Differential Geometry, Springer, Berlin, 1972.
[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, I, Interscience, New York, 1963.
[11] H. Kunita, Some extensions of Itô's formula, Séminaire des Probabilités, XV (ed. by J. Azéma and M. Yor), Lecture Notes in Math., **850**, Springer, 1981, pp. 118-141.
[12] H. Kunita, On the decomposition of solutions of stochastic differential equations, Stochastic Integrals, Proc. LMS Durham Symp. 1980 (ed. by D. Williams), Lecture Notes in Math., **851**, Springer, 1981, pp. 213-255.
[13] A. Nijenhuis, Geometric aspects of formal differential operators on tensor fields, Proc. Internat. Congr. Math., 1958, Cambridge Univ. Press, Cambridge, 1960, pp. 463-469.
[14] S. Salvioli, On the theory of geometric objects, J. Diff. Geom., **7** (1972), 257-278.
[15] K. Yano, The Theory of Lie Derivatives and Its Applications, North-Holland, Amsterdam, 1955.

Hiroshi AKIYAMA
Department of
Applied Mathematics and Physics
Faculty of Engineering
Kyoto University
Kyoto 606, Japan