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Gauss-Manin connection of integral of difference products

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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0. Let $x_1, \cdots, x_p$ be real distinct numbers. As a function of $x_1, \cdots, x_p$ the integral

$$F(x_1, \cdots, x_p)=\int \prod_{1\leq i<j\leq N}(x_i-x_j)^{\lambda_{i,j}}dx_{p+1}\wedge\cdots\wedge dx_N \tag{0.1}$$

for $2\leqq p\leqq N$, over a suitable cycle satisfies an integrable analytic differential system (called Gauss-Manin connection in analytic geometry or holonomic system from micro-local point of view). *In this note we want to give an explicit formula of it.* In the sequel we denote by Φ the product $\prod_{1\leq i<j\leq N}(x_i-x_j)^{\lambda_{i,j}}$.

Roughly speaking, our method is as follows. The structure of the integral (0.1) is of fibre type. This enables us to give a recurrent relation for integration over each variable $x_{p+1}, \cdots, x_N$ in the reverse order. Namely we first integrate (0.1) over x_N. Then we get the function of $x_1, \cdots, x_{N-1}$ satisfying a certain Gauss-Manin connection of classical Jordan-Pochhammer type. Next we do it over x_{N-1} and get a differential equation of similar nature and so on. Finally $F(x_1, \cdots, x_p)$ satisfies a Gauss-Manin connection which can be computed in inductive way.

We assume from now on that $x_1<x_2<\cdots<x_p$ and that the point $(x_{p+1}, \cdots, x_N)$ lies in $\boldsymbol{R}^{N-p}$. We denote by $\varDelta$ the closure of any of relatively compact components of the open set: $x_{p+\nu}\neq x_j$, $1\leqq j\leqq p$ and $x_{p+\mu}\neq x_{p+\nu}$ for $\mu\neq\nu$ in $\boldsymbol{R}^{N-p}$. If $\lambda_{i,j}$ are all positive, the integral over each domain $\varDelta$ has a definite meaning. If some of $\lambda_{i,j}$ are negative we have to replace $\varDelta$ by its regularized cycle $\varDelta^{\mathrm{reg}}$ (which is called "renormalized" by physicists and which is essentially the same as "finite part of divergent integrals" in the sense of J. Hadamard), such that $\int_{\varDelta^{\mathrm{reg}}}$ is an analytic continuation of the original $\int_{\varDelta}$ considered as function of the variables $\lambda=(\lambda_{i,j})_{i<j}$ (For the way of construction, see [**A2**] or [**T**] pp. 314-318). The regularized cycle $\varDelta^{\mathrm{reg}}$ defines a twisted homological $(N-p)$-cycle in the affine algebraic variety $X=\boldsymbol{C}^{N-p}-\bigcup(x_i=x_j)$ where $1\leqq i\leqq N$,

$p+1\leqq j\leqq N$ and $i<j$, with coefficients in the dual $\mathcal{S}_\lambda^*$ of the local system $\mathcal{S}_\lambda$ defined by the monodromy group of the function $\Phi: \Delta^{\mathrm{reg}}\in H_{N-p}(X, \mathcal{S}_\lambda^*)$. The integral defines the canonical pairing between the twisted homology and twisted de Rham cohomology:

$$\text{(0.2)} \qquad \begin{aligned} H_{N-p}(X, \mathcal{S}_\lambda^*)\times H^{N-p}(X, \nabla_\omega) &\longrightarrow \boldsymbol{C} \\ (\Delta, \varphi) &\longmapsto \int_\Delta \Phi\varphi \end{aligned}$$

where φ denotes a rational differential form on $\boldsymbol{C}^{N-p}$ with poles only on $\bigcup_{\substack{1\leq i\leq p,\\ p+1\leq j\leq N}} (x_i=x_j) \bigcup_{p+1\leq j<k\leq N} (x_j=x_k)$. The covariant differentiation ∇_ω is defined by $\nabla_\omega\varphi=d\varphi+\omega\wedge\varphi$ for the logarithmic form $\omega=d\log\Phi$.

We assume the following condition:

(C.1) i) Take an arbitrary r such that $p+1\leqq r\leqq N$ and fix it. For any $h\leqq r-1$, the sum $\sum_{j=1}^{s}\lambda_{h,\nu_j}+\sum_{1\leq j<k\leq s}\lambda_{p+\nu_j,\, p+\nu_k}$ is different from 1, 2, 3, $\cdots$ where $p+\nu_1$, $\cdots$, $p+\nu_s$ denotes an arbitrary sequence such that $r\leqq p+\nu_1<\cdots<p+\nu_s\leqq N$.
ii) Under the same circumstance as above, the sum, $-\sum_{h=1}^{r-1}\sum_{j=1}^{s}\lambda_{h,\, p+\nu_j}-\sum_{1\leq j<k\leq s}\lambda_{p+\nu_j,\, p+\nu_k}$ is different from 1, 2, 3, $\cdots$.

Then

PROPOSITION 1. *The twisted de Rham cohomology $H^{N-p}(X, \nabla_\omega)$ is spanned by the logarithmic forms*

$$\langle i_{p+1}, \cdots, i_N\rangle = d\log(p+1, i_{p+1})\wedge\cdots\wedge d\log(N, i_N)$$

for $i_{p+1}\leqq p, \cdots, i_N\leqq N-1$, where we denote by (i, j) the difference x_i-x_j.

The number of the above forms is equal to $p(p+1)\cdots(N-1)$. There are among these forms, $(N-p)p(p+1)\cdots(N-2)$ fundamental relations which will be given in § 2. In other words, $H^{N-p}(X, \nabla_\omega)$ is isomorphic to the quotient of the tensor product $\boldsymbol{C}^p\otimes\cdots\otimes\boldsymbol{C}^N/\sim$, by identifying each ν-th component $d\log(p+\nu, i_{p+\nu})$ of $\langle i_{p+1}, \cdots, i_N\rangle$ with an element of $\boldsymbol{C}^{p+\nu-1}$.

THEOREM 1. *In addition to* (C.1) *we assume the following condition:*

(C.2) *For each $j\geqq p+1$, there exists at least one non-zero $\lambda_{i,j}$.*

Then the rank of $H^{N-p}(X, \nabla_\omega)$ is equal to $(p-1)p\cdots(N-2)$ which will be denoted by μ. A basis of $H^{N-p}(X, \nabla_\omega)$ can be chosen as

$$d\log(p+1, i_{p+1})\wedge\cdots\wedge d\log(N, i_N) \qquad \textit{for } \nu-1\geqq i_\nu>1.$$

Remark that the condition (C.2) is implied by the stronger one:

(C.2)′ Under the same circumstance as (C.1), the sum $\sum_{j=1}^{s}\lambda_{h,\, p+\nu_j}+\sum_{1\leq j<k\leq s}\lambda_{p+\nu_j,\, p+\nu_k}$ is different from 0.

We denote by $\tilde{\varphi}$ the integral $\int \Phi\varphi$ for an arbitrary $(N-p)$-form φ. As a basis of the twisted homology $H_{N-p}(X, \mathcal{S}_\lambda^*)$, can be chosen the regularized cycles Δ^{reg} of each relative one Δ, the closures of relatively compact connected domains of the real part of X. It is easily seen that the number of Δ^{reg} is equal to $(p-1)p\cdots(N-2)$. (See (1.16), or [**A5**] and its references.) Then we have the following:

THEOREM 2. *The integrals $\widetilde{\langle i_{p+1}, \cdots, i_N\rangle}$ over each regularized cycle, as functions of $x_1, \cdots, x_p$, satisfy the following Gauss-Manin connection:*

$$(0.3)\quad d\widetilde{\langle i_{p+1}, \cdots, i_N\rangle}$$
$$= \sum_{s=1}^{N-p} \sum_{0<\nu_1<\cdots<\nu_s} d\log(i_{p+\nu_1}, i'_{p+\nu_1})\lambda_{p+\nu_s, i'_{p+\nu_s}} \widetilde{\left\langle i_{p+1}, \cdots, \begin{Bmatrix} i_{p+\nu_1} \\ i'_{p+\nu_1} \end{Bmatrix}, \cdots, \begin{Bmatrix} i_{p+\nu_s} \\ i'_{p+\nu_s} \end{Bmatrix}, \cdots, i_N \right\rangle}$$
$$+ \sum_{1\le j<k\le p} \lambda_{j,k} d\log(j, k)\widetilde{\langle i_{p+1}, \cdots, i_N\rangle},$$

where the symbol $\widetilde{\left\langle \cdots \begin{Bmatrix} i \\ i' \end{Bmatrix}, \cdots, \{\ \}, \cdots \right\rangle}$ *is defined inductively by the difference*

$$\widetilde{\langle \cdots, i, \cdots, \{\ \}, \cdots\rangle} - \widetilde{\langle \cdots, i', \cdots, \{\ \}, \cdots\rangle}.$$

The sequence $p+\nu_1<p+\nu_2<\cdots<p+\nu_s$ and the indices $i'_{p+\nu_1}, \cdots, i'_{p+\nu_s}$ are determined as follows: First take an arbitrary pair (α, β) such that $\alpha \neq \beta \leqq p$. Let $p+\nu_1$ be the smallest number such that $i_{p+\nu_1}$ is equal to α or β. Then we put $\alpha = i_{p+\nu_1}$ and $\beta = i'_{p+\nu_1}$. Next let $p+\nu_2>p+\nu_1$ be the smallest number such that $i_{p+\nu_2} \in (\alpha, \beta, p+\nu_1) - \{i_{p+\nu_1}\}$. Then we take as $i'_{p+\nu_2}$ the unique index from the set $\{\alpha, \beta, p+\nu_1\} - \{i_{p+\nu_1}, i_{p+\nu_2}\}$ and so on. Namely we take out $p+\nu_1, p+\nu_2, \cdots$ and $i'_{p+\nu_1}, i'_{p+\nu_2}, \cdots$ by the following procedure:

$$(0.4)\quad \begin{aligned} &\{\alpha, \beta, p+\nu_1, \cdots, p+\nu_{t-1}\} - \{i_{p+\nu_1}, \cdots, i_{p+\nu_{t-1}}\} \ni i_{p+\nu_t} \\ &\{\alpha, \beta, p+\nu_1, \cdots, p+\nu_{t-1}\} - \{i_{p+\nu_1}, \cdots, i_{p+\nu_t}\} \ni i'_{p+\nu_t}. \end{aligned}$$

We finish this process if there does not exist an $i_{p+\nu}$, $\nu>\nu_s$ such that $i_{p+\nu} \in (\alpha, \beta, p+\nu_1, \cdots, p+\nu_s)$. $i_{p+\nu_1}, \cdots, i_{p+\nu_s}$ (or more precisely $i_{p+\nu_1}, \cdots, i_{p+\nu_s}, i'_{p+\nu_s}$) are all different from each other. The sequence $p+\nu_1, \cdots, p+\nu_s$ makes a $(p+\nu_1)$-segment in a cluster attached to the sequence $(i_{p+1}, \cdots, i_N)$. (The definition of "cluster" will be given in § 2.)

The indices $i_{p+\nu_1}, \cdots, i_{p+\nu_s}, i'_{p+\nu_1}, \cdots, i'_{p+\nu_s}$ are also determined by the following rule: For arbitrary $\mu=\nu_s$, $\mu \geqq 1$ and $\gamma < p+\mu$ different from $i_{p+\nu_s}$, first we choose $i'_{p+\nu_s}$ as γ. Then define successively $p+\nu_t$, $i_{p+\nu_t}$, $i'_{p+\nu_t}$ in decreasing order for $t=s, s-1, \cdots$ as follows:

$$\text{(0.4)}' \qquad \begin{aligned} \max(i_{p+\nu_t}, i'_{p+\nu_t}) &= p+\nu_{t-1} \\ \min(i_{p+\nu_t}, i'_{p+\nu_t}) &= i'_{p+\nu_{t-1}}. \end{aligned}$$

Finally we arrive at the pair $(i_{p+\nu_1}, i'_{p+\nu_1})$ such that $i_{p+\nu_1}, i'_{p+\nu_1} \leqq p$.

REMARK. We see that $F(x_1, \cdots, x_p)$ of (0.1) coincides with $\langle 1, \widetilde{\cdots}, 1\rangle$ if $\lambda_{1,p+1}, \cdots, \lambda_{1,N}$ are replaced by $\lambda_{1,p+1}+1, \cdots, \lambda_{1,N}+1$ respectively. Hence F together with the other $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle$ satisfy the Gauss-Manin connection (0.3).

We now restrict ourselves to the symmetric case of the integral (0.1). Namely we assume the following condition:

$$\text{(C.3)} \qquad \lambda_{i,j}=0 \text{ for } i, j \leqq p, \quad \lambda_{i,j}=\lambda'_i \text{ for } i \leqq p,\ j \geqq p+1 \\ \text{and} \quad \lambda_{i,j}=\lambda \text{ for } i, j \geqq p+1.$$

Then Φ is invariant under the action of the symmetric group of $(N-p)$-degree $\Gamma = \mathfrak{S}_{N-p}$, provided the branch of Φ at each point of X is suitably chosen: $\sigma^*\Phi = \Phi$.

Integrands $\Phi\varphi$ and domains of integration Δ and therefore the cohomology $H^{N-p}(X, \nabla_\omega)$ and the homology $H_{N-p}(X, \mathcal{S}_\lambda^*)$ also admit of the action of Γ. If a domain of integration G is invariant in homological sense, then (0.1) is invariant under the action of Γ. We have then

$$\text{(0.5)} \qquad \begin{aligned} \int_G \Phi\langle i_{p+1}, \cdots, i_N\rangle &= \int_G \Phi\sigma^*\langle i_{p+1}, \cdots, i_N\rangle \\ &= \frac{1}{(N-p)!}\sum_{\sigma\in\Gamma}\int_G \Phi\sigma^*\langle i_{p+1}, \cdots, i_N\rangle \end{aligned}$$

where $\sigma^*\langle i_{p+1}, \cdots, i_N\rangle$ denotes the transformed $(N-p)$-form $\langle i_{\sigma(p+1)}, \cdots, i_{\sigma(N)}\rangle$ by σ. This fact makes the structure of the Gauss-Manin system of (0.1) much simpler.

PROPOSITION 2. *The invariant part* $[H^{N-p}(X, \nabla_\omega)]^\Gamma$ *is spanned by the symmetrized logarithmic forms*

$$\frac{1}{(N-p)!}\sum_{\sigma\in\Gamma}\sigma^*\langle i_{p+1}, \cdots, i_N\rangle \qquad \textit{for all } i_{p+\nu} \leqq p.$$

Let ν_j, $1 \leqq j \leqq p$ be the number of arguments i_t such that $i_t = j$. Then a symmetrized logarithmic form corresponds one-to-one to the sequence $\nu_1, \cdots, \nu_p$ such that $\sum_{j=1}^p \nu_j = N-p$. If we denote it by $\langle 1^{\nu_1}, 2^{\nu_2}, \cdots, p^{\nu_p}\rangle$, the integral $\langle 1^{\nu_1}, \widetilde{\cdots}, p^{\nu_p}\rangle$ is equal to $\langle \underbrace{1, \cdots, 1}_{\nu_1}, \underbrace{\widetilde{2, \cdots, 2}}_{\nu_2}, \cdots, \underbrace{p, \cdots, p}_{\nu_p}\rangle$. The fundamental relations among these are then simplified as follows:

$$\sum_{j=1}^{p}\left(\frac{\lambda}{2}\nu_j+\lambda'_j\right)\langle 1^{\nu_1}, \widetilde{\cdots, j^{\nu_j+1}}, \cdots, p^{\nu_p}\rangle = 0. \tag{0.6}$$

Hence we have

COROLLARY. *If* $(\lambda/2)(N-p)+\lambda'_1\neq 0$, *then* $[H^{N-p}(X, \nabla_\omega)]^\Gamma$ *is spanned by* $(p-1)p\cdots(N-2)/(N-p)!$ *linearly independent forms* $\langle 2^{\nu_2}, \cdots, p^{\nu_p}\rangle$ *for* $\nu_2+\cdots+\nu_p=|\nu|=N-p$.

THEOREM 3. *The integrals* $\langle 2^{\nu_2}, \widetilde{\cdots}, p^{\nu_p}\rangle$, *as functions of* $x_1, \cdots, x_p$, *have the fundamental relations* (0.6) *and satisfy the Gauss-Manin system*

$$\begin{aligned} d\langle 1^{\nu_1}, \widetilde{\cdots, p^{\nu_p}}\rangle = \sum_{1\leq i<j\leq p} d\log(i, j)\Big[\lambda\nu_i\nu_j\Big(&\langle 1^{\nu_1}, \widetilde{\cdots, p^{\nu_p}}\rangle \\ -\frac{1}{2}\langle 1^{\nu_1}, \cdots, i^{\nu_i-1}, \widetilde{\cdots}, j^{\nu_j+1}, \cdots, p^{\nu_p}\rangle-&\frac{1}{2}\langle 1^{\nu_1}, \cdots, i^{\nu_i+1}, \widetilde{\cdots}, j^{\nu_j-1}, \cdots, p^{\nu_p}\rangle\Big) \\ +\lambda'_i\nu_j(\langle 1^{\nu_1}, 2^{\nu_2}, \widetilde{\cdots}, p^{\nu_p}\rangle-&\langle 1^{\nu_1}, \cdots, i^{\nu_i+1}, \widetilde{\cdots}, j^{\nu_j-1}, \cdots, p^{\nu_p}\rangle) \\ +\lambda'_j\nu_i(\langle 1^{\nu_1}, 2^{\nu_2}, \widetilde{\cdots}, p^{\nu_p}\rangle-&\langle 1^{\nu_1}, \cdots, i^{\nu_i-1}, \widetilde{\cdots}, j^{\nu_j+1}, \cdots, p^{\nu_p}\rangle)\Big]. \end{aligned} \tag{0.7}$$

More generally we denote by $\langle 1^{\nu_1}, \widetilde{\cdots}, p^{\nu_p}\rangle$ for $\nu=\nu_1+\cdots+\nu_p\leqq N-p$, the integral

$$\int_G \Phi\frac{dx_{p+1}\wedge\cdots\wedge dx_N}{\prod_{1\leq j\leq\nu_1}(p+j, 1)\prod_{1\leq j\leq\nu_2}(p+\nu_1+j, 2)\cdots\prod_{1\leq j\leq\nu_p}(p+\nu_1+\cdots+\nu_{p-1}+j, p)}. \tag{0.8}$$

Then we have the recurrence formula:

PROPOSITION 3. *For* $\nu_1\geqq 1$,

$$\begin{aligned} 0 = \Big[1+\lambda'_1+\cdots+\lambda'_p+\frac{\lambda}{2}(N-p-|\nu|)+\lambda(|\nu|-1)\Big]&\langle 1^{\nu_1-1}, \widetilde{2^{\nu_2}, \cdots}, p^{\nu_p}\rangle \\ +\sum_{j=2}^{p}(j, 1)\Big(\lambda'_j+\frac{1}{2}\lambda\nu_j\Big)&\langle 1^{\nu_1-1}, \widetilde{\cdots, j^{\nu_j+1}}, \cdots, p^{\nu_p}\rangle. \end{aligned} \tag{0.9}$$

Successive applications of this proposition give us

THEOREM 4. *For* $\nu_1=0$, $|\nu|\leqq N-p$,

$$\begin{aligned} &\langle 2^{\nu_2}, \widetilde{\cdots}, p^{\nu_p}\rangle(-1)^{N-p-|\nu|-1}\prod_{t=1}^{N-p-|\nu|}\Big\{1+\lambda'_1+\cdots+\lambda'_p+\frac{\lambda}{2}t+\lambda(N-p-t-1)\Big\} \\ &= \sum_{N-p-|\nu|=\rho_2+\cdots+\rho_p}\frac{(N-p-|\nu|)!}{\rho_2!\cdots\rho_p!}(2, 1)^{\rho_2}\cdots(p, 1)^{\rho_p} \\ &\qquad\times\prod_{j=2}^{p}\prod_{t=0}^{\rho_j-2}\Big[\lambda'_j+\frac{1}{2}\lambda(\nu_j+t)\Big]\langle 2^{\nu_2+\rho_2}, \widetilde{\cdots}, p^{\nu_p+\rho_p}\rangle. \end{aligned} \tag{0.10}$$

In particular

$$\int_G \Phi\, dx_{p+1}\wedge\cdots\wedge dx_N(-1)^{N-p-1}\prod_{t=1}^{N-p}\left\{1+\sum_{j=1}^{p}\lambda'_j+\lambda\left(N-p-\frac{t}{2}-1\right)\right\} \tag{0.11}$$
$$=\sum_{N-p=\rho_2+\cdots+\rho_p}\frac{(N-p)!}{\rho_2!\cdots\rho_p!}\prod_{j=2}^{p}\prod_{t=0}^{\rho_j-1}\left[\lambda'_j+\frac{1}{2}\lambda(\nu_j+t)\right]$$
$$\times(2,1)^{\rho_2}\cdots(p,1)^{\rho_p}\langle\widetilde{2^{\rho_2},\cdots,p^{\rho_p}}\rangle.$$

By using this formula, we can derive explicitly the maximally overdetermined linear difference system with respect to the variables $\lambda'_1,\cdots,\lambda'_p$ for (0.1).

Proofs of Theorems 2~4 will be given in §3.

REMARK. When $p=2$, the integral (0.11) is expressed by

$$\int_G \prod_{j=3}^{N}(x_j-x_1)^{\lambda'_1}(x_j-x_2)^{\lambda'_2}\prod_{3\leq i<j\leq N}(x_i-x_j)^{\lambda}dx_3\wedge\cdots\wedge dx_N. \tag{0.12}$$

This is a constant multiple of $(x_2-x_1)^M$ for $M=(\lambda'_1+\lambda'_2+1)(N-2)+(N-2)(N-3)\lambda/2$. This constant is given by the celebrated formula due to A. Selberg (see [**A6**] and [**S**].)

When $\lambda=2$, the integral (0.11) is intimately related to orthogonal polynomials with the density $\prod_{j=1}^{p}(x-x_j)^{\lambda'_j}dx$. Professor M. Jimbo at R.I.M.S. has informed me that it satisfies a 2nd order non-linear differential equation of Painlevé (see [**J**] and [**O**]). It seems to be interesting to ask if it still satisfies a finite-order non-linear differential equation of similar type for general λ. It also seems to be interesting to study (0.1) further in case where $\lambda_{i,j}$ have special values, especially rational numbers, in view of recent results by A. Tsuchiya and Y. Kanie about Fock representation of the Virasoro algebra (see [**T**]).

Finally a few questions are posed about the integral (0.1):

(Q1) To evaluate the Wronskian of (0.1). Namely let the basis $\{\varphi_1,\cdots,\varphi_\mu\}$ of $H^{N-p}(X,\nabla_\omega)$ be as in Theorem 1 and $\{\gamma_1,\cdots,\gamma_\mu\}$ be a basis of $H_{N-p}(X,\mathcal{S}^*_\lambda)$, for example, as in (1.16). Then the Wronskian of (0.1) is simply defined by the determinant of $\left(\left(\int_{\gamma_k}\Phi\varphi_j\right)\right)_{1\leq j,k\leq\mu}$. It is obvious that this value coincides with the Wronskian of the differential system (0.3) apart from a Γ-factor depending only on $\lambda_{i,j}$ $(1\leqq i<j\leqq N)$.

(Q2) Under the condition (C.3) we have only considered the fixed part of the Γ-action. The question is generally to decompose (0.3) into irreducible parts as Γ-modules.

1. Recurrent system of ordinary differential equations of Fuchsian type.

DEFINITION. Let $\{i_{p+1}, \cdots, i_N\}$ be a sequence of $(N-p)$ arguments such that $i_\nu \leqq \nu-1$. We shall call such a sequence and the corresponding form $\langle i_{p+1}, \cdots, i_N\rangle$ defined in Proposition 1 "admissible" in the sequel. We say that for α, β such that $p+1\leqq\alpha<\beta$, "β precedes α" and write $\alpha\prec\beta$ if there exists a sequence $p+\nu_1<\cdots<p+\nu_s$ such that $\alpha=p+\nu_1$, $\beta=p+\nu_s$ and $i_{p+\nu_t}=p+\nu_{t-1}$ for $2\leqq t\leqq s$, and $2\leqq s\leqq N-p$. Further for an arbitrary α such that $p+1\leqq\alpha\leqq N$, we denote by K_α and call "α-cluster" (similar terminology like the one used in statistical mechanics) the set of all $\beta\in\{p+1, \cdots, N\}$ preceding α or equal to α: $K_\alpha=\{\beta ; \alpha\preceq\beta\}$. We denote by $|K_\alpha|$ the number of elements in K_α. If $i_\alpha\leqq p$, then we call the α-cluster K_α "maximal".

The following lemma is an immediate consequence of Definition.

LEMMA 1.1. *For each admissible sequence $\{i_{p+1}, \cdots, i_N\}$, the set of arguments $\{p+1, \cdots, N\}$ is divided into several maximal clusters. This correspondence is bijective.*

Each cluster K defines a directed tree. For arbitrary $\alpha, \beta\in K$ such that $\alpha\prec\beta$ there exists the unique maximal sequence $p+\nu_1, \cdots, p+\nu_s$ satisfying the following two properties: 1) $\alpha=p+\nu_1$, $\beta=p+\nu_s$ and 2) $p+\nu_t$ precedes $p+\nu_{t-1}$ for $1<t\leqq s$. In such a case this sequence is called "segment in K": In particular if there exists no element preceding β, this segment is called "α-segment".

LEMMA 1.2. *For $(N-p-1)$ fixed arguments $i_{p+1}, \cdots, i_{r-1}, i_{r+1}, \cdots, i_N$ for $i_\nu\leqq\nu-1$ and $p+1\leqq r\leqq N$, we have the cohomological identity*

$$(1.1)\quad \sum_{j=1}^{r-1}\lambda_{r,j}\langle i_{p+1}, \cdots, i_{r-1}, j, i_{r+1}, \cdots, i_N\rangle$$
$$+\sum_{s=1}^{N-r}\sum_{r-p<\nu_1<\cdots<\nu_s}\lambda_{p+\nu_s, i'_{p+\nu_s}}\left\langle i_{p+1}, \cdots, i_{r-1}, i'_r, \cdots, \begin{Bmatrix} i_{\nu_1+p} \\ i'_{\nu_1+p}\end{Bmatrix}, \cdots, \begin{Bmatrix} i_{\nu_s+p} \\ i'_{\nu_s+p}\end{Bmatrix}, \cdots, i_N\right\rangle \sim 0$$

where $\nu_1, \cdots, \nu_s$ run over all the sequences satisfying the following properties: For an arbitrary number $j=i'_r\leqq r-1$, we choose inductively $p+\nu_1, \cdots, p+\nu_s$, $i'_{p+\nu_1}, \cdots, i'_{p+\nu_s}$ in such a way that

$$(1.2)\quad \begin{aligned} & i_{p+\nu_1} \in \{r, j\} \\ & i'_{p+\nu_1} \in \{r, j\}-\{i_{p+\nu_1}\} \\ & \cdots\cdots\cdots \\ & i_{p+\nu_t} \in \{j, r, p+\nu_1, \cdots, p+\nu_{t-1}\}-\{i_{p+\nu_1}, \cdots, i_{p+\nu_{t-1}}\} \\ & i'_{p+\nu_t} \in \{j, r, p+\nu_1, \cdots, p+\nu_{t-1}\}-\{i_{p+\nu_1}, \cdots, i_{p+\nu_t}\} \end{aligned}$$

for $t\leqq s$, until there is no more $i_{p+\nu}$, $\nu>\nu_s$ such that $i_{p+\nu}\in\{j, r, p+\nu_1, \cdots, p+\nu_s\}$.

PROOF. By Stokes formula

$$0 \sim \nabla_\omega[(-1)^{r-p-1} d\log(p+1, i_{p+1}) \wedge \cdots \wedge d\log(r-1, i_{r-1}) \wedge d\log(r+1, i_{r+1}) \wedge \cdots \wedge d\log(N, i_N)] = \sum_{\substack{s=1 \\ s\neq r}}^{N} \lambda_{r,s} \langle i_{p+1}, \cdots, i_{r-1}, s, i_{r+1}, \cdots, i_N \rangle = \sum_{s<r} + \sum_{s>r}. \tag{1.3}$$

Each form in the second member of the right hand side is not admissible, but can be written as a linear representation of admissible ones. In fact if we multiply by $1/(r, s)$ for $r<s$ the admissible fraction

$$\frac{1}{(r+1, i_{r+1}) \cdots (s, i_s)} \qquad \text{namely for } r+1 \geqq i_{r+1}, \cdots, s \geqq i_s,$$

then by partial fraction

$$\frac{1}{(r, s)(r+1, i_{r+1}) \cdots (s, i_s)} = \frac{1}{(r, i_s)} \left\{ \frac{1}{(r+1, i_{r+1}) \cdots (s, i_s)} - \frac{1}{(r+1, i_{r+1}) \cdots (s-1, i_{s-1})(s, r)} \right\}. \tag{1.4}$$

Both sequences $\{i_{r+1}, \cdots, i_s\}$ and $\{i_{r+1}, \cdots, i_{s-1}, r\}$ are admissible. Being (r, s) replaced by (r, i_s) and by induction hypothesis in s, each member can be written as a linear combination of admissible fractions. We repeat this until the sequence $s \to i_s \to \cdots$ arrives at an argument smaller than or equal to r. This procedure is nothing else than the one explained in (1.2). Lemma 1.2 is thus proved.

COROLLARY. *If $\lambda_{p+1,1} \cdots \lambda_{N,1} \neq 0$, then the differential form $\langle i_{p+1}, \cdots, i_N \rangle$ for some $i_\nu = 1$ can be described as a linear combination of the ones for $i_\nu > 1$. This number is just equal to $(p-1)p \cdots (N-2)$, so that we can choose as a basis of $H^{N-p}(X, \nabla_\omega)$ the forms $\langle i_{p+1}, \cdots, i_N \rangle$ for $i_\nu > 1$.*

It has been stated in [**A1**] and [**A5**] that for an admissible sequence $I = \{i_{p+1}, \cdots, i_N\}$ the integral $\widetilde{\langle i_{p+1}, \cdots, i_N \rangle}$ satisfies the logarithmic Gauss-Manin connection

$$d\langle \tilde{I} \rangle = \sum_{1 \leq j < k \leq p} \sum_{J \text{ admissible}} d\log(x_j - x_k)\, U_{j,k}^{(p)} \binom{J}{I} \langle \tilde{J} \rangle \tag{1.5}$$

where each constant matrix $U_{j,k}^{(p)} = U_{k,j}^{(p)}$ represents a linear endomorphism in $\boldsymbol{C}^p \otimes \cdots \otimes \boldsymbol{C}^{N-1}$. The proof is straightforward by induction in $N-p$ by using the generalized Pochhammer differential equation given in [**A1**]. So we omit it. The formula (1.5) is also a degenerate case of the one (A, 5) proved in [**A4**].

Furthermore we put $U_{j,k}^{(N)}$ to be $\lambda_{j,k}$. The matrices $U_{j,k}^{(p)}$ are determined by recursive formulae in the following manner. We fix p and similarly define $U_{j,k}^{(r)}$ for $p\leqq r\leqq N$. Then

LEMMA 1.3. *Each $U_{j,k}^{(r)}$ is described as the matrix $((u_{l,m}))_{1\leqq l,m\leqq r}$ where $u_{l,m}$ represents a linear endomorphism in $C^{r+1}\otimes\cdots\otimes C^{N-1}$. Each matrix $u_{l,m}$ is equal to*

$$
\begin{aligned}
u_{l,l} &= U_{j,k}^{(r+1)} \qquad \text{if } l\neq j,k\\
u_{j,j} &= U_{j,k}^{(r+1)}+U_{k,r+1}^{(r+1)}\\
u_{j,k} &= \qquad -U_{j,r+1}^{(r+1)}\\
u_{k,j} &= \qquad -U_{k,r+1}^{(r+1)}\\
u_{k,k} &= U_{j,k}^{(r+1)}+U_{j,r+1}^{(r+1)}\\
u_{l,m} &= 0 \qquad \text{otherwise.}
\end{aligned}
\tag{1.6}
$$

It is important to remark that *this formula corresponds to the infinitesimal version of the pure braid transformation around the locus $x_j=x_k$.* $U_{j,k}^{(r)}$ satisfy the well-known relation of Lie bracket defining holonomy Lie algebra which is also called "classical Yang-Baxeter relation":

$$
\begin{aligned}
&[U_{i,k}^{(r)}+U_{j,k}^{(r)},\ U_{i,j}^{(r)}]=0\\
&[U_{i,j}^{(r)}, U_{k,l}^{(r)}]=0
\end{aligned}
\tag{1.7}
$$

for 4 different indices $i,j,k,l\leqq p$ (see [**A1**] or [**K1**]).

We sketch the way of proof of Lemma 1.3 which has been developed in [**A1**]. The following is elementary.

LEMMA 1.4. *Let $U_1,\cdots,U_m$ be matrices in $\mathfrak{gl}_n(C)$. Let the Fuchsian differential equation of order n*

$$
\frac{dy}{dx}=\sum_{i=1}^{m} y\frac{U_i}{(x-a_i)}
\tag{1.8}
$$

on P^1 with regular singularities $x=a_1,\cdots,a_m,\infty$ be given. We denote by Y and ω the fundamental solutions of matrices and $Y^{-1}dY$ respectively. We put $U_0=-\sum_{i=1}^{m}U_i$. Assume the following:

(C.4) *Each U_i, $i=0,1,\cdots,n$, has no eigenvalues $1,2,3,\cdots$.*

Then the cohomology $H^1(C-\{a_1,\cdots,a_m\},\nabla_\omega)$ is spanned by C^n-valued logarithmic forms $d\log(x-a_j)\otimes e_\mu$, $1\leqq\mu\leqq n$, where $\{e_\mu\}$ denotes a basis of C^n. The fundamental relation is given by

$$\sum_{j=1}^{m} d\log(x-a_j)\otimes e_\mu U_j \sim 0. \tag{1.9}$$

Hence we have the isomorphism

$$H^1(\boldsymbol{C}-\{a_1, \cdots, a_m\}, \nabla_\omega) = \boldsymbol{C}^{nm}/(U_1, \cdots, U_m)\cdot \boldsymbol{C}^n. \tag{1.10}$$

PROOF. See [**A1**] and [**A5**].

COROLLARY. *If*

$$\bigcap_{i=1}^{m} \operatorname{Ker} U_i = (0) \tag{C.5}$$

then

$$\operatorname{rank} H^1(\boldsymbol{C}-\{a, \cdots, a\}, \nabla_\omega) = (n-1)m.$$

Let ρ be the monodromic representation of $\pi_1(\boldsymbol{C}-\{a_1, \cdots, a_m\}, b)$ with a base point b given by Y. Any twisted cycle γ for the integral $\int_\gamma Y\varphi$ is represented by a linear combination of m loops $L_1, \cdots, L_m$ with the base point b encircling a_j respectively: $\gamma=\sum_{j=1}^{m} v_j\otimes L_j$ for $v_j\in \boldsymbol{C}^n$, such that

$$\sum_{j=1}^{m} v_j\otimes(\rho(L_j)-1) = 0. \tag{1.11}$$

If all the eigenvalues of U_j, $j=1, 2, \cdots, m$ are greater than -1, γ can also be chosen in the form $\sum_{j=1}^{m-1} v'_j\otimes\overline{a_j a_{j+1}}$ using the segments $\overline{a_j a_{j+1}}$ connecting a_j and a_{j+1}, such that

$$v_j = (v'_{j-1}-v'_j)(1-\rho(L_j))^{-1}. \tag{1.12}$$

A similar approach has been used in [**D**] for the uniformization problem associated to Pochhammer integrals.

We now consider the sequence of integrals $\{F_r\}$ for $p\leqq r<N$:

$$F_r = \int \Phi \frac{dx_{r+1} \cdots dx_N}{(r+1, i_{r+1}) \cdots (N, i_N)} \tag{1.13}$$

and $F_N=\Phi$. The F_r satisfies as a function of x_r the ordinary differential equation of Fuchsian type of order $r \cdots (N-1)$:

$$\frac{dy}{dx_r} = \sum_{j=1}^{r-1} y \frac{U_{r,j}^{(r)}}{x_r-x_j}. \tag{1.14}$$

These are related to one another by the recurrence formulae:

$$F_r = \int F_{r+1} \frac{dx_{r+1}}{(r+1, i_{r+1})} \tag{1.15}$$

for some $i_{r+1} \leqq r$. If all $\lambda_{j,k} > 0$, then successive applications of Lemma 1.3 over each variable $x_N, \cdots, x_{r+1}$ enable us to construct the cycles $\Delta(j_{r+1}, \cdots, j_N)$:

$$\begin{aligned} & x_{j_{r+1}-1} \leqq x_{r+1} \leqq x_{j_{r+1}} \\ & \cdots\cdots\cdots\cdots \\ & x_{j_N-1} \leqq x_N \leqq x_{j_N} \end{aligned} \tag{1.16}$$

for $j_{r+1} \leqq r, \cdots, j_N \leqq N-1$.

LEMMA 1.5. *If F_{r+1} satisfies the differential equation of* (1.14), *p being replaced by $r+1$, then F_r satisfies the same one, p being replaced by r.*

PROOF. This is proved by partial fractions, using the rule of exchange of derivation and integration

$$\int \frac{\partial}{\partial x_k}\left(F_{r+1} \frac{dx_{r+1}}{(r+1, j)}\right) = \frac{\partial}{\partial x_k} \int F_{r+1} \frac{dx_{r+1}}{(r+1, j)} \tag{1.17}$$

for $k \leqq r$ and $j \geqq r+1$.

Now Lemma 1.3 immediately follows from Proposition 3 in [**A1**] using Lemmas 1.4 and 1.5.

LEMMA 1.6. *Let $l \in \boldsymbol{Z}^+ = \{0, 1, 2, \cdots\}$ be given. Take an h such that $1 \leqq h \leqq p$. Suppose for arbitrary $\nu_1, \cdots, \nu_s$, $1 \leqq s \leqq N-p$, such that $1 \leqq \nu_1 < \cdots < \nu_s \leqq N-p$,*

$$\sum_{j=1}^{s} \lambda_{h,\nu_j} + \sum_{1 \leqq j < k \leqq s} \lambda_{p+\nu_j, p+\nu_k} \neq l \tag{1.18}$$

hold. Then for an arbitrary r such that $p+1 \leqq r \leqq N$, none of the matrices of order $r(r+1) \cdots (N-1)$:

$$\sum_{j=1}^{s} U_{h, p+\nu_j}^{(r)} + \sum_{1 \leqq j < k \leqq s} U_{p+\nu_j, p+\nu_k}^{(r)} \tag{1.19}$$

have the eigenvalue l, for an arbitrary sequence $p+1 \leqq \nu_1 < \cdots < \nu_s \leqq r$.

PROOF. We prove this by induction in decreasing r from N. For $r=N$ it is trivial by assumption. Assume that it holds for $r+1$. Let $w = {}^t(w_1, \cdots, w_r) \in \boldsymbol{C}^r \otimes \cdots \otimes \boldsymbol{C}^{N-1}$ for $w_j \in \boldsymbol{C}^{r+1} \otimes \cdots \otimes \boldsymbol{C}^{N-1}$ be an eigenvector:

$$\left(\sum_{j=1}^{s} U_{h, p+\nu_j}^{(r)} + \sum_{1 \leqq j < k \leqq s} U_{p+\nu_j, p+\nu_k}^{(r)}\right) w = l w. \tag{1.20}$$

Then comparing the g-th component of both sides for $1 \leqq g \leqq p$, we have

$$(1.21)\qquad \Big(\sum_{j=1}^{s} U_{h,p+\nu_j}^{(r+1)} + \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r+1)}\Big) w_g = l w_g \qquad \text{for } h\neq g,$$

and similarly

$$(1.22)\qquad \sum_{j=1}^{s} \{(U_{h,p+\nu_j}^{(r+1)} + U_{p+\nu_j,r+1}^{(r+1)}) w_h - U_{h,r+1}^{(r+1)} w_{p+\nu_j}\} + \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r+1)} w_h = l w_h,$$

$$(1.23)\qquad \Big(\sum_{j=1}^{s} U_{h,p+\nu_j}^{(r+1)} + \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r+1)}\Big) w_{p+\mu} = l w_{p+\mu} \qquad \text{for } \mu\neq\nu_1,\cdots,\nu_s.$$

$$(1.24)\qquad \Big(\sum_{j=1}^{s} U_{h,p+\nu_j}^{(r+1)} + U_{h,r+1}^{(r+1)} + \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r+1)} + \sum_{\substack{j=1\\ j\neq f}}^{s} U_{p+\nu_f,r+1}^{(r+1)}\Big) w_{p+\nu_f}$$
$$- U_{p+\nu_f,r+1}^{(r+1)} w_h - \sum_{\substack{j=1\\ j\neq f}}^{s} U_{p+\nu_k,r+1}^{(r+1)} w_{p+\nu_j} = l w_{p+\nu_f}$$

for $1\leqq f\leqq s$. By induction hypothesis, (1.21) and (1.23) imply that $w_g = w_{p+\mu} = 0$ for $g\neq h$ and $\mu\neq\nu_1,\cdots,\nu_s$. On the other hand, summing up (1.22) and (1.24) we have

$$(1.25)\qquad \Big(\sum_{j=1}^{s} U_{h,p+\nu_j}^{(r+1)} + \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r+1)}\Big)\Big(w_h + \sum_{j=1}^{s} w_{p+\nu_j}\Big) = l\Big(w_h + \sum_{j=1}^{s} w_{p+\nu_j}\Big).$$

By induction hypothesis this means

$$(1.26)\qquad w_h + \sum_{j=1}^{s} w_{p+\nu_j} = 0.$$

Substituting this w_h into (1.24) we have

$$(1.27)\qquad \Big(\sum_{j=1}^{s+1} U_{h,p+\nu_j}^{(r+1)} + \sum_{1\le j<k\le s+1} U_{p+\nu_j,p+\nu_k}^{(r+1)}\Big) w_{p+\nu_f} = l w_{p+\nu_f}$$

if we put $\nu_{s+1} = r+1$. Again by induction hypothesis we conclude that $w_{p+f}=0$, whence $w_h=0$. Lemma has thus been proved. In the same way as above one can prove

LEMMA 1.7. *Let $l\in \boldsymbol{Z}^+$ be arbitrarily given. Suppose for an arbitrary sequence $p+1\leqq\nu_1<\cdots<\nu_s\leqq N-p$*

$$(1.28)\qquad -\sum_{h=1}^{p}\sum_{j=1}^{s}\lambda_{h,p+\nu_j} - \sum_{1\le j<k\le s}\lambda_{p+\nu_j,p+\nu_k}$$

are all different from l. Then none of the matrices

$$(1.29)\qquad -\sum_{h=1}^{p}\sum_{j=1}^{s} U_{h,p+\nu_j}^{(r)} - \sum_{1\le j<k\le s} U_{p+\nu_j,p+\nu_k}^{(r)}$$

for $p+1\leqq r\leqq N$ *have the eigenvalue* l, *where* $\nu_1, \cdots, \nu_s$ *denotes an arbitrary sequence such that* $p+1\leqq\nu_1<\cdots<\nu_s\leqq r$.

COROLLARY. *If* (C.1) (*and* (C.2)′) *holds, then for each* $j\leqq p$, $p+1\leqq r\leqq N$, *none of the matrices* (1.19) *and* (1.29) *have the eigenvalues* $1, 2, 3, \cdots$ ($0, 1, 2, \cdots$ *respectively*).

LEMMA 1.8.

$$\bigcap_{j=1}^{r-1}\operatorname{Ker}U_{j,r}^{(r)}=(0).$$

PROOF. This follows from Corollary of Lemma 1.7. In fact there we have only to put $s=1$ and $\nu_1=r$.

We take and fix r for $p+1\leqq r\leqq N$. Lemmas 1.6-1.8 imply, in particular, that $U_{j,r}^{(r)}$ for $1\leqq j\leqq r-1$ and $-\sum_{j=1}^{r-1}U_{j,r}^{(r)}$ satisfy (C.4) and (C.5) respectively. This fact therefore makes us possible to apply successively Lemma 1.4 for the corresponding differential equations (1.8), starting from $r=N$ to $r=p+1$. This will be done in the final section.

2. Symmetric case.

We specialize $\lambda=(\lambda_{i,j})$ as in (C.3). In this section we assume that all the domains of integration G are invariant under the action of Γ. As an immediate consequence of it, we have

LEMMA 2.1. *If for some* $\sigma\in\Gamma$, *the equality holds:*

$$\langle i_{\sigma(p+1)}, \cdots, i_{\sigma(N)}\rangle=-\langle i_{p+1}, \cdots, i_N\rangle \tag{2.1}$$

then $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle=0$.

LEMMA 2.2. *The integral* $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle$ *for some* $i_{p+\nu}\geqq p+1$ *is a linear combination of the ones such that all* $i_{p+\nu}\leqq p$.

This lemma follows from Lemma 2.5. To prove Lemma 2.5 we begin by proving

LEMMA 2.3. *Let a sequence* $p+\nu_1<p+\nu_2<\cdots<p+\nu_s$ *be the* $(p+\nu_0)$-*cluster of the sequence* $\{i_{p+1}, \cdots, i_N\}$

$$K_{p+\nu_0}\ :\ p+\nu_0\ \begin{array}{l}\longleftarrow p+\nu_1\\ \longleftarrow p+\nu_2\\ \qquad\vdots\\ \longleftarrow p+\nu_s\end{array}$$

such that $i_{p+\nu_t}=p+\nu_0$, $1\leqq t\leqq s$. *Then*

$$\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle = \frac{1}{s+1}\langle i_{p+1}, \cdots, i_{p+\nu_0}, \widetilde{\cdots}, i_{p+\nu_0}, \cdots, i_N\rangle \tag{2.2}$$

where in the right hand side each $i_{p+\nu_t}$ *is replaced by* $i_{p+\nu_0}$ *for* $1\leqq t\leqq s$.

PROOF. During the proof of this lemma, not losing any generality, we may suppose that $\nu_t=p+t+1$ and $p+s=N$. Then $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle$ is simply equal to $\langle i_{p+1}, \underbrace{p+\widetilde{1, \cdots}, p+1}_{s}\rangle$ for $i_{p+1}\leqq p$. We prove (2.2) by induction in s. For $s=0$ we have nothing to prove. We assume that it holds true for $s-1$. Then by partial fraction and argument of symmetry with respect to transposition $p+1$ and $p+2$, we have

$$\begin{aligned}&\langle i_{p+1}, p+\widetilde{1, \cdots}, p+1\rangle \\ &= \langle i_{p+1}, i_{p+1}, \widetilde{p+1}, \cdots, p+1\rangle - \langle i_{p+1}, p+1, \widetilde{p+2}, \cdots, p+2\rangle.\end{aligned} \tag{2.3}$$

By induction hypothesis, the right hand side is equal to

$$\frac{1}{s}(\langle i_{p+1}, \widetilde{\cdots}, i_{p+1}\rangle - \langle i_{p+1}, p+\widetilde{1, \cdots}, p+1\rangle). \tag{2.4}$$

By solving this equation for $\langle i_{p+1}, p+\widetilde{1, \cdots}, p+1\rangle$ we have (2.3).

In the same way we can prove

LEMMA 2.4. *Let a sequence* $p+\nu_0<p+\nu_1<\cdots<p+\nu_s$ *be the* $(p+\nu_1)$*-cluster of* $\{i_{p+1}, \cdots, i_N\}$ *such that* $i_{p+\nu_t}=p+\nu_{t-1}$, $1\leqq t\leqq s$.

$$K_{p+\nu_0} : p+\nu_0 \longleftarrow p+\nu_1 \longleftarrow \cdots \longleftarrow p+\nu_s. \tag{2.5}$$

Then

$$\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle = \frac{1}{(s+1)!}\langle i_{p+1}, \cdots, \widetilde{i_{p+\nu_0}}, \cdots, i_{p+\nu_0}, \cdots\rangle \tag{2.6}$$

where in the right hand side all $i_{p+\nu_t}$ *are replaced by* $i_{p+\nu_0}$.

Successive applications of these two lemmas show

LEMMA 2.5. *Let a sequence* $p+\nu_0<\cdots<p+\nu_s$ *be a general* $(p+\nu_0)$*-cluster* $K_{p+\nu_0}$. *Then*

$$\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle = \frac{1}{|K_{p+\nu_t}|}\langle i_{p+1}, \cdots, i_{p+\nu_0}, \widetilde{\cdots}, i_{p+\nu_0}, \cdots, i_N\rangle \tag{2.7}$$

where in the right hand side each argument $i_{p+\nu_t}$ *in* $K_{p+\nu_t}$ *is replaced by* $i_{p+\nu_0}$.

3. Proof of the theorems.

We begin by the

PROOF OF THEOREM 2. By applying the formulae (1.5) and (1.6) for $r=p, p+1, \cdots$ we can compute the differential equations satisfied by $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle$ in successive manner. We denote by $U^{(r)}_{i,j}\begin{pmatrix} j_{r+1}, \cdots, j_N \\ i_{r+1}, \cdots, i_N\end{pmatrix}$ for $1 \leqq i, j \leqq r-1$ the matrix elements of the endomorphism $U^{(r)}_{i,j}$ on $C^r \otimes \cdots \otimes C^N$ with respect to the basis $\langle i_{p+1}, \widetilde{\cdots}, i_n\rangle$. The total variation with respect to the variables $x_1, \cdots, x_p$ can then be expressed as:

$$
\begin{aligned}
(3.1) \qquad & d\langle \widetilde{i_{p+1}, \cdots}, i_N\rangle \\
&= \sum_{\substack{1 \leq k < h \leq p \\ k, h \neq i_{p+1}}} d\log(k, h)\, U^{(p+1)}_{k,h}\begin{pmatrix} j_{p+2}, \cdots, j_N \\ i_{p+2}, \cdots, i_N\end{pmatrix} \cdot \langle i_{p+1}, \widetilde{j_{p+2}, \cdots}, j_N\rangle \\
&\quad + \sum_{\substack{h=1 \\ h \neq i_{p+1}}}^{p} d\log(i_{p+1}, h)\, U^{(p+1)}_{h, p+1}\begin{pmatrix} j_{p+2}, \cdots, j_N \\ i_{p+2}, \cdots, i_N\end{pmatrix} \cdot \langle i_{p+1}, \widetilde{j_{p+2}, \cdots}, j_N\rangle \\
&\quad - \sum_{\substack{h=1 \\ h \neq i_{p+1}}}^{p} d\log(i_{p+1}, h)\, U^{(p+1)}_{h, p+1}\begin{pmatrix} j_{p+2}, \cdots, j_N \\ i_{p+2}, \cdots, i_N\end{pmatrix} \cdot \langle h, \widetilde{j_{p+2}, \cdots}, j_N\rangle
\end{aligned}
$$

where owing to (1.6), $U^{(p+1)}_{j,k}$ itself can be described by $U^{(p+2)}_{l,m}$. Hence the right hand side of (3.1) is equal to:

$$
\begin{aligned}
(3.2) \qquad & \sum_{\substack{1 \leq k < h \leq p \\ k, h \neq i_{p+1}, i_{p+2}}} d\log(k, h) U^{(p+2)}_{k,h} \langle i_{p+1}, \widetilde{i_{p+2}, \cdots}\rangle \\
&+ \sum_{\substack{1 \leq i'_{p+1} \leq p \\ i'_{p+1} \neq i_{p+1}}} d\log(i_{p+1}, i'_{p+1})\, U^{(p+2)}_{i'_{p+1}, p+1} \langle i_{p+1}, \widetilde{i_{p+2}, \cdots}\rangle \\
&+ \sum_{\substack{1 \leq i'_{p+2} \leq p \\ i'_{p+2} \neq i_{p+2}}} d\log(i_{p+2}, i'_{p+2})\, U^{(p+2)}_{i'_{p+2}, p+2} \langle i_{p+1}, \widetilde{i_{p+2}, \cdots}\rangle \\
&+ \sum_{\substack{1 \leq i'_{p+1} \leq p \\ i'_{p+1} \neq i_{p+1}}} d\log(i_{p+1}, i'_{p+1})\, U^{(p+2)}_{i'_{p+2}, p+2} \left\langle \left\{ \begin{matrix} i_{p+1} \\ i'_{p+1} \end{matrix} \right\}, \widetilde{\left\{ \begin{matrix} i_{p+2} \\ i'_{p+2} \end{matrix} \right\}, \cdots} \right\rangle
\end{aligned}
$$

where in the last term $p+1=\max(i_{p+2}, i'_{p+2})$ and $i'_{p+1}=\min(i_{p+2}, i'_{p+2})$. In the same manner $U^{(p+2)}_{j,k}$ is expressed in terms of $U^{(p+3)}_{j,k}$ and so on. Finally the formula (0.3) is obtained.

Proof of Proposition 2 is immediate, by means of Lemma 2.2 and the equality (0.5). So we omit it.

We now come to Theorem 3.

PROOF OF THEOREM 3. We apply the formula (0.3) for $\langle i_{p+1}, \widetilde{\cdots}, i_N\rangle = \langle 1^{\nu_1}, \widetilde{\cdots}, p^{\nu_p}\rangle$. Then since $\lambda_{j,k}=0$ for $j, k \leqq p$, the second member of the right hand side of (0.3) vanishes. On the other hand, (0.4) shows that $\{k, h\} \ni i_{p+\nu_1}, i'_{p+\nu_1}$ and $\{k, h, p+\nu_1\} \ni i_{p+\nu_2}$. Since $i_{p+\nu_2} \leqq p$, and $i_{p+\nu_2}$ is different from $i_{p+\nu_1}$, $i_{p+\nu_2}$ must be equal to $i'_{p+\nu_1}$. Hence the set $\{i_{p+\nu_1}, i_{p+\nu_2}\}$ coincides with $\{k, h\}$, so that if $s \geqq 2$ then the complement $\{k, h, p+\nu_1, \cdots, p+\nu_s\} - \{i_{p+\nu_1}, \cdots, i_{p+\nu_s}\}$ has no element j such that $j \leqq p$. This means that (0.3) is simplified as follows:

$$
(3.3) \quad \sum_{\nu_1=1}^{N-p} \sum_{i'_{p+\nu_1}} d\log(i_{p+\nu_1}, i'_{p+\nu_1}) \lambda_{p+\nu_1, i'_{p+\nu_1}} \left\langle i_{p+1}, \cdots, \widetilde{\begin{Bmatrix} i_{p+\nu_1} \\ i'_{p+\nu_1} \end{Bmatrix}}, \cdots, i_N \right\rangle
$$
$$
+ \sum_{1 \leqq \nu_1 < \nu_2 \leqq N-p,} \sum_{i'_{p+\nu_1}, i'_{p+\nu_2}} d\log(i_{p+\nu_1}, i'_{p+\nu_2}) \lambda_{p+\nu_1, i'_{p+\nu_1}} \left\langle i_{p+1}, \cdots, \begin{Bmatrix} i_{p+\nu_1} \\ i'_{p+\nu_1} \end{Bmatrix}, \widetilde{\cdots}, \begin{Bmatrix} i_{p+\nu_2} \\ i'_{p+\nu_2} \end{Bmatrix}, \cdots, i_N \right\rangle
$$

where in the second part $i'_{p+\nu_2} = p+\nu_1$ and $i_{p+\nu_2} = i'_{p+\nu_1}$. On the other hand, Lemma 2.3 shows

$$
\begin{aligned}
(3.4) \quad & \left\langle i_{p+1}, \cdots, \begin{Bmatrix} i_{p+\nu_1} \\ i'_{p+\nu_1} \end{Bmatrix}, \widetilde{\cdots}, \begin{Bmatrix} i_{p+\nu_2} \\ i'_{p+\nu_2} \end{Bmatrix}, \cdots, i_N \right\rangle \\
&= \langle i_{p+1}, \cdots, i_{p+\nu_1}, \widetilde{\cdots}, i_{p+\nu_2}, \cdots \rangle - \langle i_{p+1}, \cdots, i_{p+\nu_2}, \widetilde{\cdots}, i_{p+\nu_2}, \cdots \rangle \\
&\quad - \langle i_{p+1}, \cdots, i_{p+\nu_1}, \widetilde{\cdots}, p+\nu_1, \cdots \rangle + \langle i_{p+1}, \cdots, i_{p+\nu_2}, \widetilde{\cdots}, p+\nu_1, \cdots \rangle \\
&= \langle i_{p+1}, \cdots, i_{p+\nu_1}, \widetilde{\cdots}, i_{p+\nu_2}, \cdots \rangle \\
&\quad - \frac{1}{2} \langle i_{p+1}, \cdots, i_{p+\nu_1}, \widetilde{\cdots}, i_{p+\nu_1}, \cdots \rangle - \frac{1}{2} \langle i_{p+1}, \cdots, i_{p+\nu_2}, \widetilde{\cdots}, i_{p+\nu_2}, \cdots \rangle .
\end{aligned}
$$

(0.7) is thus obtained.

PROOF OF PROPOSITION 3. Assume $\nu_1 \geqq 1$. Let $\langle i_{p+1}, \cdots, i_N \rangle$ be equal to $\langle 1^{\nu_1}, \cdots, p^{\nu_p} \rangle$ with $i_{p+1} = 1$. For all $i_\nu \leqq p$, $p+1 \leqq \nu < N$ and $p+2 \leqq r \leqq N$, by Stokes formula and partial fractions and the equalities (2.1) and (2.2), we have

$$
\begin{aligned}
(3.5) \quad 0 &\sim \nabla_\omega \left\{ \frac{(p+1, 1) d(p+2, i_{p+2}) \wedge \cdots \wedge d(N, i_N)}{(r, i_r)(r+1, i_{r+1}) \cdots (N, i_N)} \right\} \\
&= \left[1 + \sum_{j=1}^{p} \lambda'_j + \frac{\lambda}{2}(r-p-2) + \lambda(N-r+1) \right] \langle i_r, \cdots, i_N \rangle \\
&\quad + \sum_{j=2}^{p} (j, 1) \lambda'_j \langle j, i_r, \cdots, i_N \rangle \\
&\quad + \frac{\lambda}{2} \sum_{s=r}^{N} (i_s, 1) \langle i_s, i_r, \cdots, i_{r-1}, i_s, i_{s+1}, \cdots, i_N \rangle
\end{aligned}
$$

which implies the proposition in terms of $\nu_1, \cdots, \nu_p$ such that $|\nu| = N-r+2$.

Theorem 3 follows immediately from Proposition 3.

Errata in [A5].

Each term in (2.8), (5.10), (5.16) and (5.26) should be read as follows respectively:

$$\varphi(\partial_\kappa(i_0, I)) \longrightarrow \lambda_{i_0}\varphi(\partial_\kappa(i_0, I))$$

where the summation should also be taken over i_0.

$$x_1^{\nu_1}\cdots x_n^{\nu_n} \longrightarrow \frac{x_1^{\nu_1}\cdots x_n^{\nu_n}}{\nu_1!\cdots\nu_n!}.$$

$$\begin{matrix}\lambda_\sigma-1, \lambda'_\sigma-1\\ \lambda_\sigma-1, \lambda'_\sigma-1\end{matrix} \longrightarrow \begin{matrix}\lambda_\sigma-1, \lambda_\sigma-1\\ \lambda'_\sigma-1, \lambda'_\sigma-1\end{matrix}.$$

$$U^{(p+1)}_{\sigma, p+1}+\beta\mathbf{1}_{N_{p+1}} \longrightarrow U^{(p+1)}_{\sigma, p+1}.$$

$$U^{(p+1)}_{\tau, p+1}+\beta\mathbf{1}_{N_{p+1}} \longrightarrow U^{(p+1)}_{\tau, p+1}.$$

respectively.

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